TOPOLOGICAL AND DYNAMICAL ASPECTS OF SOME SPECTRAL INVARIANTS OF CONTACT MANIFOLDS WITH A CIRCLE ACTION

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Abstract. We study analytic torsion and eta like invariants on contact manifolds admitting a CR structure invariant under a transverse circle action, and equipped with a unitary representation. We show that, when defined using the spectrum of relevant operators arising in this geometry, the spectral series involved can been interpreted in their whole, both from a topological viewpoint, and as purely dynamical functions of the Reeb flow.

1. Introduction

This paper deals with the study of some spectral series associated to geometric invariants on particular compact contact CR manifolds M, those who admit a transverse locally free circle action preserving the structure.

That means the generator T of this action is the Reeb field of an invariant contact form θ and preserves an integrable complex structure J on $H = \ker\theta$. In that case, the orbifold $N = M/\mathbb{S}^1$ appears to be a Kähler V-manifold in the sense of Satake [14]. It is a stratified space endowed with a smooth open dense Kähler structure, corresponding to generically free orbits, and singular strata, corresponding to exceptional fibers with non trivial isotopy groups.

Independent of this circle action, we also equip M with a unitary representation $\rho : \pi_1(M) \to U(d)$. This broadens the framework to twisted spectral invariants associated to the flat bundles of these representations, and provides us with dynamical data using the holonomies induced by the closed orbits of the Reeb flow.

1.1. **Around the contact analytic torsion.**

We will be concerned with spectral series associated to two typical spectral invariants. The first one is the 'contact' analytic torsion, as defined in [20]. This analytic determinant is associated to the contact de Rham complex $(\mathcal{E}^*, d_Q)$, a hypoelliptic complex, homotopic to the usual Hodge-de Rham one, but benefiting from better contact homogeneity when rescaling θ in $k\theta$. See Section 2.1 for a presentation of this construction. This resolution of constants starts on functions with $d_Q = d_H$, the usual differential, but restricted to the horizontal vectors H in the contact distribution. The price for this however, is the appearance of a second order differential $D = d_Q : \mathcal{E}^n \to \mathcal{E}^{n+1}$ in 'middle degree', with $\dim M = 2n + 1$. In order to preserve homogeneity, this in turns leads to using *fourth-order* Laplacians $\boldsymbol{\Delta}_Q$ in all degrees;

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see Section 2.3. These self-adjoint operators are hypoelliptic. They possess discrete spectrum and smooth heat kernels on compact contact manifolds.

This allows to consider our first spectral series which is related to the analytic torsion of the contact complex. In the Riemannian setting, the analytic torsion was introduced by Ray and Singer in [17] as an infinite dimensional analogue of the Reidemeister-Franz torsion of a finite dimensional complexes. It is defined by an appropriate combination of analytic determinants of the Hodge-de Rham Laplacians using their zeta functions.

In [20], the authors proposed to adapt the construction on contact manifolds. Starting from heat kernels, one considers for $t > 0$

$$\vartheta(t) = \sum_{k=0}^{n} (-1)^k (n+1-k) \operatorname{Tr}(e^{-t\boldsymbol{\Delta}_Q} \mid \mathcal{E}^k). \tag{1}$$

This particular combination leads to the definition of the analytic torsion of the contact complex. Briefly, taking Mellin transform leads to zeta functions

$$\zeta(\boldsymbol{\Delta}_Q)(s) = \operatorname{Tr}^*(\boldsymbol{\Delta}_Q^{-s}) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \operatorname{Tr}^*(e^{-t\boldsymbol{\Delta}_Q}) t^{s-1} dt\,,$$

where Tr^* denotes the trace over the non zero spectrum of $\boldsymbol{\Delta}_Q$. These functions are well defined for $\operatorname{Re}(s)$ large and meromorphic with (at worst) simple poles occurring at $s \in S = \{\frac{n+1-j}{2} \mid j \in \mathbb{N}\} \setminus (-\mathbb{N})$; see e.g. [20, Section 3.1] for references. Following [20], we define then the contact torsion zeta function

$$Z(s) = \sum_{k=0}^{n} (-1)^k (n+1-k) \zeta(\boldsymbol{\Delta}_Q \mid \mathcal{E}^k)(s)\,. \tag{2}$$

Then, the analytic torsion of the contact complex is defined by

$$T_Q(M,\rho) = \exp\big(-\frac{1}{2} Z'(0)\big)\,. \tag{3}$$

It is shown in [20] that it coincides with Ray-Singer analytic torsion on three dimensional CR Seifert manifolds. An explicit formula is given in this case. On general contact manifolds, this particular combination of zeta functions is, up to a multiplicative factor, the only one such that the variation of $Z'(0)$ is given by integrals of local terms, up to cohomological factors; see [20, Remark 3.6]. More recently, Albin and Quan proved in [1] that the logarithms of the Riemannian and contact analytic torsions differ by integral of (unknown) local terms.

Our first main results relate this ϑ series to three other expressions, one using topological data, another to an explicit geometric sum over the orbifold N, and the last one to dynamical properties of the Reeb flow.

1.2. The heat analytic torsion as an index series.

We summarise the main steps toward the topological expression. As we shall see, it turns out that the spectrum in the combination of trace in ϑ is highly symmetric. Much of the plus and minus contributions cancel each other out, except on a simple residual spectrum we describe.

Let $\Omega^* H$ denote the bundle of horizontal forms on M with coefficients in V, the flat bundle associated to the representation $\rho : \pi_1(M) \to U(d)$, and consider the horizontal part of the differential, d_H, acting on $\Omega^* H$, its formal adjoint δ_H and the operator

$$D_H = d_H + \delta_H \,.$$

It exchanges $\Omega^{ev} H$ and $\Omega^{odd} H$. We denote $\ker D_H$ by $\mathcal{H}$. We will show that

$$\vartheta(t) = \mathrm{Tr}(e^{tT^2} \mid \mathcal{H}^{ev}) - \mathrm{Tr}(e^{tT^2} \mid \mathcal{H}^{odd}).$$

As we shall see in Section 3.1, the space $\mathcal{H}$ is infinite dimensional, and contains forms built using CR (holomorphic) functions and their conjugate. We will show however that it can be split into finite dimensional pieces using a twisted circle action on V and the spectrum of the Reeb flow T, namely

$$V = \bigoplus_{\lambda \in \mathrm{Spec}(iT)} V_\lambda \,, \tag{4}$$

where each component V_λ can be seen as a V-bundle over the orbifold $N = M/\mathbb{S}^1$. Then the spectral ϑ finally reduces to a renormalised index series

$$\vartheta(t) = \vartheta^{top}(t) = \sum_{\lambda \in \mathrm{Spec}(iT)} \mathrm{ind}(D_H^{ev} \mid V_\lambda) e^{-t\lambda^2} \,, \tag{5}$$

with $D_H^{ev} = D_H : \Omega^{ev} H \to \Omega^{odd} H$. The index terms can be explicitly computed using Kawasaki's index formula for V-manifold; see Section 3.2. This will link ϑ to two other expressions, one using explicit geometric data over N and the other as a dynamical series over all closed orbits.

1.3. Geometric and dynamical viewpoints on the heat analytic torsion.

We now turn to the geometric and dynamical aspects of the series ϑ. We first would like to express it using data over the orbifold N. This stratified space splits into connected components of its strata associated to various isotopic groups of the circle action. We note $N = \bigsqcup_{i \in \mathcal{S}} N_i$ this decomposition. Open strata correspond to generic closed primitive orbits with trivial isotopy, while singular points are associated to exceptional orbits of non trivial one. We pick one primitive orbit f_i in each N_i.

Recall that from the unitary representation $\rho : \pi_1(M) \to U(d)$, each $\gamma \in \pi_1(M)$ induces an holonomy map $\rho(\gamma)$ and a character value

$$\chi_\rho(\gamma) = \mathrm{Tr}(\rho(\gamma)) \,.$$

For any closed orbit we shall also need its algebraic length $\ell(\gamma) = \int_\gamma \theta$. Note that since $i_T d\theta = 0$, this length is always constant through orbit deformation in contact geometry.

At last, the geometric expression will also rely on some weight on the character. For $\gamma = f_i$, we define $V_\gamma^x = \ker(\rho(\gamma) - e^{2i\pi x}\,\mathrm{id})$ and

$$\chi_\rho^\theta(\gamma)(t) = \sum_{e^{2i\pi x} \in \mathrm{Spec}\,\rho(\gamma)} \dim V_\gamma^x \, \theta(x, 4\pi^2 t/\ell(\gamma)^2) \tag{6}$$

where

$$\theta(x,t) = \sum_{n\in\mathbb{Z}} e^{-t(n+x)^2}$$

is a Jacobi theta function. As we shall see, this weighted character is related to the average of the holonomies of closed random loops on the circle γ.

Starting from the topological series (5), we will show the following explicit geometric expression

$$\vartheta = \vartheta^{geo} = \sum_{i\in\mathcal{S}} \chi(N_i)\chi_\rho^\theta(f_i)\,, \tag{7}$$

where $\chi(N_i)$ is the Euler characteristic of the strata N_i.

This identity is also related to a Selberg-type trace formula giving an expression of ϑ using data over the set $\mathcal{C}$ of free homotopy classes γ of *all* the closed orbits of the Reeb flow, including the constant loop, together with their inverse, meaning the closed orbits of the reverse flow. These are associated to powers of the primitive closed orbits f_i. The projection to N of an homotopy class $\gamma \in \mathcal{C}$ is the closure of a stratum N_{i_γ}. We define then

$$e(\gamma) = \ell(f_{i_\gamma})\chi^{orb}(N_{i_\gamma})\,, \tag{8}$$

where $\chi^{orb}(N_{i_\gamma}) = \displaystyle\int_{N_{i_\gamma}} e(TN_{i_\gamma})$ is the rational Euler class of N_{i_γ}.

Generalising a result obtained in dimension 3 in [20], we shall prove that

$$\vartheta(t) = \vartheta^{dyn}(t) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma\in\mathcal{C}} \chi_\rho(\gamma)e(\gamma)e^{-\ell(\gamma)^2/4t}\,. \tag{9}$$

All trace formulae given here are invariant under the rescaling $\theta \mapsto k\theta$ and $t \mapsto k^2 t$, which is specific to the use of the contact de Rham complex instead of the usual Riemannian one. It holds without particular assumption on the curvature or symmetry of the Kähler orbifold N.

As we shall see in Section 3.5, these identities have counterparts using zeta type spectral functions instead of heat ones. They will lead in Corollary 3.9 to explicit Lefschetz type and dynamical formulae for the analytic contact torsion.

We now turn to the second spectral series we will be concerned with.

1.4. **Around the eta invariant in the contact setting.**

In Riemannian geometry of dimension $4k-1$, the eta invariant is defined using the odd signature operator

$$S = (-1)^k(*d + d*)w$$

acting (for convenience here) on odd forms, with $*$ the Hodge star and $w = (-1)^p$ on $\Omega^{2p-1}M$; see [4, p. 63]. This operator is self-adjoint and $S^2 = \Delta$ is Hodge–de Rham Laplacian. The eta invariant is given by the value at $s = 0$ of the meromorphic function

$$\eta(S)(s) = \mathrm{Tr}(S|S|^{-2s-1}) = \frac{1}{\Gamma(s+1/2)} \int_0^{+\infty} \mathrm{Tr}(\sqrt{t}Se^{-t\Delta})t^{s-1}dt.$$

As S maps $\Omega^{2p-1}M$ to $\Omega^{2(2k-p)-1}M \oplus \Omega^{2(2k-p+1)-1}M$, only forms in 'middle degree' $\Omega^{2k-1}M$ contribute to the trace in $\eta(S)$, so that

$$\eta(S)(s) = \frac{1}{\Gamma(s+1/2)} \int_0^{+\infty} \operatorname{Tr}(\sqrt{t}(*d)e^{-t\Delta} \mid \Omega^{2k-1}M)t^{s-1}dt\,.$$

Now, it has been shown that $\eta(S)(0)$ is related to the eta invariant of its contact second order counterpart $*D$ acting on $\mathcal{E}^{2k-1}$; see [7] for the three dimensional Seifert case and Albin-Quan's more recent work [1, §6] on general contact manifolds. The difference is given by the integral of (unknown) universal curvature polynomial. Although it captures the eta invariant, the operator $*D$ itself does not have good analytic properties, due to its infinite dimensional kernel. It is not hypoelliptic. It needs to be completed by some extra term like in the signature Riemannian operator S. Possible choices could be $P = *D \pm d_Q\delta_Q$ on $\mathcal{E}^{2k-1}$. One has $P^2 = \mathbf{\Delta}_Q$ and $\eta(P) = \eta(*D) \pm \zeta(d_Q\delta_Q)$, where the zeta series of the positive operator $d_Q\delta_Q$ contributes to an alternating sum of cohomological dimensions up to some local term at $s = 0$. In higher dimension however, these choices of 'extensions' of $*D$ don't seem to be the most natural ones in terms of spectral symmetry.

We will consider instead the operator defined by

$$S_Q = \begin{cases} *D + (d_Q + \delta_Q)\sigma\delta_Q & \text{on } \mathcal{E}^{2k-1} \\ (d_Q + \delta_Q)\sigma(d_Q + \delta_Q) & \text{on } \bigoplus_{1\le p\le k-1} \mathcal{E}^{2k-1-2p} \end{cases} \tag{10}$$

where $\sigma = (-1)^p J$ on $\mathcal{E}^{2p}$ and $J = i^{a-b}$ on forms $\mathcal{E}^{a,b}$ of bidegree (a,b) with respect to the complex structure. We shall see in Proposition 4.2 that when the CR structure has a transverse symmetry, $S_Q^2 = \mathbf{\Delta_Q}$ and still

$$\eta(S_Q) = \eta(*D) + \sum \pm\, \zeta(\Delta_Q)$$

leading again to adding cohomological dimensions and local terms at $s = 0$. The advantage of this choice of signature operator lies in its extra symmetry with respect to $\sigma = (-1)^p J$ on $\mathcal{E}^{2p-1}$. It splits into

$$S_Q = \sigma T + P\,,$$

with $\sigma P = -P\sigma$ while $\sigma T = T\sigma$ and $TP = PT$, so that the spectrum of S_Q is symmetric except on (an infinite dimensional space) $\ker P = \mathcal{H}_S$ on which $S_Q = \sigma T$.

1.5. **The contact eta trace as topological and dynamical series.**

As in the previous case of the analytic torsion, the spectral series involved in $\eta(S_Q)$ have both closed topological and dynamical expressions. Let

$$\vartheta_S(t) = \operatorname{Tr}(\sqrt{t}S_Q e^{-t\mathbf{\Delta}_Q})\,. \tag{11}$$

The domain of S_Q

$$\mathcal{E}_S = \bigoplus_{1\le p\le k} \mathcal{E}^{2p-1}$$

splits into $\mathcal{E}_S^+ \oplus \mathcal{E}_S^-$ with respect to the involution $\tau = i\sigma$, as does $\mathcal{H}_S = \mathcal{H}_S^+ \oplus \mathcal{H}_S^-$. The operator P exchanges this splitting and we set $P^+ = P : \mathcal{E}_S^+ \to \mathcal{E}_S^-$. By the previous discussion $\vartheta_S(t)$

reduces on $\mathcal{H}_S$ and we have that

$$\vartheta_S(t) = -\operatorname{Tr}(i\sqrt{t}Te^{tT^2} \mid \mathcal{H}_S^+) + \operatorname{Tr}(i\sqrt{t}Te^{tT^2} \mid \mathcal{H}_S^-)\,.$$

Using the same splitting of V through the circle action as in (4), we will finally get

$$\vartheta_S(t) = \vartheta_S^{top}(t) = -\sqrt{t} \sum_{\lambda \in \operatorname{Spec}(iT)} \operatorname{ind}(P^+ \mid V_\lambda)\lambda e^{-t\lambda^2}\,,$$

where, as we shall see in Section 4.1, the index there is the signature of the (non flat) bundle V_λ over the orbifold $N = M/\mathbb{S}^1$. This is the first interpretation of ϑ_S as an index series. These indices can be computed using Kawasaki's index formula. We will give an explicit geometric expression in the case the orbifold N has only a finite number of singular points.

