Les Diablerets, 7-11 January, 2019 Anne Moreau, University of Lille

EXERCISES ON THE LECTURE "VERTEX ALGEBRAS AND ASSOCIATED VARIETIES"

Unless otherwise specified, the notations are those of the lectures.

Part 1.

Exercise 1 (On the translation axiom). Let V be a $\mathbb{C}$-vector space.

(1) Assume that V is a vertex algebra, and fix $a \in V$. Verify that for all $n \in \mathbb{Z}$,
$$[T, a_{(n)}] = -na_{(n-1)}, \quad (Ta)_{(n)} = -na_{(n-1)},$$
and deduce from this that
$$Ta = a_{(-2)}|0\rangle.$$

(2) Conversely, verify that if the vector space V is endowed with a vector $|0\rangle \in V$ and a linear map $F \to \mathscr{F}(V)$, $a \mapsto a(z)$ such that the vacuum and the locality axioms hold, then the linear map
$$V \to V, \quad a \mapsto a_{(-2)}|0\rangle$$
satisfies the translation axiom. *This shows that the translation operator T is in fact a redondant datum in the definition of a vertex algebra.*

Hints for Exercise 1. (1) Use the translation axiom.

(2) Use the vacuum axiom.

(3) Compare $(\partial_z a(z))_{(-1)}|0\rangle$ and $a_{(-2)}|0\rangle$, and compute $[T, a(z)]|0\rangle|_{z=0}$.

Exercise 2 (Commutative algebras equipped with a derivation are commutative vertex algebras). Show that there is a unique structure of a commutative vertex algebra on a commutative algebra R equipped with a derivation ∂ such that the vacuum vector is the unit, and
$$a(z)b = \left(e^{z\partial}a\right) b = \sum_{n\geqslant 0} \frac{z^n}{n!}(\partial^n a)b \quad \text{for all} \quad \in R.$$

Hints for Exercise 2. Notice that the locality axiom is automatically satisfied by the OPE.

Exercise 3 (Center of a vertex algebra). For V a vertex algebra, its (vertex) center $\mathcal{Z}(V)$ is defined by:
$$\mathcal{Z}(V) := \{a \in V \mid [b(z), a(w)] = 0 \text{ for all } b \in V\}.$$
Show that the following are equivalent:

(i) $a \in \mathcal{Z}(V)$,

(ii) $[b_{(m)}, a_{(n)}] = 0$ for all $b \in V$ and all $m, n \in \mathbb{Z}$,

(iii) $b(z)a \in V[[z]]$ for all $b \in V$,

(iv) $b_{(m)}a = 0$ for all $b \in V$ and all $m \in \mathbb{Z}_{\geqslant 0}$.

Hints for Exercise 3. First, note that the equivalences (i) $\iff$ (ii) and (iii) $\iff$ (iv) are clear. To show (i) $\iff$ (iii), observe that $b(z)a = b(z)a(w)|0\rangle|_{w=0}$.

Exercise 4 (On the center of the universal affine vertex algebra). Let us consider the universal affine vertex algebra $V^k(\mathfrak{g})$ associated with a simple Lie algebra $\mathfrak{g}$ at level $k \in \mathbb{C}$.

(1) Show that $\mathcal{Z}(V^k(\mathfrak{g})) = V^k(\mathfrak{g})^{\mathfrak{g}[[t]]}$, that is,
$$\mathcal{Z}(V^k(\mathfrak{g})) = \{a \in V^k(\mathfrak{g} \mid x_{(m)}a = 0 \text{ for all } x \in \mathfrak{g},\ m \in \mathbb{Z}_{\geqslant 0}\}.$$

(2) Show that we have the following isomorphism of commutative $\mathbb{C}$-algebras (the product on the commutative vertex algebra $\mathcal{Z}(V^k(\mathfrak{g}))$ is the normally ordered product):

$$\mathcal{Z}(V^k(\mathfrak{g})) \cong \mathrm{End}_{\widehat{\mathfrak{g}}}(V^k(\mathfrak{g})).$$

We shall first prove that $\mathcal{Z}(V^k(\mathfrak{g}))$ naturally embeds into $\mathrm{End}_{\widehat{\mathfrak{g}}}(V^k(\mathfrak{g}))$.

