

LECTURES NOTES ON VERTEX ALGEBRAS AND ASSOCIATED VARIETIES

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ABSTRACT. This note is an introduction to the notion of associated variety of a vertex algebra. It includes some basics on vertex algebras and Poisson vertex algebras. It is written in preparation to the winter school in Les Diablerets in Switzerland, 7-11 January, 2019.

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Introduction

The goal of this lecture is to introduce the theory of vertex algebras and affine W -algebras, which are certain vertex algebras, with emphasis on their geometrical aspects.

Roughly speaking, a vertex algebra is a vector space V , endowed with a distinguished vector, the vacuum vector, and the vertex operator map from V to the space of formal Laurent series with linear operators on V as coefficients. These data satisfy a number of axioms and have some fundamental properties as, for example, an analogue to the Jacobi identity, locality and associativity. Although the definition is purely algebraic, the above axioms have deep geometric meaning. They reflect the fact that vertex algebras give an algebraic framework of the two-dimensional conformal field theory. The connections of this topic with other branches of mathematics and physics include algebraic geometry (moduli spaces), representation theory (modular representation theory, geometric Langlands correspondence), two dimensional conformal field theory, string theory (mirror symmetry) and four dimensional gauge theory (AGT conjecture).

The nicest vertex algebras are those which are both rational and lisse. The rationality means the completely reducibility of modules. The lisse condition is a certain finiteness condition as explained next paragraph. If a vertex algebra V is rational and lisse, then it gives rise to a rational conformal field theory. In particular, the characters of simple V -modules form vector valued modular functions, and moreover, the category of V -modules forms a modular tensor category, so that one can associate with it an invariant of knots.

To each vertex algebra V one can naturally attach a certain Poisson variety X_V called the associated variety of V . For an affine Poisson variety X , a vertex algebra V such that $X_V \cong X$ is called a *chiral quantization* of X . A vertex algebra V is called lisse if $\dim X_V = 0$. Lisse vertex algebras are natural generalizations of finite-dimensional algebras and possess remarkable properties. For instance, the modular invariance of characters still holds without the rationality assumption.

In fact the geometry of the associated variety often reflects some algebraic properties of the vertex algebras V . For example, vertex algebras whose associated variety has only finitely symplectic leaves are also of great interest for several reasons that will be addressed in the lecture.

Important examples of vertex algebras are those coming from affine Kac-Moody algebras, which are called affine vertex algebras. They play a crucial role in the representation theory of affine Kac-Moody algebras, and of W -algebras. In the case that V is a simple affine vertex algebra, its associated variety is an invariant and conic subvariety of the corresponding simple Lie algebra. It plays a role analogous to that of the associated variety of primitive ideals of the enveloping algebra of simple Lie algebras. However, associated varieties of affine vertex algebras are not necessarily contained in the nilpotent cone and it is difficult to describe them in general.

In fact, although associated varieties seem to be significant also in connection with the recent study of four dimensional superconformal field theory, their general description is fairly open, except in a few cases.

The affine W -algebras are certain vertex algebras associated with nilpotent elements of simple Lie algebras. They can be regarded as affinizations of finite W -algebras (introduced by Premet), and can also be considered as generalizations of affine Kac-Moody algebras and Virasoro algebras. They quantize the arc space of the Slodowy slices associated with nilpotent elements. The study of affine W -algebras began with the work of Zamolodchikov in 1985. Mathematically, affine W -algebras are defined by the method of quantized Drinfeld-Sokolov reduction that was discovered by Feigin and Frenkel in the 1990s. The general definition of affine W -algebras were given by Kac, Roan and Wakimoto in 2003. Affine W -algebras are related with integrable systems, the two-dimensional conformal field theory and the geometric Langlands program. The most recent developments in representation theory of affine W -algebras were done by Kac-Wakimoto and Arakawa.

Since they are not finitely generated by Lie algebras, the formalism of vertex algebras is necessary to study them. In this context, associated varieties of W -algebras, and their quotients, are important tools to understand some properties, such as the lisse condition and even the rationality condition. In general, associated varieties of W -algebras are related to the singularities of nilpotent Slodowy slices.

It is only quite recently that the study of associated varieties of vertex algebras and their arc spaces has been more intensively developed (recent developments include works of Arakawa, van Ekeren, Heluani, Kawasetsu, Linshaw, and the author). This lecture aims to highlight this aspect of the theory of vertex algebras. It includes open problems raised by my recent works with Arakawa.

Part 1. Vertex algebras, definitions, first properties and examples

The best general references for this part are [46, 60]. Vertex algebras were introduced by Borcherds in 1986 [33]. They give the mathematical formalism of two-dimensional conformal field theory (CFT). In fact, vertex algebras are an algebraic version of what physicists called chiral algebras whose rigorous definition has been given by Beilinson and Drinfeld [30].

1.1. DEFINITION OF VERTEX ALGEBRAS AND OPERATOR PRODUCT EXPANSION

Let V be a vector space over \mathbb{C} . We denote by $(\text{End } V)[[z, z^{-1}]]$ the set of all formal Laurent series in the variable z with coefficients in the space $\text{End } V$. We call elements $a(z)$ of $(\text{End } V)[[z, z^{-1}]]$ a *series on V* . For a series $a(z)$ on V , we set

$$a_{(n)} = \text{Res}_{z=0} a(z) z^n$$

so that the expansion of $a(z)$ is

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}.$$

The coefficient $a_{(n)}$ is called a *Fourier mode* of $a(z)$. We write

$$a(z)b = \sum_{n \in \mathbb{Z}} a_{(n)} b z^{-n-1}$$

for $b \in V$.

Definition 1. A series $a(z) \in (\text{End } V)[[z, z^{-1}]]$ is called a *field* on V if for any $b \in V$, $a(z)b \in V((z))$, that is, for any $b \in V$, $a_{(n)}b = 0$ for large enough n .

In the sequel, $\mathcal{F}(V)$ stands for the space of all fields on V .

1.1.1. **Definition.** A *vertex algebra* is a vector space V equipped with the following data:

- (the *vacuum vector*) a vector $|0\rangle \in V$,
- (the *vertex operator*) a linear map

$$Y: V \rightarrow \mathcal{F}(V), \quad a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} = a(z),$$

- (the *translation operator*) a linear map $T: V \rightarrow V$.

These data are subject to the following axioms:

- (the *vacuum axiom*) $|0\rangle(z) = \text{Id}_V$. Furthermore, for all $a \in V$,

$$a(z)|0\rangle \in V[[z]]$$

and $\lim_{z \rightarrow 0} a(z)|0\rangle = a$. In other words, $a_{(n)}|0\rangle = 0$ for $n \geq 0$ and $a_{(-1)}|0\rangle = a$,

- (the *translation axiom*) we have $T|0\rangle = 0$ and for any $a \in V$,

$$[T, a(z)] = \partial_z a(z), \quad (Ta)(z) = \partial_z a(z)$$

- (the *locality axiom*) for all $a, b \in V$,

$$(z-w)^{N_{a,b}} [a(z), b(w)] = 0$$

for some $N_{a,b} \in \mathbb{Z}_{\geq 0}$.

When two fields $a(z), b(z)$ on a vector space V verify the condition of the locality axiom, we say that there are *mutually local*.

The vacuum axiom implies that the map $V \rightarrow \text{End}(V)$ defined by the formula $a \mapsto a_{(-1)}$ is injective. Namely, we have $a = a_{(-1)}|0\rangle$. Therefore the map $a \mapsto a(z)$ is also injective.

Exercise 1 (On the translation axiom). Let V be a \mathbb{C} -vector space.

- (1) Assume that V is a vertex algebra, and fix $a \in V$. Verify that for all $n \in \mathbb{Z}$,

$$[T, a_{(n)}] = -n a_{(n-1)}, \quad (Ta)_{(n)} = -n a_{(n-1)},$$

and deduce from this that

$$Ta = a_{(-2)}|0\rangle.$$

- (2) Conversely, verify that if the vector space V is endowed with a vector $|0\rangle \in V$ and a linear map $F \rightarrow \mathcal{F}(V)$, $a \mapsto a(z)$ such that the vacuum and the locality axioms hold, then the linear map

$$V \rightarrow V, \quad a \mapsto a_{(-2)}|0\rangle$$

satisfies the translation axiom. *This shows that the translation operator T is in fact a redundant datum in the definition of a vertex algebra.*

Hints for Exercise 1. (1) Use the translation axiom.

(2) Use the vacuum axiom.

(3) Compare $(\partial_z a(z))_{(-1)}|0\rangle$ and $a_{(-2)}|0\rangle$, and compute $[T, a(z)]|0\rangle|_{z=0}$.

1.1.2. Operator product expansion (OPE).

Proposition 2. *Fix two fields $a(z), b(z)$ on a vector space V . The following assertions are equivalent:*

- (i) $(z-w)^{N_{a,b}}[a(z), b(w)] = 0$ for some $N_{a,b} \in \mathbb{Z}_{\geq 0}$.
(ii)

$$[a(z), b(w)] = \sum_{n=0}^{N_{a,b}-1} (a_{(n)}b)(w) \frac{1}{n!} \partial_w^n \delta(z-w),$$

where $\delta(z-w) := \sum_{n \in \mathbb{Z}} w^n z^{-n-1} \in \mathbb{C}[[z, w, z^{-1}, w^{-1}]]$ is the formal delta-function.

(iii)

$$a(z)b(w) = \sum_{n=0}^{N_{a,b}-1} (a_{(n)}b)(w) \tau_{z,w} \left(\frac{1}{(z-w)^{n+1}} \right) + :a(z)b(w):,$$

and

$$b(w)a(z) = \sum_{n=0}^{N_{a,b}-1} (a_{(n)}b)(w) \tau_{w,z} \left(\frac{1}{(z-w)^{n+1}} \right) + :a(z)b(w):,$$

where $:a(z)b(w):$ and the maps $\tau_{z,w}$ and $\tau_{w,z}$ are defined below.

For $a(z), b(z) \in \mathcal{F}(V)$,

$$:a(z)b(w): = a(z)_+ b(w) + b(w) a(z)_-,$$

where

$$a(z)_+ = \sum_{n < 0} a_{(n)} z^{-n-1}, \quad a(z)_- = \sum_{n \geq 0} a_{(n)} z^{-n-1}.$$

The *normally ordered product* on a vertex algebra V is defined as $:ab := a_{(-1)}b$. Thus

$$:ab : (z) = :a(z)b(z) :.$$

The normally ordered product is neither commutative nor associative. By definition, $:a(z)b(z)c(z):$ stands for $:a(z) : b(z)c(z) : :$

The two maps $\tau_{z,w}$ and $\tau_{w,z}$ are the homomorphisms of algebras defined by:

$$\begin{aligned} \tau_{z,w} : \mathbb{C}[z, w, z^{-1}, w^{-1}, \frac{1}{z-w}] &\rightarrow \mathbb{C}((z))((w)), & \frac{1}{z-w} &\mapsto \frac{1}{z} \sum_{n \geq 0} \left(\frac{w}{z}\right)^n = \delta(z-w)_-, \\ \tau_{w,z} : \mathbb{C}[z, w, z^{-1}, w^{-1}, \frac{1}{z-w}] &\rightarrow \mathbb{C}((w))((z)), & \frac{1}{z-w} &\mapsto -\frac{1}{z} \sum_{n > 0} \left(\frac{z}{w}\right)^n = -\delta(z-w)_+. \end{aligned}$$

Thus the map $\tau_{z,w}$ is the expansion of $\frac{1}{z-w}$ in $|z| > |w|$ and $\tau_{w,z}$ is the expansion of $\frac{1}{z-w}$ in $|w| > |z|$.

Proof of the implication (ii) \Rightarrow (i). We prove only this implication, and we refer to [46, Chap. 3] for the other implications.

Let us write

$$\delta(z-w) = \underbrace{\frac{1}{z} \sum_{n \geq 0} \left(\frac{w}{z}\right)^n}_{\delta(z-w)_-} + \underbrace{\frac{1}{z} \sum_{n > 0} \left(\frac{z}{w}\right)^n}_{\delta(z-w)_+},$$

so that when $|z| > |w|$, the series $\delta(z-w)_-$ converges to the meromorphic function $\frac{1}{z-w}$ and when $|z| < |w|$, the series $\delta(z-w)_+$ converges to the meromorphic function $-\frac{1}{z-w}$.

We have

$$\delta(z-w) = \tau_{z,w} \left(\frac{1}{z-w} \right) - \tau_{w,z} \left(\frac{1}{z-w} \right).$$

Both homomorphisms $\tau_{z,w}$ and $\tau_{w,z}$ commute with ∂_w and ∂_z . Therefore,

$$\frac{1}{j!} \partial_w^j \delta(z-w) = \tau_{z,w} \left(\frac{1}{(z-w)^{j+1}} \right) - \tau_{w,z} \left(\frac{1}{(z-w)^{j+1}} \right),$$

whence

$$\begin{aligned} (z-w)^{n+1} \frac{1}{n!} \partial_w^n \delta(z-w) &= (z-w)^{n+1} \left(\tau_{z,w} \left(\frac{1}{(z-w)^{n+1}} \right) - \tau_{w,z} \left(\frac{1}{(z-w)^{n+1}} \right) \right) \\ &= \tau_{z,w}(1) - \tau_{w,z}(1) = 0. \end{aligned}$$

The implication (ii) \Rightarrow (i) is then clear. Note that (iii) \Rightarrow (ii) also follows. \square

1.1.3. Borcherds identities. Consequences of the definition are the following relations, called *Borcherds identities*:

$$(1) \quad [a_{(m)}, b_{(n)}] = \sum_{i \geq 0} \binom{m}{i} (a_{(i)} b)_{(m+n-i)},$$

$$(2) \quad (a_{(m)} b)_{(n)} = \sum_{j \geq 0} (-1)^j \binom{m}{j} (a_{(m-j)} b_{(n+j)} - (-1)^m b_{(m+n-j)} a_{(j)}),$$

for $m, n \in \mathbb{Z}$. In the above formulas, the notation $\binom{m}{i}$ for $i \geq 0$ and $m \in \mathbb{Z}$ means

$$\binom{m}{i} = \frac{m(m-1) \times \cdots \times (m-i+1)}{i(i-1) \times \cdots \times 1}.$$

1.2. COMMUTATIVE VERTEX ALGEBRAS

A vertex algebra V is called *commutative* if all vertex operators $a(z)$, $a \in V$, commute each other (i.e., we have $N_{a,b} = 0$ in the locality axiom). This condition is equivalent to that

$$[a_{(m)}, b_{(n)}] = 0 \quad \text{for all } a, b \in V, m, n \in \mathbb{Z}.$$

Hence if V is a commutative vertex algebra, then $a(z) \in \text{End } V[[z]]$ for all $a \in V$, that is, $a_{(n)} = 0$ for $n \geq 0$ in $\text{End } V$ for all $a \in V$.

Then a commutative vertex algebra has a structure of a unital commutative algebra with the product:

$$a \cdot b = : ab := a_{(-1)} b,$$

where the unit is given by the vacuum vector $|0\rangle$. The translation operator T of V acts on V as a derivation with respect to this product:

$$T(a \cdot b) = (Ta) \cdot b + a \cdot (Tb).$$

Therefore a commutative vertex algebra has the structure of a *differential algebra*, that is, a unital commutative algebra equipped with a derivation.

The converse holds according to the following exercise.

Exercise 2 (Commutative algebras equipped with a derivation are commutative vertex algebras). Show that there is a unique structure of a commutative vertex algebra on a commutative algebra R equipped with a derivation ∂ such that the vacuum vector is the unit, and

$$a(z)b = (e^{z\partial}a)b = \sum_{n \geq 0} \frac{z^n}{n!} (\partial^n a)b \quad \text{for all } a, b \in R.$$

Hints for Exercise 2. Notice that the locality axiom is automatically satisfied by the OPE (cf. Proposition 2, (ii) \Rightarrow (i)).

This correspondence gives the following result.

Theorem 3 ([33]). *The category of commutative vertex algebras is the same as that of differential algebras.*

One important example of commutative vertex algebras are obtained by considering the function sheaf over arc spaces of a scheme (see Sect. 2.1).

Exercise 3 (Center of a vertex algebra). For V a vertex algebra, its (vertex) center $\mathcal{Z}(V)$ is defined by:

$$\mathcal{Z}(V) := \{a \in V \mid [b(z), a(w)] = 0 \text{ for all } b \in V\}.$$

Show that the following are equivalent:

- (i) $a \in \mathcal{Z}(V)$,
- (ii) $[b_{(m)}, a_{(n)}] = 0$ for all $b \in V$ and all $m, n \in \mathbb{Z}$,
- (iii) $b(z)a \in V[[z]]$ for all $b \in V$,
- (iv) $b_{(m)}a = 0$ for all $b \in V$ and all $m \in \mathbb{Z}_{\geq 0}$.

Hints for Exercise 3. First, note that the equivalences (i) \iff (ii) and (iii) \iff (iv) are clear. To show (i) \iff (iii), observe that $b(z)a = b(z)a(w)|_0|_{w=0}$.

1.3. UNIVERSAL AFFINE VERTEX ALGEBRAS

1.3.1. Affine Kac-Moody algebras. Let \mathfrak{g} be a complex simple Lie algebra. Hence \mathfrak{g} is the Lie algebra of a certain linear algebraic group G . The *Killing form* of \mathfrak{g} ,

$$\kappa_{\mathfrak{g}}: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}, \quad (x, y) \mapsto \text{Tr}(\text{ad } x \text{ ad } y),$$

is a nondegenerate symmetric bilinear form of \mathfrak{g} which is G -invariant. Since \mathfrak{g} is semisimple, any other such bilinear form is a nonzero multiple of the Killing form.

We define the normalized bilinear form (\mid) on \mathfrak{g} by:

$$(\mid) = \frac{1}{2h^\vee} \kappa_{\mathfrak{g}},$$

where h^\vee is the dual Coxeter number. For example, if $\mathfrak{g} = \mathfrak{sl}_n$, realized as the set of traceless n -size square matrices with Lie bracket $[x, y] = xy - yx$, then $h^\vee = n$ and for all $x, y \in \mathfrak{g}$, $(x|y) = \text{Tr}(xy)$.

We define the *affine Kac-Moody algebra* as the vector space $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$, with the commutation relations:

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \quad [K, \widehat{\mathfrak{g}}] = 0,$$

for all $x, y \in \mathfrak{g}$ and all $m, n \in \mathbb{Z}$, where $\delta_{i,j}$ is the Kronecker symbol.

Consider the Lie subalgebra $\mathfrak{g}[t] \oplus \mathbb{C}K$ of $\widehat{\mathfrak{g}}$. It is a parabolic subalgebra of $\widehat{\mathfrak{g}}$ since it contains the Borel subalgebra $\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}$, where

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$$

is a triangular decomposition of \mathfrak{g} , and

$$\begin{aligned} \widehat{\mathfrak{n}} &:= (\mathfrak{n}_- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t] \oplus \mathfrak{n} \otimes \mathbb{C}[t] = \mathfrak{n} + t\mathfrak{g}[t], \\ \widehat{\mathfrak{h}} &:= (\mathfrak{h} \otimes 1) \oplus \mathbb{C}K = \mathfrak{h} + \mathbb{C}K. \end{aligned}$$

Fix $k \in \mathbb{C}$, and consider the one-dimensional representation \mathbb{C}_k of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts by 0 and K acts as a multiplication by the scalar k .

Definition 4. We define the *universal vacuum representation of level k of $\widehat{\mathfrak{g}}$* as the representation induced from \mathbb{C}_k :

$$V^k(\widehat{\mathfrak{g}}) = \text{Ind}_{\mathfrak{g}[t] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}_k = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k.$$

It can be viewed as a generalized Verma module.

1.3.2. Level of a representation. The representation $V^k(\mathfrak{g})$ is a highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, with Λ_0 is the highest weight of the *basic* representation (it corresponds to $k = 1$) and highest weight vector v_k , where v_k denotes the image of $1 \otimes 1$ in $V^k(\mathfrak{g})$. We will often write $|0\rangle$ for the vector v_k (the notation will be justified §1.3.4). According to the well-known Schur Lemma, any central element of a Lie algebra acts as a scalar on a simple finite dimensional representation. As the Schur Lemma extends to a representation with countable dimension¹, the result holds for highest weight $\widehat{\mathfrak{g}}$ -modules.

Definition 5. A representation M is said to be of *level k* if K acts as $k\text{Id}$ on M .

Then $V^k(\mathfrak{g})$ is by construction of level k .