We now turn to the link with dynamical data. The objective is to single out the contribution of each homotopy class of closed orbit like in the analytic torsion case (9). Let

$$\mathbf{c} = c_1(L) = -\frac{d\theta}{2\pi}$$

be the first Chern class of M seen as the circle bundle of a complex line bundle over $N = M/\mathbf{S}^1$. Let $\mathcal{L}(N)$ be Hirzebruch L-genus of the (smooth part of the) orbit space N.

We shall see in Section 4.2 that in the case singular orbits are finite, one has

$$\vartheta_S(t) = \vartheta_S^{dyn}(t) = \frac{1}{\sqrt{4\pi}} \sum_{\gamma \in \mathcal{C}} \chi_\rho(\gamma)\sigma(\gamma)(t),$$

where powers of a generic orbit $\gamma = f^n$ contribute to

$$\sigma(\gamma)(t) = \frac{i\ell(f)}{2t} \left\langle (\ell(\gamma) + i\mathbf{c})e^{-(\ell(\gamma)+i\mathbf{c})^2/4t} \wedge \mathcal{L}(N), [N_{smooth}] \right\rangle,$$

while powers of a singular orbit f_i of order α_i, $\gamma = f_i^k$ with $k \not\equiv 0 \mod \alpha_i$, contribute to

$$\sigma(\gamma)(t) = \frac{i\ell(f_i)}{2t} \ell(\gamma) e^{-\ell(\gamma)^2/4t}\, \nu(\gamma)$$

with

$$\nu(\gamma) = i(-1)^k \prod_{j=1}^{2k-1} \cot(\theta_j/2)\,.$$

Here θ_j are the angles of the action of γ on the horizontal space H. Following Atiyah-Bott's work [3], $\nu(\gamma)$ arises in the Lefschetz fixed point formula for the signature operator.

This will eventually lead in Sections 4.3 and 4.5 to explicit expressions of the twisted eta invariant $\eta(S_Q)(0)$ in terms of these topological and dynamical data.

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2. Review of basic constructions and miscellaneous formulae

To make the paper as self contained as possible we will start by discussing the contact de Rham complex because it plays an important role here. We will also review miscellaneous formulae around it. Much of this material can be found in other places; see e.g. [18, 19, 20]

Let M be a smooth manifold of dimension $2n+1$. A $2n$-dimensional sub-bundle $H \subset TM$ is a *contact distribution* if a 1-form θ such that $H = \ker \theta$ satisfies the non integrability condition $\theta \wedge d\theta^n \neq 0$. Such a form is called a *contact form*. Associated to a choice of θ is the transverse *Reeb field* T; it is the unique vector field satisfying $\theta(T) = 1$ and $\mathcal{L}_T\theta = i_T d\theta = 0$, where $\mathcal{L}_T$ is Lie derivative along T.

The exterior algebra of M splits into horizontal and vertical forms

$$\Omega^* M = \Omega^* H \oplus \theta \wedge \Omega^* H$$

where $\Omega^* H$ are forms vanishing on T. The exterior differential d on $\Omega^* M$ satisfies

$$d(\alpha_H + \theta \wedge \alpha_T) = (d_H \alpha_H + d\theta \wedge \alpha_T) + \theta \wedge (T\alpha_H - d_H \alpha_T)$$

using the notation $T = \mathcal{L}_T$ on forms, that is in matrix form

$$d = \begin{pmatrix} d_H & L \\ T & -d_H \end{pmatrix}, \tag{12}$$

where $d_H = \Pi_{\Omega^* H} d$ is the horizontal part of d (that skips the differential along T), and $L\alpha = d\theta \wedge \alpha$. From $d^2 = 0$, one gets

$$d_H^2 = -LT \ , \quad [L, T] = 0 = [L, d_H] \, . \tag{13}$$

Note that $(\Omega^* H, d_H)$ is not a complex, and moreover that the splitting of $\Omega^* M$ and d_H depend on the choice of a contact form θ. According to [18] it is possible to construct another sequence of operators that avoid this. We discuss this now.

2.1. **The contact complex.**

Let $\mathcal{I}^*$ be the ideal in $\Omega^* M$ generated by θ and $d\theta$

$$\mathcal{I}^* = \{\alpha \in \Omega^* M \mid \alpha = \theta \wedge \beta + d\theta \wedge \gamma\} \, ,$$

and $\mathcal{J}^*$ its annihilator

$$\mathcal{J}^* = \{\alpha \in \Omega^* M \mid \theta \wedge \alpha = d\theta \wedge \alpha = 0\} \, .$$

They are independent on the choice of contact form and stable under d. From e.g. [22], L is injective on $\Omega^k H$ for $k \leq n-1$ and surjective onto $\Omega^k H$ for $k \geq n+1$. Hence $\mathcal{I}^k = \Omega^k M$ for $k \geq n+1$ and $\mathcal{J}^k = 0$ for $k \leq n+1$. Then the de Rham exterior differential induces a quotiented complex

$$\Omega^0 M \xrightarrow{d_Q} \Omega^1 M / \mathcal{I}^1 \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \Omega^n M / \mathcal{I}^n$$

and a subcomplex

$$\mathcal{J}^{n+1} \xrightarrow{d_Q} \mathcal{J}^{n+2} \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \mathcal{J}^{2n+1} .$$

These can be joined using the following:

Lemma 2.1. [18, p. 286] *Let* $\alpha \in \Omega^n M/\{\theta \wedge \Omega^* H\}$. *Then there exists a unique lift* $\overline{\alpha}$ *of* α *in* $\Omega^n M$ *such that* $d\overline{\alpha} \in \mathcal{J}^{n+1}$. *Moreover* $d\overline{\alpha} = 0$ *if* $\alpha = d\theta \wedge \beta$.

One defines then $D : \Omega^n M/\mathcal{I}^n \to \mathcal{J}^{n+1}$ by $D\alpha = d\overline{\alpha}$. Note that D is a *second order* operator, taking T as a second order one in our contact setting by (13). Given a choice of contact form, the formula for D reads

$$D\alpha = d(\alpha_H - \theta \wedge L^{-1} d_H \alpha_H) = \theta \wedge (T + d_H L^{-1} d_H)\alpha_H \,, \tag{14}$$

if α_H is the representative of α in $\Omega^n H$. The so-called contact complex is then

$$\Omega^0 M \xrightarrow{d_Q} \Omega^1 M/\mathcal{I}^1 \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \Omega^n M/\mathcal{I}^n \xrightarrow{D} \mathcal{J}^{n+1} \xrightarrow{d_Q} \mathcal{J}^{n+2} \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \mathcal{J}^{2n+1}.$$

We have:

Proposition 2.2. [18, p. 286] *The contact complex is a resolution of the constant sheaf and hence its cohomology coincides with de Rham cohomology of* M. *Moreover the canonical projections* $\pi : \Omega^k M \to \Omega^k M/\mathcal{I}^k$ *for* $k \leq n$ *and injections* $i : \mathcal{J}^k \to \Omega^k M$ *for* $k \geq n+1$ *induce an isomorphism between the two cohomologies.*

The arguments being purely local, these results also apply on twisted version of the complexes with a flat bundle V, as coming from a representation $\rho : \pi_1(M) \to U(d)$.

Using a complex structure J on H such that $d\theta(\cdot, J\cdot)$ is Hermitian positive definite, one defines a Riemannian metric on M

$$g = d\theta(\cdot, J\cdot) + \theta^2 \,.$$

Let then $\Lambda = L^*$ be the adjoint of $L : \Omega^k H \to \Omega^{k+2} H$ where $L\alpha = d\theta \wedge \alpha$, and $\Omega_0^* H = \ker \Lambda$ be the bundle of primitive horizontal forms. We will identify in the sequel the quotient spaces $\Omega^k M/\mathcal{I}^k$ in the lower-half of the contact complex with $\Omega_0^k H$. Let

$$\mathcal{E}^k = \begin{cases} \Omega_0^k H & \text{if } k \leq n \\ \mathcal{J}^k & \text{if } k \geq n+1. \end{cases} \tag{15}$$

be the definition spaces of the contact complex in this identification. Note that Hodge star operator $*$ exchanges $\mathcal{E}^k$ and $\mathcal{E}^{2n+1-k}$.

2.2. Miscellaneous formulae.

We gather now some useful identities; see e.g. [19, Section 4] for more details. The first ones are similar to basic formulae from Kählerian geometry, see [22]. At the algebraic level, it holds on the Hermitian space H that

$$[\Lambda, L] = n - p \quad \text{on } \Omega^p H \,. \tag{16}$$

Moreover, following [22, Thm. 3] for instance, $\Omega^* H$ splits under the Lefschetz decomposition

$$\Omega^* H = \bigoplus_{0 \leq k \leq q \leq n} L^k \Omega_0^{n-q} H = \bigoplus_{0 \leq k \leq q \leq n} L^k \mathcal{E}^{n-q} \,. \tag{17}$$

At the level of first order operators, one has

$$[\Lambda, d_H] = -\delta_H^J \,. \tag{18}$$

where δ_H is the formal adjoint of d_H, $\delta_H^J = J^{-1}\delta_H J$ and $J\alpha(X_1,\cdots,X_p) = \alpha(JX_1,\cdots,JX_p)$ on $\Omega^p H$.

This leads to the action of d_H with respect to Lefschetz decomposition. Thanks to (16) and (18), it holds on $\Omega_0^p H$ that

$$d_H = d_Q - \frac{L}{n-p+1}\delta_Q^J\,, \tag{19}$$

where δ_Q is the formal adjoint of d_Q, $\delta_Q^J = J^{-1}\delta_Q J$ and the convention here that $d_Q = 0$ on $\Omega_0^n H$. This extends to $L^k\Omega_0^p H$ using $[L, d_H] = 0$ by (13).

We now come to second order relations on the contact complex. From (13) (14) and (19), one gets:

$$T = \begin{cases} \frac{1}{n-p}\delta_Q^J d_Q + \frac{1}{n-p+1} d_Q\delta_Q^J & \text{on } \mathcal{E}^p \text{ for } p<n \\ i_T D + d_Q\delta_Q^J & \text{on } \mathcal{E}^n\,. \end{cases} \tag{20}$$

In order to get rid of the multiplicative coefficients in formulae as above, we will normalise the differentials d_Q as in [19, p. 418]. Namely on $\mathcal{E}^p$ for $p < n$, we shall use from now on

$$\frac{1}{\sqrt{n-p}}d_Q \quad \text{instead of} \quad d_Q\,. \tag{21}$$

We will keep the same notation d_Q for this normalised differential in the sequel since we will only use them. Hence (20) reads now

$$T = \begin{cases} \delta_Q^J d_Q + d_Q\delta_Q^J & \text{on } \mathcal{E}^p \text{ for } p<n \\ i_T D + d_Q\delta_Q^J & \text{on } \mathcal{E}^n\,. \end{cases} \tag{22}$$

Using this and $JL = LJ$, $d\theta$ being a $(1,1)$ form, one deduces that on $\Omega^* H$

$$T^* = -T^J := -J^{-1}TJ \quad \text{and} \quad [\Lambda, T] = 0\,. \tag{23}$$

So far, all identities here hold for any calibrated complex structure J, i.e. satisfying that $d\theta(\cdot, J\cdot)$ is positive Hermitian. In the sequel, we will assume moreover that J is *integrable*, meaning that $[H^{1,0}, H^{1,0}] \subset H^{1,0}$. In that case both d_H and d_Q split into two components

$$d_H = \partial_H + \overline{\partial}_H \quad \text{and} \quad d_Q = \partial_Q + \overline{\partial}_Q\,, \tag{24}$$

where $\partial_{H,Q}$ increases the bidegree by $(1,0)$ and $\overline{\partial}_{H,Q}$ by $(0,1)$. Developing $d_Q^2 = 0$ on $\mathcal{E}^p$ for $p \le n-2$, first gives

$$\partial_Q^2 = \overline{\partial}_Q^2 = 0 = \partial_Q\overline{\partial}_Q + \overline{\partial}_Q\partial_Q = d_Q d_Q^J + d_Q^J d_Q\,. \tag{25}$$

We can also get other second order relations between the Q-differentials by developing (22) on $\mathcal{E}^p$ with $p < n$. One can split T into its components $T^{0,0}$, $T^{1,-1}$ and $T^{-1,1}$ relatively to its action on the bidegree (p,q) of forms in $\Omega^* H$. One gets

$$\begin{cases} \Delta_{\overline{\partial}_Q} - \Delta_{\partial_Q} = iT^{0,0} \\ \overline{\partial}_Q^*\partial_Q + \partial_Q\overline{\partial}_Q^* = iT^{1,-1} \\ \partial_Q^*\overline{\partial}_Q + \overline{\partial}_Q\partial_Q^* = -iT^{-1,1} \end{cases} \tag{26}$$

where

$$\Delta_{\overline{\partial}_Q} = \overline{\partial}_Q^*\overline{\partial}_Q + \overline{\partial}_Q\overline{\partial}_Q^* \quad \text{and} \quad \Delta_{\partial_Q} = \partial_Q^*\partial_Q + \partial_Q\partial_Q^* \,.$$

At this point we note that

$$T - T^J = -J^{-1}(\mathcal{L}_T J) = (1+i)T^{1,-1} + (1-i)T^{-1,1}$$

is a zero order algebraic operator that vanishes when the Reeb flow preserves the complex structure, thus the metric. In conclusion we have the following:

Proposition 2.3. [19, p. 418] *Suppose that the complex structure on a CR contact manifold is integrable and preserved by the Reeb flow.*

Then the second order Q-Laplacian $\Delta_Q = d_Q\delta_Q + \delta_Q d_Q$ commutes with J on $\mathcal{E}^p$ for $p < n$ and satisfies

$$\Delta_Q = \Delta_{\partial_Q} + \Delta_{\overline{\partial}_Q} \quad \text{with} \quad \Delta_{\overline{\partial}_Q} - \Delta_{\partial_Q} = iT \,. \tag{27}$$

Remark 2.4. We recall that these results apply to the renormalised differentials as defined in (21).

2.3. **The middle degree case.**

At this point we still miss a Q-Laplacian in middle degree. The differential $D : \mathcal{E}^n \to \mathcal{E}^{n+1}$ here is second order. Hence a starting expression for a positive Laplacian is the fourth order D^*D. A natural way to complete it, is to set on $\mathcal{E}^n$

$$\boldsymbol{\Delta}_Q = (d_Q\delta_Q)^2 + D^*D \,. \tag{28}$$

An important feature of this choice lies in its commuting property with J, as in the lower degree case.