(3) Prove that if $k \neq -h^\vee$, then $\mathcal{Z}(V^k(\mathfrak{g})) = \mathbb{C}|0\rangle$.
For $k = -h^\vee$, the center $\mathcal{Z}(V^{-h^\vee}(\mathfrak{g})) =: \mathfrak{z}(\widehat{\mathfrak{g}})$ is "huge", and it is usually referred as the Feigin-Frenkel center: we have $\mathrm{gr}\,\mathfrak{z}(\widehat{\mathfrak{g}}) \cong \mathbb{C}[J_\infty(\mathfrak{g}/\!/G)]$, with $\mathfrak{g}/\!/G = \mathrm{Spec}\,\mathbb{C}[\mathfrak{g}]^G$.

Hints for Exercise 4. (1) Follows from Exercise 3.

(2) Apply the "Frobenius reciprocity", which asserts that

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}}(U(\widehat{\mathfrak{g}}) \otimes_{\mathfrak{g}[t]\oplus\mathbb{C}K} \mathbb{C}_k, V^k(\mathfrak{g})) \cong \mathrm{Hom}_{\mathfrak{g}[t]\oplus\mathbb{C}K}(\mathbb{C}_k, V^k(\mathfrak{g})).$$

(3) Use the Segal-Sugawara conformal vector ω.

Part 2.

Exercise 5 (Poisson structure on the Zhu's C_2-algebra of the universal affine vertex algebra). Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with a simple Lie algebra $\mathfrak{g}$ at level $k \in \mathbb{C}$.

(1) Show that the map

$$\begin{aligned} \mathbb{C}[\mathfrak{g}^*] \cong S(\mathfrak{g}) &\longmapsto V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}) \\ x_1 \dots x_r &\longmapsto (x_1t^{-1}) \dots (x_rt^{-1})|0\rangle + t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}), \qquad x_1, \dots, x_r \in \mathfrak{g}. \end{aligned}$$

defines an isomorphism of commutative algebras, the product on the right-hand side being given by:

$$\big((x_1t^{-1}) \dots (x_rt^{-1})|0\rangle\big) . \big((y_1t^{-1}) \dots (y_st^{-1})|0\rangle\big) = (x_1t^{-1}) \dots (x_rt^{-1})(y_1t^{-1}) \dots (y_st^{-1})|0\rangle,$$

for $x_i, y_j \in \mathfrak{g}$.

(2) Verify that

$$R_{V^k(\mathfrak{g})} = V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}),$$

and show that the Poisson bracket on $R_{V^k(\mathfrak{g})}$ is the one induced from the isomorphism of (1).

Hints for Exercise 5. (1) Use the PBW basis to show the bijectivity, the rest of the verifications are clear.

(2) Just verify using the commuting relations that for $x, y \in \mathfrak{g}$,

$$\{x, y\} = [x, y] = \bar{x}_{(0)}\bar{y},$$

where $\bar{x}$ stands for the image of x, viewed as an element of $\mathfrak{g} \cong V^k(\mathfrak{g})_1$, in $R_{V^k(\mathfrak{g})}$.

Exercise 6 (Zhu's C_2-algebra and associated variety of the universal Virasoro vertex algebra). Let Vir^c be the universal Virasoro vertex algebra of central charge $c \in \mathbb{C}$.

(1) Show that $\mathrm{gr}^F \mathrm{Vir}^c \cong \mathbb{C}[L_{-2}, L_{-3}, \dots]$, where F is the Li filtration.

(2) Deduce from (1) that $R_{\mathrm{Vir}^c} \cong \mathbb{C}[x]$, where x is the image of $L := L_{-2}|0\rangle$ in R_{Vir^c}, with the trivial Poisson structure, and that $X_{\mathrm{Vir}^c} = \mathbf{A}^1$ is the affine line.

(3) Show that one can endow $\mathrm{gr}^F \mathrm{Vir}^c$ with a non-trivial Poisson vertex algebra structure such that

$$\overline{L}_{-1}\overline{L} = \overline{L}_{(0)}\overline{L} = T\overline{L} \quad \text{and} \quad \overline{L}_0\overline{L} = \overline{L}_{(1)}\overline{L} = 2\overline{L}, \quad \text{with} \quad \overline{L} := \sigma_0(L).$$

Hints for Exercise 6. (1) Describe $F^p\mathrm{Vir}^c_\Delta$, where $\Delta \in \mathbb{Z}_{\geqslant 0}$, using the PBW Theorem.

(2) Just use (1).