1.3.3. PBW basis and grading. By the Poincaré-Birkhoff-Witt Theorem, the direct sum decomposition (as a vector space)

$$\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{g}[t] \oplus \mathbb{C}K)$$

gives us the isomorphism of vector spaces

$$U(\widehat{\mathfrak{g}}) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes U(\mathfrak{g}[t] \oplus \mathbb{C}K),$$

whence

$$V^k(\widehat{\mathfrak{g}}) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]).$$

Let $\{x^1, \dots, x^d\}$, where $d = \dim \mathfrak{g}$, be an ordered basis of \mathfrak{g} . For any $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$, set

$$x_{(n)} := x \otimes t^n = xt^n \in \mathfrak{g}[t, t^{-1}].$$

Then $\{K, x_{(n)}^i, i = 1, \dots, d, n \in \mathbb{Z}\}$ forms a basis of $\widehat{\mathfrak{g}}$ and $\{K, x_{(n)}^i, i = 1, \dots, d, n \in \mathbb{Z}_{\geq 0}\}$ forms a basis of $\mathfrak{g}[t] \oplus \mathbb{C}K$. By the PBW Theorem, $V^k(\mathfrak{g})$ has a PBW basis of monomials of the form

$$x_{(n_1)}^{i_1} \dots x_{(n_m)}^{i_m} |0\rangle,$$

where $n_1 \leq n_2 \leq \dots \leq n_m < 0$, and if $n_j = n_{j+1}$, then $i_j \leq i_{j+1}$.

The space $V^k(\mathfrak{g})$ is naturally graded, $V^k(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} V^k(\mathfrak{g})_{\Delta}$, where the grading is defined by

$$\deg x_{(n_1)}^{i_1} \dots x_{(n_m)}^{i_m} |0\rangle = - \sum_{i=1}^m n_i, \quad \deg |0\rangle = 0.$$

We have $V^k(\mathfrak{g})_0 = \mathbb{C}|0\rangle$, and we identify \mathfrak{g} with $V^k(\mathfrak{g})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$.

Any graded quotient V of $V^k(\mathfrak{g})$ (i.e., a quotient by a proper submodule of $V^k(\mathfrak{g})$) is again a highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, and of level k . In particular, $V^k(\mathfrak{g})$ has a unique maximal proper graded submodule N_k and so

$$L_k(\mathfrak{g}) := V^k(\mathfrak{g})/N_k$$

is an irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, and of level k . Thus, as a $\widehat{\mathfrak{g}}$ -representation, we have

$$L_k(\mathfrak{g}) \cong L(k\Lambda_0),$$

where for $\lambda \in \widehat{\mathfrak{h}}^*$, $L(\lambda)$ stands for the highest weight representation of $\widehat{\mathfrak{g}}$ of highest weight λ .

¹i.e., it admits a countable set of generators.

1.3.4. **Vertex algebra structure.** Recall that $|0\rangle = v_k$ is the image of $1 \otimes 1 \in U(\widehat{\mathfrak{g}}) \otimes \mathbb{C}$ in $V^k(\mathfrak{g})$. Then $(V^k(\mathfrak{g}), |0\rangle, T, Y)$ is a vertex algebra, where the translation operator T is given by

$$T|0\rangle = 0, \quad [T, x_{(n)}] = -nx_{(n-1)}, \quad x \in \mathfrak{g}, n \in \mathbb{Z},$$

and the vertex operators are defined inductively by:

$$\begin{aligned} Y(|0\rangle, z) &= \text{Id}_{V^k(\mathfrak{g})}, & Y(x_{(-1)}^i|0\rangle, z) &= x^i(z) = \sum_{n \in \mathbb{Z}} x_{(n)}^i z^{-n-1}, \\ Y(x_{(n_1)}^{i_1} \dots x_{(n_m)}^{i_m}|0\rangle, z) &= \frac{1}{(-n_1 - 1)! \dots (-n_m - 1)!} : \partial_z^{-n_1-1} x^{i_1}(z) \dots \partial_z^{-n_m-1} x^{i_m}(z) : \end{aligned}$$

We have thus obtained:

Theorem-Definition 6. *The vector space $V^k(\mathfrak{g})$ is a $\mathbb{Z}_{\geq 0}$ -graded vertex algebra, called the universal affine vertex algebra associated with \mathfrak{g} at level k .*

Proof. We prove only the locality axiom. It is enough to check the locality on generator fields by Dong's lemma, which says that if $a(z), b(z), c(z)$ are three mutually local fields on a vector space V , then the fields $:a(z)b(z):$ and $c(z)$ are also mutually local.

Let $x, y \in \mathfrak{g}$. Then

$$\begin{aligned} [x(z), y(w)] &= \sum_{n, m \in \mathbb{Z}} [x_{(n)}, y_{(m)}] z^{-n-1} w^{-m-1} \\ &= \sum_{n, m \in \mathbb{Z}} [x, y]_{(n+m)} z^{-n-1} w^{-m-1} + \sum_{n \in \mathbb{Z}} n(x|y)k z^{-n-1} w^{n-1} \\ &= \sum_{l \in \mathbb{Z}} [x, y]_{(l)} \left(\sum_{n \in \mathbb{Z}} z^{-n-1} w^n \right) w^{-l-1} + (x|y)k \sum_{n \in \mathbb{Z}} n z^{-n-1} w^{n-1} \\ &= [x, y](w) \delta(z-w) + (x|y)k \partial_w \delta(z-w). \end{aligned}$$

Then it follows that for all $x, y \in \mathfrak{g}$,

$$(z-w)^2 [x(z), y(w)] = 0,$$

so the locality axiom holds for these fields. □

Remark 7. The equality

$$[x(z), y(w)] = [x, y](w) \delta(z-w) + (x|y)k \partial_w \delta(z-w)$$

is equivalent to the commutation relations in the Lie algebra $\widehat{\mathfrak{g}}$.

Remark 8. The above construction can be generalized to the Kac-Moody affinization

$$\widehat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$$

of an arbitrary Lie algebra \mathfrak{a} , endowed with a symmetric invariant bilinear form κ , with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}\kappa(x, y)\mathbf{1}, \quad x, y \in \mathfrak{a}, m, n \in \mathbb{Z}, \quad [\mathbf{1}, \widehat{\mathfrak{a}}] = 0.$$

We show similarly that the induced representation,

$$V^\kappa(\mathfrak{a}) := U(\widehat{\mathfrak{a}}) \otimes_{U(\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C},$$

where \mathbb{C} is a one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{a}[t]$ acts trivially and $\mathbf{1}$ acts as the identity, has the structure of a vertex algebra, called the *universal affine vertex algebra $V^\kappa(\mathfrak{a})$ associated with \mathfrak{a} and κ* .

In particular, when $\mathfrak{a} \cong \mathbb{C}$ is a one-dimensional Lie algebra and κ is any non-degenerate bilinear form on \mathfrak{a} , then $V^\kappa(\mathfrak{a})$ is the *Heisenberg vertex algebra*.

1.4. THE VIRASORO VERTEX ALGEBRA

Let $Vir = \mathbb{C}((t))\partial_t \oplus \mathbb{C}C$ be the *Virasoro Lie algebra*, with the commutation relations

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m,0}C, \\ [C, Vir] &= 0, \end{aligned}$$

where $L_n := -t^{n+1}\partial_t$ for $n \in \mathbb{Z}$.

Given $c \in \mathbb{C}$, we define the induced representation

$$Vir^c = \text{Ind}_{\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C}^{Vir} \mathbb{C}_c = U(Vir) \otimes_{\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C} \mathbb{C}_c,$$

where C acts as multiplication by c and $\mathbb{C}[[t]]\partial_t$ acts by 0 on the one-dimensional module \mathbb{C}_c .

By the PBW Theorem, Vir^c has a basis of the form

$$L_{j_1} \dots L_{j_m} |0\rangle, \quad j_1 \leq \dots \leq j_m \leq -2,$$

where $|0\rangle$ is the image of $1 \otimes 1$ in Vir^c . Then $(Vir^c, |0\rangle, T, Y)$ is a vertex algebra, called the *universal Virasoro vertex algebra with central charge c* , such that $T = L_{-1}$ and:

$$\begin{aligned} Y(|0\rangle, z) &= \text{Id}_{Vir^c}, & Y(L_{-2}|0\rangle, z) &= \sum_{n \in \mathbb{Z}} L_n z^{-n-2} =: L(z), \\ Y(L_{j_1} \dots L_{j_m} |0\rangle, z) &= \frac{1}{(-j_1-2)! \dots (-j_m-2)!} : \partial_z^{-j_1-2} L(z) \dots \partial_z^{-j_m-2} L(z) : \end{aligned}$$

Moreover, Vir^c is $\mathbb{Z}_{\geq 0}$ -graded by $\deg |0\rangle = 0$ and $\deg L_n |0\rangle = -n$.

1.5. CONFORMAL VERTEX ALGEBRAS

Definition 9. A vertex algebra V is called *conformal* if there exists a vector ω , called the *conformal vector*, such that the corresponding field

$$\omega(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

satisfies the following conditions:

- (1) $[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m,0}c$, where c is a constant called *the central charge* of V ,
- (2) $\omega_{(1)} = L_0$ acts semisimply on V ,
- (3) $\omega_{(0)} = L_{-1} = T$.

For a conformal vertex algebra V we set $V_\Delta = \{v \in V \mid L_0 v = \Delta v\}$ so that

$$V = \bigoplus_{\Delta} V_\Delta.$$

For $a \in V_\Delta$, the *conformal weight* of a is $\Delta_a := \Delta$. A \mathbb{Z} -graded conformal vertex algebra such that $V_\Delta = 0$ for sufficiently small Δ is also called a *vertex operator algebra*.

Example 10. The Virasoro vertex algebra Vir^c is clearly conformal with central charge c and conformal vector $\omega = L_{-2}|0\rangle$.

Example 11. The universal affine vertex algebra $V^k(\mathfrak{g})$ has a natural conformal vector, called the *Segal-Sugawara vector* ω , with central charge

$$c(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee},$$

provided that $k \neq -h^\vee$ (cf. [46, §3.4.8]). It is defined by

$$\omega = \frac{1}{2(k + h^\vee)} \sum_{i=1}^d x_{i,(-1)} x_{(-1)}^i |0\rangle,$$

where $\{x_i, \dots, x_d\}$ is the dual basis of $\{x^i, \dots, x^d\}$ with respect to the bilinear form $(\cdot | \cdot)$, and

$$x^i(z) = \sum_{n \in \mathbb{Z}} x_{(n)}^i z^{-n-1}, \quad x_i(z) = \sum_{n \in \mathbb{Z}} x_{i,(n)} z^{-n-1}.$$

Note that we have

$$[L_m, x_{(n)}] = -n x_{(m+n)} \quad x \in \mathfrak{g}, \quad m, n \in \mathbb{Z}.$$

1.6. MODULES OVER VERTEX ALGEBRAS

1.6.1. Definition. A *module* over a vertex algebra V is a vector space M together with a linear map

$$V \rightarrow \mathcal{F}(M), \quad a \mapsto a^M(z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-n-1},$$

which satisfies the following axioms:

$$(3) \quad \begin{aligned} &|0\rangle(z) = \text{Id}_M, \\ &(Ta)^M(z) = \partial_z a^M(z), \\ &\sum_{j \geq 0} \binom{m}{j} (a_{(n+j)} b)_{(m+k-j)}^M \\ &= \sum_{j \geq 0} (-1)^j \binom{n}{j} (a_{(m+n-j)}^M b_{(k+j)}^M - (-1)^n b_{(n+k-j)}^M a_{(m+j)}^M). \end{aligned}$$

Notice that (3) is equivalent to (1) and (2) for $M = V$.

The axioms imply that V is a module over itself (called the *adjoint module*). We have naturally the notions of *submodules*, *quotient module* and *vertex ideals*. Note that vertex ideals are the same as submodules of adjoint modules. For example, any graded quotient of the universal affine vertex algebra $V^k(\mathfrak{g})$ inherits a vertex algebra structure. Such quotients are called *affine vertex algebras*.

A module whose only submodules are 0 and itself is called *simple*. In particular, the vertex algebra V is said to be *simple* if it is simple as a module over itself. For example, the irreducible $\widehat{\mathfrak{g}}$ -representation $L(k\Lambda_0) \cong L_k(\mathfrak{g})$ is simple as a vertex algebra.

1.6.2. Modules of the universal affine vertex algebra. In the case that V is the universal affine vertex algebra $V^k(\mathfrak{g})$ associated with \mathfrak{g} at level $k \in \mathbb{C}$, V -modules play a crucial role in the representation theory of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$.

A $\widehat{\mathfrak{g}}$ -module M of level k is called *smooth* if $x(z)$ is a field on M for $x \in \mathfrak{g}$, that is, given any $m \in M$, there is $N > 0$ such that $(xt^n)m = 0$ for all $x \in \mathfrak{g}$ and $n \geq N$.

Any $V^k(\mathfrak{g})$ -module M is naturally a smooth $\widehat{\mathfrak{g}}$ -module of level k . Conversely, any smooth $\widehat{\mathfrak{g}}$ -module of level k can be regarded as a $V^k(\mathfrak{g})$ -module. It follows that a $V^k(\mathfrak{g})$ -module is the same as a smooth $\widehat{\mathfrak{g}}$ -module of level k .

More specifically, we have the following result.

Proposition 12 (See [46, §5.1.18] for a proof). *There is an equivalence of category between the category of $V^k(\mathfrak{g})$ -modules and the category of smooth $\widehat{\mathfrak{g}}$ -modules of level k .*

1.6.3. Center of the universal affine vertex algebra. The following exercise gives a description of the vertex center (cf. Exercise 3) of $V^k(\mathfrak{g})$ which has a priori nothing to do the vertex algebra structure.

Exercise 4 (On the center of the universal affine vertex algebra). Let us consider the universal affine vertex algebra $V^k(\mathfrak{g})$ associated with a simple Lie algebra \mathfrak{g} at level $k \in \mathbb{C}$.

(1) Show that $\mathcal{Z}(V^k(\mathfrak{g})) = V^k(\mathfrak{g})^{\mathfrak{g}[[t]]}$, that is,

$$\mathcal{Z}(V^k(\mathfrak{g})) = \{a \in V^k(\mathfrak{g}) \mid x_{(m)}a = 0 \text{ for all } x \in \mathfrak{g}, m \in \mathbb{Z}_{\geq 0}\}.$$

(2) Show that we have the following isomorphism of commutative \mathbb{C} -algebras (the product on the commutative vertex algebra $\mathcal{Z}(V^k(\mathfrak{g}))$ is the normally ordered product):

$$\mathcal{Z}(V^k(\mathfrak{g})) \cong \text{End}_{\widehat{\mathfrak{g}}}(V^k(\mathfrak{g})).$$

We shall first prove that $\mathcal{Z}(V^k(\mathfrak{g}))$ naturally embeds into $\text{End}_{\widehat{\mathfrak{g}}}(V^k(\mathfrak{g}))$.

- (3) Prove that if $k \neq -h^\vee$, then $\mathcal{Z}(V^k(\mathfrak{g})) = \mathbb{C}|0\rangle$.

For $k = -h^\vee$, the center $\mathcal{Z}(V^{-h^\vee}(\mathfrak{g})) =: \mathfrak{z}(\widehat{\mathfrak{g}})$ is “huge”, and it is usually referred as the Feigin-Frenkel center²: we have $\text{gr } \mathfrak{z}(\widehat{\mathfrak{g}}) \cong \mathbb{C}[J_\infty(\mathfrak{g}/G)]$, with $\mathfrak{g}/G = \text{Spec } \mathbb{C}[\mathfrak{g}]^G$.

Hints for Exercise 4. (1) Follows from Exercise 3.

- (2) Apply the “Frobenius reciprocity”, which asserts that

$$\text{Hom}_{\widehat{\mathfrak{g}}}(U(\widehat{\mathfrak{g}}) \otimes_{\mathfrak{g}[t] \oplus \mathbb{C}K} \mathbb{C}_k, V^k(\mathfrak{g})) \cong \text{Hom}_{\mathfrak{g}[t] \oplus \mathbb{C}K}(\mathbb{C}_k, V^k(\mathfrak{g})).$$

- (3) Use the Segal-Sugawara conformal vector ω (cf. Example 11).

² See Example 17 for more details about the scheme $J_\infty(\mathfrak{g}/G)$.

Part 2. Poisson vertex algebras, arc spaces, and associated varieties

As we will see in this part, any vertex algebra is naturally filtered and the corresponding graded algebra is a Poisson vertex algebra. A nice way to construct Poisson vertex algebras is to consider the coordinate ring of the arc space of an affine Poisson variety. Actually, strong relations exist, at least conjecturally, between the arc space of the associated variety and the singular support of a vertex algebra, that is, the spectrum of the corresponding graded algebra.

For all these reasons we start by reviewing some standard facts on jet schemes and arc spaces.

2.1. JET SCHEMES AND ARC SPACES

For more details on jet schemes and arc spaces, we refer to [86, 42, 55].

2.1.1. Definitions. Let X be an object of the category Sch of schemes of finite type over \mathbb{C} , and fix $m \in \mathbb{Z}_{\geq 0}$.

Definition 13. An m -jet of X is a morphism

$$\mathrm{Spec} \mathbb{C}[t]/(t^{m+1}) \longrightarrow X.$$

The set of all m -jets of X carries the structure of a scheme $J_m(X)$, called the m -th jet scheme of X . It is a scheme of finite type over \mathbb{C} characterized by the functorial property that for every scheme Z over \mathbb{C} ,

$$\mathrm{Hom}_{Sch}(Z, J_m(X)) = \mathrm{Hom}_{Sch}(Z \times_{\mathrm{Spec} \mathbb{C}} \mathrm{Spec} \mathbb{C}[t]/(t^{m+1}), X).$$

The \mathbb{C} -points of $J_m(X)$ are thus the $\mathbb{C}[t]/(t^{m+1})$ -points of X . From Definition 13, we have for example that $J_0(X) \simeq X$ and that $J_1(X) \simeq TX$, where TX denotes the total tangent bundle of X .

The canonical projection $\mathbb{C}[t]/(t^{m+1}) \rightarrow \mathbb{C}[t]/(t^{n+1})$, $m \geq n$, induces a *truncation morphism*

$$\pi_{m,n}: J_m(X) \rightarrow J_n(X).$$

The canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[t]/(t^{m+1})$ induces a morphism $\iota_m: X \rightarrow J_m(X)$, and we have $\pi_{m,0} \circ \iota_{m,0} = \mathrm{Id}_X$. Hence ι_m is injective and $\pi_{m,0}$ is surjective.

Define the (formal) disc as

$$D := \mathrm{Spec} \mathbb{C}[[t]].$$

The projections $\pi_{m,n}$ yield a projective system $\{J_m(X), \pi_{m,n}\}_{m \geq n}$ of schemes.

Definition 14. Denote by $J_\infty(X)$ its projective limit in the category of schemes,

$$J_\infty(X) = \varprojlim J_m(X).$$

It is called the *arc space*, or *the infinite jet scheme* of X .

Thus elements of $J_\infty(X)$ are the morphisms

$$\gamma: D \rightarrow X,$$

and for every scheme Z over \mathbb{C} ,

$$\mathrm{Hom}_{Sch}(Z, J_m(X)) = \mathrm{Hom}_{Sch}(Z \widehat{\times}_{\mathrm{Spec} \mathbb{C}} D, X),$$

where $Z \widehat{\times}_{\mathrm{Spec} \mathbb{C}} D = Z \widehat{\times} D$ is the completion of $Z \times D$ with respect to the subscheme $Z \times \{0\}$. In other words, the contravariant functor

$$Sch \rightarrow Set, \quad Z \mapsto \mathrm{Hom}_{Sch}(Z \widehat{\times} D, X)$$

is represented by the scheme $J_\infty(X)$. The reason why we need the completion $Z \widehat{\times} D$ in the definition is that, for A an algebra, $A \otimes \mathbb{C}[[t]] \not\cong A[[t]] = A \widehat{\otimes} \mathbb{C}[[t]]$ in general.