Proposition 2.5. [18, p. 312] *$\boldsymbol{\Delta}_\mathbf{Q}$ preserves the bidegree of forms in $\mathcal{E}^n$ when the complex structure is integrable and invariant through the Reeb flow.*

Proof. From (23) one has $T^* = -T$ when $\mathcal{L}_T J = 0$, and then by (20)

$$\begin{aligned} D^*D = (i_T D)^*(i_T D) &= (-T - d_Q^J\delta_Q)(T - d_Q\delta_Q^J) \\ &= -T^2 + T(-d_Q^J\delta_Q + d_Q\delta_Q^J) + d_Q^J\delta_Q d_Q\delta_Q^J \,. \end{aligned}$$

There T and $-d_Q^J\delta_Q + d_Q\delta_Q^J$ commute with J, whereas by Proposition 2.3

$$\begin{aligned} d_Q^J\delta_Q d_Q\delta_Q^J = d_Q^J(\Delta_Q - d_Q\delta_Q)\delta_Q^J &= d_Q^J\Delta_Q^J\delta_Q^J - d_Q^J d_Q\delta_Q\delta_Q^J \\ &= (d_Q^J\delta_Q^J)^2 - d_Q^J d_Q\delta_Q\delta_Q^J \,. \end{aligned}$$

The last term preserves the bidegree by (25). Finally adding $(d_Q\delta_Q)^2$ gives that $\boldsymbol{\Delta}_Q$ commutes with J. Note that $\boldsymbol{\Delta}_Q$ has no $(2,-2)$ (and $(-2,2)$) component neither since it can come only from combination of type

$$\partial_Q\overline{\partial}_Q^*\partial_Q\overline{\partial}_Q^* = -\partial_Q\partial_Q\overline{\partial}_Q^*\overline{\partial}_Q^* = 0 \,,$$

by (26) and (25). □

In order to get fourth order Laplacians for all degrees we shall define

$$\boldsymbol{\Delta}_Q = \Delta_Q^2 = (d_Q\delta_Q)^2 + (\delta_Q d_Q)^2 \quad \text{on } \mathcal{E}^p \text{ for } p < n\,. \tag{29}$$

We extend these operators to all of $\Omega^* H$ using the Lefschetz decomposition (17) and requiring that

$$L\boldsymbol{\Delta}_Q = \boldsymbol{\Delta}_Q L\,.$$

In this way, the Laplacian $\boldsymbol{\Delta}_Q$ commutes with all the algebra of operators we face here: L, J, d_Q, d_H and their adjoints. It will play the role of a "Casimir" operator in our situation.

To conclude this part, we mention the main analytic result on the Q-Laplacian.

Theorem 2.6. [18, p. 290] *The Q-Laplacians Δ_Q and $\boldsymbol{\Delta}_Q$ are maximally hypoelliptic on any compact contact manifold.*

Roughly speaking, that means that these operators control as many horizontal derivatives as possible: two for Δ_Q and four for $\boldsymbol{\Delta}_Q$. See for instance the discussion in [20, Section 3.1] for a presentation of main properties and references about this analytic notion. A consequence for this paper is that on compact contact manifolds, the Q-Laplacians are self-adjoint and possess pure point spectrum with smooth eigenvectors. Moreover the associated heat kernels $e^{-t\boldsymbol{\Delta}_Q}$ are smooth hence trace class in this setting.

3. Results on the contact analytic torsion

We now state our main result on the torsion function. We first define the contact manifolds we will be concerned with.

Definition 3.1. Let M be a compact contact manifold. We shall say that M is a *CR Seifert manifold* if it admits a circle action generated by a Reeb field T that preserves moreover an integrable complex (CR) structure J on H.

The orbit space $N = M/\mathbf{S^1}$ of such a manifold inherits a structure of a Kählerian orbifold with cyclic quotient singularities; see e.g. [9, Thm 7.1.3]. Note that these manifold are particular cases of Sasakian manifolds, called quasi-regular, whose Reeb field have closed orbits.

We also endow M with a unitary representation $\rho : \pi_1(M) \to U(d)$ (that can be trivial). This possibly allows to twist the contact complex by taking values in the associated flat bundle V. That will highlight the role of each individual homotopy class of closed orbit in the dynamical expression.

Using notations introduced in Sections 1.2 and 1.3 we recall that the initial heat torsion function is spectral and given by (1)

$$\vartheta(t) = \sum_{k=0}^{n} (-1)^k (n+1-k) \operatorname{Tr}(e^{-t\boldsymbol{\Delta}_Q} \mid \mathcal{E}^k).$$

We also define the topological theta function

$$\vartheta^{top}(t) = \sum_{\lambda \in \operatorname{Spec}(iT)} \operatorname{ind}(D_H^{ev} \mid V_\lambda) e^{-t\lambda^2}\,,$$

the geometric expression over the strata of N

$$\vartheta^{geo} = \sum_{i\in\mathcal{S}} \chi(N_i)\chi_\rho^\theta(f_i)\,,$$

and the dynamical series

$$\vartheta^{dyn}(t) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma\in\mathcal{C}} \chi_\rho(\gamma)e(\gamma)e^{-\ell(\gamma)^2/4t}\,.$$

We shall prove the following.

Theorem 3.2. *Let M be a CR Seifert manifold endowed with a unitary representation. Then it holds that*

$$\vartheta = \vartheta^{top} = \vartheta^{geo} = \vartheta^{dyn}\,.$$

3.1. **From spectral to topological torsion series.**

We start with the proof of the first identity $\vartheta = \vartheta^{top}$. Recall that by Lefschetz decomposition (17)

$$\Omega^* H = \bigoplus_{0\leq p\leq q\leq n} L^p\mathcal{E}^{n-q}\,.$$

Therefore Ω^*H contains $n+1-k$ copies of each $\mathcal{E}^k$ for $0\leq k\leq n$. This is the multiplicity of $\mathcal{E}^k$ in ϑ. Moreover, since $L = d\theta\wedge\cdot$ preserves the parity, the parity of forms in Ω^*H and their components in $\mathcal{E}^*$ coincides. Thus the plus and minus parts of ϑ combine to give

$$\vartheta(t) = \operatorname{Tr}(e^{-t\boldsymbol{\Delta}_Q} \mid \Omega^{ev}H) - \operatorname{Tr}(e^{-t\boldsymbol{\Delta}_Q} \mid \Omega^{odd}H)\,.$$

Now $D_H = d_H + \delta_H$ exchanges $\Omega^{ev}H$ and $\Omega^{odd}H$ and commutes with $\boldsymbol{\Delta}_Q$ by Section 2.3. Hence, setting $\mathcal{H} = \ker D_H$, one has

$$\vartheta(t) = \operatorname{Tr}(e^{-t\boldsymbol{\Delta}_Q} \mid \mathcal{H}^{ev}) - \operatorname{Tr}(e^{-t\boldsymbol{\Delta}_Q} \mid \mathcal{H}^{odd})\,.$$

We need the following Lemma to identify the residual spectrum on $\mathcal{H}$.

Lemma 3.3. *On CR Seifert manifolds, one has $\boldsymbol{\Delta}_Q = -T^2$ on $\mathcal{H} = \ker D_H$.*

Note that $\mathcal{H} = \ker D_H$ should not be confused with $\ker\Delta_H = \ker d_H \cap \ker\delta_H$ since d_H is not a complex by (13). The next proposition will lead us to a useful description of $\mathcal{H}$.

Proposition 3.4. *Let $P = L+\Lambda$ acting on Ω^*H and $U = e^{i\pi P/4}$. Then on a contact manifold with an integrable J, it holds that*

$$\begin{cases} [P, D_H] = d_H^J - \delta_H^J \\ [P, d_H^J - \delta_H^J] = D_H \end{cases}$$

and

$$U^{-1}D_H U = \sqrt{2}(\overline{\partial}_H + \overline{\partial}_H^*)\,.$$

Proof. The first identities come from (13) and (18) giving $[L, d_H] = 0$ and $[\Lambda, d_H] = -\delta_H^J$ and conjugated relations. Therefore for any angle φ

$$\begin{aligned} e^{i\varphi\,\mathrm{ad}(P)} D_H &= \cos\varphi D_H + i\sin\varphi(d_H^J - \delta_H^J) \\ &= \mathrm{Ad}(e^{i\varphi P}) D_H = e^{i\varphi P} D_H e^{-i\varphi P}\,. \end{aligned}$$

This yields the last identity using $\varphi = -\pi/4$ and the splitting of $d_H = \partial_H + \overline{\partial}_H$ for integrable J by (24). $\square$

Let $\mathcal{H}_{\overline{\partial}_H} = \ker(\overline{\partial}_H + \overline{\partial}_H^*)$. By the previous proposition

$$\mathcal{H} = \ker D_H = U(\mathcal{H}_{\overline{\partial}_H})\,,$$

with the advantage that on CR Seifert manifolds $\overline{\partial}_H^2 = 0$ from (13). Hence one has also $\mathcal{H}_{\overline{\partial}_H} = \ker \Delta_{\overline{\partial}_H}$ with

$$\Delta_{\overline{\partial}_H} = \overline{\partial}_H \overline{\partial}_H^* + \overline{\partial}_H^* \overline{\partial}_H\,.$$

Now by definition $\Delta_{\overline{\partial}_H}$ preserves the bi-degree of forms, but also the spaces $L^k\Omega_0^p H$ in the Lefschetz decomposition. This is due to the commutation relation

$$\Delta_{\overline{\partial}_H} L = L(\Delta_{\overline{\partial}_H} - iT)$$

that comes from (18) when $\mathcal{L}_T J = 0$. Indeed, from

$$[L, \overline{\partial}_H] = 0 \quad \text{and} \quad [L, \overline{\partial}_H^*] = -i\partial_H$$

one gets

$$\Delta_{\overline{\partial}_H} L - L\Delta_{\overline{\partial}_H} = i(\partial_H\overline{\partial}_H + \overline{\partial}_H\partial_H) = -iLT$$

by (13). This leads eventually to the splitting of $\mathcal{H}_{\overline{\partial}_H}$ into forms of pure bidegree and type in the Lefschetz decomposition. Using (19) we first find.

Proposition 3.5. *On a CR Seifert manifold, one has*

$$\ker\overline{\partial}_H = \begin{cases} \ker\overline{\partial}_Q \cap \ker\partial_Q^* & \text{on } L^k\mathcal{E}^{n-q} \text{ for } 0 \le k < q \le n \\ \ker\partial_Q^* & \text{on } L^q\mathcal{E}^{n-q} \end{cases}$$

and

$$\ker\overline{\partial}_H^* = \begin{cases} \ker\overline{\partial}_Q^* \cap \ker\partial_Q & \text{on } L^k\mathcal{E}^{n-q} \text{ for } 1 \le k \le q \le n \\ \ker\overline{\partial}_Q^* & \text{on } \mathcal{E}^p \text{ for } 0 \le p \le n\,. \end{cases}$$

In conclusion

$$\mathcal{H}_{\overline{\partial}_H} = \begin{cases} \ker\Delta_Q & \text{on } L^k\mathcal{E}^{n-q} \text{ for } 1 \le k < q \le n \\ \ker\overline{\partial}_Q \cap \ker\partial_Q^* \cap \ker\overline{\partial}_Q^* & \text{on } \mathcal{E}^p \text{ for } p < n \\ \ker\partial_Q^* \cap \ker\overline{\partial}_Q^* \cap \ker\partial_Q & \text{on } L^q\mathcal{E}^{n-q} \text{ for } 0 < q \le n \\ \ker\partial_Q^* \cap \ker\overline{\partial}_Q^* & \text{on } \mathcal{E}^n\,. \end{cases}$$

This leads to the following results on $\mathcal{H}_{\overline{\partial}_H}$:

- On $L^k\mathcal{E}^{n-q}$ for $0 \le k < q \le n$, $\Delta_Q = \Delta_Q^J = \mathcal{L}_T = 0$ by (27),
- On $\mathcal{E}^p$ for $p < n$, $\Delta_{\overline{\partial}_Q} = 0$, so that by (27) $\Delta_Q = -iT$ and $\boldsymbol{\Delta}_Q = \Delta_Q^2 = -T^2$,
- On $L^q\mathcal{E}^{n-q}$ for $0 < q \le n$, $\Delta_{\partial_Q} = 0$, so that $\Delta_Q = iT$ and $\boldsymbol{\Delta}_Q = \Delta_Q^2 = -T^2$,

- On $\mathcal{E}^n$, $\delta_Q = 0 = \delta_Q^J = 0$, so that $i_T D = T$ and $\mathbf{\Delta}_Q = -T^2$.

This yields Lemma 3.3 because the isometry $U = e^{i\frac{\pi}{4}(L+\Lambda)}$ mapping $\mathcal{H}_{\overline{\partial}_H}$ to $\mathcal{H} = \ker D_H$ in Proposition 3.4 commutes with $\mathbf{\Delta}_Q$ and T, hence preserves the equation $\mathbf{\Delta}_Q = -T^2$.

To conclude with this study of $\mathcal{H}$ we observe that it is infinite dimensional since it contains images by U of CR functions (satisfying $\overline{\partial}_H f = 0$).

We have obtained that

$$\vartheta(t) = \mathrm{Tr}(e^{tT^2} \mid \mathcal{H}^{ev}) - \mathrm{Tr}(e^{tT^2} \mid \mathcal{H}^{odd})\,. \tag{30}$$

Note that both traces converge since they are parts of traces of the heat operators $e^{-t\mathbf{\Delta}_Q}$. The next step will be to split this as a series of indices using a twisted Fourier decomposition along the circle action by T. We discuss this briefly and refer to [20, Section 4.1] for a more detailed discussion. We will suppose that M is connected in the sequel of this section. The general case reduces to it by summing the contributions of each connected component.

The flat bundle V over M associated to the unitary representation $\rho : \pi_1(M) \to U(d)$ is the quotient of the trivial bundle $\widetilde{M} \times \mathbb{C}^d$ by the deck transformations $\gamma.(m, v) = (\tau(\gamma)m, \rho(\gamma)v)$. The circle action φ_t induced by T on M may be lifted to V by parallel transport using the flat connection ∇^ρ. On V we don't have a circle action since

$$\varphi_{2\pi} = \rho(f)^{-1}$$

where $f = \varphi_{[0,2\pi]}(m)$ is a generic closed primitive orbit. However over each connected component of M, we can split V into bundles V^x on which

$$\rho(f) = e^{2i\pi x}$$

with $x \in]0, 1]$ by convention. Note that the bundles V^x are flat since the homotopy class of f lies in the center of $\pi^1(M)$ as seen using the flow. We recover a circle action on V^x by setting

$$\psi_t = e^{itx}\varphi_t\,.$$

We can still perform a Fourier decomposition of sections $s \in \mathbf{V}^x = \Gamma(M, V^x)$. One has

$$s = \sum_{n\in\mathbb{Z}} \pi_n s \quad \text{with} \quad \pi_n s = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\psi_t(s)dt\,.$$

Since $\psi_t(\pi_n s) = e^{int}\pi_n s$ one gets that $\nabla^\rho_T(\pi_n s) = i(n - x)\pi_n s$ and the spectrum of $iT = i\nabla^\rho_T$ on $\mathbf{V}^x$ is the shifted $x + \mathbb{Z}$. For $\lambda = x + n$ we shall note

$$\mathbf{V}_\lambda = \pi_{-n}(\mathbf{V}^x) = \mathbf{V}^x \cap \{iT = \lambda\}. \tag{31}$$

Since T commutes with all our geometric operators including D_H, we can split the V-valued bundle

$$\mathcal{H} = \ker D_H = \bigoplus_{\lambda\in\mathrm{Spec}(iT)} \mathcal{H} \cap \mathbf{V}_\lambda\,.$$

Therefore (30) eventually reads

$$\begin{aligned}\vartheta(t) &= \sum_{\lambda\in\mathrm{Spec}(iT)} \big(\dim(\mathcal{H}^{ev}\cap\mathbf{V}_\lambda) - \dim(\mathcal{H}^{odd}\cap\mathbf{V}_\lambda)\big)e^{-t\lambda^2} \\ &= \sum_{\lambda\in\mathrm{Spec}(iT)} \mathrm{ind}(D_H^{ev}\mid\mathbf{V}_\lambda)e^{-t\lambda^2} = \vartheta^{top}(t)\,,\end{aligned}$$

with $D_H^{ev} : \Omega^{ev}H \to \Omega^{odd}H$. This is the topological version of the ϑ series. We shall see that these indices can be interpreted as coming from operators and bundles over the orbifold $N = M/\mathbf{S}^1$

3.2. **Orbifold structure and index computations.**

These indices in the previous formula can be computed using the index theorem for V-(orbi)bundles over V-manifolds (orbifolds) as developed by Kawasaki in [14]. One has first to interpret the index of D_H^{ev} on $\mathbf{V}_\lambda$ as the index of an operator over the orbifold $N = M/\mathbf{S}^1$ acting on a V-bundle.