(3) Remember that when the Poisson structure is trivial, one can go one step further, and then compute $\sigma_1(\overline{L}_{(0)}\overline{L})$, $\sigma_0(\overline{L}_{(1)}\overline{L})$ using the commuting relations.

Part 3.

Exercise 7 (Simple affine vertex algebras associated with $\mathfrak{sl}_2$). Let N be the proper maximal ideal of $V^k(\mathfrak{sl}_2)$ so that $L_k(\mathfrak{g}) = V^k(\mathfrak{sl}_2)/N$. Let I be the image of N in $R_{V^k(\mathfrak{sl}_2)} = \mathbb{C}[\mathfrak{g}]$ so that $R_{L_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}]/I$. It is known that either N is trivial, that is, $V^k(\mathfrak{sl}_2)$ is simple, or N is generated by a singular vector v whose image $\overline{v}$ in I is nonzero. We assume in this exercise that N is non trivial. Thus, $N = U(\widehat{\mathfrak{sl}_2})v$.

(1) Using Kostant's Separation Theorem show that, up to a nonzero scalar,

$$v = \Omega^m e^n,$$

for some $m, n \in \mathbb{Z}_{>0}$, where $\Omega = 2ef + \frac{1}{2}h^2$ is the Casimir element of the symmetric algebra of $\mathfrak{sl}_2$.

(2) Deduce from this that

$$X_{L_k(\mathfrak{g})} \subset \mathcal{N}.$$

It is known that N is nontrivial if and only k is an admissible level for $\mathfrak{sl}_2$, or $k = -2$ is critical. Thus we have shown that $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ if and only if $k = -2$ or k is admissible, i.e., $k = -2 + \frac{p}{q}$, with $(p, q) = 1$ and $p \geqslant 2$. This was proven by Feigin and Malikov.

Hints for Exercise 7. (1) For $\mathfrak{g} = \mathfrak{sl}_2$, Kostant's Separation Theorem says that $S = ZH$, where $Z \cong \mathbb{C}[\Omega]$ is the center of the symmetric algebra S of $\mathfrak{sl}_2$, and H is the space of invariant harmonic polynomials which decomposes, as an $\mathfrak{sl}_2$-module, as $H = \bigoplus_{\lambda\in\mathbb{Z}} V_\lambda^{m_\lambda}$, with $m_\lambda = 1$ for all λ since $\mathfrak{g} = \mathfrak{sl}_2$. Therefore, $S^{\mathrm{ad}\, e} = \bigoplus_{\lambda\in\mathbb{Z}} Z V_\lambda^{\mathrm{ad}\, e}$. To conclude, observe that, v being a singular vector, it has a fixed weight and, hence, a fixed degree.

(2) Note that from (1), $\Omega e \in \sqrt{I}$ and, so, $\Omega\mathfrak{g} \in \sqrt{I}$, whence $\Omega \in \sqrt{I}$. But in $\mathfrak{sl}_2$, $\mathcal{N}$ is the zero locus of Ω.

Exercise 8 (An explicit computation of an associated variety). The aim of this exercice is to compute $X_{L_{-3/2}(\mathfrak{sl}_3)}$. It was shown by Perše that the proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the singular vector v given by:

$$v := \frac{1}{3}\left((h_1t^{-1})(e_{1,3}t^{-1})|0\rangle - (h_2t^{-1})(e_{1,3}t^{-1})|0\rangle\right) + (e_{1,2}t^{-1})(e_{2,3}t^{-1})|0\rangle - \frac{1}{2}e_{1,3}t^{-2}|0\rangle,$$

where $h_1 := e_{1,1} - e_{2,2}$, $h_2 := e_{2,2} - e_{3,3}$ and $e_{i,j}$ is the elementary matrix of the coefficient (i, j) in $\mathfrak{sl}_3$ identified with the set of traceless 3-size square matrices.

(1) Verify that v is indeed a singular vector for $\widehat{\mathfrak{sl}_3}$, that is, $e_{i,i+1}v = 0$ for $i = 1, 2$ and $(e_{3,1}t)v = 0$.

(2) Let $\mathfrak{h} := \mathbb{C}h_1 + \mathbb{C}h_2$ be the usual Cartan subalgebra of $\mathfrak{sl}_3$. Show that $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h} = \{0\}$, and deduce from this that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ is contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{sl}_3$.

(3) Show that $\mathcal{N}$ is not contained in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.