We denote by π_∞ the canonical projection:

$$\pi_\infty: J_\infty(X) \rightarrow X.$$

2.1.2. **The affine case.** In the case where $X = \text{Spec } \mathbb{C}[x^1, \dots, x^N] \cong \mathbb{A}^N$, is an N -dimensional affine space, we have the following explicit description of $J_\infty X$. Giving a morphism $\gamma: D \rightarrow \mathbb{A}^N$ is equivalent to giving a morphism $\gamma^*: \mathbb{C}[x^1, \dots, x^N] \rightarrow \mathbb{C}[[t]]$, or to giving

$$\gamma^*(x^i) = \sum_{j \geq 0} \gamma_{(-j-1)}^i t^j, \quad i = 1, \dots, N.$$

Then

$$J_\infty \mathbb{A}^N = \text{Spec } \mathbb{C}[x_{(-j-1)}^i; i = 1, \dots, N, j \geq 0],$$

where for $i = 1, \dots, N$ and $j \geq 0$,

$$x_{(-j-1)}^i(\gamma) = j! \gamma_{(-j-1)}^i.$$

Define a derivation T of the algebra $\mathbb{C}[x_{(-j-1)}^i; i = 1, \dots, N, j \geq 0]$ by

$$T x_{(-j)}^i = j x_{(-j-1)}^i, \quad j > 0.$$

Here we identify x^i with $x_{(-1)}^i$.

More generally, if $X \subset \mathbb{A}^N$ is an affine subscheme defined by an ideal $I = (f_1, \dots, f_r)$ of $\mathbb{C}[x^1, \dots, x^N]$, that is, $X = \text{Spec } R$ with

$$R = \mathbb{C}[x^1, x^2, \dots, x^N]/(f_1, f_2, \dots, f_r),$$

then its arc space $J_\infty X$ is the affine scheme $\text{Spec}(J_\infty R)$, where

$$(4) \quad J_\infty R := \frac{\mathbb{C}[x_{(-j-1)}^i; i = 1, 2, \dots, N, j \geq 0]}{(T^j f_i; i = 1, \dots, r, j \geq 0)},$$

and T is as above.

The derivation T acts on the quotient ring $J_\infty R$ given by (4). Hence for an affine scheme $X = \text{Spec } R$, the coordinate ring $J_\infty R = \mathbb{C}[J_\infty X]$ of its arc space is a differential algebra, hence is a commutative vertex algebra by Theorem 3.

Remark 15 ([42]). The differential algebra $(J_\infty(R), T)$ is universal in the following sense. We have a \mathbb{C} -algebra homomorphism $j: R \rightarrow J_\infty(R)$ such that if (A, ∂) is another differential algebra, and if $f: R \rightarrow A$ is a \mathbb{C} -algebra homomorphism, then there is a unique differential algebra homomorphism³ $h: J_\infty(R) \rightarrow A$ making the following diagram commutative.

$$\begin{array}{ccc} R & \xrightarrow{j} & (J_\infty(R), T) \\ & \searrow f & \swarrow h \\ & (A, \partial) & \end{array}$$

Lemma 16 ([42]). *Given any $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and any open subset U of X , $J_m(U) = \pi_m^{-1}(U)$.*

Then for a general scheme Y of finite type with an affine open covering $\{U_i\}_{i \in I}$, its arc space $J_\infty(Y)$ is obtained by glueing $J_\infty(U_i)$ (see [42, 55]). In particular, the structure sheaf $\mathcal{O}_{J_\infty(Y)}$ is a sheaf of commutative vertex algebras.

The natural projection $\pi_\infty: J_\infty(X) \rightarrow X$ corresponds to the embedding $R \hookrightarrow J_\infty(R)$, $x^i \rightarrow x_{(-1)}^i$ in the case where $X = \text{Spec } R$ is affine. In terms of arcs, $\pi_\infty(\alpha) = \alpha(0)$ for $\alpha \in \text{Hom}_{\text{Sch}}(D, X)$, where 0 is the unique closed point of the formal disc D .

³A differential algebra homomorphism is a \mathbb{C} -algebra homomorphism which commutes with the derivations.

2.1.3. Basic properties. The map from a scheme to its jet schemes or its arc space is functorial. If $f: X \rightarrow Y$ is a morphism of schemes, then we naturally obtain a morphism $J_m f: J_m(X) \rightarrow J_m(Y)$ making the following diagram commutative,

$$\begin{array}{ccc} J_m(X) & \xrightarrow{J_m f} & J_m(Y) \\ \pi_{m,0} \downarrow & & \downarrow \pi_{m,0} \\ X & \xrightarrow{f} & Y \end{array}$$

In terms of arcs, it means that $J_m f(\alpha) = f \circ \alpha$ for $\alpha \in J_m(X)$. This also holds for $m = \infty$.

In addition, for every $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and every schemes X, Y ,

$$(5) \quad J_m(X \times Y) \cong J_m(X) \times J_m(Y).$$

Indeed, for any scheme Z in Sch ,

$$\begin{aligned} \mathrm{Hom}(Z, J_m(X \times Y)) &= \mathrm{Hom}(Z \times_{\mathrm{Spec} \mathbb{C}} \mathbb{C}[t]/(t^{m+1}), X \times Y) \\ &\cong \mathrm{Hom}(Z \times_{\mathrm{Spec} \mathbb{C}} \mathbb{C}[t]/(t^{m+1}), X) \times \mathrm{Hom}(Z \times_{\mathrm{Spec} \mathbb{C}} \mathbb{C}[t]/(t^{m+1}), Y) \\ &= \mathrm{Hom}(Z, J_m(X)) \times \mathrm{Hom}(Z, J_m(Y)) \\ &\cong \mathrm{Hom}(Z, J_m(X) \times J_m(Y)). \end{aligned}$$

For $m = \infty$, just replace $\mathbb{C}[t]/(t^{m+1})$ with $\mathbb{C}[[t]]$ and take the completion in the product $Z \widehat{\times} \mathrm{Spec} \mathbb{C}[[t]] = Z \widehat{\times} D$.

If \mathbf{G} is a group scheme over \mathbb{C} , then $J_m(\mathbf{G})$ is also a group scheme over \mathbb{C} . Moreover, by (5), if \mathbf{G} acts on X , then $J_m(\mathbf{G})$ acts on $J_m(X)$.

Example 17. Consider the algebra

$$\mathfrak{g}_\infty := \mathfrak{g}[[t]] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t]] \cong J_\infty(\mathfrak{g}).$$

It is naturally a Lie algebra, with Lie bracket:

$$[xt^m, yt^n] = [x, y]t^{m+n}, \quad x, y \in \mathfrak{g}, m, n \in \mathbb{Z}_{\geq 0}.$$

The arc space $J_\infty(G)$ of the algebraic group G is naturally a proalgebraic group⁴. Regarding $J_\infty(G)$ as the set of $\mathbb{C}[[t]]$ -points of G , we have $J_\infty(G) = G[[t]]$. As Lie algebras, we have

$$\mathfrak{g}_\infty \cong \mathrm{Lie}(J_\infty(G)).$$

The adjoint action of G on \mathfrak{g} induces an action of $J_\infty(G)$ on \mathfrak{g}_∞ , and the coadjoint action of G on \mathfrak{g}^* induces an action of $J_\infty(G)$ on $J_\infty(\mathfrak{g}^*)$, and so on $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$.

Let \mathcal{N} be the *nilpotent cone* of \mathfrak{g} , that is, the set of nilpotent elements of \mathfrak{g} . It is well-known that \mathcal{N} is the reduced scheme of \mathfrak{g} defined by the equations p_1, \dots, p_r , where p_1, \dots, p_r are homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$. Hence, $J_\infty(\mathcal{N})$ is the subscheme of \mathfrak{g}_∞ defined by the equations $T^j p_i$, $i = 1, \dots, r$ and $j \geq 0$.

Furthermore, following deep result was obtained independently by Raïs-Tauvel, Beilinson-Drinfeld, Eisenbud-Frenkel [91, 31, 43]:

$$J_\infty(\mathfrak{g} // G) \cong J_\infty \mathfrak{g} // J_\infty G,$$

where $J_\infty \mathfrak{g} // J_\infty G = \mathrm{Spec} \mathbb{C}[J_\infty \mathfrak{g}]^{J_\infty G}$. In other words, the invariant ring $\mathbb{C}[J_\infty \mathfrak{g}]^{J_\infty G}$ is the polynomial ring

$$\mathbb{C}[J_\infty(\mathfrak{g} // G)] = \mathbb{C}[T^j p_i, i = 1, \dots, r, j \geq 0],$$

since $\mathbb{C}[\mathfrak{g} // G] = \mathbb{C}[p_1, \dots, p_r]$. In particular,

$$J_\infty(\mathcal{N}) = \mathrm{Spec} \mathbb{C}[J_\infty \mathfrak{g}] / \mathbb{C}[J_\infty \mathfrak{g}]_+^{J_\infty G},$$

where $\mathbb{C}[J_\infty \mathfrak{g}]_+^{J_\infty G}$ is the augmentation ideal of $\mathbb{C}[J_\infty \mathfrak{g}]^{J_\infty G}$.

⁴A *proalgebraic group* is an inverse limit of algebraic groups.

2.1.4. **Geometrical results.** As we have seen, the jet schemes $J_m(X)$ and the arc space $J_\infty(X)$ share several properties. For geometrical aspects, arc spaces behave rather differently. The main reason is that $\mathbb{C}[[t]]$ is a domain, contrary to $\mathbb{C}[t]/(t^{m+1})$. Thereby, although $J_\infty(X)$ is not of finite type in general, its geometric properties are somehow simpler than those of the finite jet schemes $J_m(X)$.

Lemma 18. *The natural morphism $X_{\text{red}} \rightarrow X$ induces an isomorphism $J_\infty X_{\text{red}} \xrightarrow{\cong} (J_\infty X)_{\text{red}}$ of topological spaces, where X_{red} stands for the reduced scheme of X .*

Proof. We may assume that $X = \text{Spec } R$. An arc α of X corresponds to a ring homomorphism $\alpha^*: R \rightarrow \mathbb{C}[[t]]$. Since $\mathbb{C}[[t]]$ is an integral domain, it decomposes as $\alpha^*: R \rightarrow R/\sqrt{0} \rightarrow \mathbb{C}[[t]]$. Thus, α is an arc of X_{red} . \square

Similarly, if $X = X_1 \cup \dots \cup X_r$, where all X_i are closed in X , then

$$J_\infty(X) = J_\infty(X_1) \cup \dots \cup J_\infty(X_r).$$

(Note that Lemma 18 is false for the schemes $J_m(X)$.)

If X is a point, then $J_\infty(X)$ is also a point, because $\text{Hom}(D, X) = \text{Hom}(\mathbb{C}, \mathbb{C}[[t]])$ consists of only one element. Thus, Lemma 18 implies the following.

Corollary 19. *If X is zero-dimensional, then $J_\infty(X)$ is also zero-dimensional.*

Theorem 20 ([74]). *The scheme $J_\infty(X)$ is irreducible if X is irreducible.*

Theorem 20 is false for the jet schemes $J_m(X)$: see for instance [85] for counter-examples in the setting of nilpotent orbit closures. We refer to *loc. cit.*, and the references given there, for more about existing relations between the geometry of the jet schemes $J_m(X)$, $m \in \mathbb{Z}_{\geq 0}$, and the singularities of X .

2.2. POISSON VERTEX ALGEBRAS

Let V be a commutative vertex algebra (cf. §1.2), or equivalently, a unital commutative algebra equipped with a derivation. Recall that this means: $a_{(n)} = 0$ in $\text{End}(V)$ for all $n \geq 0$.

2.2.1. **Definition.** A commutative vertex algebra V is called a *Poisson vertex algebra* if it is also equipped with a linear operation,

$$V \rightarrow \text{Hom}(V, z^{-1}V[z^{-1}]), \quad a \mapsto a_-(z),$$

such that

$$(6) \quad (Ta)_{(n)} = -na_{(n-1)},$$

$$(7) \quad a_{(n)}b = \sum_{j \geq 0} (-1)^{n+j+1} \frac{1}{j!} T^j(b_{(n+j)}a),$$

$$(8) \quad [a_{(m)}, b_{(n)}] = \sum_{j \geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)},$$

$$(9) \quad a_{(n)}(b \cdot c) = (a_{(n)}b) \cdot c + b \cdot (a_{(n)}c)$$

for all $a, b, c \in V$ and all $n, m \geq 0$. Here, by abuse of notations, we have set

$$a_-(z) = \sum_{n \geq 0} a_{(n)}z^{-n-1}$$

so that the $a_{(n)}$, $n \geq 0$, are “new” operators, the “old” ones given by the field $a(z)$ being zero for $n \geq 0$, for V is commutative.

The equation (9) says that $a_{(n)}$, $n \geq 0$, is a derivation of the ring V . (Do not confuse $a_{(n)} \in \text{Der}(V)$, $n \geq 0$, with the multiplication $a_{(n)}$ as a vertex algebra, which should be zero for a commutative vertex algebra.) Note that (7), (8) and (9) are equivalent to the “skewsymmetry”, the “Jacobi identity” and the “left Leibniz rule” in [61, §5.1].

It follows from the definition that we also have the “right Leibniz rule” ([61, Exercise 4.2]):

$$(a \cdot b)_{(n)}c = \sum_{i \geq 0} (b_{(-i-1)}a_{(n+i)}c + a_{(-i-1)}b_{(n+i)}c),$$

for all $a, b, c \in V$, $n \in \mathbb{Z}_{\geq 0}$.

2.2.2. Poisson vertex structure on arc spaces. Arc spaces over an affine Poisson scheme naturally give rise to a vertex Poisson algebras, as shows the following result.

Theorem 21 ([6, Prop. 2.3.1]). *Given an affine Poisson scheme X , that is, $X = \text{Spec } R$ for some Poisson algebra R , there is a unique Poisson vertex algebra structure on $J_\infty(R) = \mathbb{C}[J_\infty(X)]$ such that*

$$a_{(n)}b = \begin{cases} \{a, b\} & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

for all $a, b \in R$.

Proof. The uniqueness is clear by (6) since $J_\infty(R)$ is generated by R as a differential algebra. We leave it to the reader to check the well-definedness. Since $J_\infty(R)$ is generated by R , the formula $a_{(n)}b = \delta_{n,0}\{a, b\}$ for all $a, b \in R$ is sufficient to define the fields on $J_\infty(R)$ by formulas (6)–(9), whence the existence. \square

Remark 22. More generally, given a Poisson scheme X , not necessarily affine, the structure sheaf $\mathcal{O}_{J_\infty(X)}$ carries a unique Poisson vertex algebra structure such that

$$f_{(n)}g = \delta_{n,0}\{f, g\}$$

for all $f, g \in \mathcal{O}_X \subset \mathcal{O}_{J_\infty(X)}$, see [16, Lem. 2.1.3.1].

Example 23. Recall that $\mathbb{C}[\mathfrak{g}^*]$ has naturally a Poisson structure induced from the Kirillov-Kostant-Souriau Poisson structure on \mathfrak{g}^* . Namely, for all $f, g \in \mathbb{C}[\mathfrak{g}^*]$ and all $x \in \mathfrak{g}^*$,

$$\{f, g\}(x) = \langle x, [d_x f, d_x g] \rangle,$$

where $d_x f, d_x g$ are the differentials of f, g , respectively, at $x \in \mathfrak{g}^*$ viewed as elements of $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. In particular, for $f, g \in \mathfrak{g} = (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*]$,

$$\{f, g\} = [f, g].$$

By §2.1.2,

$$J_\infty(\mathfrak{g}^*) = \text{Spec } \mathbb{C}[x_{(-n)}^i ; i = 1, \dots, d, n \geq 1],$$

where $\{x^1, \dots, x^d\}$ is a basis of \mathfrak{g} . So by Theorem 21, $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ inherits a Poisson vertex algebra from that of $\mathbb{C}[\mathfrak{g}^*]$.

We may identify $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ with the symmetric algebra $S(\mathfrak{g}[t^{-1}]t^{-1})$ via

$$x_{(-n)} \mapsto xt^{-n}, \quad x \in \mathfrak{g}, n \geq 1.$$

For $x \in \mathfrak{g}$, identify x with $x_{(-1)}|0\rangle = (xt^{-1})|0\rangle$, where $|0\rangle$ stands for the unit element in $S(\mathfrak{g}[t^{-1}]t^{-1})$. Then (8) gives that

$$[x_{(m)}, y_{(n)}] = (x_{(0)}y)_{m+n} = \{x, y\}_{(m+n)} = [x, y]_{(m+n)},$$

for all $x, y \in \mathfrak{g}$ and all $m, n \in \mathbb{Z}_{\geq 0}$. So the Lie algebra $J_\infty(\mathfrak{g}) = \mathfrak{g}[[t]]$ acts on $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ by:

$$\mathfrak{g}[[t]] \rightarrow \text{End}(\mathbb{C}[J_\infty(\mathfrak{g}^*)]), \quad xt^n \mapsto x_{(n)}, \quad n \geq 0,$$

where $x_{(n)}$, $n \geq 0$, is the endomorphism of $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ given by the Poisson vertex structure on $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$. This action coincides with that obtained by differentiating the action of $J_\infty(G) = G[[t]]$ on $J_\infty(\mathfrak{g}^*)$ induced by the coadjoint action of G (see Example 17). In other words, the Poisson vertex algebra structure of $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ comes from the $J_\infty(G)$ -action on $J_\infty(\mathfrak{g}^*)$.

2.2.3. Canonical filtration and Poisson vertex structure. Our second basic example of Poisson vertex algebras comes from the graded vertex algebra associated with the canonical filtration, that is, the *Li filtration*.

Haisheng Li [79] has shown that every vertex algebra is canonically filtered. For a vertex algebra V , let $F^p V$ be the subspace of V spanned by the elements

$$a_{(-n_1-1)}^1 a_{(-n_2-1)}^2 \cdots a_{(-n_r-1)}^r |0\rangle$$

with $a^1, a^2, \dots, a^r \in V$, $n_i \geq 0$, $n_1 + n_2 + \cdots + n_r \geq p$. Then

$$V = F^0 V \supset F^1 V \supset \dots$$

It is clear that $TF^p V \subset F^{p+1} V$.

Set

$$(F^p V)_{(n)} F^q V := \text{span}_{\mathbb{C}}\{a_{(n)} b ; a \in F^p V, b \in F^q V\}.$$

Note that $F^1 V = \text{span}_{\mathbb{C}}\{a_{(-2)} b \mid a, b \in V\}$. We also write $C_2(V)$ for the space $F^1 V$.

Lemma 24. *We have*

$$F^p V = \sum_{j \geq 0} (F^0 V)_{(-j-1)} F^{p-j} V.$$

Proposition 25. (1) $(F^p V)_{(n)} (F^q V) \subset F^{p+q-n-1} V$. Moreover, if $n \geq 0$, $(F^p V)_{(n)} (F^q V) \subset F^{p+q-n} V$. Here we have set $F^p V = V$ for $p < 0$.

(2) The filtration $F^\bullet V$ is separated, that is, $\bigcap_{p \geq 0} F^p V = \{0\}$, if V is positively graded.

The verifications are straightforward and are left to the reader. (Part (2) also follows from Lemma 37 below.)

In this note we always assume that a vertex algebra V is conformal and positively graded, $V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n$, so that the filtration $F^\bullet V$ is separated. We will also assume that $V_0 = \mathbb{C}|0\rangle \cong \mathbb{C}$.

Set

$$\text{gr}^F V = \bigoplus_{p \geq 0} F^p V / F^{p+1} V.$$

We denote by $\sigma_p: F^p V \mapsto F^p V / F^{p+1} V$, for $p \geq 0$, the canonical quotient map. When the filtration F is obvious, we often briefly write $\text{gr} V$ for the space $\text{gr}^F V$.

Proposition 26 ([79]). *The space $\text{gr}^F V$ is a Poisson vertex algebra by*

$$\begin{aligned} \sigma_p(a) \cdot \sigma_q(b) &:= \sigma_{p+q}(a_{(-1)} b), \\ T\sigma_p(a) &:= \sigma_{p+1}(Ta), \\ \sigma_p(a)_{(n)} \sigma_q(b) &:= \sigma_{p+q-n}(a_{(n)} b), \end{aligned}$$

for all $a \in F^p V \setminus F^{p-n} V$, $b \in F^q V$, $n > 0$.

Set

$$R_V := F^0 V / F^1 V = V / C_2(V) \subset \text{gr} V.$$

Definition 27. The algebra R_V is called the *Zhu C_2 -algebra* of V . The algebra structure is given by:

$$\bar{a} \cdot \bar{b} := \overline{a_{(-1)} b},$$

where $\bar{a} = \sigma_0(a)$.

Proposition 28 ([95, 79]). *The restriction of the vertex Poisson structure on $\text{gr}^F V$ gives to Zhu's C_2 -algebra R_V a Poisson algebra structure, that is, R_V is a Poisson algebra by*

$$\bar{a} \cdot \bar{b} := \overline{a_{(-1)} b}, \quad \{\bar{a}, \bar{b}\} = \overline{a_{(0)} b},$$

where $\bar{a} = \sigma_0(a)$.