We follow the discussion in [20, section 5.2]. We first introduce the relevant V-(orbi)bundle here. Given a point $p\in N$, we consider the vector space $V_\lambda(p)$ of sections of V^x along the orbit $\mathbf{S}^1(p)$ in M satisfying $iTs = \lambda s$. Call V_λ this family of spaces. One has by definition

$$\mathbf{V}_\lambda = \Gamma(N, V_\lambda)\,,$$

and V_λ is a vector bundle of dimension $\dim V^x$ over the generic non singular points of N since the circle action is free there.

We now describe the orbifold structure of V_λ near a singular point p_j corresponding to an exceptional closed primitive orbit f_j of order α_j. That means that the isotopy group of the circle action on f_j is the cyclic group of order α_j. From the Slice Theorem, locally over p_j the bundle V^x is isomorphic to the quotient of the trivial bundle $\mathbb{C}^n\times\mathbb{R}\times\mathbb{C}^{\dim V^x}$ by the deck transformation

$$F_j : (p,t,v) \mapsto (M_j(p), t + 2\pi/\alpha_j, \rho(f_j)v) \tag{32}$$

where $M_j(p)\in U(n)$ generates a cyclic group of order α_j. Now let $\widetilde{V}_\lambda$ be the trivial bundle over $\mathbb{C}^n$ whose fiber over p consists of functions $s_p : \mathbb{R}\to\mathbb{C}^{\dim V_x}$ satisfying $s_p(t) = e^{-i\lambda t}s_p(0)$. Since $(iT)s_p = \lambda s_p$, one sees that a section $s : p\mapsto s_p$ of $\widetilde{V}_\lambda$ near $0\in\mathbb{C}^n$ descends to $\mathbf{V}_\lambda$ if it is invariant under the deck transform F_j above. This means that $(p, s_p(0))$ is invariant by

$$F_{j,\lambda} : (p,v) \mapsto (M_j(p), e^{-2i\pi\lambda/\alpha_j}\rho(f_j)v)\,. \tag{33}$$

Since $f_j^{\alpha_j}\sim f$ in $\pi_1(M)$, one has $\rho(f_j)^{\alpha_j} = \rho(f) = e^{2i\pi x}$ and $F_{j,\lambda}^{\alpha_j} = \mathrm{id}$.

This shows that V_λ is a V-bundle (orbi-bundle) over N since locally over p_j it is the quotient of $\widetilde{V}_\lambda$ by the finite group $\Gamma_j \simeq \mathbb{Z}/\alpha_j\mathbb{Z}$ generated by $F_{j,\lambda}$.

At this point we can identify $\mathrm{ind}(D_H^{ev}\mid\mathbf{V}_\lambda)$ as being the index of the Dirac operator D_H^{ev} acting on sections of the V-bundle V_λ over N. Note that even though d_H is not a complex, the operator D_H is elliptic on N as $D_H^2 = \Delta_H - LT + T\Lambda = \Delta_H$ up to order 0 terms on V_λ.

More concretely from Proposition 3.4, D_H is unitarily conjugated to $\bar{\partial\!\!\!/}_H = \overline{\partial}_H + \overline{\partial}_H^*$ hence

$$\begin{aligned}\operatorname{ind}(D_H^{ev} \mid V_\lambda) &= \operatorname{ind}(\bar{\partial\!\!\!/}_H \mid \Omega^{ev} H \otimes V_\lambda)\\ &= \sum_p (-1)^p \chi_{\overline{\partial}_H}(N, \Omega^{p,0} H \otimes V_\lambda)\end{aligned}$$

where $\chi_{\overline{\partial}_H}(N, \Omega^{p,0} H \otimes V_\lambda)$ is the holomorphic Euler characteristic of the V-bundle $\Omega^{p,0} H \otimes V_\lambda$. Recall that $\overline{\partial}_H^2 = 0$ on CR Seifert manifolds by (13). From this discussion, one can compute these indices using Kawasaki's Riemann–Roch theorem for complex V-manifolds [14]. It reads here

$$\operatorname{ind}(D_H^{ev} \mid V_\lambda) = \langle \operatorname{ch}(V_\lambda) \wedge e(N)_{orb}, [N] \rangle_{orb} \,. \tag{34}$$

This pairing on the orbifold N splits into a usual smooth contribution over the generic orbits of the characteristic classes and an average of equivariant classes over components of exceptional orbits in various singular strata; see [14, 3]. To be more explicit, we need first to discuss the stratification of N.

The orbifold N splits into the connected components N_j of its strata that are determined by the possible isotopy groups $\Gamma \simeq \mathbb{Z}/\alpha Z$ of the circle action. We note $N = \bigsqcup_{j \in \mathcal{S}} N_j$ this decomposition and α_j the order of the isotopy group in N_j. Pick then one primitive orbit f_j in each N_j. By (32), one has locally around f_j that N identifies with the quotient of $\mathbb{C}^n$ by a cyclic group of order α_j generated by a unitary 'isotopy' matrix M_j. Let

$$\sigma_j = \operatorname{Spec}(M_j)$$

be its spectrum. It consists in α_j-th roots of unity. Let d_λ denotes the order of $\lambda \in \sigma_j$. It divides α_j and the lower common multiple of these d_λ is α_j.

Locally the stratum N_j containing f_j is given by $\ker(M_j - \mathrm{id})$. Other strata N_k around f_j are given by the sets of points of order *exactly* d in

$$\ker(M_j^d - \mathrm{id}) = \bigoplus_{\{\lambda \in \sigma_j,\ d_\lambda \mid d\}} \ker(M_j - \lambda\, \mathrm{id}) \,, \tag{35}$$

for some divisors d of α_j, when these sets are non empty. This happens when d is the lower common multiple of $\{\lambda \in \sigma_j,\ d_\lambda \mid d\}$. In these cases $f_k \in N_k$ is homotopic to f_j^d. Note that from this local model, the set $\mathcal{S}$ labelling the connected components of strata is finite on the compact N. Moreover the closure of a stratum consists of the union of the connected components of strata it contains.

Using this orbifold structure, the action of the isotopy group on V_λ given in (33) and Kawasaki's formula, one finds that (34) gives here

$$\operatorname{ind}(D_H^{ev} \mid V_\lambda) = \sum_{j \in \mathcal{S}} \frac{1}{\alpha_j} \sum_{k \in K_j \cap [0\,..\,\alpha_j - 1]} e^{-2i\pi k\lambda/\alpha_j} \chi_\rho(f_j^k) \int_{N_j} e(TN_j) \nu(f_j^k) \,, \tag{36}$$

where

$$K_j = \{k \in \mathbb{Z} \text{ such that } \ker(M_j^k - \mathrm{id}) = \ker(M_j - \mathrm{id})\} \,,$$

$e(TN_j)$ is the Euler class of N_j and $\nu(f_j^k)$ is the Lefschetz index of the Poincaré return map along f_j^k acting on the normal bundle of N_j. Since $df_j^k \in U(n)$, one has $\nu(f_j^k) = 1$ here so that (36) reduces to

$$\operatorname{ind}(D_H^{ev} \mid V_\lambda) = \sum_{j\in\mathcal{S}} \frac{\chi^{orb}(N_j)}{\alpha_j} \sum_{k\in K_j\cap[0\,..\,\alpha_j-1]} e^{-2i\pi k\lambda/\alpha_j}\chi_\rho(f_j^k)\,, \tag{37}$$

where $\chi^{orb}(N_j) = \int_{N_j} e(TN_j)$ is the rational (orbifold) Euler characteristic of N_j. Note that in this sum, the case $k = 0$ only appears for $\alpha_j = 1$ and corresponds to the contribution of the generic regular orbits.

Our next step is to express this sum using integral data from N and ρ. We shall prove the following.

Proposition 3.6. *Let M be a connected CR Seifert manifold. Then*

$$\operatorname{ind}(D_H^{ev} \mid V_\lambda) = \sum_{j\in\mathcal{S}} \chi(N_j) \dim\bigl(\ker(\rho(f_j) - e^{2i\pi\lambda/\alpha_j}\,\mathrm{id})\bigr)\,,$$

where $\chi(N_j)$ is the usual (topological) Euler characteristic of N_j.

We first observe that following (35) the summation set K_j in (36) satisfies

$$K_j = \Bigl(\bigcup_{\lambda\in\sigma_j\setminus\{1\}} \mathbb{Z}d_\lambda\Bigr)^c.$$

Therefore the indicator function of K_j can be written

$$\mathbf{1}_{K_j} = \sum_{A\subset\sigma_j^*} (-1)^{|A|}\mathbf{1}_{\mathbb{Z}d_A}\,,$$

where for a given subset A in $\sigma_j^* = \sigma_j \setminus \{1\}$, d_A denotes the lower common multiple of the d_λ for $\lambda \in A$, with the convention here that $d_\emptyset = 1$. Given one strata N_j, one can then express in (37) the sum

$$\begin{aligned}\sum_{k\in K_j\cap[0,\,..\,\alpha_j-1]} e^{-2i\pi k\lambda/\alpha_j}\chi_\rho(f_j^k) &= \sum_{A\subset\sigma_j^*}(-1)^{|A|}\sum_{k=0}^{\alpha_j/d_A-1} e^{-2i\pi d_A k\lambda/\alpha_j}\chi_\rho(f_j^{kd_A})\\ &= \sum_{A\subset\sigma_j^*}(-1)^{|A|}\frac{\alpha_j}{d_A}\dim\bigl(\ker(\rho(f_j^{d_A}) - e^{2i\pi d_A\lambda/\alpha_j}\,\mathrm{id})\bigr)\,,\end{aligned}$$

because the linear maps $\varphi = \rho(f_j^{d_A})e^{-2i\pi d_A\lambda/\alpha_j}$ satisfy $\varphi^{\alpha_j/d_A} = \rho(f)e^{-2i\pi\lambda} = \mathrm{id}$, so that

$$\sum_{k=0}^{\alpha_j/d_A-1} \operatorname{Tr}(\varphi^k) = \frac{\alpha_j}{d_A}\dim\ker(\varphi - \mathrm{id})\,.$$

Inserting in (37) gives

$$\operatorname{ind}(D_H^{ev} \mid V_\lambda) = \sum_{j\in\mathcal{S}}\sum_{A\subset\sigma_j^*}(-1)^{|A|}\frac{\chi^{orb}(N_j)}{d_A}\dim\bigl(\ker(\rho(f_j^{d_A}) - e^{2i\pi d_A\lambda/\alpha_j}\,\mathrm{id})\bigr)\,.$$

In view of Proposition 3.6, we have to gather in this sum the couples (j, A) such that $f_j^{d_A}$ is homotopic to a given orbit f_i in a stratum N_i. From the stratification structure we previously described, those correspond to strata N_j such that $N_j \subset \overline{N}_i$, that we will note $j \prec i$ in the sequel. One has in such a case

$$\ker(M_i - \mathrm{id}) = \ker(M_j^{d_A} - \mathrm{id})\,.$$

We will note $\Sigma_{j,i}$ the set of such parts A of σ_j^*. Note also that $d_A = \alpha_j/\alpha_i$ can be seen as the relative order of f_j inside $\overline{N}_i$, meaning taking $\overline{N}_i$ itself as a CR Seifert manifold, with a rescaled circle action such that f_i has a trivial isotopy.

The following lemma then completes the proof of Proposition 3.6.

Lemma 3.7. *Let N be a connected CR Seifert manifold and $N = \sqcup_{i\in\mathcal{S}} N_i$ be its decomposition into connected strata, with N_0 being the open dense one. Then it holds that*

$$\chi(N_0) = \sum_{i\in\mathcal{S}} \sum_{A\in\Sigma_{i,0}} (-1)^{|A|} \frac{\chi^{orb}(N_i)}{\alpha_i}\,.$$

Proof. For any orbifold N with even dimensional strata N_i, Satake's Gauss Bonnet formula expresses the orbifold Euler characteristic as

$$\chi^{orb}(N_0) = \sum_{i\in\mathcal{S}} \frac{\chi(N_i)}{\alpha_i}\,,$$

see [21] or [9, p. 121]. We need here to 'invert' this expression. Starting from the non singular case, where $\chi(N) = \chi^{orb}(N)$, we proceed by induction in the depth of the stratification of N. One has

$$\begin{aligned}
\chi(N_0) &= \chi^{orb}(N_0) - \sum_{i\neq 0} \frac{\chi(N_i)}{\alpha_i} \\
&= \chi^{orb}(N_0) - \sum_{i\neq 0} \frac{1}{\alpha_i} \sum_{j\prec i} \sum_{A\in\Sigma_{j,i}} (-1)^{|A|} \frac{\chi^{orb}(N_j)}{\alpha_j/\alpha_i} \quad \text{by induction,} \\
&= \chi^{orb}(N_0) - \sum_{j\neq 0} \frac{\chi^{orb}(N_j)}{\alpha_j} \sum_{\substack{i\neq 0\\ j\prec i}} \sum_{A\in\Sigma_{j,i}} (-1)^{|A|}\,.
\end{aligned}$$

Now, the last double sum amounts to summing over all the parts A of σ_j^* except those in $\Sigma_{j,0}$. Since $\sum_{A\subset\sigma_j^*} (-1)^{|A|} = 0$, this gives the result.

□

3.3. End of proof of the geometric formula for ϑ.

We complete now the proof of the geometric identity

$$\vartheta = \vartheta^{geo} = \sum_{j\in\mathcal{S}} \chi(N_j)\chi_\rho^\theta(f_j)\,.$$

We suppose that M is connected and the general case will follow. From Proposition 3.6 and the index series formula (5) for ϑ, we already know that

$$\vartheta(t) = \sum_{j\in\mathcal{S}} \chi(N_j) \sum_{\lambda\in\mathrm{Spec}(iT)} \dim\bigl(\ker(\rho(f_j) - e^{2i\pi\lambda/\alpha_j}\,\mathrm{id})\bigr)e^{-t\lambda^2}\,, \tag{38}$$

where we recall that $\mathrm{Spec}(iT)$ splits into $\mathbb{Z}+x$ on $V^x = \ker(\rho(f) - e^{2i\pi x}\,\mathrm{id})$, where f is the generic regular orbit.

Since $\rho(f_j)^{\alpha_j} = \rho(f)$, the spectrum of $\rho(f_j)$ on V^x consists in complex numbers $e^{2i\pi x_{j,k}}$ with $\alpha_j x_{j,k} = x + n_{j,k}$ for some integers $n_{j,k}$. Hence if $\lambda = x+n$ one finds that

$$e^{2i\pi x_{j,k}} = e^{2i\pi\lambda/\alpha_j} \Leftrightarrow x_{j,k} \equiv \frac{x+n}{\alpha_j} \mathrm{mod}\ \mathbb{Z} \Leftrightarrow n \equiv n_{j,k} \bmod \alpha_j\mathbb{Z}\,,$$

so that $\lambda = \alpha_j x_{j,k} + \alpha_j p$ with $p\in\mathbb{Z}$. Let $V_{f_j}^{x_{j,k}} = \ker(\rho(f_j) - e^{2i\pi x_{j,k}}\,\mathrm{id})$. The contribution of f_j to (38) reads

$$\begin{aligned}
\sum_{\lambda\in\mathrm{Spec}(iT)} & \dim\bigl(\ker(\rho(f_j) - e^{2i\pi\lambda/\alpha_j}\,\mathrm{id})\bigr)e^{-t\lambda^2} \\
&= \sum_{e^{2i\pi x_{j,k}}\in\mathrm{Spec}(\rho(f_j))} \dim V_{f_j}^{x_{j,k}} \sum_{p\in\mathbb{Z}} e^{-t\alpha_i^2(x_{j,k}+p)^2} \\
&= \sum_{e^{2i\pi x_{j,k}}\in\mathrm{Spec}(\rho(f_j))} \dim V_{f_j}^{x_{j,k}} \theta(x_{j,k}, \alpha_j^2 t) = \chi_\rho^\theta(f_j)(t)\,.
\end{aligned}$$

Note that we replace α_j by $2\pi/\ell(f_j)$ in the definition of $\chi_\rho^\theta(f_j)(t)$ in order to preserve its homogeneity in the rescaling $\theta \mapsto c^2\theta$ and $\ell(f_j) \mapsto c\ell(f_j)$. This gives the geometric expression for ϑ. □

Before coming to the general dynamical expression for ϑ, one can already observe that the weighted characters $\chi_\rho^\theta(f_j)$ have a simple probabilistic interpretation.