(4) Denoting by $\mathbb{O}_{min}$ the minimal nilpotent orbit of $\mathfrak{sl}_3$, conclude that

$$X_{L_{-3/2}(\mathfrak{sl}_3)} = \overline{\mathbb{O}_{min}}.$$

Hints for Exercise 8. (1) Just use the commuting relations in $V^{-3/2}(\mathfrak{sl}_3)$.

(2) Observe that the image I of the maximal proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the vector $\bar{v}$ as an $(\mathrm{ad}\,\mathfrak{sl}_3)$-module, where

$$\bar{v} = \frac{1}{3}\left(h_1 - h_2\right) e_{1,3} + e_{1,2}e_{2,3}$$

is the image of v in $R_{V^{-3/2}(\mathfrak{sl}_3)} \cong \mathbb{C}[h_i, e_{k,l}\,;\, i = 1, 2, k \neq l]$. Verify that

$$(\mathrm{ad}\, e_{3,2})(\mathrm{ad}\, e_{2,1})\bar{v} = -e_{1,2}e_{2,1} + e_{1,3}e_{3,1} + \frac{1}{3}\left(2h_1 + h_2\right) h_2,$$

$$(\mathrm{ad}\, e_{2,1})(\mathrm{ad}\, e_{3,2})\bar{v} = -e_{2,3}e_{3,2} + e_{1,3}e_{3,1} + +\frac{1}{3}\left(h_1 + 2h_2\right) h_1,$$

and deduce from this that the intersection $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h}$ is zero. For the last part, remember that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ is closed, invariant, and conical.

(3) Verify that $e_{1,2} + e_{2,3}$ is not in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.

(4) Observe that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ cannot be reduced to zero.

Part 4.

Exercise 9 (A preliminary result for the BRST reduction). Let V be a vertex *superalgebra*, that is, a vector superspace $V = V_0 \oplus V_1$ satisfying the same axioms as a vertex algebra except that, in the locality axiom, the bracket $[a(z), b(w)]$ stands for

$$[a(z), b(w)] = a(z)b(w) - (-1)^{|a||b|} b(w)a(z).$$

Fix an odd element Q of V such that $Q_{(n)}Q = 0$ for all $n \geqslant 0$.

(1) Show that $Q_{(0)}^2 = 0$.

(2) Show that the quotient $\dfrac{\ker Q_{(0)}}{\operatorname{im} Q_{(0)}}$ is naturally a vertex algebra, provided it is nonzero.

Hints for Exercise 9. (1) Remember that Q is odd and, hence, that $Q_{(0)}^2 = \dfrac{1}{2}[Q_{(0)}, Q_{(0)}]$. Then use the Borcherds identity.

(2) Show that $\ker Q_{(0)}$ is a vertex subalgebra of V, and that $\operatorname{im} Q_{(0)}$ is a vertex ideal of it.

Exercise 10 (Definition of the W-algebra associated with $\mathfrak{sl}_2$ and a principal nilpotent element). Set

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that $\mathfrak{sl}_2 = \operatorname{span}_{\mathbb{C}}(e, h, f)$. The aim of this exercice is to define the W-algebra $\mathcal{W}^k(\mathfrak{sl}_2, f)$ associated with $\mathfrak{sl}_2$ and f at level $k \in \mathbb{C}$. Set $\mathfrak{n} := \mathbb{C}e$.

(1) Let $\hat{Cl}$ be the Clifford algebra associated with $\mathfrak{n}[t, t^{-1}] \oplus \mathfrak{n}^*[t, t^{-1}]$ and the symmetric bilinear form $(\ |\ )$ given by:

$$(et^m | et^n) = (e^*t^m | e^*t^n) = 0, \quad (et^m | e^*t^n) = \delta_{m+n,0}.$$

We write ψ_m for $et^m \in \hat{Cl}$ and ψ_m^* for $e^*t^m \in \hat{Cl}$, $m \in \mathbb{Z}$, so that $\hat{Cl}$ is the associative superalgebra with odd generators ψ_m, ψ_m^*, $m \in \mathbb{Z}$, and relations:

$$[\psi_m, \psi_n] = [\psi_m^*, \psi_n^*] = 0, \quad [\psi_m, \psi_n^*] = \delta_{m+n,0}.$$