Proof. It is straightforward from Proposition 26. □

We say that a vertex algebra V is *finitely strongly generated* if R_V is finitely generated as a ring.

In this note all vertex algebras are assumed to be finitely strongly generated.

Exercise 5 (Poisson structure on the Zhu C_2 -algebra of the universal affine vertex algebra). Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with a simple Lie algebra \mathfrak{g} at level $k \in \mathbb{C}$.

(1) Show that the map

$$\begin{aligned} \mathbb{C}[\mathfrak{g}^*] \cong S(\mathfrak{g}) &\longmapsto V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}) \\ x_1 \dots x_r &\longmapsto (x_1 t^{-1}) \dots (x_r t^{-1})|0\rangle + t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}), \quad x_1, \dots, x_r \in \mathfrak{g}. \end{aligned}$$

defines an isomorphism of commutative algebras, the product on the right-hand side being given by:

$$((x_1 t^{-1}) \dots (x_r t^{-1})|0\rangle) \cdot ((y_1 t^{-1}) \dots (y_s t^{-1})|0\rangle) = (x_1 t^{-1}) \dots (x_r t^{-1})(y_1 t^{-1}) \dots (y_s t^{-1})|0\rangle,$$

for $x_i, y_j \in \mathfrak{g}$.

(2) Verify that

$$R_{V^k(\mathfrak{g})} = V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}),$$

and show that the Poisson bracket on $R_{V^k(\mathfrak{g})}$ is the one induced from the isomorphism of (1).

Hints for Exercise 5. (1) Use the PBW basis to show the bijectivity, the rest of the verifications are clear.

(2) Just verify using the commuting relations that for $x, y \in \mathfrak{g}$,

$$\{x, y\} = [x, y] = \bar{x}_{(0)}\bar{y},$$

where \bar{x} stands for the image of x , viewed as an element of $\mathfrak{g} \cong V^k(\mathfrak{g})_1$, in $R_{V^k(\mathfrak{g})}$.

Remark 29. Suppose that the Poisson structure of R_V is trivial. Then the Poisson vertex algebra structure of $J_\infty(R_V)$ is trivial, and so is that of $\text{gr}^F V$ by Theorem 31. This happens if and only if

$$(F^p V)_{(n)}(F^q V) \subset F^{p+q-n+1}V \quad \text{for all } n \geq 0.$$

If this is the case, one can give $\text{gr}^F V$ yet another Poisson vertex algebra structure by setting

$$\sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n+1}(a_{(n)}b) \quad \text{for all } n \geq 0.$$

(We can repeat this procedure if this Poisson vertex algebra structure is again trivial.)

Exercise 6 (Zhu's C_2 -algebra and associated variety of the universal Virasoro vertex algebra). Let Vir^c be the universal Virasoro vertex algebra of central charge $c \in \mathbb{C}$.

(1) Show that $\text{gr}^F \text{Vir}^c \cong \mathbb{C}[L_{-2}, L_{-3}, \dots]$, where F is the Li filtration.

(2) Deduce from (1) that $R_{\text{Vir}^c} \cong \mathbb{C}[x]$, where x is the image of $L := L_{-2}|0\rangle$ in R_{Vir^c} , with the trivial Poisson structure.

(3) Show that one can endow $\text{gr}^F \text{Vir}^c$ with a non-trivial Poisson vertex algebra structure such that

$$\bar{L}_{-1}\bar{L} = \bar{L}_{(0)}\bar{L} = T\bar{L} \quad \text{and} \quad \bar{L}_0\bar{L} = \bar{L}_{(1)}\bar{L} = 2\bar{L}, \quad \text{with } \bar{L} := \sigma_0(L).$$

Hints for Exercise 6. (1) Describe $F^p \text{Vir}^c_\Delta$, where $\Delta \in \mathbb{Z}_{\geq 0}$, using the PBW Theorem.

(2) Just use (1).

(3) Remember that by Remark 29, one can go one step further, and then compute $\sigma_1(\bar{L}_{(0)}\bar{L})$, $\sigma_0(\bar{L}_{(1)}\bar{L})$ using the commuting relations.

2.3. ASSOCIATED VARIETY OF A VERTEX ALGEBRA

We now in a position to define the main object of study of this lecture note.

2.3.1. Associated variety and singular support.

Definition 30. Define the *associated scheme* \tilde{X}_V and the *associated variety* X_V of a vertex algebra V as

$$\tilde{X}_V := \text{Spec } R_V, \quad X_V := \text{Specm } R_V = (\tilde{X}_V)_{\text{red}}.$$

It was shown in [79, Lem. 4.2] that $\text{gr}^F V$ is generated by the subring R_V as a differential algebra. Thus, we have a surjection $J_\infty(R_V) \rightarrow \text{gr}^F V$ of differential algebras by Remark 15 since R_V generates $J_\infty(R_V)$ as a differential algebra, too.

This is in fact a homomorphism of Poisson vertex algebras.

Theorem 31 ([79, Lem. 4.2], [6, Prop. 2.5.1]). *The identity map $R_V \rightarrow R_V$ induces a surjective Poisson vertex algebra homomorphism*

$$J_\infty(R_V) = \mathbb{C}[J_\infty(\tilde{X}_V)] \rightarrow \text{gr}^F V.$$

Definition 32. Define the *singular support* of a vertex algebra V as

$$SS(V) := \text{Spec}(\text{gr}^F V) \subset J_\infty(\tilde{X}_V).$$

Note that the equality $SS(V)_{\text{red}} = (J_\infty(X_V))_{\text{red}}$ as topological spaces hold in many examples (there is no known counter-example so far).

However, the equality $SS(V) = J_\infty(\tilde{X}_V)$ as schemes is not true in general. Van Ekeren and Heluani [41] showed that this may fail for the *minimal series representation of the Virasoro algebra* (cf. §3.2.2). Arakawa et Linshaw [15] found a counter-example in the context of W -algebras. Finally, Arakawa and the author discovered a counter-example for a non *quasi-lisse* (cf. Definition 46) affine vertex algebra. Note that all these counter-examples were discovered only recently.

Theorem 33. *We have $\dim SS(V) = 0$ if and only if $\dim X_V = 0$.*

Proof. The “only if” part is obvious since $\pi_\infty(SS(V)) = \tilde{X}_V$. The “if” part follows from Corollary 19. \square

2.3.2. The lisse condition.

Definition 34. A vertex algebra V is called *lisse* (or C_2 -cofinite) if $R_V = V/C_2(V)$ is finite dimensional.

Thus by Theorem 33 we get:

Lemma 35. *The vertex algebra V is lisse if and only if $\dim X_V = 0$, that is, if and only if $\dim SS(V) = 0$.*

Remark 36. By our assumption that $V = \bigoplus_{i \geq 0} V_i$ is $\mathbb{Z}_{\geq 0}$ -graded with $V_0 = \mathbb{C}|0\rangle$, the algebras $\text{gr}^F V$ and R_V are equipped with the induced grading:

$$\begin{aligned} \text{gr}^F V &= \bigoplus_{i \geq 0} (\text{gr}^F V)_i, & (\text{gr}^F V)_0 &= \mathbb{C}, \\ R_V &= \bigoplus_{i \geq 0} (R_V)_i, & (R_V)_0 &= \mathbb{C}. \end{aligned}$$

So the following conditions are equivalent:

- (1) V is lisse,
- (2) $X_V = \{\text{point}\}$,
- (3) the image of any vector $a \in V_i$ for $i \geq 1$ in $\text{gr}^F V$ is nilpotent,
- (4) the image of any vector $a \in V_i$ for $i \geq 1$ in R_V is nilpotent.

Thus, lisse vertex algebras can be regarded as a generalization of finite-dimensional algebras.

2.3.3. Comparison with weight-depending filtration. There is another natural filtration of V defined as follows [78]. Let $G_p V$ be the subspace of V spanned by the vectors

$$a_{(-n_1-1)}^1 a_{(-n_2-1)}^2 \cdots a_{(-n_r-1)}^r |0\rangle$$

with $a^i \in V$ homogeneous, $\Delta_{a^1} + \cdots + \Delta_{a^r} \leq p$. Then $G_\bullet V$ defines an increasing filtration of V :

$$0 = G_{-1} V \subset G_0 V \subset \cdots \subset G_1 V \subset \cdots, \quad V = \bigcup_p G_p V.$$

Moreover we have

$$\begin{aligned} TG_p V &\subset G_p V, \\ (G_p)_{(n)} G_q V &\subset G_{p+q} V \quad \text{for } n \in \mathbb{Z}, \\ (G_p)_{(n)} G_q V &\subset G_{p+q-1} V \quad \text{for } n \in \mathbb{Z}_{\geq 0}, \end{aligned}$$

It follows that $\text{gr}_G V = \bigoplus G_p V / G_{p-1} V$ is naturally a Poisson vertex algebras.

Lemma 37 ([6, Prop. 2.6.1]). *We have*

$$F^p V_\Delta = G_{\Delta-p} V_\Delta,$$

where $F^p V_\Delta = V_\Delta \cap F^p V$, $G_p V_\Delta = V_\Delta \cap G_p V$. Therefore

$$\mathrm{gr}^F V \cong \mathrm{gr}_G V$$

as Poisson vertex algebras.

2.3.4. Universal affine vertex algebras. Consider the universal affine vertex algebra $V^k(\mathfrak{g})$ associated with a simple Lie algebra \mathfrak{g} at level $k \in \mathbb{C}$. Recall that $F^1 V^k(\mathfrak{g}) = \mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g})$ and that $R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*]$ as Poisson algebras (cf. Exercice 5). Thus, identifying \mathfrak{g}^* with \mathfrak{g} through (|), we get

$$\tilde{X}_{V^k(\mathfrak{g})} = \mathfrak{g}^* \cong \mathfrak{g}.$$

On the other hand,

$$G_p V^k(\mathfrak{g}) = U_p(\mathfrak{g}[t^{-1}]t^{-1})|0\rangle,$$

where $\{U_p(\mathfrak{g}[t^{-1}]t^{-1})\}_p$ is the PBW filtration of $U(\mathfrak{g}[t^{-1}]t^{-1})$, and we have the isomorphisms (cf. Example 23)

$$\mathrm{gr} U(\mathfrak{g}[t^{-1}]t^{-1}) \cong S(\mathfrak{g}[t^{-1}]t^{-1}) \cong \mathbb{C}[J_\infty(\mathfrak{g}^*)].$$

As a consequence of Lemma 37, we get

$$SS(V^k(\mathfrak{g})) = J_\infty(\mathfrak{g}^*).$$

(The equality holds as schemes and, hence, $SS(V^k(\mathfrak{g})) = J_\infty(\tilde{X}_{V^k(\mathfrak{g})})$.)

Given any quotient V of $V^k(\mathfrak{g})$, one can set

$$R_V = V/t^{-2}\mathfrak{g}[t^{-1}]V,$$

and we get a surjective homomorphism of Poisson algebras,

$$(10) \quad \begin{aligned} \mathbb{C}[\mathfrak{g}^*] &\longrightarrow R_V = V/t^{-2}\mathfrak{g}[t^{-1}]V \\ x_1 \dots x_r &\longmapsto \overline{(x_1 t^{-1}) \dots (x_r t^{-1})|0\rangle} + t^{-2}\mathfrak{g}[t^{-1}]V \quad (x_i \in \mathfrak{g}), \end{aligned}$$

the Poisson algebra structure on R_V being defined as before. This map is surjective but not an isomorphism in general. The associated variety X_V is then the zero locus in \mathfrak{g}^* of the kernel of the map (10). In particular, for $V = L_k(\mathfrak{g})$ the simple quotient, we get that $X_{L_k(\mathfrak{g})}$ is a closed G -invariant conic subvariety of \mathfrak{g}^* .

Part 3. Examples and properties of lisse and quasi-lisse vertex algebras

In this part, we give examples of lisse and quasi-lisse vertex algebras, essentially coming from affine vertex algebras. We will see other important examples in the setting of W -algebras next part. To motivate our examples, we start with a short digression on primitive ideals.

3.1. DIGRESSION ON PRIMITIVE IDEALS

3.1.1. Associated variety of primitive ideals. We remind that $\mathfrak{g} = \text{Lie } G$ is a simple Lie algebra. Let I be a two-sided ideal of $U(\mathfrak{g})$. The PBW filtration on $U(\mathfrak{g})$ induces a filtration on I , so that $\text{gr } I$ becomes a graded Poisson ideal in $\mathbb{C}[\mathfrak{g}^*]$. The zero locus $\mathcal{V}(I)$ of $\text{gr } I$ in \mathfrak{g}^* ,

$$\mathcal{V}(I) = \text{Specm } \mathbb{C}[\mathfrak{g}^*]/\text{gr } I \subset \mathfrak{g}^*,$$

is usually referred to as the *associated variety* of I . Identifying \mathfrak{g}^* with \mathfrak{g} through $(\cdot | \cdot)$, we shall often view associated varieties of two-sided ideals of $U(\mathfrak{g})$ as subsets of \mathfrak{g} .

A proper two-sided ideal I of $U(\mathfrak{g})$ is called *primitive* if it is the annihilator of a simple left $U(\mathfrak{g})$ -module. Let us mention two important results on primitive ideals of $U(\mathfrak{g})$.

Theorem 38 (Duflo Theorem [40]). *Any primitive ideal in $U(\mathfrak{g})$ is the annihilator $\text{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)$ of some irreducible highest weight representation $L_{\mathfrak{g}}(\lambda)$, where $\lambda \in \mathfrak{h}^*$, of \mathfrak{g} .*

Theorem 39 (Irreducibility Theorem [34, 73, 57]). *The associated variety $\mathcal{V}(I)$ of a primitive ideal I in $U(\mathfrak{g})$ is irreducible, specifically, it is the closure $\overline{\mathbb{O}}$ of some nilpotent orbit \mathbb{O} in \mathfrak{g} .*

In particular, the associated variety of a primitive ideal is contained in the nilpotent cone, which is a crucial property. Theorem 39 was first partially proved (by a case-by-case argument) in [34], and in a more conceptual way in [73] and [57] (independently), using many earlier deep results due to Joseph, Gabber, Lusztig, Vogan and others.

It is possible that different primitive ideals share the same associated variety. At the same time, not all nilpotent orbit closures appear as associated variety of some primitive ideal of $U(\mathfrak{g})$.

3.1.2. Analogs for affine Kac-Moody algebras? We have seen that $V^k(\mathfrak{g})$ plays a role similar to that of the enveloping algebra of \mathfrak{g} for the representation theory of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ (cf. §1.6.2). Because of this, it would be nice to have analogs of the associated varieties of primitive ideals in this context. Unfortunately, one cannot expect exactly the same theory. One of the main reasons is that the center of $U(\widehat{\mathfrak{g}})$ is trivial (unless for the *critical level* $k = -h^\vee$), and so we do not have analog of the nilpotent cone (for the critical level, the analog is played by the arc space of the nilpotent cone, see Exercice 4 and Example 17). So we need some replacements.

In this context, the associated variety of the highest weight irreducible representation $L(k\Lambda_0) = L_k(\mathfrak{g})$ of $\widehat{\mathfrak{g}}$, $k \in \mathbb{C}$, viewed as a vertex algebra⁵, is a better analog. We will see next paragraphs some analogies between the associated variety of $L_k(\mathfrak{g})$ and the associated variety of primitive ideals. But there are also substantial differences. For example, since $L_k(\mathfrak{g}) \cong V^k(\mathfrak{g})$ for $k \notin \mathbb{Q}$ (cf. [62]), we see that $X_{V^k(\mathfrak{g})}$ is not always contained in the nilpotent cone \mathcal{N} . See Remark 57 for other examples where $X_{V^k(\mathfrak{g})}$ is not contained in \mathcal{N} .

3.2. LISSE VERTEX ALGEBRAS

Recall that a vertex algebra V is called *lisse* if $\dim X_V = 0$, or equivalently, if R_V is finite-dimensional (cf. §2.3.2). Below are some examples.

⁵More generally, there is a notion of an associated variety for any module over a vertex algebra [6], and in particular, the associated variety of any irreducible highest representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ makes sense, too.

3.2.1. Integrable representations of affine Kac-Moody algebras. The irreducible \mathfrak{g} -representation $L_{\mathfrak{g}}(\lambda)$, where $\lambda \in \mathfrak{h}^*$, is finite-dimensional if and only if its associated variety $\mathcal{V}(\text{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{g}}(\lambda)))$ is zero. Contrary to irreducible highest weight representations of \mathfrak{g} , the irreducible $\widehat{\mathfrak{g}}$ -representation $L(\lambda)$, where $\lambda \in \widehat{\mathfrak{h}}^*$, is finite-dimensional if and only if $\lambda = 0$, that is, $L(\lambda)$ is the trivial representation.

The notion of finite-dimensional representations has to be replaced by the notion of *integrable representations* in the category \mathcal{O} . The category \mathcal{O} for $\widehat{\mathfrak{g}}$ is defined in the similar way that for \mathfrak{g} , except that we do not require that the object are finitely generated by \mathfrak{g} (cf. [83]).

Definition 40. Given a triangular decomposition of $\widehat{\mathfrak{g}}$,

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}},$$

a representation M of $\widehat{\mathfrak{g}}$ is said to be *integrable* if

- (1) M is $\widehat{\mathfrak{h}}$ -diagonalisable,
- (2) for $\lambda \in \widehat{\mathfrak{h}}^*$, M_{λ} is finite-dimensional,
- (3) for $i = 0, \dots, r$, E_i and F_i act locally nilpotently on M , where E_i, H_i, F_i are Chevalley generators⁶ of $\widehat{\mathfrak{g}}$.

Remark 41. As an \mathfrak{a}_i -module, $i = 0, \dots, r$, an integrable representation M decomposes into a direct sum of finite dimensional irreducible $\widehat{\mathfrak{h}}$ -invariant modules, where $\mathfrak{a}_i \cong \mathfrak{sl}_2$ is the Lie algebra generated by the Chevalley generators E_i, F_i, H_i . Hence the action of \mathfrak{a}_i on M can be “integrated” to the action of the group $SL_2(\mathbb{C})$.

The character of the simple integrable representations in the category \mathcal{O} satisfy remarkable combinatorial identities (related to Macdonald identities).

Theorem 42 ([39]). $L_k(\mathfrak{g})$ is lisse if and only if $L_k(\mathfrak{g})$ is integrable as a $\widehat{\mathfrak{g}}$ -module (which happens if and only if $k \in \mathbb{Z}_{\geq 0}$).

The last equivalence in parenthesis of the statement is well-known. We explain below the “if” part of Theorem 42.

Lemma 43. Let (R, ∂) be a differential algebra over \mathbb{Q} , and let I be a differential ideal of R , i.e., I is an ideal of R such that $\partial I \subset I$. Then $\partial\sqrt{I} \subset \sqrt{I}$.

Proof. Let $a \in \sqrt{I}$, so that $a^m \in I$ for some $m \in \mathbb{Z}_{\geq 0}$. Since I is ∂ -invariant, we have $\partial^m a^m \in I$. But

$$\partial^m a^m \equiv m!(\partial a)^m \pmod{\sqrt{I}}.$$

Hence $(\partial a)^m \in \sqrt{I}$, and therefore, $\partial a \in \sqrt{I}$. □

Recall that a *singular vector* of a \mathfrak{g} -module M is a vector $v \in M$ such that $\mathfrak{n}.v = 0$, that is, $e_i.v = 0$ for $i = 1, \dots, r$. A *singular vector* of a $\widehat{\mathfrak{g}}$ -representation M is a vector $v \in M$ such that $\widehat{\mathfrak{n}}.v = 0$, that is, $e_i.v = 0$ for $i = 1, \dots, r$, and $(f_{\theta}t).v = 0$ (θ is the highest positive root). In particular, regarding $V^k(\mathfrak{g})$ as a $\widehat{\mathfrak{g}}$ -representation, a vector $v \in V^k(\mathfrak{g})$ is singular if and only if $\widehat{\mathfrak{n}}.v = 0$.