This is based on the usual Poisson formula relating the Gaussian and Jacobi theta function. It reads

$$\theta(x,t) = \sum_{n\in\mathbb{Z}} e^{-t(n+x)^2} = \sqrt{\frac{\pi}{t}} \sum_{n\in\mathbb{Z}} e^{2i\pi nx} e^{-\pi^2 n^2/t}\,. \tag{39}$$

This gives here that

$$\chi_\rho^\theta(f_j)(t) = \frac{\ell(f_j)}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \chi_\rho(f_j^n) e^{-\ell^2(f_j^n)/4t}\,. \tag{40}$$

Thus $\chi_\rho^\theta(f_j)(t)$ is a weighted sum of holonomies of ρ along all the closed curves f_j^n. This series has a simple probabilistic meaning. We recall that the function

$$k_t(y) = \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}$$

is the heat kernel on $\mathbb{R}$. Now, given a random (Brownian) curve $y(s)$ on the circle orbit f_j, it is closed at time t if its lift $\widetilde{y}$ in $\mathbb{R}$ satisfies $\widetilde{y}(t) - \widetilde{y}(0) = n\ell(f_j)$. In such a case, the rotation index of $y_{[0,t]}$ is n. Therefore its holonomy along ρ is $\chi_\rho(f_j^n)$. Moreover the probability density of such a displacement is $k_t(\ell(f_j^n)) = \frac{1}{\sqrt{4\pi t}} e^{-\ell^2(f_j^n)/4t}$, which arises in (40).

3.4. ϑ from the dynamical viewpoint.

It remains to link the geometric formula for ϑ to its dynamical one ϑ^{dyn}. We start from the topological series ϑ^{top} and the index formula (37).

Poisson formula (39) at $x' = x + i\pi k/t\alpha_j$ gives that

$$\sum_{n\in\mathbb{Z}} e^{2i\pi k(n+x)/\alpha_j} e^{-t(n+x)^2} = \sqrt{\frac{\pi}{t}} \sum_{n\in\mathbb{Z}} e^{2i\pi nx} e^{-\frac{\pi^2}{t}(k/\alpha_j+n)^2} .$$

Then, using that $f^n \sim f_j^{n\alpha_j}$ in $\pi^1(M)$ for the generic orbit f and $\ell(f_j^{k+n\alpha_i}) = 2\pi(k/\alpha_j + n)$, one obtains by (37) that

$$\theta^{top}(t) = \frac{1}{\sqrt{4\pi t}} \sum_{j\in\mathcal{S}} \sum_{k\in K_j\cap[0\,..\,\alpha_j-1]} \ell(f_j)\chi^{orb}(N_j)\chi_\rho(f_j^{k+n\alpha_j}) e^{-\ell(f_j^{k+n\alpha_j})^2/4t} . \tag{41}$$

By (36), for each given connected strata N_j, the index set K_j gives the indices k such that, in projection to N, the free homotopy class of f_j^k is not larger than the one of f_j, i.e. $\overline{N}_j$. Therefore in (41), each free homotopy class of closed orbit is counted only once. These are all the $f_j^{k+n\alpha_j} \sim f_j^k f^n$ in $\pi^1(M)$ for $n \in \mathbb{Z}$ and $k \in K_j$.

This gives the required dynamical series (9) over all free homotopy classes of closed orbits, their inverse, and the constant loop.

3.5. Zeta functions viewpoint.

We now turn to identities on the contact analytic torsion from the viewpoint of zeta functions. On the spectral side, let

$$Z(s) = \sum_{k=0}^{n} (-1)^k (n+1-k)\zeta(\mathbf{\Delta}_Q \mid \mathcal{E}^k)(s) ,$$

and on the dynamical side

$$Z^{dyn}(s) = \sum_{\gamma\in\mathcal{C}^*} \chi_\rho(\gamma) e(\gamma) |\ell(\gamma)|^{2s-1} ,$$

where $e(\gamma)$ is defined by (8), and γ ranges over all free homotopy classes of non trivial closed orbits together with their inverse.

On the geometric side at last we consider

$$Z^{geo}(s) = \sum_{j\in\mathcal{S}} \chi(N_j)\chi_\rho^\zeta(f_j)(s) \tag{42}$$

with

$$\begin{aligned} \alpha_j^{2s}\chi_\rho^\zeta(f_j)(s) = \sum_{e^{2i\pi x}\in \mathrm{Spec}^*(\rho(f_j))} & \dim V_{f_j}^x (\zeta(2s,x) + \zeta(2s,1-x)) \\ & + 2\zeta(2s)\dim V_{f_j}^1 \end{aligned}$$

where $V^x_{f_j} = \ker(\rho(f_j) - e^{2i\pi x}\,\mathrm{id})$, ζ is the Riemann zeta function and

$$\zeta(s,x) = \sum_{n\geq 0} \frac{1}{(n+x)^s} \tag{43}$$

is the Hurwitz zeta function.

Theorem 3.8. *On CR Seifert manifolds the functions Z and Z^{geo} are meromorphic and equal, with a simple pole at $s = 1/2$.*

The function Z^{dyn} is meromorphic with a simple pole at $s = 0$, and one has

$$\Gamma(s)Z(s) = \frac{2^{-2s}}{\sqrt{\pi}}\Gamma(\frac{1}{2} - s)Z^{dyn}(s)\,. \tag{44}$$

Moreover

$$\begin{cases} Z(0) = \mathrm{Res}_0(Z^{dyn}) = -\chi'_Q(M,\rho) \\ \mathrm{Res}_{1/2}(Z) = -\dfrac{1}{2\pi}Z^{dyn}(1/2) = \chi^{orb}(N)\dim V\,, \end{cases}$$

with

$$\chi'_Q(M,\rho) = \sum_{k=0}^{n}(-1)^k(n+1-k)\dim H^k(M,\rho)\,,$$

using the cohomology groups of the flat bundle V over M.

This extends results obtained in [20] in the three dimensional case. Note also that Kitaoka computed in [15] the spectral series $Z(s)$ on lens spaces endowed with their symmetric metric and found it reduces to a Hurwitz zeta function.

3.6. **Proof of Theorem 3.8.**

We know from the spectral expression of ϑ that

$$\vartheta(t) = \sum_{k=0}^{n}(-1)^k(n+1-k)\dim\ker(\boldsymbol{\Delta}_Q \mid \mathcal{E}^k) + O(e^{-Ct})$$

for some $C > 0$ when $t \to +\infty$. According to Proposition 2.2 the limiting constant is the topological number $\chi'_Q(M,\rho)$.

From $\vartheta = \vartheta^{geo}$, one gets using $t \to +\infty$ that

$$\chi'_Q(M,\rho) = \sum_{j\in\mathcal{S}} \chi(N_j)\dim\ker(\rho(f_j) - \mathrm{id})\,. \tag{45}$$

Whereas when $t \to 0^+$, the dynamical expression (9) yields

$$\vartheta(t) = \sqrt{\frac{\pi}{t}}\chi^{orb}(N)\dim V + O(e^{-C/t})\,.$$

Let us consider the function

$$\vartheta_0(t) = \vartheta(t) - c_0\mathbf{1}_{[1,+\infty[}(t) - \frac{c_{-1/2}}{\sqrt{t}}\mathbf{1}_{]0,1]}(t)$$

with

$$c_0 = \chi'_Q(M,\rho) \quad \text{and} \quad c_{-1/2} = \sqrt{\pi}\chi^{orb}(N)\dim V\,.$$

From the previous discussion, the integral

$$I(s)=\int_0^{+\infty}\vartheta_0(t)t^{s-1}dt$$

coming from Mellin transform defines a holomorphic function over $\mathbb{C}$. Writing

$$\vartheta_0(t)=\vartheta(t)-c_0+(c_0-\frac{c_{-1/2}}{\sqrt{t}})\mathbf{1}_{]0,1]}(t)$$

one finds first for $\Re(s)>1/2$ that

$$I(s)=\Gamma(s)Z(s)+\frac{c_0}{s}-\frac{c_{-1/2}}{s-1/2}=\Gamma(s)Z^{geo}(s)+\frac{c_0}{s}-\frac{c_{-1/2}}{s-1/2}\,.$$

Using

$$\vartheta_0(t)=\vartheta^{dyn}(t)-\frac{c_{-1/2}}{\sqrt{t}}+(\frac{c_{-1/2}}{\sqrt{t}}-c_0)\mathbf{1}_{]1,+\infty]}(t)$$

yields for $\Re(s)<0$ that

$$I(s)=\frac{2^{-2s}}{\sqrt{\pi}}\Gamma(\frac{1}{2}-s)Z^{dyn}(s)+\frac{c_0}{s}-\frac{c_{-1/2}}{s-1/2}\,.$$

This proves Theorem 3.8.

3.7. The contact analytic torsion from the Lefschetz and dynamical viewpoints.

Recall that from (3) the contact analytic torsion is defined by $T_Q(M,\rho)=\exp(-Z'(0)/2)$. We shall now compute it using the previous formulae. This extends results obtained in [20] in the three dimensional case.

Corollary 3.9. *Let M be a CR Seifert manifold endowed with a unitary representation $\rho:\pi_1(M)\to U(d)$. Let $N=\sqcup_{j\in\mathcal{S}}N_j$ be the decomposition of $N=M/\mathbf{S}^1$ into connected strata. Given a primitive orbit f_j in N_j, define*

$$V^1_{f_j}=\ker(\rho(f_j)-\mathrm{id})$$

and denote by $\rho(f_j)^\perp$ the restriction of these holonomies to $(V^1_{f_j})^\perp$.

Then it holds that

$$T_Q(M,\rho)=(2\pi)^{\chi'_Q(M,\rho)}\prod_{j\in\mathcal{S}}\Big(\big|\det(\rho(f_j)^\perp-\mathrm{id})\big|\,\alpha_j^{-\dim(V^1_{f_j})}\Big)^{\chi(N_j)}\,.$$

Moreover one has

$$-2\ln(T_Q(M,\rho))=Z'(0)=\lim_{s\to0}\big(Z^{dyn}(s)+\frac{\chi'_Q(M,\rho)}{s}\big)\,.$$

The first expression coincides with that found, via topological methods, for the Reidemeister–Franz torsion by Fried [13, p. 198], in the case of an *acyclic representation*, i.e. $H^*(M,\rho)=0$. Therefore, the contact analytic torsion also coincides with the (Riemannian) Ray–Singer analytic torsion in that case, from works of Cheeger and Müller [10, 16]. The only new factor for general representations is the cohomological term $(2\pi)^{\chi'_Q(M,\rho)}$. Note that by (45) one can also write

$$T_Q(M,\rho)=\prod_{j\in\mathcal{S}}\Big(\big|\det(\rho(f_j)^\perp-\mathrm{id})\big|\,\ell(f_j)^{\dim(V^1_{f_j})}\Big)^{\chi(N_j)}\,.$$

The dynamical expression for the analytic torsion also extends a result proved by Fried in the acyclic case on Seifert manifolds [12, Thm 5.1], [13]. We have there

$$Z'(0) = Z^{dyn}(0) = \sum_{\gamma\in\mathcal{C}^*} \chi_\rho(\gamma)e(\gamma)/|\ell(\gamma)|\,,$$

which is known as the total twisted Fuller measure of periodic orbits. It has a formal invariance by deformation of the flow, as long as the orbit periods stay bounded; see [12, Section 4].

Proof of Corollary 3.9.

Since $Z = Z^{geo}$ is an explicit function by Theorem 3.8, we compute $(Z^{geo})'(0)$. We follow the proof in [20, p. 771] which we recall for completeness.

From [23, p. 271], one has for $x \in]0,1[$, $\zeta(0,x) = \frac{1}{2} - x$, hence $\zeta(0,x) + \zeta(0,1-x) = 0$. Also by Lerch's formula $\partial_s\zeta(s,x)_{s=0} = \ln\Gamma(x) - \frac{1}{2}\ln(2\pi)$, it holds that $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$ and

$$\begin{aligned}\partial_s\zeta(s,x)_{s=0} + \partial_s\zeta(s,1-x)_{s=0} &= \ln(\Gamma(x)\Gamma(1-x)/2\pi) = -\ln(2\sin(\pi x))\\ &= -\ln|1-e^{2i\pi x}|\,.\end{aligned}$$

Hence from (42), one finds that

$$\chi^\zeta_\rho(f_j)'(0) + 2\ln(\alpha_j)\chi^\zeta_\rho(f_j)(0) = -2\ln|\det(\rho(f_j)^\perp - \mathrm{id})| - 2\dim V^1_{f_j}\ln(2\pi)\,.$$

Summing these results using the formula (42) for Z^{geo} and (45) gives the first result for $T_Q(M,\rho)$.

For the second one, it is useful to observe that

$$2\Gamma(2s)\cos(\pi s)Z = Z^{dyn}\,, \tag{46}$$

as comes by multiplying (44) by $\Gamma(s+1/2)$ and using the classical identities

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\sqrt{\pi}\Gamma(2s) \quad\text{and}\quad \Gamma(s+\frac{1}{2})\Gamma(s-\frac{1}{2}) = \frac{\pi}{\cos\pi s}\,.$$

One knows from Theorem 3.8 that $Z(0) = -\chi'_Q(M,\rho)$. Hence developing (46) when $s\to 0$ gives the dynamical formula for $T_Q(M,\rho)$. ☐

4. Results on the contact eta trace.

We now turn to the study of the contact eta invariant and trace as introduced in Section 1.4. Following (10) we consider the contact signature operator on contact manifolds of dimension $2n+1 = 4k-1$

$$S_Q = \begin{cases} *D + (d_Q+\delta_Q)\sigma\delta_Q & \text{on } \mathcal{E}^{2k-1}\\ (d_Q+\delta_Q)\sigma(d_Q+\delta_Q) & \text{on } \bigoplus_{1\le p\le k-1}\mathcal{E}^{2k-1-2p}\end{cases}$$

where $\sigma = (-1)^pJ$ on $\mathcal{E}^{2p}$ and $\mathcal{E}^{2p-1}$. As claimed, the advantage of this choice over others lies in the following algebraic properties.

Proposition 4.1.