Define the *charged fermion Fock space* as

$$\mathcal{F} := \frac{\hat{Cl}}{\sum_{m \geqslant 0} \hat{Cl}\psi_m + \sum_{n \geqslant 1} \hat{Cl}\psi_n^*}.$$

Show that there is a unique vertex (super)algebra structure on $\mathcal{F}$ such that the image of 1 is the vacuum $|0\rangle$, and

$$\psi(z) := Y(\psi_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \qquad \psi^*(z) := Y(\psi_0^*|0\rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}.$$

Let $V^k(\mathfrak{sl}_2)$ be the universal affine vertex algebra associated with $\mathfrak{sl}_2$ at level k, and set

$$\mathcal{C}^k(\mathfrak{sl}_2) := V^k(\mathfrak{sl}_2) \otimes \mathcal{F}.$$

Define a gradation $\mathcal{F} = \bigoplus_{p \in Z} \mathcal{F}^p$ by setting $\deg \psi_m = -1$, $\deg \psi_n^* = 1$ for all $m, n \in \mathbb{Z}$ and $\deg |0\rangle = 0$. Then set $\mathcal{C}^{k,p}(\mathfrak{sl}_2) := V^k(\mathfrak{sl}_2) \otimes \mathcal{F}^p$. Define a vector $\hat{Q}$ of degree 1 in $\mathcal{C}^{k,1}(\mathfrak{sl}_2)$ by:

$$\hat{Q}(z) := (e(z) + 1) \otimes \psi^*(z).$$

(2) Verify that $\hat{Q}_{(n)}\hat{Q} = 0$ for all $n \geqslant 0$, and deduce from Exercice 9 that the cohomology $H^\bullet(\mathcal{C}^k(\mathfrak{sl}_2), \hat{Q}_{(0)})$ inherits a vertex algebra structure from that of $\mathcal{C}^k(\mathfrak{sl}_2)$, provided that it is nonzero.
The W-algebra $\mathcal{W}^k(\mathfrak{sl}_2, f)$ associated with $(\mathfrak{sl}_2, f)$ at level $k \in \mathbb{C}$ is defined by:

$$\mathcal{W}^k(\mathfrak{sl}_2, f) := H^0(\mathcal{C}^k(\mathfrak{sl}_2), \hat{Q}_{(0)}).$$

This definition of $\mathcal{W}^k(\mathfrak{sl}_2, f)$ is due to Feigin and Frenkel. It can be generalized to any simple Lie algebra $\mathfrak{g}$ and to any nilpotent element.

(3) Assume that $k \neq -2$. Show that there exists a unique vertex algebra homomorphism

$$\mathrm{Vir}^{c(k)} \to \mathcal{W}^k(\mathfrak{sl}_2, f), \qquad \text{where} \qquad c(k) := 1 - \frac{6(k+1)^2}{k+2}.$$

It can be shown that the above homomorphism is actually an isomorphism.

Hints for Exercise 10. (1) The main thing to be verified is the locality axiom.

(2) Observe that $\hat{Q} = (e_{(-1)}|0\rangle + |0\rangle) \otimes e^*_{(0)}|0\rangle$ and then compute $\hat{Q}(z)\hat{Q} = 0$.

(3) This is a very difficult question! We give the necessary guidance. Set

$$L(z) = L_{sug}(z) + \frac{1}{2}h(z) + L_{\mathcal{F}}(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-1},$$

where

$$L_{sug}(z) = \frac{1}{2(k+2)}\left(: e(z)f(z) : + : f(z)e(z) : + \frac{1}{2}h(z)^2 \right) \quad \text{and} \quad L_{\mathcal{F}}(z) =: \partial_z \psi(z)\psi^*(z) :,$$

and verify that $\hat{Q}_{(0)})L = 0$ so that L defines an element of $\mathcal{W}^k(\mathfrak{sl}_2, f)$. Then check that $L_{-1} = T$, that L_0 acts semisimply on $\mathcal{W}^k(\mathfrak{sl}_2, f)$ by

$$L_0|0\rangle = 0, \qquad [L_0, h_{(n)}] = -nh_{(n)},$$
$$[L_0, e_{(n)}] = (1-n)e_{(n)}, \qquad [L_0, f_{(n)}] = (-1-n)f_{(n)},$$
$$[L_0, \psi^*_{(n)}] = (-1-n)\psi^*_{(n)}, \qquad [L_0, \psi_{(n)}] = (1-n)\psi_{(n)},$$

and that the L_n's verify the Virasoro relations.