Proof of the “if” part of Theorem 42. Suppose that $L_k(\mathfrak{g})$ is integrable. This condition is equivalent to that $k \in \mathbb{Z}_{\geq 0}$, and the maximal submodule $N_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_{\theta}t^{-1})^{k+1}|0\rangle$ ([59]). The exact sequence $0 \rightarrow N_k(\mathfrak{g}) \rightarrow V^k(\mathfrak{g}) \rightarrow L_k(\mathfrak{g}) \rightarrow 0$ induces the exact sequence

$$0 \rightarrow I_k \rightarrow R_{V^k(\mathfrak{g})} \rightarrow R_{L_k(\mathfrak{g})} \rightarrow 0,$$

where I_k is the image of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$, and so, $R_{L_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_k$. The image of the singular vector in I_k is given by e_{θ}^{k+1} . Therefore, $e_{\theta} \in \sqrt{I_k}$. On the other hand, by Lemma 43, $\sqrt{I_k}$ is preserved by the adjoint action of \mathfrak{g} . Since \mathfrak{g} is simple, $\mathfrak{g} \subset \sqrt{I_k}$. This proves that $X_{L_k(\mathfrak{g})} = \{0\}$ as required. □

The proof of the “only if” part follows from [39]. It can also be proven using W-algebras.

In view of Theorem 42, one may regard the lisse condition as a generalization of the integrability condition to an arbitrary vertex algebra.

⁶Namely, $E_i = e_i \otimes 1$, $F_i = f_i \otimes 1$, $H_i = f_i \otimes 1$, for $i = 1, \dots, r$, where e_i, f_i, h_i are the Chevalley generators of \mathfrak{g} , and $E_0 = e_0 \otimes t$, $F_0 = f_0 \otimes t^{-1}$, with $f_0 \in \mathfrak{g}_{\theta}$, $e_0 \in \mathfrak{g}_{-\theta}$ such that $(f_0|e_0) = 1$. Here θ is the highest positive root.

3.2.2. Minimal series representations of the Virasoro algebra. Let N_c be the unique maximal submodule of the Virasoro vertex algebra Vir^c , and let $\text{Vir}_c := \text{Vir}^c/N_c$ be the simple quotient. By [6, Prop. 3.4.1], the following are equivalent:

- (i) Vir_c is lisse,
- (ii) $c = 1 - \frac{6(p-q)^2}{pq}$ for some $p, q \in \mathbb{Z}_{\geq 2}$ such that $(p, q) = 1$. (These are precisely the central charge of the *minimal series representations* of the Virasoro algebra Vir .)

3.2.3. On the lisse and the rational conditions. It is known that lisse vertex algebras have various nice properties. Let us mention a remarkable result.

Theorem 44 ([1, 95, 82]). *Let V be a $\mathbb{Z}_{\geq 0}$ -graded conformal lisse vertex algebra.*

- (1) *Any simple V -module is a positive energy representation, that is, a positively graded V -module. Therefore the number of isomorphic classes of simple V -modules is finite.*
- (2) *Let M_1, \dots, M_s be representatives of these classes, and let for $i = 1, \dots, s$,*

$$\chi_{M_i}(\tau) = \text{Tr}_{M_i}(q^{L_0 - \frac{c}{24}}) = \sum_{n \geq 0} \dim(M_i)_n q^{n - \frac{c}{24}}, \quad q = e^{2i\pi\tau},$$

be the normalized character of M_i . Then $\chi_{M_i}(\tau)$ converges in the domain $\{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$, and the vector space generated by $SL_2(\mathbb{Z}) \cdot \chi_{M_i}(\tau)$ is finite-dimensional.

Definition 45. A conformal vertex algebra V is called *rational* if every $\mathbb{Z}_{\geq 0}$ -graded V -modules is completely reducible (i.e., isomorphic to a direct sum of simple V -modules).

It is known ([38]) that the rationality condition implies that V has finitely many simple $\mathbb{Z}_{\geq 0}$ -graded modules and that the graded components of each of these $\mathbb{Z}_{\geq 0}$ -graded modules are finite dimensional. If V is as in Theorem 44 and also rational, it is known [54] that under some mild assumptions, the category of V -modules forms a modular tensor category, which for instance yields an invariant of 3-manifolds, see [26]. It is actually conjectured by Zhu in [95] that rational vertex algebras must be lisse (this conjecture is still open).

However, there are significant vertex algebras that do not satisfy the lisse condition. For instance, an *admissible* affine vertex algebra $L_k(\mathfrak{g})$ (see below) has a complete reducibility property ([8]), and the modular invariance property ([65]) in the category \mathcal{O} still holds, although it is not lisse unless it is integrable.

So it is natural to try to relax the lisse condition. This is the purpose of next section.

3.3. QUASI-LISSE VERTEX ALGEBRAS

3.3.1. Symplectic stratification. Recall that X_V is a Poisson variety.

If X_V is smooth, then one may view X_V as a complex-analytic manifold equipped with a holomorphic Poisson structure, and for each point $x \in X_V$ there is a well-defined symplectic \mathcal{S}_x leaf through x , which is the set of points that can be reached from x by going along Hamiltonian flows.

If X_V is not necessarily smooth, let $\text{Sing}(X_V)$ be the singular locus of X , and for any $k \geq 1$ define inductively $\text{Sing}^k(X_V) := \text{Sing}(\text{Sing}^{k-1}(X_V))$. We get a finite partition

$$X_V = \bigsqcup_k X_V^k,$$

where the strata $X_V^k := \text{Sing}^{k-1}(X_V) \setminus \text{Sing}^k(X_V)$ are smooth analytic varieties (by definition we put $X_V^0 = X_V \setminus \text{Sing}(X_V)$). It is known (cf. e.g., [36]) that each X_V^k inherits a Poisson structure. So for any point $x \in X_V^k$ there is a well-defined symplectic leaf $\mathcal{S}_x \subset X_V^k$. In this way one defines symplectic leaves on an arbitrary Poisson variety.

For example, the symplectic leaves of \mathfrak{g}^* are the (co)adjoint orbits $G\xi$, $\xi \in \mathfrak{g}^* \cong \mathfrak{g}$.

3.3.2. Definition and properties of quasi-lisse vertex algebras.

Definition 46 ([14]). A vertex algebra is called *quasi-lisse* if X_V has only finitely many symplectic leaves.

Clearly, lisse vertex algebras are quasi-lisse. We have already noticed that lisse vertex algebras are very nice (see Lemma 35 and Theorem 44). It turns out that quasi-lisse vertex algebras have remarkable properties, too.

Theorem 47 ([14]). *A quasi-lisse vertex operator algebra has only finitely many simple ordinary representations. Here, a V -module is called ordinary if it is a positive energy representation and each homogeneous space is finite-dimensional, so that the normalized character*

$$\chi_M(\tau) = \text{Tr}_M(q^{L_0 - \frac{c}{24}})$$

is well-defined.

Moreover, the normalized character of any ordinary module has a modular invariance property, in the sense that it satisfies a modular linear differential equation.

A quasi-lisse vertex algebras quantizes the arc space of some Poisson variety as we recently demonstrated.

Theorem 48 ([21]). *Assume that V is a quasi-lisse vertex algebra. Then*

$$SS(V)_{\text{red}} \cong (J_\infty X_V)_{\text{red}}$$

as topological spaces. Moreover, the reduced singular support $SS(V)_{\text{red}}$ have finitely many irreducible components, and each of them is a chiral symplectic cores closure⁷.

We now intend to give various examples of quasi-lisse vertex algebras.

Lemma 49. *The simple affine vertex algebra $L_k(\mathfrak{g})$ is quasi-lisse if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$.*

Proof. Symplectic leaves in $X_{L_k(\mathfrak{g})}$ are the adjoint G -orbits contained in $X_{L_k(\mathfrak{g})} \subset \mathfrak{g}$. It is well-known that the nilpotent cone \mathcal{N} of the simple Lie algebra is a finite union of adjoint orbits. Hence, if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ then $L_k(\mathfrak{g})$ is quasi-lisse. This proves the converse implication.

To show the direct implication, first observe that if x is semisimple, then the closed G -invariant cone $\overline{GC^*x}$ generated by x contains infinitely many symplectic leaves. Assume now that $X_{L_k(\mathfrak{g})}$ contains a non-nilpotent element x , with Jordan decomposition $x = x_s + x_n$. If $x_n = 0$, then $X_{L_k(\mathfrak{g})}$ contains $\overline{GC^*x}$ since $X_{L_k(\mathfrak{g})}$ is a closed G -invariant cone of \mathfrak{g} . So $X_{L_k(\mathfrak{g})}$ cannot be quasi-lisse. If $x_n \neq 0$, then given an \mathfrak{sl}_2 -triplet (x_n, h, y_n) of \mathfrak{g} , we have

$$\rho(t)x = x_s + t^2x_n,$$

where $\rho: \mathbb{C}^* \rightarrow G$ is the one-parameter subgroup generated by $\text{ad } h$. Taking the limit when t goes to 0, we deduce that $x_s \in X_{L_k(\mathfrak{g})}$ and, hence, by the first case, $X_{L_k(\mathfrak{g})}$ cannot be quasi-lisse. \square

3.3.3. Admissible representations. Let $\hat{\Delta}^{re}$ be the set of real roots of $\hat{\mathfrak{g}}$, and $\hat{\Delta}_+^{re}$ the set of real positive roots with respect to the triangular decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+.$$

Definition 50 ([65, 67]). A weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if

(1) λ is *regular dominant*, that is,

$$\langle \lambda + \hat{\rho}, \alpha^\vee \rangle \notin -\mathbb{Z}_{\geq 0} \quad \text{for all } \alpha \in \hat{\Delta}_+^{re},$$

where $\hat{\rho} = h^\vee \Lambda_0 + \rho$, with ρ the half-sum of positive roots of \mathfrak{g} ,

(2) $\mathbb{Q}\hat{\Delta}_\lambda = \mathbb{Q}\hat{\Delta}^{re}$, where $\hat{\Delta}_\lambda := \{\alpha \in \hat{\Delta}^{re} \mid \langle \lambda + \hat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\}$.

The irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ with highest weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if λ is admissible. An irreducible integrable representation of $\hat{\mathfrak{g}}$ is admissible. The simple affine vertex algebra $L_k(\mathfrak{g})$ is called *admissible* if it is admissible as a $\hat{\mathfrak{g}}$ -module. This happens if and only if k satisfies one of the following conditions:

⁷The notion of *chiral symplectic core* is introduced in [21]: it is an affine analog to the notion of *symplectic core* ([36]).

- (1) $k = -h^\vee + \frac{p}{q}$, where $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$ and $p \geq h^\vee$,
- (2) $k = -h^\vee + \frac{p}{q}$, where $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$, $(q, r^\vee) = r^\vee$ and $p \geq h_{\mathfrak{g}}$.

Here r^\vee is the *lacety* of \mathfrak{g} (i.e., $r^\vee = 1$ for the types A, D, E , $r^\vee = 2$ for the types B, C, F and $r^\vee = 3$ for the type G_2), and $h_{\mathfrak{g}}$ is the Coxeter number.

Definition 51. If k satisfies one of the above conditions (1) or (2) we say that k is an *admissible level*.

The following fact was conjectured by Feigin and Frenkel and proved for the case that $\mathfrak{g} = \mathfrak{sl}_2$ by Feigin and Malikov [45].

Theorem 52 ([7]). *If k is admissible, then $SS(L_k(\mathfrak{g})) \subset J_\infty(\mathcal{N})$ or, equivalently, the associated variety $X_{L_k(\mathfrak{g})}$ is contained in \mathcal{N} .*

In fact, a stronger result holds.

Theorem 53 ([7]). *Assume that k is admissible. Then*

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k},$$

where \mathbb{O}_k is a nilpotent orbit which only depends on q , with q as in Definition 51.

Remark 54. Let us describe explicitly the nilpotent orbit \mathbb{O}_k of Theorem 53 in the case where $\mathfrak{g} = \mathfrak{sl}_n$. Recall that the nilpotent orbits of \mathfrak{sl}_n are parameterized by the partitions of n . Let k be an admissible level for \mathfrak{sl}_n , that is, $k = -n + \frac{p}{q}$, with $p \in \mathbb{Z}$, $p \geq n$, and $(p, q) = 1$. Then

$$X_{L_k(\mathfrak{g})} = \{x \in \mathfrak{g} \mid (\text{ad } x)^{2q} = 0\} = \overline{\mathbb{O}_k},$$

where \mathbb{O}_k is the nilpotent orbit corresponding to the partition (n) is $q \geq n$, and to the partition $(q, q, \dots, q, s) = (q^m, s)$, where m and s are the quotient and the rest of the Euclidean division of n by q , respectively, if $q < n$.

Next exercise gives a proof of Theorem 53 for $\mathfrak{g} = \mathfrak{sl}_2$. It is based on Feigin and Malikov approach (see also [7, Theo. 5.6]).

Exercise 7 (Simple affine vertex algebras associated with \mathfrak{sl}_2). Let N be the proper maximal ideal of $V^k(\mathfrak{sl}_2)$ so that $L_k(\mathfrak{g}) = V^k(\mathfrak{sl}_2)/N$. Let I be the image of N in $R_{V^k(\mathfrak{sl}_2)} = \mathbb{C}[\mathfrak{g}]$ so that $R_{L_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}]/I$. It is known that either N is trivial, that is, $V^k(\mathfrak{sl}_2)$ is simple, or N is generated by a singular vector v whose image \bar{v} in I is nonzero ([64, 81]).

We assume in this exercise that N is non trivial. Thus, $N = U(\widehat{\mathfrak{sl}_2})v$.

- (1) Using Kostant's Separation Theorem show that, up to a nonzero scalar,

$$v = \Omega^m e^n,$$

for some $m, n \in \mathbb{Z}_{>0}$, where $\Omega = 2ef + \frac{1}{2}h^2$ is the Casimir element of the symmetric algebra of \mathfrak{sl}_2 .

- (2) Deduce from this that

$$X_{L_k(\mathfrak{g})} \subset \mathcal{N}.$$

It is known that N is nontrivial if and only if k is an admissible level for \mathfrak{sl}_2 , or $k = -2$ is critical. Thus we have shown that $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ if and only if $k = -2$ or k is admissible, i.e., $k = -2 + \frac{p}{q}$, with $(p, q) = 1$ and $p \geq 2$.

Hints for Exercise 7. (1) For $\mathfrak{g} = \mathfrak{sl}_2$, Kostant's Separation Theorem [76, Th. 0.2 and 0.11] says that $S = ZH$, where $Z \cong \mathbb{C}[\Omega]$ is the center of the symmetric algebra S of \mathfrak{sl}_2 , and H is the space of invariant harmonic polynomials which decomposes, as an \mathfrak{sl}_2 -module, as $H = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda^{m_\lambda}$, with $m_\lambda = 1$ for all λ since $\mathfrak{g} = \mathfrak{sl}_2$. Therefore, $S^{\text{ad } e} = \bigoplus_{\lambda \in \mathbb{Z}} ZV_\lambda^{\text{ad } e}$. To conclude, observe that, v being a singular vector, it has a fixed weight and, hence, a fixed degree.

- (2) Note that from (1), $\Omega e \in \sqrt{I}$ and, so, $\Omega \mathfrak{g} \in \sqrt{I}$, whence $\Omega \in \sqrt{I}$. But in \mathfrak{sl}_2 , \mathcal{N} is the zero locus of Ω .

3.3.4. Exceptional Deligne series. There was actually a “strong Feigin-Frenkel conjecture” stating that k is admissible if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ (provided that k is not critical, that is, $k \neq -h^\vee$ in which case it is known that $X_{L_k(\mathfrak{g})} = \mathcal{N}$). Such a statement would be interesting because it would give a geometrical description of the admissible representations $L_k(\mathfrak{g})$.

As seen in Exercise 7, the equivalence holds for $\mathfrak{g} = \mathfrak{sl}_2$. The stronger conjecture is wrong in general, as shown the following result.

Theorem 55 ([18]). *Assume that \mathfrak{g} belongs to the Deligne exceptional series,*

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

and that $k = -\frac{h^\vee}{6} - 1 + n$, where $n \in \mathbb{Z}_{\geq 0}$ is such that $k \notin \mathbb{Z}_{\geq 0}$. Then

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}},$$

where \mathbb{O}_{min} is the minimal nilpotent orbit of \mathfrak{g} , that is, the unique nilpotent orbit of \mathfrak{g} of minimal dimension $2h^\vee - 2$.

Note that the level $k = -\frac{h^\vee}{6} - 1$ is not admissible for the types D_4, E_6, E_7, E_8 (it equals $-2, -3, -4, -6$, respectively). Theorem 55 provides the first known examples of associated varieties contained in the nilpotent cone corresponding to non-admissible levels. The proof of this result is closely related to the *Joseph primitive ideal*⁸ [56], and its description by Gan and Savin [49], associated with the minimal nilpotent orbit.

Note that the condition $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ implies that $L_k(\mathfrak{g})$ has only finitely many simple objects in the category \mathcal{O} , and one can describe them thanks to Joseph’s classification [58] of irreducible highest weights representation $L_{\mathfrak{g}}(\lambda)$ whose associated variety is $\overline{\mathbb{O}_{min}}$.

The following exercise explains how to compute the associated variety in a concrete example exploiting a singular vector. This example is covered by both Theorem 53 and Theorem 55.

Exercise 8 (An explicit computation of an associated variety). The aim of this exercise is to compute $X_{L_{-3/2}(\mathfrak{sl}_3)}$. It was shown by Perše [89] that the proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the singular vector v given by:

$$v := \frac{1}{3} ((h_1 t^{-1})(e_{1,3} t^{-1})|0\rangle - (h_2 t^{-1})(e_{1,3} t^{-1})|0\rangle) + (e_{1,2} t^{-1})(e_{2,3} t^{-1})|0\rangle - \frac{1}{2} e_{1,3} t^{-2}|0\rangle,$$

where $h_1 := e_{1,1} - e_{2,2}$, $h_2 := e_{2,2} - e_{3,3}$ and $e_{i,j}$ is the elementary matrix of the coefficient (i, j) in \mathfrak{sl}_3 identified with the set of traceless 3-size square matrices.

- (1) Verify that v is indeed a singular vector for $\widehat{\mathfrak{sl}_3}$, that is, $e_{i,i+1}v = 0$ for $i = 1, 2$ and $(e_{3,1}t)v = 0$.
- (2) Let $\mathfrak{h} := \mathbb{C}h_1 + \mathbb{C}h_2$ be the usual Cartan subalgebra of \mathfrak{sl}_3 . Show that $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h} = \{0\}$, and deduce from this that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ is contained in the nilpotent cone \mathcal{N} of \mathfrak{sl}_3 .
- (3) Show that \mathcal{N} is not contained in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.
- (4) Denoting by \mathbb{O}_{min} the minimal nilpotent orbit of \mathfrak{sl}_3 , conclude that

$$X_{L_{-3/2}(\mathfrak{sl}_3)} = \overline{\mathbb{O}_{min}}.$$

Hints for Exercise 8. (1) Just use the commuting relations in $V^{-3/2}(\mathfrak{sl}_3)$.

- (2) Observe that the image I of the maximal proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the vector \bar{v} as an $(\text{ad } \mathfrak{sl}_3)$ -module, where

$$\bar{v} = \frac{1}{3} (h_1 - h_2) e_{1,3} + e_{1,2} e_{2,3}$$

is the image of v in $R_{V^{-3/2}(\mathfrak{sl}_3)} \cong \mathbb{C}[h_i, e_{k,l}; i = 1, 2, k \neq l]$. Verify that

$$\begin{aligned} (\text{ad } e_{3,2})(\text{ad } e_{2,1})\bar{v} &= -e_{1,2}e_{2,1} + e_{1,3}e_{3,1} + \frac{1}{3}(2h_1 + h_2)h_2, \\ (\text{ad } e_{2,1})(\text{ad } e_{3,2})\bar{v} &= -e_{2,3}e_{3,2} + e_{1,3}e_{3,1} + \frac{1}{3}(h_1 + 2h_2)h_1, \end{aligned}$$

⁸For \mathfrak{g} not of type A , there is the unique primitive ideal \mathcal{J}_0 of $U(\mathfrak{g})$ whose associated variety $\mathcal{V}(\mathcal{J}_0)$ is the minimal nilpotent orbit closure.

and deduce from this that the intersection $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h}$ is zero. For the last part, resume the arguments of the proof of Lemma 49.

- (3) Verify that $e_{1,2} + e_{2,3}$ is not in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.
- (4) Observe that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ cannot be reduced to zero.

Remark 56. There are other examples of simple quasi-lisse affine vertex algebras $L_k(\mathfrak{g})$, at non-admissible level k , in type D_r , $r \geq 5$, and in type B_r , $r \geq 3$; see [18, 19, 20]. Except for $\mathfrak{g} = \mathfrak{sl}_2$, the classification problem of quasi-lisse affine vertex algebras is wide open.