- *The operator S_Q is symmetric on any contact manifold of dimension $4k-1$ endowed with a calibrated complex structure J.*

- *If moreover J is integrable and invariant by T, then it holds that $S_Q^2 = \mathbf{\Delta}_Q$ and*
$$S_Q = \sigma T + P \text{ with } P\sigma = -\sigma P \text{ and } [T,\sigma] = [P,T] = 0\,.$$
- *The non-zero spectrum of S_Q splits as follows*
$$\mathrm{Spec}^*(S_Q) = \mathrm{Spec}^*(*D) \bigsqcup_{0\le p\le k-1} \mathrm{Spec}^*\big(\sigma\Delta_Q \mid \mathcal{E}^{2p}\big)\,.$$

Proof. • The first statement follows easily from the definition and the facts that by (20)
$$(-1)^k * D = J(i_T D) = JT - Jd_Q J^{-1}\delta_Q J$$
with $J^* = (-1)^p J$ on $\mathcal{E}^p$ and $T^* = -T^J$ on calibrated J by (23). Note that using the σ symmetry this can be written

$$*D = \sigma T - \sigma^{-1} d_Q \sigma \delta_Q \sigma\,. \tag{47}$$

• By Proposition 2.3, one knows that Δ_Q commutes with σ on $\mathcal{E}^p$ for $p < n = 2k-1$ on CR manifolds with transverse symmetry. Then using that the sequence (d_Q, D) is a complex, one finds on $\mathcal{E}^{2k-1-2p}$ for $p \ge 2$ that
$$\begin{aligned} S_Q^2 &= (d_Q+\delta_Q)\sigma(d_Q+\delta_Q)^2\sigma(d_Q+\delta_Q)\\ &= (d_Q+\delta_Q)\sigma\Delta_Q\sigma(d_Q+\delta_Q) = (d_Q+\delta_Q)\Delta_Q(d_Q+\delta_Q)\\ &= \Delta_Q^2 = \mathbf{\Delta}_Q\,. \end{aligned}$$

In degree $2k-3$
$$\begin{aligned} S_Q^2 &= (*D + (d_Q+\delta_Q)\sigma\delta_Q)(d_Q\sigma d_Q)\\ &\quad + (d_Q+\delta_Q)\sigma(d_Q+\delta_Q)((d_Q+\delta_Q)\sigma\delta_Q + \delta_Q\sigma d_Q)\\ &= (d_Q+\delta_Q)\sigma\Delta_Q\sigma d_Q + (d_Q+\delta_Q)\sigma\Delta_Q\sigma\delta_Q\\ &= (d_Q+\delta_Q)\Delta_Q d_Q + (d_Q+\delta_Q)\Delta_Q\delta_Q = \Delta_Q^2 = \mathbf{\Delta}_Q \end{aligned}$$

In degree $2k-1$
$$\begin{aligned} S_Q^2 &= (*D)^2 + ((d_Q+\delta_Q)\sigma\delta_Q)(d_Q\sigma\delta_Q) + (d_Q+\delta_Q)\sigma(d_Q+\delta_Q)\delta_Q\sigma\delta_Q\\ &= D^*D + (d_Q+\delta_Q)\sigma\Delta_Q\sigma\delta_Q\\ &= D^*D + (d_Q+\delta_Q)\Delta_Q\delta_Q = D^*D + (d_Q\delta_Q)^2 = \mathbf{\Delta}_Q\,. \end{aligned}$$

We determine the invariant part $(S_Q)^\sigma$ of S_Q through σ. One observes first from (25) that $d_Q\sigma d_Q$ adds $(1,1)$ to the bidegree, hence preserves J and anti-commutes with σ. The same holds for $\delta_Q\sigma\delta_Q$. Therefore

$$(S_Q)^\sigma = \begin{cases} (*D + d_Q\sigma\delta_Q)^\sigma & \text{on } \mathcal{E}^{2k-1}\\ (d_Q\sigma\delta_Q + \delta_Q\sigma d_Q)^\sigma & \text{on } \bigoplus_{1\le p\le k-1}\mathcal{E}^{2k-1-2p}\,. \end{cases} \tag{48}$$

Then (47) gives in degree $2k-1$ that
$$(S_Q)^\sigma = (\sigma T - \sigma^{-1} d_Q\sigma\delta_Q\sigma + d_Q\sigma\delta_Q)^\sigma = \sigma T\,.$$

In degree $2k-1-2p$ with $p \geq 1$, we know from (22) that $T = \delta_Q^J d_Q + d_Q \delta_Q^J$, thus

$$\sigma T = \delta_Q \sigma d_Q + \sigma^{-1} d_Q \sigma \delta_Q \sigma \,.$$

By (48), we look for

$$(d_Q \sigma \delta_Q + \delta_Q \sigma d_Q)^\sigma = (\sigma T - \sigma^{-1} d_Q \sigma \delta_Q \sigma + d_Q \sigma \delta_Q)^\sigma = \sigma T \,,$$

as needed.

- To relate the spectrum of S_Q to the one of $*D$ we first observe that $S_Q = *D$ on $\mathcal{E}_Q^n = \mathcal{E}^n \cap \ker \delta_Q$. Then

$$\mathrm{Spec}^*(S_Q) = \mathrm{Spec}^*(*D) \bigsqcup \mathrm{Spec}^*(S_Q \mid (\mathcal{E}_Q^n)^\perp) \,.$$

Moreover on $(\mathcal{E}_Q^n)^\perp$, $(d_Q + \delta_Q) S_Q = \Delta_Q \sigma (d_Q + \delta_Q)$, with the convention that $d_Q = 0$ in degree n. Therefore

$$\mathrm{Spec}^*(S_Q \mid (\mathcal{E}_Q^n)^\perp) = \bigsqcup_{0 \leq p \leq k-1} \mathrm{Spec}^*\big(\sigma \Delta_Q \mid \mathcal{E}^{2p}\big) \,.$$

□

From this, we can compare the eta trace functions of S_Q and $*D$.

Proposition 4.2. *On a CR Seifert manifold of given dimension $4k-1$, it holds that*

$$\eta(S_Q)(0) - \eta(*D)(0) + \sum_{0 \leq a+b=2p \leq 2(k-1)} (-1)^p i^{a-b} \dim H^{a,b}(M, \rho)$$

is the integral over M of some universal polynomial of local invariants of the metric.

Proof. From Proposition 4.1, it holds on CR Seifert manifolds that

$$\eta(S_Q)(s) = \eta(*D)(s) + \sum_{0 \leq p \leq k-1} \eta(\sigma \Delta_Q \mid \mathcal{E}^{2p})(s) \,. \tag{49}$$

Moreover $\sigma \Delta_Q$ splits through bidegree and

$$\eta(\sigma \Delta_Q \mid \mathcal{E}^{2p})(s) = \sum_{a+b=2p} (-1)^p i^{a-b} \zeta(\Delta_Q \mid \mathcal{E}^{a,b})(s) \,.$$

Now, for positive hypoelliptic operators like Δ_Q, one knows that

$$\zeta(\Delta_Q)(0) + \dim \ker \Delta_Q$$

is the constant term in the development of $e^{-t\Delta_Q}$ and is given by the integral over M of some universal polynomial of local invariant of the pseudo-hermitian metric; see [20, Section 3.1] or [1] for discussion and references. This proves the result since the kernels of Δ_Q are isomorphic to the cohomology groups $H^*(M, \rho)$. □

Remarks 4.3. In dimension three with a trivial representation ρ, $H^0(M, \rho) = \mathbb{R}$ is given by the constant functions. Hence the cohomological sum is 1 and $\eta(S_Q)(0) - \eta(*D)(0)$ is *not* given by a local term a priori. The two eta invariants are not equivalent up to local terms.

Note also that in dimension 3, one has $S_Q = *D + d_Q \delta_Q$, as studied for instance in previous work [7], but the expression differs in higher dimensions.

The advantage of working with S_Q instead of $*D$ or $*D \pm d_Q\delta_Q$ in general lies in its spectral symmetry with respect to σ. This should lead to a simpler expression of the eta invariant, as will be confirmed and discussed in Section 4.4.

4.1. **The contact eta trace from the topological viewpoint.**

We start the study of the eta trace spectral series involved in the eta invariant S_Q. From (11) it reads

$$\vartheta_S(t) = \mathrm{Tr}(\sqrt{t} S_Q e^{-t\mathbf{\Delta}_Q})\,.$$

We complete arguments already sketched in Section 1.5. From Proposition 4.1, one has $\sigma P S_Q = -S_Q \sigma P$ with $\sigma P = -P\sigma$. Therefore the spectrum of $S_Q = P + \sigma T$ is symmetric with respect to zero, except on $\mathcal{H}_S = \ker P$ and the eta trace ϑ_S retracts on it with $S_Q = \sigma T$. Hence

$$\mathbf{\Delta}_Q = S_Q^2 = -T^2 \quad \text{on} \quad \mathcal{H}_S\,.$$

One can split the domain $\mathcal{E}_S$ of S_Q and $\mathcal{H}_S$ with respect to the action of the involution $\tau = i\sigma = \pm 1$. We have

$$\vartheta_S(t) = -\sqrt{t}(\mathrm{Tr}(iTe^{tT^2} \mid \mathcal{H}_S^+) - \mathrm{Tr}(iTe^{tT^2} \mid \mathcal{H}_S^-))\,.$$

On CR Seifert manifolds, the V-valued forms in $\mathcal{H}_S$ split under Fourier decomposition as in Sections 3.1–3.2. Hence we obtain the identity.

Theorem 4.4. *One has on connected CR Seifert manifolds*

$$\vartheta_S(t) = \vartheta_S^{top}(t) = -\sqrt{t} \sum_{\lambda\,\mathrm{Spec}(iT)} \mathrm{ind}(P^+ \mid V_\lambda) \lambda e^{-t\lambda^2}\,,$$

where $P^+ = P : \mathcal{E}_S^+ \to \mathcal{E}_S^-$.

We shall now describe the index involved.

4.2. **From the topological to the geometric expression for ϑ_S.**

Let $*_H$ denotes the $*$ operator on the Hermitian space H. From [22, Thm. 2] one has

$$*_H L^r \alpha = L^r((-1)^{p(p+1)/2} J\alpha) = L^r(\sigma\alpha)$$

on $\Omega_0^p H = \mathcal{E}^p$ when $p + 2r = n$. That means that the involution $\tau = i\sigma$ on $\mathcal{E}_S$ is conjugated to $\tau_H = i*_H$ on $\Omega^n H$ through the Lefschetz decomposition

$$\mathcal{E}_S \simeq \Omega^n H \quad \text{with} \quad \mathcal{E}_S^{\tau=\pm 1} \simeq (\Omega^n H)^{\tau_H = \pm 1}\,.$$

From these isomorphisms $P^+ : \Omega^{n,+} H \to \Omega^{n,-} H$. Hence its elliptic symbol class as seen from N is associated to the signature index on N; see [5, Section 6]. Let then $\mathcal{L}(N)_{orb}$ denote the Hirzebruch L-genus of the orbifold N. As in Section 3.2 we have by the Kawasaki index theorem and [5]

$$\mathrm{ind}(P^+ \mid V_\lambda) = \langle \mathrm{ch}(V_\lambda) \wedge \mathcal{L}(N)_{orb}, [N]\rangle_{orb}\,,$$

and $\vartheta_S(t)$ can be written

$$\vartheta_S(t) = -\langle \mathrm{ch}(V)^\theta_{odd}(t) \wedge \mathcal{L}(N)_{orb}, [N]\rangle_{orb} \tag{50}$$

using the notation

$$\operatorname{ch}(V)^{\theta}_{odd}(t) = \sum_{\lambda\in\operatorname{Spec}(iT)} \sqrt{t}\lambda \operatorname{ch}(V_\lambda) e^{-t\lambda^2}\,. \tag{51}$$

We need to study this "θ-regularised" Chern character of V, which is seen as an infinite dimensional bundle from N by (4). It is also useful to consider its even version

$$\operatorname{ch}(V)^{\theta}_{ev}(t) = \sum_{\lambda\in\operatorname{Spec}(iT)} \operatorname{ch}(V_\lambda) e^{-t\lambda^2}\,. \tag{52}$$

Note that its zero degree part already showed up in (34) in the study of the topological series for the torsion function, that satisfies

$$\vartheta^{top}(t) = \langle \operatorname{ch}(V)^{\theta}_{ev}(t) \wedge e(N)_{orb}, [N]\rangle_{orb}\,.$$

From (50) we need now to compute these differential forms in their entirety.

The expressions for the G-equivariant Chern class and Hirzebruch L-genus being quite involved in general, see [6, Thm. 6.12], we will restrict ourself to the case where the singular set of N is finite, i.e. the exceptional orbits f_i are isolated.

This is always satisfied in dimension three because the singular set is at least of codimension two. In higher dimension, following Section 3.2, that means that all the eigenvalues of the 'isotopy' matrices M_j of the strata are of order α_j exactly. In that case one has $\ker(M_j^k - \mathrm{id}) = \ker(M_j - \mathrm{id})$ unless $k \in \mathbb{Z}\alpha_j$ and the summation sets K_j used in Kawasaki's formula, see (36), reduces to $(\mathbb{Z}\alpha_j)^c$. Using Atiyah–Bott's work [3, Section 6], one can identify the contribution of the isolated singular orbits, and the orbifold pairing in (50) decomposes as follows

$$\begin{aligned}\langle \operatorname{ch}(V)^{\theta}_{odd}(t) \wedge \mathcal{L}(N)_{orb}, [N]\rangle_{orb} &= \langle \operatorname{ch}(V)^{\theta}_{odd}(t) \wedge \mathcal{L}(N), [N_{smooth}]\rangle \\ &\quad + \sum_j \frac{1}{\alpha_j} \sum_{r=1}^{\alpha_j-1} \operatorname{ch}(V)^{\theta}_{odd}(f_j^r)(t)\nu(f_j^r)\,,\end{aligned} \tag{53}$$

where, for $\gamma = f_j^n$ with $n \not\equiv 0 \mod \alpha_j$,

$$\nu(\gamma) = i(-1)^k \prod_{m=1}^{2k-1} \cot(\theta_m(\gamma)/2)$$

for angles $\theta_m(\gamma)$ associated to the unitary return map $d\tau_\gamma$ along γ in H_{p_j} and $\dim N = 4k-2$. Note that these angles are non zero since the singular orbits f_j are isolated.

It remains to express the regularised Chern characters of V. To compute first the Chern character of V_λ, as seen from N, we use the following twisted connection

$$\nabla^\lambda = \nabla^\rho + i\lambda\theta\,,$$

where ∇^ρ is the flat connection on V. Since $\nabla^\lambda_T s = 0$ on sections of $\mathbf{V}_\lambda$, this connection descends to M as a connection on V_λ. Its curvature form is given for $X, Y \in H$ by

$$\begin{aligned} R_{\nabla^\lambda}(X,Y) &= \nabla^\lambda_X\nabla^\lambda_Y - \nabla^\lambda_Y\nabla^\lambda_X - \nabla^\lambda_{[X,Y]} \\ &= R_{\nabla^\rho}(X,Y) + i\lambda d\theta(X,Y) \\ &= i\lambda d\theta(X,Y)\,. \end{aligned}$$

Hence the bundle $(V_\lambda, \nabla_\lambda)$ has curvature form $\Omega_\lambda = i\lambda d\theta \otimes \mathrm{id}_{V_\lambda}$. Let then

$$\mathbf{c} = -\frac{d\theta}{2\pi}\,.$$

Note that $\mathbf{c} = c_1(L)$ is the first Chern class of the line bundle L over N whose circle bundle is M. Using the V-bundle structure of V_λ as given in (33), one obtains that

$$\mathrm{ch}(V_\lambda) = \mathrm{Tr}(e^{\frac{i\Omega_\lambda}{2\pi}}) = \dim V^x e^{\lambda\mathbf{c}}$$

over smooth points of N, while the equivariant Chern character at f_j^r is

$$\mathrm{ch}(V_\lambda)(f_j^r) = e^{-2i\pi r\lambda/\alpha_j}\chi_\rho(f_j^r \mid V^x)\,.$$

This leads to the geometrical and dynamical expressions for the θ-regularised Chern characters $\mathrm{ch}(V)^\theta_{ev}$ and $\mathrm{ch}(V)^\theta_{odd}$ using the classical Jacobi theta function $\theta(x,t) = \sum_{n\in\mathbb{Z}} e^{-t(n+x)^2}$ and Poisson formula (39) as in Section 3.4.