Remark 57. It may happen that the associated variety $X_{L_k(\mathfrak{g})}$ is neither the whole \mathfrak{g} , nor contained in the nilpotent cone. For example [19, Th. 1.1], for $n \geq 4$,

$$X_{L_{-1}(\mathfrak{sl}_n)} = \overline{G\mathbb{C}^* \tilde{\omega}_1} \not\subset \mathcal{N},$$

where $\tilde{\omega}_1$ is the fundamental co-weight associated with α_1 if $\alpha_1, \dots, \alpha_{n-1}$ are the simple roots of \mathfrak{sl}_n , and for $m \geq 2$,

$$X_{L_{-m}(\mathfrak{sl}_{2m})} = \overline{G\mathbb{C}^* \tilde{\omega}_m} \not\subset \mathcal{N},$$

where $\tilde{\omega}_m$ is the fundamental co-weight associated with α_m .

3.3.5. Chiral differential operators. So far, all our examples of quasi-lisse, non lisse, vertex algebras are affine vertex algebras. We will see other examples next part in the context of W -algebras by taking the quantized Drinfeld-Sokolov reduction of quasi-lisse affine vertex algebras. There are other expected examples coming from four dimensional $N = 2$ superconformal field theories, see Sect. 3.4.

Here is another type of example.

Example 58. Given a smooth affine variety X , the global section of the *chiral differential operators* \mathcal{D}_X^{ch} ([80, 53, 30]) is quasi-lisse because its associated scheme is canonically isomorphic to the cotangent bundle T^*X . As a consequence of [21, Cor. 9.3], the vertex algebra \mathcal{D}_X^{ch} is simple, since the associated scheme is smooth, reduced and symplectic. In particular, the global section of the chiral differential operators $\mathcal{D}_{G,k}^{ch}$ on the group G ([52, 25]) is simple at any level k . This example is important since $\mathcal{D}_{G,-h^\vee}^{ch}$ appears in the 4d/2d duality for the class \mathcal{S} theory (cf. [13] or, here, Sect. 3.4).

3.3.6. Irreducibility conjecture. In view of the above results, and other ones, particularly, on associated varieties of simple affine W -algebras (cf. Part 4), we formulate a conjecture.

Conjecture 59 ([19, Conj. 1]). *Let $V = \bigoplus_{d \geq 0} V_d$ be a simple, finitely strongly generated, positively graded conformal vertex operator algebra such that $V_0 = \mathbb{C}$. Assume that X_V has finitely many symplectic leaves, that is, V is quasi-lisse. Then X_V is irreducible. In particular, if $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$, then $X_{V_k(\mathfrak{g})}$ is the closure of some nilpotent orbit.*

The conjecture is a natural affine analog of the irreducibility theorem (cf. Theorem 39) for the associated variety of primitive ideals of $U(\mathfrak{g})$, which has been generalized to a large class of Noetherian algebras by Ginzburg [50]:

Theorem 60 ([50]). *Let A be a filtered unital \mathbb{C} -algebra. Assume furthermore that $\text{gr } A \cong \mathbb{C}[X]$ is the coordinate ring of a reduced irreducible affine algebraic variety X , and assume that the Poisson variety $\text{Spec}(\text{gr } A)$ has only finitely many symplectic leaves. Then for any primitive ideal $I \subset A$, the zero locus $\mathcal{V}(I)$ of $\text{gr } I$ in X is the closure of a single symplectic leaf. In particular, it is irreducible.*

Unfortunately, the algebras we consider in Conjecture 59 are not Noetherian. The reader is referred to Remark 72 for more about this conjecture in the context of W -algebras.

3.4. HIGGS BRANCH AND FOUR DIMENSIONAL $N = 2$ SUPERCONFORMAL FIELD THEORIES

There are other known examples of quasi-lisse vertex algebras: apart from the above examples, it is the case when V is the (generalized) Drinfeld-Sokolov reduction of a quasi-lisse affine vertex algebra provided that it is nonzero [7] (cf. §§4.2.2 and 4.2.3).

This is also expected to happen for the vertex algebras obtained from four dimensional $N = 2$ superconformal field theories (4d $N = 2$ SCFTs), where the associated variety is expected to coincide with the spectrum of the chiral ring of the Higgs branch of the four dimensional theory [29].

More precisely, the physicists Beem, Rastelli et al [29] showed that there is a remarkable map

$$\Phi: \{4d\ N = 2\ \text{SCFTs}\} \rightarrow \{\text{vertex algebras}\},$$

such that the character of the vertex algebra $\Phi(\mathcal{T})$ coincides with the *Schur index* of the corresponding 4d $N = 2$ SCFT \mathcal{T} , which is an important invariant. Now, there is another important invariant, called the *Higgs branch*, which we denote by $Higgs(\mathcal{T})$. The Higgs branch $Higgs(\mathcal{T})$ is an affine algebraic variety that has a hyperKähler structure in its smooth part. In particular, $Higgs(\mathcal{T})$ is a (possibly singular) symplectic variety. Note the meaning of this is not completely clear since there is no mathematical definition of the Higgs branch in general.

The main examples of vertex algebras considered in [29] are the affine vertex algebras $L_k(\mathfrak{g})$ of types D_4 , E_6 , E_7 , E_8 at level $k = -\frac{h^\vee}{6} - 1$, which are non-rational, non-admissible quasi-lisse affine vertex algebras that appear in Theorem 55.

Let \mathcal{T} be one of the 4d $N = 2$ SCFTs studied in [29] such that $\Phi(\mathcal{T}) = L_k(\mathfrak{g})$ with $k = -\frac{h^\vee}{6} - 1$ for \mathfrak{g} of type D_4 , E_6 , E_7 , E_8 as above. It is known that the $Higgs(\mathcal{T}) = \overline{\mathbb{O}_{min}}$, which equals to $X_{L_k(\mathfrak{g})}$ by Theorem 55. It is expected that this is not just a coincidence.

Conjecture 61 ([28]). *For \mathcal{T} any 4d $N = 2$ SCFT, we have*

$$Higgs(\mathcal{T}) = X_{\Phi(\mathcal{T})}.$$

Conjecture 61 has been recently proved by Arakawa for the theory of class \mathcal{S} [13], a particular class of 4d $N = 2$ SCFT's for which the Higgs branches has been mathematically defined in terms of two-dimensional topological quantum field theories [84, 35]. This includes the above examples. In particular, Arakawa's result reproves Theorem 55 for $\mathfrak{g} = D_4$, $k = -2$ and $\mathfrak{g} = E_6$, $k = -3$. We refer to the recent surveys [11, 12] for more details about this conjecture.

It is expected by physicists that the Higgs branch of 4d $N = 2$ SCFT's is an irreducible, normal, (possibly singular) symplectic variety. Hence, by a result of Kaledin [68, Th. 2.3] it implies that the Higgs branch has expectedly only finitely many symplectic leaves. In other words, physical intuition predicts that vertex algebras that come from 4d $N = 2$ SCFTs via the map Φ are quasi-lisse. In this context, our conjecture 59 would give an evidence to Beem-Rastelli conjecture.

Part 4. Affine W -algebras

Given a nilpotent element f of a simple Lie algebra \mathfrak{g} , the W -algebra associated with (\mathfrak{g}, f) at level k is a certain vertex algebra defined by the generalized quantized Drinfeld-Sokolov reduction, which is a certain quantized Hamiltonian reduction. Affine W -algebras are natural affinizations of the finite W -algebras introduced by Premet [90] in the sense that Zhu's algebras⁹ of W -algebras are finite W -algebras. The later are certain generalizations of the enveloping algebra of a simple Lie algebra. On the other hand, as discussed later, affine W -algebras are chiral quantizations of Slodowy slices associated with nilpotent elements.

In this part, we indicate how to define concretely the affine W -algebra associated with $\mathfrak{g} = \mathfrak{sl}_2$ and a nonzero nilpotent element f via the BRST cohomology (see Exercice 10). However we do not give the general definition in this note. We refer to [10] and the references given there for more details. Then we will restrict our attention to associated varieties of quantized Drinfeld-Sokolov reductions, and discuss some applications.

4.1. POISSON STRUCTURE ON SLOWOWY SLICES

In this section, we review some important properties of Slodowy slices. Continue to assume that $\mathfrak{g} = \text{Lie}(G)$ is a simple Lie algebra.

4.1.1. Slodowy slices. Fix a nilpotent element f of \mathfrak{g} that we embed into an \mathfrak{sl}_2 -triple (e, h, f) of \mathfrak{g} . Let $\phi: \mathfrak{g} \rightarrow \mathfrak{g}^*$ be the isomorphism induced from the non-degenerate bilinear form $(\cdot | \cdot)$, and set

$$\chi := \phi(f) = (f | \cdot) \in \mathfrak{g}^*.$$

Then define the *Slodowy slice associated with (e, h, f)* to be

$$\mathcal{S}_f := \phi(f + \mathfrak{g}^e) = \chi + \phi(\mathfrak{g}^e) \subset \mathfrak{g}^*.$$

Denote by \mathfrak{g}_i the i -eigenspace of $\text{ad}(h)$ for $i \in \mathbb{Z}$,

$$\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [h, x] = ix\}, \quad i \in \mathbb{Z}.$$

The restriction of the antisymmetric bilinear form,

$$\omega_\chi: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}, \quad (x, y) \mapsto (f | [x, y]),$$

to $\mathfrak{g}_{\frac{1}{2}} \times \mathfrak{g}_{\frac{1}{2}}$ is nondegenerate. This results from the pairing between $\mathfrak{g}_{\frac{1}{2}}$ and $\mathfrak{g}_{-\frac{1}{2}}$, and from the injectivity of the map $\text{ad } f: \mathfrak{g}_{\frac{1}{2}} \rightarrow \mathfrak{g}_{-\frac{1}{2}}$. It is called the *Kirillov form associated with f* . Let ℓ be a Lagrangian subspace of $\mathfrak{g}_{\frac{1}{2}}$, that is, ℓ is maximal isotropic which means $\omega_\chi(\ell, \ell) = 0$ and $\dim \ell = \frac{1}{2} \dim \mathfrak{g}_{\frac{1}{2}}$. Set

$$\mathfrak{m} = \mathfrak{m}_{\chi, \ell} := \ell \oplus \bigoplus_{j > \frac{1}{2}} \mathfrak{g}_j.$$

Then \mathfrak{m} is an ad-nilpotent¹⁰, ad h -graded subalgebra, of \mathfrak{g} , and we have:

- (χ 1) $\chi([\mathfrak{m}, \mathfrak{m}]) = (f | [\mathfrak{m}, \mathfrak{m}]) = 0$,
- (χ 2) $\mathfrak{m} \cap \mathfrak{g}^f = \{0\}$,
- (χ 3) $\dim \mathfrak{m} = \frac{1}{2} \dim G.f$.

Denote by M the unipotent subgroup of G corresponding to \mathfrak{m} .

4.1.2. Contracting \mathbb{C}^* -action. The embedding $\text{span}_{\mathbb{C}}\{e, h, f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ exponentiates to a homomorphism $SL_2 \rightarrow G$. By restriction to the one-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup $\rho: \mathbb{C}^* \rightarrow G$. Thus $\rho(t)x = t^{2j}x$ for any $x \in \mathfrak{g}_j$. For $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, set

$$(11) \quad \tilde{\rho}(t)x := t^2 \rho(t)(x).$$

So, for any $x \in \mathfrak{g}_j$, $\tilde{\rho}(t)x = t^{2+2j}x$. In particular, $\tilde{\rho}(t)f = f$ and the \mathbb{C}^* -action of $\tilde{\rho}$ stabilizes \mathcal{S}_f . Moreover, it is contracting to f on \mathcal{S}_f , that is,

$$\lim_{t \rightarrow 0} \tilde{\rho}(t)(f + x) = f$$

⁹The *Zhu algebra* of a graded vertex algebra is a certain quotient of the vertex algebra which naturally has the structure of a filtered associative algebra.

¹⁰i.e., \mathfrak{m} only consists of nilpotent elements of \mathfrak{g} .

for any $x \in \mathfrak{g}^e$, because $\mathfrak{g}^e \subset \mathfrak{m}^\perp \subseteq \mathfrak{g}_{>-1}$. The same lines of arguments show that the action $\tilde{\rho}$ stabilizes $f + \mathfrak{m}^\perp$ and it is contracting to f on $f + \mathfrak{m}^\perp$, too.

Next theorem asserts that affine space \mathcal{S}_f is a “slice”.

Theorem 62. *The affine space \mathcal{S}_f is transversal to the coadjoint orbits of \mathfrak{g}^* . More precisely, given any $\xi \in \mathcal{S}_f$, we have $T_\xi(G.\xi) + T_\xi(\mathcal{S}_f) = \mathfrak{g}^*$. An analogue statement holds for the affine variety $\chi + \mathfrak{m}^\perp$.*

Proof. We have to show that $[\mathfrak{g}, x] + \mathfrak{g}^e = \mathfrak{g}$ for any $x \in f + \mathfrak{g}^e$ since $T_x(G.x) = [\mathfrak{g}, x]$ and $T_x(f + \mathfrak{g}^e) = \mathfrak{g}^e$. It suffices to verify that the map

$$\eta: G \times (f + \mathfrak{g}^e) \rightarrow \mathfrak{g}$$

is a submersion at any point (g, x) of $G \times (f + \mathfrak{g}^e)$, that is, the differential $d\eta_{(g,x)}$ of η at (g, x) is surjective for any point (g, x) of $G \times (f + \mathfrak{g}^e)$. The differential of η is the linear map $\mathfrak{g} \times \mathfrak{g}^e \rightarrow \mathfrak{g}$, $(v, w) \mapsto g([v, x]) + g(w)$. So $d\eta_{(\text{Id}, f)}(v, w) = [v, f] + w$ and, hence, $d\eta_{(\text{Id}, f)}$ is surjective, for $[\mathfrak{g}, f] + \mathfrak{g}^e = \mathfrak{g}$. Thus $d\eta_{(\text{Id}, x)}$ is surjective for any x in an open neighborhood Ω of f in $f + \mathfrak{g}^e$. Because the morphism η is G -equivariant for the action by left multiplication, we deduce that $d\eta_{(g,x)}$ is surjective for any $g \in G$ and any $x \in \Omega$. In particular, for any $x \in \Omega$, we get

$$\mathfrak{g} = [\mathfrak{g}, x] + \mathfrak{g}^e$$

We now use the contracting \mathbb{C}^* -action ρ on $f + \mathfrak{g}^e$ to show that η is actually a submersion at any point of $G \times (f + \mathfrak{g}^e)$. \square

4.1.3. An isomorphism. Consider the adjoint map

$$M \times (f + \mathfrak{m}^\perp) \rightarrow \mathfrak{g}, \quad (g, x) \mapsto g.x$$

Its image is contained in $f + \mathfrak{m}^\perp$. Indeed, for any $x \in \mathfrak{n}$ and any $y \in \mathfrak{m}^\perp$, $\exp(\text{ad } x)(f + y) \in f + \mathfrak{m}^\perp$ since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$ and $\chi([\mathfrak{m}, \mathfrak{m}]) = 0$. This is enough to conclude because, \mathfrak{m} being ad-nilpotent, M is generated by the elements $\exp(\text{ad } x)$ for x running through \mathfrak{m} . As a result, by restriction, we get a map

$$\alpha: M \times \mathcal{S}_f \rightarrow f + \mathfrak{m}^\perp.$$

Theorem 63 ([48, §2.3]). *The map $\alpha: M \times \mathcal{S}_f \rightarrow f + \mathfrak{m}^\perp$ is an isomorphism of affine varieties.*

Proof. We have a contracting \mathbb{C}^* -action on $M \times \mathcal{S}_f$ defined by:

$$t.(g, x) := (\rho(t^{-1})g\rho(t), \tilde{\rho}(t)x) \quad \text{for all } t \in \mathbb{C}^*, g \in M, x \in \mathcal{S}_f.$$

The morphism α is \mathbb{C}^* -equivariant with respect to this contracting \mathbb{C}^* -action, and the \mathbb{C}^* -action $\tilde{\rho}$ on $f + \mathfrak{m}^\perp$. This finishes the proof, thanks to the following fact, formulated in [48, Proof of Lem. 2.1]:

“a \mathbb{C}^* -equivariant morphism $\alpha: X_1 \rightarrow X_2$ of smooth affine \mathbb{C}^* -varieties with contracting \mathbb{C}^* -actions which induces an isomorphism between the tangent spaces of the \mathbb{C}^* -fixed points is an isomorphism.” \square

As a consequence of this result, we get the isomorphism:

$$\mathbb{C}[\mathcal{S}_f] \cong \mathbb{C}[f + \mathfrak{m}^\perp]^M.$$

4.1.4. Hamiltonian reduction. The connected Lie group M acts on the Poisson variety \mathfrak{g}^* by the coadjoint action. The action is Hamiltonian and the moment map,

$$\mu: \mathfrak{g}^* \rightarrow \mathfrak{m}^*,$$

is the restriction of functions from \mathfrak{g} to \mathfrak{m} . Since $\chi|_{\mathfrak{m}}$ is a character on \mathfrak{m} , it is fixed by the coadjoint action of M . As a consequence, the set

$$\mu^{-1}(\chi|_{\mathfrak{m}}) = \{\xi \in \mathfrak{g}^* \mid \mu(\xi) = \chi|_{\mathfrak{m}}\}$$

is M -stable.

Lemma 64. $\chi|_{\mathfrak{m}}$ is a regular value for the restriction of μ to each symplectic leaf of \mathfrak{g}^* .

Proof. Note that $\mu^{-1}(\chi|_{\mathfrak{m}}) = \chi + \mathfrak{m}^\perp$. Then we have to prove that for any $\xi \in \chi + \mathfrak{m}^\perp$, the map

$$d_\xi \mu: T_\xi(G.\xi) \rightarrow T_{\chi|_{\mathfrak{m}}}(\mathfrak{m}^*)$$

is surjective. But $T_\xi(G.\xi) \simeq [\mathfrak{g}, \xi]$ while $T_{\chi|_{\mathfrak{m}}}(\mathfrak{m}^*) = \mathfrak{m}^*$. Since $\chi + \mathfrak{m}^\perp$ is transversal to the coadjoint orbits in \mathfrak{g}^* (cf. Theorem 62), we have

$$\mathfrak{g} = [\mathfrak{g}, \xi] + \mathfrak{m}^\perp.$$

Fix $\gamma \in \mathfrak{m}^*$ and write $\gamma = x + x'$, with $x \in [\mathfrak{g}, \xi]$ and $x' \in \mathfrak{m}^\perp$, according to the above decomposition of \mathfrak{g} . Then $\mu(x) = \gamma$. \square

Since the map

$$M \times \mathcal{S}_f \longrightarrow \chi + \mathfrak{m}^\perp$$

is an isomorphism of affine varieties (cf. Theorem 63),

$$\mathcal{S}_f \cong (\chi + \mathfrak{m}^\perp)/M.$$

Therefore, by [94, Th. 7.31] or [77, Prop. 5.39 and Def. 5.9], we get a symplectic structure on \mathcal{S}_f . In fact, thanks to Lemma 64, we have shown that the symplectic form on each leaf on \mathcal{S}_f is obtained by symplectic reduction from the symplectic form of the corresponding leaf of \mathfrak{g}^* . The Poisson structure on \mathcal{S}_f is described as follows. Let $\pi: \chi + \mathfrak{m}^\perp \rightarrow (\chi + \mathfrak{m}^\perp)/M \simeq \mathcal{S}_f$ be the natural projection map, and $\iota: \chi + \mathfrak{m}^\perp \hookrightarrow \mathfrak{g}^*$ be the natural inclusion. Then for any $f, g \in \mathbb{C}[\mathcal{S}_f]$,

$$\{f, g\}_{\mathcal{S}_f} \circ \pi = \{\tilde{f}, \tilde{g}\} \circ \iota$$

where \tilde{f}, \tilde{g} are arbitrary extensions of $f \circ \pi, g \circ \pi$ to \mathfrak{g}^* .

4.1.5. BRST reduction. The Hamiltonian reduction obtained in §4.1.4 can also be described by means of the *BRST cohomology* (the letters BRST refers to the physicists Becchi, Rouet, Stora and Tyutin).

We briefly outline below the construction in the special case where $f = f_{prin}$ is a principal nilpotent element, that is, such that Gf is of maximal dimension $\dim \mathfrak{g} - r$, with $r = \text{rk } \mathfrak{g}$. See [10, §2.4] for more details¹¹, and see [17] for the construction in a more general setting; see also e.g. [9, Sect. 2].

One can suppose that $f_{prin} = \sum_i e_{-\alpha_i}$, where $e_{-\alpha_1}, \dots, e_{-\alpha_r}$ are the opposite simple roots vectors associated with the simple roots $\alpha_1, \dots, \alpha_r$ with respect to the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}.$$

Then \mathfrak{m} is just the nilpotent subalgebra \mathfrak{n} .

Consider the *Clifford algebra* Cl associated with the vector space $\mathfrak{n} \oplus \mathfrak{n}^*$ and the non-degenerate bilinear forms $(\mid \)$ defined by $(\phi + x \mid \psi + y) = (\phi \mid \psi) + (\psi \mid x)$ for $\phi, \psi \in \mathfrak{n}^*, x, y \in \mathfrak{n}$. Specifically, Cl is the unital \mathbb{C} -superalgebra that is isomorphic to $\Lambda(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*)$ as \mathbb{C} -vector spaces, and

$$[x, \phi] = \phi(x), \quad x \in \mathfrak{n} \subset \Lambda(\mathfrak{n}), \quad \phi \in \mathfrak{n}^* \subset \Lambda(\mathfrak{n}^*).$$

(Note that $[x, \phi] = x\phi + \phi x$ since x, ϕ are odd.)

Define an increasing filtration on Cl by setting $Cl_p := \Lambda^{\leq p}(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*)$. We have

$$0 = Cl_{-1} \subset Cl_0 \subset Cl_1 \subset \dots \subset Cl_N = Cl,$$

where $N = \dim \mathfrak{n} = \frac{1}{2} \dim Gf$, and

$$Cl_p.Cl_q \subset Cl_{p+q}, \quad [Cl_p, Cl_q] \subset Cl_{p+q-1}.$$

As a consequence, the associated graded algebra,

$$\bar{Cl} := \text{gr } Cl = \bigoplus_{p \geq 0} \frac{Cl_p}{Cl_{p+1}},$$

is naturally a graded Poisson superalgebra. We have $\bar{Cl} = \Lambda(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*)$ as a commutative superalgebra, and its Poisson (super)bracket is given by:

$$\{x, \phi\} = \phi(x), \quad \{x, y\} = 0, \quad \{\phi, \psi\} = 0, \quad x, y \in \mathfrak{n} \subset \Lambda(\mathfrak{n}), \quad \phi, \psi \in \mathfrak{n}^* \subset \Lambda(\mathfrak{n}^*).$$

¹¹The lecture [10] only deals with the case where $\mathfrak{g} = \mathfrak{sl}_n$ but the general case with principal f works similarly.

Set

$$\bar{\mathcal{C}}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \bar{\mathcal{C}}l.$$

Since $\bar{\mathcal{C}}(\mathfrak{g})$ is a tensor product of Poisson superalgebras, it is naturally a Poisson superalgebra. Set

$$\bar{Q} = \sum_{\alpha} (x_{\alpha} - \chi(x_{\alpha})) \otimes x_{\alpha}^* - 1 \otimes \frac{1}{2} \sum_{\alpha, \beta, \gamma} c_{\alpha, \beta}^{\gamma} x_{\alpha}^* x_{\beta}^* x_{\gamma},$$

where $\{x_{\alpha}\}_{\alpha \in \Delta_+}$ is the basis of \mathfrak{n} , with Δ_+ the set of positive roots with respect to the above triangular decomposition, $\{x_{\alpha}^*\}_{\alpha \in \Delta_+}$ is its dual basis, and $c_{\alpha, \beta}^{\gamma}$ is the constant structure, that is, $[x_{\alpha}, x_{\beta}] = c_{\alpha, \beta}^{\gamma} x_{\gamma}$. Then (cf. [31]) we have $\{\bar{Q}, \bar{Q}\} = 0$. Because \bar{Q} is odd it forces

$$(\text{ad } \bar{Q})^2 = 0.$$

Hence $(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q})$ is a differential graded Poisson superalgebra. Its cohomology

$$H^{\bullet}(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q}) = \bigoplus_{i \in \mathbb{Z}} H^i(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q})$$

inherits a graded Poisson superalgebra structure from that of $\bar{\mathcal{C}}(\mathfrak{g})$.

According to Kostant and Sternberg [75] the Poisson structure on $\mathbb{C}[\mathcal{S}_f]$ may be described through the following isomorphism.

Theorem 65 ([75]). *We have $H^i(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q}) = 0$ for $i \neq 0$ and*

$$H^0(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q}) \cong \mathbb{C}[\mathcal{S}_f]$$

as Poisson algebras.

Since $\mathbb{C}[\mathfrak{g}^*]^G$ is the Poisson center of $\mathbb{C}[\mathfrak{g}^*]$, the natural map $\mathbb{C}[\mathfrak{g}^*]^G \rightarrow H^0(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q})$ sending p to $p \otimes 1$ is a well-defined homomorphism of Poisson algebras. Moreover, the diagram

$$\begin{array}{ccc} & \mathbb{C}[\mathfrak{g}^*]^G & \\ \swarrow \sim & \downarrow & \\ \mathbb{C}[\mathcal{S}_f] & \xleftarrow{\sim} & H^0(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q}) \end{array}$$

commutes, which yields an isomorphism

$$\mathbb{C}[\mathfrak{g}^*]^G \xrightarrow{\sim} H^0(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q}).$$

For an arbitrary nilpotent element f , the statement has to be rephrased as follows.

Theorem 66. *Given an arbitrary nilpotent element f of \mathfrak{g} , the natural map $\mathbb{C}[\mathfrak{g}^*]^G \rightarrow H^0(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q})$ sending p to $p \otimes 1$ induces an isomorphism of Poisson algebras from $\mathbb{C}[\mathfrak{g}^*]^G$ to the Poisson center of $H^0(\bar{\mathcal{C}}(\mathfrak{g}), \text{ad } \bar{Q}) \cong \mathbb{C}[\mathcal{S}_f]$.*

Remark 67. In [20, Th. 11.1] we have stated an affine version of the above theorem. It stipulates that there is an isomorphism of Poisson vertex algebras from $\mathbb{C}[J_{\infty} \mathfrak{g}]^{J_{\infty} G}$ to the Poisson vertex center of $\mathbb{C}[J_{\infty} \mathcal{S}_f]$. It implies that the vertex center of the affine W -algebra $\mathcal{W}^{-h^{\vee}}(\mathfrak{g}, f)$ at the critical level is isomorphic to the Feigin-Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ (cf. Exercise 4).

4.2. AFFINE W -ALGEBRAS AND THEIR ASSOCIATED VARIETIES

This section looks at the associated varieties of affine W -algebras.

4.2.1. About the definition of W -algebras. Given a nilpotent element f of the simple Lie algebra \mathfrak{g} , the *universal affine W -algebra* $\mathcal{W}^k(\mathfrak{g}, f)$ associated with \mathfrak{g} and f at level $k \in \mathbb{C}$ is defined by the quantized Drinfeld-Sokolov reduction associated with (\mathfrak{g}, f) with coefficients in a $\widehat{\mathfrak{g}}$ -module M ([44, 63]). It means that $\mathcal{W}^k(\mathfrak{g}, f)$ is defined by a certain BRST cohomology,

$$\mathcal{W}^k(\mathfrak{g}, f) := H^0(\mathcal{C}^k(\mathfrak{g}), \hat{Q}_{(0)}),$$

where $\mathcal{C}^k(\mathfrak{g}) := V^k(\mathfrak{g}) \otimes \mathcal{F}^\chi$, \mathcal{F}^χ is a certain vertex superalgebra¹² which depends on f , and $(\mathcal{C}^k(\mathfrak{g}), \hat{Q}_{(0)})$ is a certain cochain complex which depends on f , too. Thus $\mathcal{W}(\mathfrak{g}, f)$ is naturally a graded vertex algebra (see Exercices 9). We briefly denote by $H_{DS,f}^\bullet(V^k(\mathfrak{g}))$ the cohomology $H^0(\mathcal{C}^k(\mathfrak{g}), \hat{Q}_{(0)})$.

Rather than discuss this in full generality, we detail (cf. Exercise 10) the case where $\mathfrak{g} = \mathfrak{sl}_2$ and $f = f_{prin}$ is principal, which is already very informative. We refer to [10] for the more general case where $\mathfrak{g} = \mathfrak{sl}_n$ and $f = f_{prin}$ in principal, and to [63] for the most general case.

We call (*affine*) *W-algebras* any graded quotient of the universal affine *W*-algebra $\mathcal{W}^k(\mathfrak{g}, f)$. The affine *W*-algebras generalize both affine vertex algebras and Virasoro vertex algebras. Indeed,

$$\mathcal{W}^k(\mathfrak{g}, 0) \cong V^k(f),$$

and (cf. Exercise 10 (3)),

$$\mathcal{W}^k(\mathfrak{sl}_2, f_{prin}) \cong \text{Vir}^{c(k)},$$

provided that $k \neq -2$, where

$$c(k) := 1 - \frac{6(k+1)^2}{k+2}.$$

Exercise 9 (A preliminary result for the BRST reduction). Let V be a vertex *superalgebra*, that is, a vector superspace $V = V_0 \oplus V_1$ satisfying the same axioms as a vertex algebra except that, in the locality axiom, the bracket $[a(z), b(w)]$ stands for

$$[a(z), b(w)] = a(z)b(w) - (-1)^{|a||b|}b(w)a(z).$$

Fix an odd element Q of V such that $Q_{(n)}Q = 0$ for all $n \geq 0$.

(1) Show that $Q_{(0)}^2 = 0$.

(2) Show that the quotient $\frac{\ker Q_{(0)}}{\text{im } Q_{(0)}}$ is naturally a vertex algebra, provided it is nonzero.

Hints for Exercise 9. (1) Remember that Q is odd and, hence, that $Q_{(0)}^2 = \frac{1}{2}[Q_{(0)}, Q_{(0)}]$. Then use the Borchers identity.

(2) Show that $\ker Q_{(0)}$ is a vertex subalgebra¹³ of V , and that $\text{im } Q_{(0)}$ is a vertex ideal of it.

Exercise 10 (Definition of the *W*-algebra associated with \mathfrak{sl}_2 and a principal nilpotent element). Set

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that $\mathfrak{sl}_2 = \text{span}_{\mathbb{C}}(e, h, f)$. The aim of this exercise is to define the *W*-algebra $\mathcal{W}^k(\mathfrak{sl}_2, f)$ associated with \mathfrak{sl}_2 and f at level $k \in \mathbb{C}$. Set $\mathfrak{n} := \mathbb{C}e$.

(1) Let $\hat{\mathcal{C}}l$ be the Clifford algebra associated with $\mathfrak{n}[t, t^{-1}] \oplus \mathfrak{n}^*[t, t^{-1}]$ and the symmetric bilinear form $(| \)$ given by:

$$(et^m | et^n) = (e^*t^m | e^*t^n) = 0, \quad (et^m | e^*t^n) = \delta_{m+n, 0}.$$

We write ψ_m for $et^m \in \hat{\mathcal{C}}l$ and ψ_m^* for $e^*t^m \in \hat{\mathcal{C}}l$, $m \in \mathbb{Z}$, so that $\hat{\mathcal{C}}l$ is the associative superalgebra with odd generators ψ_m, ψ_m^* , $m \in \mathbb{Z}$, and relations:

$$[\psi_m, \psi_n] = [\psi_m^*, \psi_n^*] = 0, \quad [\psi_m, \psi_n^*] = \delta_{m+n, 0}.$$

Define the *charged fermion Fock space* as

$$\mathcal{F} := \frac{\hat{\mathcal{C}}l}{\sum_{m \geq 0} \hat{\mathcal{C}}l\psi_m + \sum_{n \geq 1} \hat{\mathcal{C}}l\psi_n^*}.$$

Show that there is a unique vertex (super)algebra structure on \mathcal{F} such that the image of 1 is the vacuum $|0\rangle$, and

$$\psi(z) := Y(\psi_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) := Y(\psi_0^*|0\rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}.$$

¹²It is the vertex algebra of *neutral free superfermions* associated with $\mathfrak{g}_{1/2}$.

¹³The definition of *vertex subalgebra* is very natural, see [46, §1.3.4] for more details.

Let $V^k(\mathfrak{sl}_2)$ be the universal affine vertex algebra associated with \mathfrak{sl}_2 at level k , and set

$$\mathcal{C}^k(\mathfrak{sl}_2) := V^k(\mathfrak{sl}_2) \otimes \mathcal{F}.$$

Define a gradation $\mathcal{F} = \bigoplus_{p \in \mathbb{Z}} \mathcal{F}^p$ by setting $\deg \psi_m = -1$, $\deg \psi_n^* = 1$ for all $m, n \in \mathbb{Z}$ and $\deg |0\rangle = 0$. Then set $\mathcal{C}^{k,p}(\mathfrak{sl}_2) := V^k(\mathfrak{sl}_2) \otimes \mathcal{F}^p$. Define a vector \hat{Q} of degree 1 in $\mathcal{C}^{k,1}(\mathfrak{sl}_2)$ by:

$$\hat{Q}(z) := (e(z) + 1) \otimes \psi^*(z).$$

- (2) Verify that $\hat{Q}_{(n)}\hat{Q} = 0$ for all $n \geq 0$, and deduce from Exercice 9 that the cohomology $H^\bullet(\mathcal{C}^k(\mathfrak{sl}_2), \hat{Q}_{(0)})$ inherits a vertex algebra structure from that of $\mathcal{C}^k(\mathfrak{sl}_2)$, provided that it is nonzero.

The W -algebra $\mathcal{W}^k(\mathfrak{sl}_2, f)$ associated with (\mathfrak{sl}_2, f) at level $k \in \mathbb{C}$ is defined by:

$$\mathcal{W}^k(\mathfrak{sl}_2, f) := H^0(\mathcal{C}^k(\mathfrak{sl}_2), \hat{Q}_{(0)}).$$

This definition of $\mathcal{W}^k(\mathfrak{sl}_2, f)$ is due to Feigin and Frenkel. It can be generalized to any simple Lie algebra \mathfrak{g} and to any nilpotent element.

- (3) Assume that $k \neq -2$. Show that there exists a unique vertex algebra homomorphism

$$\text{Vir}^{c(k)} \rightarrow \mathcal{W}^k(\mathfrak{sl}_2, f), \quad \text{where} \quad c(k) := 1 - \frac{6(k+1)^2}{k+2}.$$

It can be shown that the above homomorphism is actually an isomorphism.

Hints for Exercise 10. (1) The main thing to be verified is the locality axiom.

- (2) Observe that $\hat{Q} = (e_{(-1)}|0\rangle + |0\rangle) \otimes e_{(0)}^*|0\rangle$ and then compute $\hat{Q}(z)\hat{Q} = 0$.

- (3) This is a very difficult question! We give the necessary guidance. Set

$$L(z) = L_{\text{sug}}(z) + \frac{1}{2}h(z) + L_{\mathcal{F}}(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-1},$$

where

$$L_{\text{sug}}(z) = \frac{1}{2(k+2)} \left(: e(z)f(z) : + : f(z)e(z) : + \frac{1}{2}h(z)^2 \right) \quad \text{and} \quad L_{\mathcal{F}}(z) = : \partial_z \psi(z) \psi^*(z) :,$$

and verify that $\hat{Q}_{(0)}L = 0$ so that L defines an element of $\mathcal{W}^k(\mathfrak{sl}_2, f)$. Then check that $L_{-1} = T$, that L_0 acts semisimply on $\mathcal{W}^k(\mathfrak{sl}_2, f)$ by

$$\begin{aligned} L_0|0\rangle &= 0, & [L_0, h_{(n)}] &= -nh_{(n)}, \\ [L_0, e_{(n)}] &= (1-n)e_{(n)}, & [L_0, f_{(n)}] &= (-1-n)f_{(n)}, \\ [L_0, \psi_{(n)}^*] &= (-1-n)\psi_{(n)}^*, & [L_0, \psi_{(n)}] &= (1-n)\psi_{(n)}, \end{aligned}$$

and that the L_n 's verify the Virasoro relations.

4.2.2. Associated variety of quantized Drinfeld-Sokolov reductions. Return to the general case, and let f be a nilpotent element of the simple Lie algebra \mathfrak{g} . The W -algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is a chiral quantization of $\mathbb{C}[\mathcal{S}_f]$. Specifically, we have [37, 7] a natural isomorphism $R_{\mathcal{W}^k(\mathfrak{g}, f)} \cong \mathbb{C}[\mathcal{S}_f]$ of Poisson algebras, so that

$$\tilde{X}_{\mathcal{W}^k(\mathfrak{g}, f)} = \mathcal{S}_f.$$

Moreover,

$$\text{gr } \mathcal{W}^k(\mathfrak{g}, f) \cong \mathbb{C}[J_\infty \mathcal{S}_f],$$

so that $\mathcal{W}^k(\mathfrak{g}, f)$ is a quantization of $\mathbb{C}[J_\infty \mathcal{S}_f]$.

Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$. Then $X_{\mathcal{W}_k(\mathfrak{g}, f)}$ is a \mathbb{C}^* -invariant Poisson subvariety of \mathcal{S}_f . Since it is \mathbb{C}^* -invariant, $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse if and only if $X_{\mathcal{W}_k(\mathfrak{g}, f)} = \{f\}$.

Theorem 68 ([7]). *For any quotient V of $V^k(\mathfrak{g})$, the associated scheme $\tilde{X}_{H_{DS, f}^0(V)}$ is isomorphic to the scheme theoretic intersection $\tilde{X}_V \times_{\mathfrak{g}^*} \mathcal{S}_f$. So $X_{H_{DS, f}^0(V)} = X_V \cap \mathcal{S}_f$.*

As a consequence of Theorem 68 and Lemma 49, we get that:

- $H_{DS, f}^0(V)$ is nonzero if and only if $f \in X_V$,

- $H_{DS,f}^0(V)$ is lisse if $\overline{G.f} = X_V$,
- $H_{DS,f}^0(V)$ is quasi-lisse if X_V is contained in the nilpotent cone \mathcal{N} and if $f \in X_V$.

The simple W -algebra algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient vertex algebra of $H_{DS,f}^0(L_k(\mathfrak{g}))$, provided it is nonzero. Conjecturally [63, 67], we have $\mathcal{W}_k(\mathfrak{g}, f) \cong H_{DS,f}^0(L_k(\mathfrak{g}))$, provided that $H_{DS,f}^0(L_k(\mathfrak{g})) \neq 0$. (This conjecture has been verified in many cases [3, 5].)

4.2.3. Lisse and quasi-lisse W -algebras. Theorem 68 implies that if $L_k(\mathfrak{g})$ is quasi-lisse and if $f \in X_{L_k(\mathfrak{g})}$, then the W -algebra $H_{DS,f}^0(L_k(\mathfrak{g}))$ is quasi-lisse as well, and so is its simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$ if $H_{DS,f}^0(L_k(\mathfrak{g}))$ is nonzero. In this way we obtain a huge number of quasi-lisse W -algebras. Furthermore, if $X_{L_k(\mathfrak{g})} = \overline{G.f}$, then $X_{H_{DS,f}^0(L_k(\mathfrak{g}))} = \{f\}$, so that $\mathcal{W}_k(\mathfrak{g}, f)$ in fact lisse. Thus, Conjecture 59 in particular says that any quasi-lisse simple affine vertex algebra produces exactly one lisse simple W -algebra.

Example 69. If k is an admissible level, then one knows that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$ for some nilpotent orbit \mathbb{O}_k (cf. Theorem 52). Picking $f \in \mathbb{O}$, we obtain that $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. Moreover, for any $f \in \overline{\mathbb{O}}$, we obtain that $\mathcal{W}_k(\mathfrak{g}, f)$ is quasi-lisse.

Example 70. By Theorem 55, there are other lisse simple W -algebras, not coming from admissible levels. Namely, fix \mathfrak{g}, k as in Theorem 55, and choose $f_{min} \in \mathbb{O}_{min}$. Then $\mathcal{W}_k(\mathfrak{g}, f_{min})$ is lisse. In [18], we actually obtained a stronger result: if $\mathfrak{g} = D_4, E_6, E_7, E_8$ and if $k = -\frac{h^\vee}{6} - 1 + n$, where $n \in \mathbb{Z}_{\geq 0}$, then $\mathcal{W}_k(\mathfrak{g}, f_{min})$ is lisse. In fact, for $n = 0$, we have that $\mathcal{W}_k(\mathfrak{g}, f_{min}) \cong \mathbb{C}$ and so $\mathcal{W}_k(\mathfrak{g}, f_{min})$ is also rational.