Proposition 4.5. • *Over smooth orbits it holds that*

$$\begin{aligned}
\mathrm{ch}(V)^\theta_{ev}(t) &= \sum_{\lambda\in\mathrm{Spec}(iT)} \dim V^x e^{\lambda\mathbf{c}-t\lambda^2}\\
&= \sum_{e^{2i\pi x}\in\mathrm{Spec}(\rho(f))} \dim V^x e^{\mathbf{c}^2/4t}\theta(x-\frac{\mathbf{c}}{2t},t)\\
&= \frac{\ell(f)}{\sqrt{4\pi t}}\sum_{n\in\mathbb{Z}}\chi_\rho(f^n)e^{-(\ell(f^n)+i\mathbf{c})^2/4t}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ch}(V)^\theta_{odd}(t) &= \sqrt{t}\sum_{\lambda\in\mathrm{Spec}(iT)} \dim V^x \lambda e^{\lambda\mathbf{c}-t\lambda^2}\\
&= \sqrt{t}\frac{d}{d\mathbf{c}}\,\mathrm{ch}(V)^\theta_{ev}(t)\\
&= -\frac{i\ell(f)}{4t\sqrt{\pi}}\sum_{n\in\mathbb{Z}}\chi_\rho(f^n)(\ell(f^n)+i\mathbf{c})e^{-(\ell(f^n)+i\mathbf{c})^2/4t},
\end{aligned}$$

in terms of formal derivation as a polynomial in $\mathbf{c}$.

• *At a singular orbit* f_j^r*, the equivariant* θ*-Chern character satisfies*

$$\begin{aligned}
\mathrm{ch}(V)^\theta_{ev}(f_j^r)(t) &= \sum_{\lambda\in\mathrm{Spec}(iT)} \chi_\rho(f_j^r \mid V^x)e^{-2i\pi r\lambda/\alpha_j - t\lambda^2}\\
&= \sum_{e^{2i\pi x}\in\mathrm{Spec}(\rho(f))} \chi_\rho(f_j^r \mid V^x)e^{-\pi^2r^2/t\alpha_j^2}\theta(x+i\frac{\pi r}{t\alpha_j},t)\\
&= \frac{\alpha_j\ell(f_j)}{\sqrt{4\pi t}}\sum_{n\in\mathbb{Z}}\chi_\rho(f_j^{r+n\alpha_j})e^{-\ell(f_j^{r+n\alpha_j})^2/4t}
\end{aligned}$$

and

$$\begin{aligned}\mathrm{ch}(V)^{\theta}_{odd}(f_j^r)(t) &= \sqrt{t}\sum_{\lambda\in\mathrm{Spec}(iT)}\chi_\rho(f_j^r\mid V^x)\lambda e^{-2i\pi r\lambda/\alpha_j - t\lambda^2}\\ &= -\frac{i\alpha_j\ell(f_j)}{4t\sqrt{\pi}}\sum_{n\in\mathbb{Z}}\chi_\rho(f_j^{r+n\alpha_j})\ell(f_j^{r+n\alpha_j})e^{-\ell(f_j^{r+n\alpha_j})^2/4t}\,.\end{aligned}$$

Note the similar expressions for regular and discrete orbits. All the homotopy classes of closed orbits of the circle action, together with their opposite, enter only once in the various contributions. We also observe that the spectral–dynamical duality in our Selberg–type trace formulae actually shows up at the level of these θ-regularised Chern characters, and only depends on the classical Poisson formula for the heat kernel on the circle.

Using Proposition 4.5 in (53), we obtain a dynamical expression for the contact eta function.

Theorem 4.6. *On a connected CR Seifert manifolds with finite singular orbits, one has*

$$\vartheta_S(t) = \vartheta_S^{dyn}(t) = \frac{1}{\sqrt{4\pi}}\sum_{\gamma\in\mathcal{C}}\chi_\rho(\gamma)\sigma(\gamma)(t)$$

where powers of the generic orbit $\gamma = f^n$, *with* $n\in\mathbb{Z}$, *contribute to*

$$\sigma(\gamma)(t) = \frac{i\ell(f)}{2t}\left\langle(\ell(\gamma)+i\mathbf{c})e^{-(\ell(\gamma)+i\mathbf{c})^2/4t}\wedge\mathcal{L}(N),[N_{smooth}]\right\rangle,$$

while powers of the singular orbits $\gamma = f_j^p$, *with* $p\not\equiv 0 \mod \alpha_j$, *contribute to*

$$\sigma(\gamma)(t) = \frac{i\ell(f_j)}{2t}\ell(\gamma)e^{-\ell(\gamma)^2/4t}\,\nu(\gamma)\,.$$

Remark 4.7. We observe that $\sigma(\gamma)(t)$ is real. Indeed for the smooth component, only the odd powers of this polynomial in $\mathbf{c}$ contribute to it since $\mathcal{L}(N)$ has components in degrees $4p$ while $\dim N = 4k-2$. Then $\sigma(\gamma^{-1})(t) = \sigma(\gamma)(t)$ so that $\vartheta_S(t)$ is real on unitary representations.

4.3. **Applications to the contact eta function $\eta(S_Q)(s)$.**

In order to get applications to the contact eta invariant we need now to study our theta–regularised contact eta function ϑ_S from its "zeta" viewpoint $\eta(S_Q)(s)$.

We first study the odd zeta regularised Chern character of V i.e. the series

$$\mathrm{ch}(V)^{\zeta}_{odd}(s) = \sum_{\lambda\in\mathrm{Spec}^*(iT)}\frac{\lambda\,\mathrm{ch}(V_\lambda)}{|\lambda|^{2s+1}} = \sum_{\lambda\in\mathrm{Spec}^*(iT)}\dim V^x\frac{\lambda e^{\lambda\mathbf{c}}}{|\lambda|^{2s+1}}$$

We proceed by taking Mellin transform of $\mathrm{ch}(V)^{\theta}_{odd}(t)$. From Proposition 4.5 its smooth part reads

$$\begin{aligned}\mathrm{ch}(V)^{\theta}_{odd}(t) &= \sqrt{t}\sum_{\lambda\in\mathrm{Spec}(iT)}\dim V^x\lambda e^{\lambda\mathbf{c}-t\lambda^2}\\ &= \sum_{\lambda\in\mathrm{Spec}^*(iT)}\dim V^x\sum_{p=0}^{2k-1}\frac{\mathbf{c}^p}{p!}\lambda^{p+1}\sqrt{t}e^{-t\lambda^2}\,,\end{aligned}$$

with $\dim N = 4k-2$. Then one finds by dominated convergence and direct computation that for $\Re(s) > k$ the following integral converges and

$$I(s) = \int_0^{+\infty} \mathrm{ch}(V)^{\theta}_{odd}(t) t^{s-1} dt = \Gamma(s+\frac{1}{2}) \sum_{p=0}^{2k-1} \frac{\mathbf{c}^p}{p!} \sum_{e^{2i\pi x} \in \mathrm{Spec}(\rho(f))} Z(p,s,x)$$

with

$$Z(p,s,x) = \begin{cases} \dim V^x(\zeta(2s-p,x) + (-1)^{p+1}\zeta(2s-p,1-x)) & \text{if } 0<x<1 \\ \dim V^1(1+(-1)^{p+1})\zeta(2s-p) & \text{if } x=1 \end{cases} \tag{54}$$

where $\zeta(s,x)$ is the Hurwitz zeta function (43). Therefore $I(s)/\Gamma(s+\frac{1}{2})$ defines a meromorphic function with possible simple poles at $s = \frac{j}{2}$ for $j \in [[1, 2k]]$. Hence we get that the series

$$\begin{aligned} \mathrm{ch}(V)^{\zeta}_{odd}(s) &= \sum_{\lambda \in \mathrm{Spec}(iT)*} \lambda \frac{\mathrm{ch}(V_\lambda)}{|\lambda|^{2s+1}} = \frac{I(s)}{\Gamma(s+\frac{1}{2})} \\ &= \sum_{p=0}^{2k-1} \frac{\mathbf{c}^p}{p!} \sum_{e^{2i\pi x} \in \mathrm{Spec}(\rho(f))} Z(p,s,x) \end{aligned} \tag{55}$$

are well defined and meromorphic on the same domain.

On the discrete part $\mathrm{ch}(V)^{\theta}_{odd}(f^r_j)$ as given in Proposition 4.5, one finds that for $\Re(s) > 1/2$

$$\mathrm{ch}(V)^{\zeta}_{odd}(f^r_j)(s) = \sum_{e^{2i\pi x} \in \mathrm{Spec}(\rho(f))} \chi_\rho(f^r_j \mid V^x) Z(f^r_j, s, x) \tag{56}$$

with

$$Z(f^r_j, s, x) = \begin{cases} e^{-2i\pi r x/\alpha_j} L(e^{-2i\pi r/\alpha_j}, 2s, x) - e^{2i\pi r(1-x)/\alpha_j} L(e^{2i\pi r/\alpha_j}, 2s, 1-x) & \text{for } 0<x<1 \\ e^{-2i\pi r/\alpha_j} L(e^{-2i\pi r/\alpha_j}, 2s, 1) - e^{2i\pi r/\alpha_j} L(e^{2i\pi r/\alpha_j}, 2s, 1) & \text{for } x=1\,. \end{cases} \tag{57}$$

where

$$L(z,s,x) = \sum_{n\geq 0} \frac{z^n}{(n+x)^s}$$

is the Lerch zeta function. In our case, $z = e^{\pm 2i\pi r/\alpha_j}$ is a root of unity and

$$L(z,s,x) = \frac{1}{\alpha_j^s} \sum_{m=0}^{\alpha_j - 1} z^m \zeta(s, \frac{m+x}{\alpha_j}) \tag{58}$$

splits into a sum of Hurwitz zeta functions. From [2, p. 255], $\zeta(s,x)$ is analytic with a simple pole at $s = 1$ and residue 1. Therefore $\mathrm{Res}_{s=1/2} L(e^{\pm 2i\pi r/\alpha_j}, 2s, 0) = 0$ and the functions $Z(f^r_j, s, x)$ are entire.

This gives an explicit geometric expression for $\mathrm{ch}(V)^{\zeta}_{odd}(s)$ that leads with (50) and (53) to the formula for $\eta(S_Q)(s)$ from

$$\begin{aligned} -\eta(S_Q)(s) &= \langle \mathrm{ch}(V)^{\zeta}_{odd}(s) \wedge \mathcal{L}(N)_{orb}, [N]\rangle_{orb} \\ &= \langle \mathrm{ch}(V)^{\zeta}_{odd}(s) \wedge \mathcal{L}(N), [N_{smooth}]\rangle + \sum_i \frac{1}{\alpha_i} \sum_{r=1}^{\alpha_i - 1} \mathrm{ch}(V)^{\zeta}_{odd}(f_i^r)(s)\nu(f_i^r)\,. \end{aligned} \tag{59}$$

Specialising at the regular value $s = 0$ gives an explicit formula for the contact eta invariant $\eta(S_Q)(0)$. Let

$$B(t,x) = \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{+\infty} B_n(x)\frac{t^n}{n!}$$

be the generating function of Bernoulli polynomials; see e.g. [2], and

$$B_{ev}(t,x) = t\frac{\cosh(t(x-\frac{1}{2}))}{2\sinh(\frac{t}{2})} = \sum_{n=0}^{+\infty} B_{2n}(x)\frac{t^{2n}}{(2n)!}$$

its even part in t. We consider the function

$$\begin{aligned} (\Delta B_{ev})(t,x) &= \frac{1}{t}(B_{ev}(t,x) - 1) = \sum_{n=1}^{+\infty} B_{2n}(x)\frac{t^{2n-1}}{(2n)!} \\ &= \frac{\cosh(t(x-\frac{1}{2}))}{2\sinh(\frac{t}{2})} - \frac{1}{t}\,. \end{aligned}$$

We shall also need

$$\widetilde{B}(t,x) = \begin{cases} B(t,x) & \text{for } 0 < x < 1 \\ \frac{1}{2}(B(t,1) + B(t,0)) & \text{for } x = 1\,. \end{cases}$$

In the following statement, we recall that $V_f^x = \ker(\rho(f) - e^{2i\pi x}\,\mathrm{id})$ for $0 < x \leq 1$.

Theorem 4.8. *On a connected CR Seifert manifold with finite singular orbits, it holds that*

$$\begin{aligned} \eta(S_Q)(0) = 2 &\sum_{e^{2i\pi x} \in \mathrm{Spec}(\rho(f))} \dim V_f^x \langle (\Delta B_{ev})(\mathbf{c}, x) \wedge \mathcal{L}(N), [N]_{smooth}\rangle \\ &+ 2 \sum_{e^{2i\pi x} \in \mathrm{Spec}(\rho(f))} \sum_j \frac{1}{\alpha_j} \sum_{r=1}^{\alpha_j - 1} \chi_\rho(f_j^r \mid V_f^x) \frac{\widetilde{B}(-2i\pi r/\alpha_j, x)}{-2i\pi r/\alpha_j} \nu(f_j^r)\,. \end{aligned}$$

Proof.

We have to evaluate $Z(0,p,x)$ as given in (54). From e.g. [2, p. 264], we know that

$$\zeta(-p,x) = -\frac{B_{p+1}(x)}{p+1} \quad \text{for } p \in \mathbb{N}\,.$$

Moreover $B_n(1-x) = (-1)^n B_n(x)$, so that $Z(0,x,p) = 0$ for p even, while for p odd

$$Z(0,x,p) = -2\dim V^x \frac{B_{p+1}(x)}{p+1}\,.$$

Hence from (55)

$$\begin{aligned}\mathrm{ch}(V)^{\zeta}_{odd}(0) &= -2 \sum_{e^{2i\pi x}\in \mathrm{Spec}(f)} \dim V_f^x \sum_{p=1\, p\, odd}^{2k-1} B_{p+1}(x) \frac{\mathbf{c}^p}{(p+1)!} \\ &= -2 \sum_{e^{2i\pi x}\in \mathrm{Spec}(f)} \dim V_f^x (\Delta B_{ev})(\mathbf{c}, x)\,,\end{aligned}$$

as needed. Note that one can replace ΔB_{ev} by ΔB in the formula for $\eta(S_Q)(0)$ since $\langle (\Delta B_{odd}(\mathbf{c}, x) \wedge \mathcal{L}(N), [N]_{smooth}\rangle = 0$ for dimensional reasons.

We compute the discrete contribution of f_j^r. Let $z = e^{\pm 2i\pi r/\alpha_j}$. From $\zeta(0, x) = 1/2 - x$ and (58), one gets

$$\begin{aligned}L(z, 0, x) &= \sum_{m=0}^{\alpha_j - 1} z^m (\frac{1}{2} - \frac{m + x}{\alpha_j}) = -\frac{1}{\alpha_j} \sum_{m=0}^{\alpha_j - 1} m z^m \\ &= \frac{1}{1 - z}\,,\end{aligned}$$

by differentiating the identity $\displaystyle\sum_{m=0}^{\alpha_i - 1} u^{m+1} = \frac{u(1 - u^{\alpha_j})}{1 - u}$ at $u = z$. Inserting it in (57) gives the result.

□

4.4. **Comparison with related results.**

Theorem 4.8 generalises an expression given in [7, Theorem 8.8] in the three dimensional case and with a trivial representation. (Note that in [7], $\zeta(P)(s)$ includes $\dim\ker P$ by convention.) There one has

$$\eta(S_Q)(0) = \eta(*D + d_Q \delta_Q)(0) = \eta_0(M, \theta) - 1\,,$$

where $\eta_0(M, \theta) = \mathrm{FP}_{\varepsilon=0}\, \eta(*_{g_\varepsilon} d)$ is the renormalised eta invariant of $*_{g_\varepsilon} d$ for the sub-Riemannian limit of metrics $g_\varepsilon = d\theta(\cdot, J\cdot) + \varepsilon^{-1}\theta^2$ when $\varepsilon \to 0$. From [7, Section 3], this also coincides with the adiabatic limit of $\eta(*_{g_\varepsilon} d)$ for $\varepsilon \to +\infty$ in our fibred case.

In general dimension, the formula for $\eta(S_Q)(0)$ in Theorem 4.8 no longer shows up global (over M) cohomological terms, in contrast then to $\eta(*D)(0)$ by Proposition 4.2. This is related to the fact that the spectrum of S_Q is more symmetric than the one of $*D$ by Proposition 4.1. While $*D$ acts on a smaller space than S_Q, it contains un-symmetrised copies of zeta functions of Laplacians, leading to additional cohomological terms at $s = 0$.

Also, from [1], one knows that $\eta(*D)(0)$ compares on general contact manifolds with the renormalised sub-Riemanian limit $\eta_0(M, \theta) = FP_{\varepsilon=0}(*g_\varepsilon d)$, up to local terms. It coincides with the adiabatic limit of the eta invariant in our CR Seifert situation. This limit has been studied in depth by other means. Building on previous works by Bismut and Cheeger [8], Dai [11] and Zhang [24] expressed $\eta_0(M, \theta)$ for circle bundles, in the case of a trivial representation and a smooth quotient N. Taking into account that their eta invariant is twice ours, which is defined using $*D$ only instead of $*D + D*$, one sees from [11, Theorem 0.3] and [24, Theorem 1.7] that

$$\eta_0(M,\theta) = 2\langle \Delta B_{ev}(\mathbf{c},1) \wedge \mathcal{L}(N), [N]\rangle + \tau \tag{60}$$

where τ is the signature of the "collapsing spectrum" in the adiabatic limit. More precisely, from [11, Section 4.3], τ is computed using the Leray spectral sequence of the fibration $\mathbf{S}^1 \to M \to N$.

We briefly present this in relation to Proposition 4.2. From [11], (12) and Section 2.2, one finds that the E_2-term of the Leray spectral sequence identifies with

$$\ker T \cap \ker d_H \cap \ker \delta_H = \ker \boldsymbol{\Delta}_Q \,,$$

where $\boldsymbol{\Delta}_Q$ is acting diagonally on horizontal, vertical and Lefschetz components of $\Omega^* M$. Then, following [11], τ is the signature of the symmetric bilinear form

$$\begin{aligned} Q : E_2^{2k-1,\theta} \otimes E_2^{2k-1,\theta} &\to \mathbb{R} \\ \alpha \otimes \beta &\mapsto \int_M \alpha \wedge d_2\beta \,. \end{aligned}$$

using the second differential of the Leray spectral sequence

$$\begin{aligned} d_2 : E_2^{2k-1,\theta} &\to E_2^{2k} \\ \theta \wedge \alpha &\mapsto L\alpha \end{aligned}$$

Now, by e.g. [22, Theorem 2], one has for $\alpha \in \mathcal{E}^{2p}$

$$*_H \alpha = \frac{1}{(2k-1-2p)!} L^{2k-1-2p} \sigma\alpha \,,$$

and therefore

$$\begin{aligned} Q(\theta \wedge L^{k-p-1}\alpha, \theta \wedge L^{k-p-1}\beta) &= \int_M \theta \wedge L^{2k-2p-1}\alpha \wedge \beta \\ &= (2k-1-2p)! \int_M (\sigma\alpha, \beta)\, \mathrm{dvol} \,, \end{aligned}$$

from which it follows that

$$\tau = \text{Signature}\,(Q) = \sum_{0 \le a+b = 2p \le 2(k-1)} (-1)^p i^{a-b} \dim H^{a,b}(M) \,.$$

This is the cohomological sum in Proposition 4.2. Hence, for smooth quotient N and trivial representation, this relates (60) to the equations

$$\eta_0(M,\theta) = \eta(S_Q)(0) + \tau = \eta(*D)(0) + \text{local terms} \,.$$

We note in conclusion that in the adiabatic viewpoint, the term $2\langle \Delta B_{ev}(\mathbf{c},1) \wedge \mathcal{L}(N), [N]\rangle$ comes from the analysis of an eta form constructed by Bismut and Cheeger [8]. It is associated to a Dirac operator for a superconnection over the fibers. Here, in this smooth case, from our viewpoint

$$2\langle \Delta B_{ev}(\mathbf{c},1) \wedge \mathcal{L}(N), [N]\rangle = \eta(S_Q)(0) \,,$$

so that this term is interpreted as being the eta invariant of a second order hypoelliptic differential operator over the whole contact manifold M.

4.5. **The contact eta function from the dynamical viewpoint.**

We eventually work out the dynamical aspect of the eta function and eta invariant. We start with the smooth part. From Proposition 4.5 we have that for some $a > 0$

$$\operatorname{ch}(V)^{\theta}_{odd}(t) = O(e^{-at}) \quad \text{when} \quad t \to +\infty \,.$$

We study the behaviour of $\operatorname{ch}(V)^{\theta}_{odd}(t)$ when $t \to 0^+$. Its dynamical expression contains terms $e^{-(\ell(\gamma)+i\mathbf{c})^2/4t}$ with $\gamma = f^n$ that can be written

$$\begin{aligned} e^{-(\ell(\gamma)+i\mathbf{c})^2/4t} &= e^{-\ell^2(\gamma)/4t} e^{(-2i\mathbf{c}\ell(\gamma)+\mathbf{c}^2)/4t} \\ &= e^{-\ell^2(\gamma)/4t} \sum_{p=0}^{2k-1} \frac{1}{p!}\Big(\frac{-2i\mathbf{c}\ell(\gamma)+\mathbf{c}^2}{4t}\Big)^p \\ &= O(e^{-\ell^2(\gamma)/8t}) \end{aligned}$$

when $t \to 0^+$ and $n \neq 0$. Therefore the divergence of $\operatorname{ch}(V)^{\theta}_{odd}(t)$ when $t \to 0$ only comes from the trivial constant orbit and one has for some $b > 0$

$$\operatorname{ch}(V)^{\theta}_{odd}(t) - \dim V \frac{\ell(f)\mathbf{c}}{4t\sqrt{\pi}} e^{\mathbf{c}^2/4t} = O(e^{-b/t}) \,,$$

with

$$\frac{\mathbf{c}}{4t} e^{\mathbf{c}^2/4t} = \sum_{p=0}^{k-1} \frac{\mathbf{c}^{2p+1}}{p!(4t)^{p+1}} \,.$$

This shows that the function

$$f(t) = \operatorname{ch}(V)^{\theta}_{odd}(t) - \dim V \frac{\ell(f)\mathbf{c}}{4t\sqrt{\pi}} e^{\mathbf{c}^2/4t} \mathbf{1}_{]0,1]}(t)$$

has an entire Mellin transform over $\mathbb{C}$. Proceeding as in Section 3.6 we first find that for $\Re(s) > k$

$$\begin{aligned} I(s) = \int_0^{+\infty} f(t) t^{s-1} dt &= \Gamma(s+\frac{1}{2}) \operatorname{ch}(V)^{\zeta}_{odd}(s) \\ &\qquad - \dim V \sum_{p=0}^{k-1} \frac{\ell(f)\mathbf{c}^{2p+1}}{p! 4^{p+1}\sqrt{\pi}} \times \frac{1}{s-p-1} \,, \end{aligned}$$

While writing

$$f(t) = \operatorname{ch}(V)^{\theta}_{odd}(t) - \dim V \frac{\ell(f)\mathbf{c}}{4t\sqrt{\pi}} e^{\mathbf{c}^2/4t} + \dim V \frac{\ell(f)\mathbf{c}}{4t\sqrt{\pi}} e^{\mathbf{c}^2/4t} \mathbf{1}_{]1,+\infty]}(t)$$

and using the dynamical expression for $\operatorname{ch}(V)^{\theta}_{odd}(t)$ one has for $\Re(s) < 0$

$$\begin{aligned} I(s) = \Gamma(1-s)\Big(\sum_{n\in\mathbb{Z}^*} &-\frac{i\ell(f)}{4^s\sqrt{\pi}} \chi_\rho(f^n)(\ell(f^n)+i\mathbf{c})\big((\ell(f^n)+i\mathbf{c})^2\big)^{s-1}\Big) \\ &- \dim V \sum_{p=0}^{k-1} \frac{\ell(f)\mathbf{c}^{2p+1}}{p! 4^{p+1}\sqrt{\pi}} \times \frac{1}{s-p-1} \,. \end{aligned}$$

Here z^s denotes the principal branch of the power function. We get then the identity of meromorphic functions through analytic continuation

$$\Gamma(s+1/2)\,\mathrm{ch}(V)^{\zeta}_{odd}(s) = \Gamma(1-s)\sum_{n\in\mathbb{Z}^*} -\frac{i\ell(f)}{4^s\sqrt{\pi}}\chi_\rho(f^n)(\ell(f^n)+i\mathbf{c})\big((\ell(f^n)+i\mathbf{c})^2\big)^{s-1}\,.$$

One finds more easily the discrete dynamical contribution of f_j^r due to rapid decay of $\mathrm{ch}(V)^{\theta}_{odd}(f_j^r)$ both at 0 and $+\infty$ from Proposition 4.5. It gives that

$$\Gamma(s+1/2)\,\mathrm{ch}(V)^{\zeta}_{odd}(f_j^r)(s) = \Gamma(1-s)\sum_{n\in\mathbb{Z}} -\frac{i\alpha_j\ell(f_j)}{4^s\sqrt{\pi}}\chi_\rho(f_j^{r+n\alpha_j})\ell(f_j^{r+n\alpha_j})\big(\ell(f_j^{r+n\alpha_j})^2\big)^{s-1}$$

is an entire function. Gathering this with (59) we get the following.

Theorem 4.9. *The following identity defines a meromorphic function* φ *with simple poles at* $s=1,\cdots,k$

$$\varphi(s) = \Gamma(s+\frac{1}{2})\eta(S_Q)(s) = \frac{\Gamma(1-s)}{4^s\sqrt{\pi}}\sum_{\gamma\in\mathcal{C}^*}\chi_\rho(\gamma)\eta(\gamma)(s)$$

with

$$\eta(\gamma)(s) = i\ell(f)\langle(\ell(\gamma)+i\mathbf{c})\big((\ell(\gamma)+i\mathbf{c})^2\big)^{s-1}\wedge\mathcal{L}(N),[N_{smooth}]\rangle \text{ if } \gamma=f^n\,,$$

and

$$\eta(\gamma)(s) = i\ell(f_j)\ell(\gamma)(\ell(\gamma)^2)^{s-1}\nu(\gamma) \text{ if } \gamma=f_j^n,\ n\not\equiv 0 \mod \alpha_j\,.$$

Moreover

$$\mathrm{Res}_{s=p}(\varphi) = \dim V\frac{\ell(f)}{\sqrt{\pi}(p-1)!4^p}\langle\mathbf{c}^{2p-1}\wedge\mathcal{L}(N),[N_{smooth}]\rangle\,.$$

The function being regular at $s=0$ we get a dynamical formula for the eta invariant.

Corollary 4.10. *On a connected CR Seifert manifold with finite singular orbits, one has*

$$\eta(S_Q)(0) = \sum_{\gamma\in\mathcal{C}^*}\chi_\rho(\gamma)\eta(\gamma)(0)$$

with

$$\eta(\gamma)(0) = \frac{i\ell(f)}{\pi}\langle\frac{1}{\ell(\gamma)+i\mathbf{c}}\wedge\mathcal{L}(N),[N_{smooth}]\rangle \quad \text{if } \gamma=f^n\,,$$

and

$$\eta(\gamma)(0) = \frac{i\ell(f_j)}{\pi\ell(\gamma)}\nu(\gamma) \quad \text{if } \gamma=f_j^n,\ n\not\equiv 0 \mod \alpha_j\,.$$

This is a decomposition of the eta invariant into its dynamical "atoms". These dynamical series are *a priori* formal expressions coming from analytic continuation. However they can be turned into convergent ones. The smooth contribution is actually an absolutely convergent series using the $\mathbf{c}\leftrightarrow-\mathbf{c}$ symmetry; see Remark 4.7. It comes as a limit of the smooth dynamical expression in Theorem 4.9 when $s\to 0^-$. The discrete contribution are semi-convergent series when gathering the orbit contributions of $\gamma=f_j^{r+n\alpha_j}$ and $\overline{\gamma}=f_j^{r-n\alpha_j}$. Using Abel's lemma one sees that it is also the limit coming from the corresponding dynamical expression of Theorem 4.9 when $s\to 0^-$.

References

[1] P. Albin and H. Quan. Sub-Riemannian limit of the differential form heat kernels of contact manifolds. *Int. Math. Res. Not.*, 2022(8):5818–5881, 2022.

[2] T. M. Apostol. *Introduction to analytic number theory.* Undergraduate Texts Math. Springer, Cham, 1976.

[3] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. II: Applications. *Ann. Math. (2)*, 88:451–491, 1968.

[4] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[5] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Philos. Soc.*, 79(1):71–99, 1976.

[6] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

[7] O. Biquard, M. Herzlich, and M. Rumin. Diabatic limit, eta invariants and Cauchy-Riemann manifolds of dimension 3. *Ann. Sci. Ecole Norm. Sup. (4)*, 40(4):589–631, 2007.

[8] J.-M. Bismut and J. Cheeger. η-invariants and their adiabatic limits. *J. Amer. Math. Soc.*, 2(1):33–70, 1989.

[9] C. P. Boyer and K. Galicki. *Sasakian geometry.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[10] J. Cheeger. Analytic torsion and the heat equation. *Ann. of Math. (2)*, 109(2):259–322, 1979.

[11] X. Dai. Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. *J. Amer. Math. Soc.*, 4(2):265–321, 1991.

[12] D. Fried. Lefschetz formulas for flows. In *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 19–69. Amer. Math. Soc., Providence, RI, 1987.

[13] D. Fried. Counting circles. In *Dynamical systems (College Park, MD, 1986–87)*, volume 1342 of *Lecture Notes in Math.*, pages 196–215. Springer, Berlin, 1988.

[14] T. Kawasaki. The Riemann-Roch theorem for complex V-manifolds. *Osaka J. Math.*, 16(1):151–159, 1979.

[15] A. Kitaoka. Ray-Singer torsion and the Rumin Laplacian on Lens spaces. *SIGMA, Symmetry Integrability Geom. Methods Appl.*, 18:paper 091, 16, 2022.

[16] W. Müller. Analytic torsion and R-torsion of Riemannian manifolds. *Adv. Math.*, 28(3):233–305, 1978.

[17] D. B. Ray and I. M. Singer. R-torsion and the Laplacian on Riemannian manifolds. *Adv. Math.*, 7:145–210, 1971.

[18] M. Rumin. Formes différentielles sur les variétés de contact. *J. Differential Geom.*, 39(2):281–330, 1994.

[19] M. Rumin. Sub-Riemannian limit of the differential form spectrum of contact manifolds. *Geom. Funct. Anal.*, 10(2):407–452, 2000.

[20] M. Rumin and N. Seshadri. Analytic torsions on contact manifolds. *Ann. Inst. Fourier (Grenoble)*, 62(2):727–782, 2012.

[21] I. Satake. The Gauss-Bonnet theorem for V-manifolds. *J. Math. Soc. Japan*, 9:464–492, 1957.

[22] A. Weil. Introduction à l'étude des variétés kählériennes. Actualités Scientifiques et Industrielles, No. 1267, Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Paris: Hermann & Cie. 175 pp. (1958)., 1958.

[23] E. T. Whittaker and G. N. Watson. *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions.* Fourth edition. Reprinted. Cambridge University Press, New York, 1962.

[24] W. P. Zhang. Circle bundles, adiabatic limits of η-invariants and Rokhlin congruences. *Ann. Inst. Fourier (Grenoble)*, 44(1):249–270, 1994.

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