The first example of simple affine W -algebra not coming from an admissible level was discovered by Kawasetsu [69]. Specifically, Kawasetsu showed that $\mathcal{W}_k(\mathfrak{g}, f_{min})$ is lisse for $\mathfrak{g} = D_4, E_6, E_7, E_8$ and $k = -\frac{h^\vee}{6}$. Furthermore, for such \mathfrak{g}, k , $\mathcal{W}_k(\mathfrak{g}, f_{min})$ is rational.

Conjecture 71. Assume that \mathfrak{g} belongs to the Deligne exceptional series and that $k = -\frac{h^\vee}{6} - 1 + n$, where $n \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{W}_k(\mathfrak{g}, f_{min})$ is rational if and only if $k \notin \mathbb{Z}_{\geq 0}$.

Conjecture 71 for admissible k , that is, for $\mathfrak{g} = A_1, A_2, G_2, F_4$ is known by Kac-Wakimoto [67]. See Conjecture 76 for another conjecture on the same theme.

4.3. NILPOTENT SLODOWY SLICES AND COLLAPSING LEVELS FOR W -ALGEBRAS

4.3.1. Singularities of nilpotent Slodowy slices. Given a nilpotent orbit \mathbb{O} in \mathfrak{g} and an \mathfrak{sl}_2 -triple (e, h, f) , the *nilpotent Slodowy slice* associated with \mathbb{O} and (e, h, f) is the intersection

$$\mathcal{S}_{\mathbb{O},f} := \overline{\mathbb{O}} \cap \mathcal{S}_f,$$

where $\mathcal{S}_f \cong f + \mathfrak{g}^e$ is the Slodowy slice of the \mathfrak{sl}_2 -triple (e, h, f) . Note that the set $\mathcal{S}_{\mathbb{O},f}$ is nonempty if and only if $f \in \overline{\mathbb{O}}$. The singularities of nilpotent Slodowy slices play a significant role in the theory of symplectic singularities. They are understood best for Gf a *minimal degeneration* of \mathbb{O} , that is, Gf is open in the boundary of \mathbb{O} in $\overline{\mathbb{O}}$. (The boundary of \mathbb{O} in $\overline{\mathbb{O}}$ is precisely the singular locus of $\overline{\mathbb{O}}$ as was shown by Namikawa [87] using results of Kaledin and Panyushev [68, 88].)

The most well-known nilpotent Slodowy slices are those associated with the principal nilpotent orbit \mathbb{O}_{prin} and a subregular nilpotent element f_{subreg} for types A, D, E in which case they have the simple singularity of the same type as G . This is the classical theory of Brieskorn and Slodowy ([92]). Kraft and Procesi studied nilpotent Slodowy slices in the classical cases [71, 72] in order to determine the generic singularities. More recently, Fu, Juteau, Levy and Sommers [47] have completed that work. They determined the generic singularities of $\overline{\mathbb{O}}$ when \mathfrak{g} is of exceptional type by studying the various nilpotent Slodowy slices $\mathcal{S}_{\mathbb{O},f}$ at minimal degenerations Gf .

Here are a few properties of nilpotent Slodowy slices. First, $\mathcal{S}_{\mathbb{O},f}$ is equidimensional of dimension $\dim \overline{\mathbb{O}} - \dim \overline{Gf}$. It is irreducible if and only if \mathbb{O} is *unibranch*¹⁴ (this happens for instance if $\overline{\mathbb{O}}$ is normal, which is always true in type A [70]). In addition, its normalization have symplectic singularities in the sense of Beauville [27] (see for instance [47, Sect. 1.2]).

¹⁴An irreducible variety X is *unibranch at x* if the normalization $\pi: (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ of (X, x) is locally a homeomorphism at x .

Remark 72. In view of Conjecture 59 we have checked that for all known cases where the associated variety $X_{L_k(\mathfrak{g})}$ is contained in the nilpotent cone, that is, $L_k(\mathfrak{g})$ is quasi-lisse, then $X_{L_k(\mathfrak{g})}$ is a nilpotent orbit closure which is besides unibranch. Therefore, the associated varieties of the corresponding quantized Drinfeld-Sokolov reductions are always irreducible (cf. §4.2.2). This is a sort of verification of our conjecture in this context, although we do not claim that our list simple affine quasi-lisse vertex algebras is exhaustive.

4.3.2. Collapsing levels for W -algebras. Let \mathfrak{g}^{\natural} be the centralizer in \mathfrak{g} of the \mathfrak{sl}_2 -triple (e, h, f) . Then \mathfrak{g}^{\natural} is a reductive algebra and it decomposes as $\mathfrak{g}^{\natural} = \bigoplus_{i=0}^s \mathfrak{g}_i^{\natural}$, where $\mathfrak{g}_0^{\natural}$ is the center of \mathfrak{g}^{\natural} , and $\mathfrak{g}_1^{\natural}, \dots, \mathfrak{g}_s^{\natural}$ are the simple factors of $[\mathfrak{g}^{\natural}, \mathfrak{g}^{\natural}]$. Define an invariant bilinear form on $\mathfrak{g}_i^{\natural}$, for $i = 0, \dots, s$, by (cf. [66])

$$(x|y)_i^{\natural} := k(x|x)_{\mathfrak{g}} - (\kappa_{\mathfrak{g}}(x, x) - \kappa_{\mathfrak{g}_0}(x, x) - \kappa_{\frac{1}{2}}(x, x))/2, \quad x, y \in \mathfrak{g}_i^{\natural},$$

where $\kappa_{\mathfrak{g}_0}$ denotes the Killing form of \mathfrak{g}_0 , and $\kappa_{\frac{1}{2}}(x, y) := \text{tr}(\text{ad}_{\mathfrak{g}_{1/2}}(x) \text{ad}_{\mathfrak{g}_{1/2}}(y))$, for $x, y \in \mathfrak{g}_0$, with $\text{ad}_{\mathfrak{g}_{1/2}}(x)$ the endomorphism of $\mathfrak{g}_{1/2}$ sending y to $(\text{ad}_x)y$. Then there exists a polynomial k_i^{\natural} in k of degree one such that

$$(\ |)_i^{\natural} = k_i^{\natural} (\ |)_i, \quad i = 1, \dots, s,$$

where $(\ |)_i$ is the normalised inner product of $\mathfrak{g}_i^{\natural}$. For $i = 0$, we choose for $(\ |)_0$ any non-degenerate bilinear form of $\mathfrak{g}_0^{\natural}$.

Definition 73. We say that the level k is *collapsing* if $\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^{\natural}}(\mathfrak{g}^{\natural})$, where

$$L_{k^{\natural}}(\mathfrak{g}^{\natural}) := \bigotimes_{i=0}^s L_{k_i^{\natural}}(\mathfrak{g}_i^{\natural}).$$

Equivalently, k is collapsing if

$$\mathcal{W}_k(\mathfrak{g}, f)^{\mathfrak{g}^{\natural}[t]} \cong \mathbb{C}.$$

For example, if $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}$, then k is collapsing.

The notion of collapsing levels for the case where $f = f_{\min}$ is a minimal nilpotent element, that is, $f_{\min} \in \mathbb{O}_{\min}$, goes back to Adamaović et al. [2]. Their motivations come from the complete reducibility of some categories of representations. There is a full classification of collapsing levels for $f = f_{\min}$, including the case where \mathfrak{g} is a simple affine Lie superalgebra. It can be summarized as follows ([2]): k is collapsing if and only if $k \neq -h^{\vee}$ and $p(k) = 0$, where p is a polynomial of degree two with coefficients in \mathbb{Q} .

Example 74. If $\mathfrak{g} = \mathfrak{sl}(m|n)$, $n \neq m$, then k is collapsing if and only if $(k+1)(k+(m-n)/2) = 0$. If \mathfrak{g} is of type E_6 , then k is collapsing if and only if $(k+3)(k+4) = 0$, etc.

Furthermore, there is a full classification of pairs (\mathfrak{g}, k) such that $\mathcal{W}_k(\mathfrak{g}, f_{\min}) \cong \mathbb{C}$. It was obtained by Arakawa and the author in [18], and then extended to the super case by Adamaović et al. in [2]. For the non super case, the statement is the following.

Theorem 75 ([18]). $\mathcal{W}_k(\mathfrak{g}, f_{\min}) \cong \mathbb{C}$ if and only if either \mathfrak{g} belongs to the Deligne exceptional series and $k = -\frac{h^{\vee}}{6} - 1$, or $\mathfrak{g} = \mathfrak{sp}_{2r}$, $r \geq 2$, $k = -\frac{1}{2}$, or $\mathfrak{g} = \mathfrak{sl}_2$ and $k+2 = \frac{2}{3}$ or $\frac{3}{2}$.

As noted previously, for $\mathfrak{g} = D_4, E_6, E_7, E_8$ and $k = -\frac{h^{\vee}}{6} - 1$, $\mathcal{W}_k(\mathfrak{g}, f_{\min}) \cong \mathbb{C}$ and so $\mathcal{W}_k(\mathfrak{g}, f_{\min})^{\mathfrak{g}^{\natural}[t]} \cong \mathbb{C}$ is lisse. Kawasetsu's description of the vertex algebra $\mathcal{W}_{-h^{\vee}/6}(\mathfrak{g}, f_{\min})$ implies that $\mathcal{W}_{k+1}(\mathfrak{g}, f_{\min})^{\mathfrak{g}^{\natural}[t]}$ is lisse (and rational). See Example 70 and Conjecture 71 for related topics.

Conjecture 76. If $\mathcal{W}_k(\mathfrak{g}, f_{\min})^{\mathfrak{g}^{\natural}[t]}$ is lisse for some k , then $\mathcal{W}_{k+n}(\mathfrak{g}, f_{\min})^{\mathfrak{g}^{\natural}[t]}$ is lisse for all $n \in \mathbb{Z}_{\geq 0}$.

To sum up, the minimal nilpotent case is quite well-understood. However, little or almost nothing is known for collapsing levels for non minimal nilpotent elements. The main reason is that for an arbitrary nilpotent element f , the commutation relations in $\mathcal{W}^k(\mathfrak{g}, f)$ are unknown, and so it is extremely difficult to predict which levels are collapsing.

In this context, the notion of associated variety and the singularities of nilpotent Slodowy slices are proving to be very useful tools to find new collapsing levels. Let us outline the main idea.

It may happen that two nilpotent Slodowy slices $\mathcal{S}_{\mathbb{O},f}$ and $\mathcal{S}_{\mathbb{O}',f'}$ (in different Lie algebras) are isomorphic. In particular, it may happen that a nilpotent Slodowy slice $\mathcal{S}_{\mathbb{O},f}$ is isomorphic to the nilpotent closure $\overline{\mathbb{O}^{\natural}}$ is the reductive Lie algebra \mathfrak{g}^{\natural} . Many examples can be exhibited from [71, 72, 47]. If $\mathcal{S}_{\mathbb{O},f} \cong \overline{\mathbb{O}^{\natural}}$ and if $\overline{\mathbb{O}}$ and $\overline{\mathbb{O}^{\natural}}$ are the associated varieties of some affine vertex algebras $L_k(\mathfrak{g})$ and $L_{k^{\natural}}(\mathfrak{g}^{\natural})$, respectively, one may ask whether the isomorphism of vertex algebras,

$$\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k'}(\mathfrak{g}'),$$

holds, that is, whether k is a collapsing level. Naturally, the knowledge of the associated varieties is far from sufficient to ensure that given vertex algebras are isomorphic, but in some favourable cases we are able to conclude, as we will show below.

Example 77. In order to illustrate our strategy, let us consider a relatively easy example which was actually the starting point of our work on collapsing levels (this example came up from a question by Creutzig and Kawasetsu).

If $\mathbb{O} = \mathbb{O}_{(3^2,1)}$ is the nilpotent orbit of \mathfrak{sl}_7 associated with the partition $(3^2, 1)$, and if Gf is the nilpotent orbit of \mathfrak{sl}_7 associated with the partition $(3, 1^4)$, then it is not difficult to show that

$$\mathcal{S}_{\mathbb{O},f} = \overline{\mathbb{O}_{(3^2,1)}} \cap \mathcal{S}_f \cong \overline{\mathbb{O}_{(3,1)}},$$

where $\mathbb{O}_{(3,1)}$ is the nilpotent orbit of \mathfrak{sl}_4 associated with the partition $(3, 1)$.

In this example, we observe that $\mathfrak{sl}_7^{\natural} \cong \mathbb{C} \times \mathfrak{sl}_4$, and we easily verify that

$$k_0^{\natural} = k + 2 \quad \text{and} \quad k_1^{\natural} = 3k + 14.$$

On the other hand, by Theorem 53 and Remark 54, $\overline{\mathbb{O}_{(3^2,1)}}$ is the associated variety of any simple affine vertex algebra $L_k(\mathfrak{sl}_7)$ at (admissible) level k of the form $k = -7 + \frac{p}{3}$, with $(p, 3) = 1$, $p \geq 7$. Similarly, $\overline{\mathbb{O}_{(3,1)}}$ is the associated variety of any simple affine vertex algebra $L_{k'}(\mathfrak{sl}_4)$ at (admissible) level k' of the form $k' = -4 + \frac{p}{3}$, with $(p, 4) = 1$, $p \geq 4$.

The condition $k_0^{\natural} = 0$ is equivalent to $k = -\frac{14}{3} = -7 + \frac{7}{3}$. With such a k , $k_1^{\natural} = -\frac{8}{3} = -4 + \frac{4}{3}$. The levels $-7 + \frac{7}{3}$ and $-4 + \frac{4}{3}$ are admissible for \mathfrak{sl}_7 and \mathfrak{sl}_4 , respectively. A natural question arising from these observations is whether

$$(12) \quad \mathcal{W}_{-7+7/3}(\mathfrak{sl}_7, f) \cong L_{-4+4/3}(\mathfrak{sl}_4),$$

that is, whether $-\frac{14}{3}$ is a collapsing level. As an evidence of the above isomorphism one can check that the central charges of the above vertex algebras coincide. Recall that the central charge of the simple affine vertex algebra $L_k(\mathfrak{g})$ is

$$c_k(\mathfrak{g}) := \frac{k \dim \mathfrak{g}}{k + h^{\vee}}.$$

On the other hand, the central charge of the simple affine W -algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is

$$c_k(\mathfrak{g}, f) := \dim \mathfrak{g}_0 - \frac{1}{2} \dim \mathfrak{g}_{1/2} - 12 \left(\frac{|\rho|^2}{k + h^{\vee}} - (\rho|h) + \frac{k + h^{\vee}}{4} |h|^2 \right).$$

Here, we find that the central charge is -30 in both sides of (12). In this example, one can show that (12) is indeed an isomorphism. This is a particular case of Theorem 80 below.

Once we have detected a possible collapsing level, as in the above example, one can sometimes conclude using the asymptotic behavior of the normalized characters for admissible levels as discovered by Kac and Wakimoto.

Proposition 78 ([65]). *Assume that k is a principal¹⁵ admissible level, and assume that V is either the simple affine vertex algebra $L_k(\mathfrak{g})$ or its Drinfeld-Sokolov reduction $H_{DS,f}^0(L_k(\mathfrak{g}))$, with f an arbitrary nilpotent*

¹⁵It means that $k = -h^{\vee} + \frac{p}{q}$, with $(p, q) = 1$, $(q, r^{\vee}) = 1$, and $p \geq h^{\vee}$. All admissible levels are principal for the types A, D, E .

element. Then,

$$\chi_V(\tau) \sim \mathcal{A}(V)e^{\pi i \mathcal{G}(V)/12\tau}, \quad \text{as } \tau \downarrow 0,$$

where $\mathcal{A}(V)$ and $\mathcal{G}(V)$ are two constants, called the amplitude and the asymptotic growth of V , respectively. Recall here that the normalised character of V is given by

$$\chi_V(\tau) = \text{Tr}_V e^{2i\pi\tau(L_0 - c/24)}, \quad \text{Im } \tau > 0.$$

The amplitude $\mathcal{A}(V)$ and the asymptotic growth $\mathcal{G}(V)$ have been described combinatorially in [65] for any V as in Proposition 78. For example, for a principal admissible level k , we have:

$$\mathcal{G}(L_k(\mathfrak{g})) = \left(1 - \frac{h^\vee}{pq}\right) \dim \mathfrak{g} \quad \text{and} \quad \mathcal{G}(H_{DS,f}^0(V)) = \mathcal{G}(L_k(\mathfrak{g})) - \dim \overline{Gf}.$$

The formulas for the amplitude are slightly more complicated. We omit them here.

Theorem 79 ([22]). *Assume that k and k^\natural are admissible levels for \mathfrak{g} and \mathfrak{g}^\natural , respectively, that $f \in X_{L_k(\mathfrak{g})}$ ($\subset \mathcal{N}$) and that*

$$\chi_{H_{DS,f}^0(L_k(\mathfrak{g}))}(\tau) \sim \chi_{L_{k^\natural}(\mathfrak{g}^\natural)}(\tau), \quad \text{as } \tau \downarrow 0,$$

that is,

$$\mathcal{G}(H_{DS,f}^0(L_k(\mathfrak{g}))) = \mathcal{G}(L_{k^\natural}(\mathfrak{g}^\natural)) \quad \text{and} \quad \mathcal{A}(H_{DS,f}^0(L_k(\mathfrak{g}))) = \mathcal{A}(L_{k^\natural}(\mathfrak{g}^\natural)).$$

Then

$$\mathcal{W}_k(\mathfrak{g}, f) \cong H_{DS,f}^0(L_k(\mathfrak{g})) \quad \text{and} \quad \mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g}^\natural).$$

In particular, $\mathcal{W}_k(\mathfrak{g}, f)$ is simple and k is a collapsing level.

In this way we discovered a large number of collapsing levels. Next theorem covers Example 77 (we have similar results for \mathfrak{so}_n and \mathfrak{sp}_n).

Theorem 80. *Let n be a positive integer that we write as $n = mq + s$, with $m, q > 0$ and $s \geq 0$. Assume that $(q, s) = 1$ and that the partition associated with the nilpotent orbit Gf is $(q^m, 1^s)$. Then*

$$\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong L_{-s+s/q}(\mathfrak{sl}_s).$$

In particular, $k = -n + \frac{n}{q}$ is a collapsing level for \mathfrak{sl}_n .

Example 81. Let us now give a few examples in the exceptional types. Below, we have written the label in the Bala-Carter classification of the nilpotent orbit Gf instead of f :

$$\begin{aligned} \mathcal{W}_{-12+12/5}(E_6, A_4) &\cong L_{-2+2/5}(A_1), & \mathcal{W}_{-18+18/3}(E_7, E_6) &\cong L_{-2+2/13}(A_1), \\ \mathcal{W}_{-18+19/12}(E_7, E_6) &\cong L_{-2+3/4}(A_1), & \mathcal{W}_{-18+18/7}(E_7, (A_5)'') &\cong L_{-4+4/7}(G_2), \\ \mathcal{W}_{-9+9/7}(F_4, B_3) &\cong L_{-2+2/7}(A_1), \quad \text{etc.} \end{aligned}$$

All these examples are obtained by exploiting [7] and [47] to detect the levels, and then applying Theorem 79 to prove that the isomorphisms indeed hold.

4.3.3. Collapsing levels in Argyres-Douglas theory. As already noted, nilpotent Slodowy slices appear as associated variety of W -algebras (cf. §4.2.2). It is also known that they appear as the Higgs branches of the *Argyres-Douglas theories* ([23, 24]) in four-dimensional $N = 2$ superconformal field theories (see e.g. [93]). These two aspects are connected by the fact that the Higgs branch of a four-dimensional $N = 2$ superconformal field theory \mathcal{T} is conjecturally [28] isomorphic to the associated variety of the vertex algebra corresponding to \mathcal{T} via the 4d/2d-duality discovered in [29] (see Sect. 3.4). As is apparent from the previous paragraph, nilpotent Slodowy slices and their singularities are further important to find collapsing levels.

In fact, we think that collapsing levels play an important role in the Argyres-Douglas theory ([22]). In the 4d/2d-duality provided by the map Φ of Sect. 3.4, typical examples of vertex algebras corresponding to the Argyres-Douglas theories are the vertex algebras,

$$L_{-h^\vee+h^\vee/q}(\mathfrak{g}), \quad \mathcal{W}_{-h^\vee+h^\vee/q}(\mathfrak{g}, f),$$

for \mathfrak{g} of type A, D, E , with $(h^\vee, q) = 1$. (It seems that the non-admissible case where $(h^\vee, q) \neq 1$ also occurs.) Notice that such examples have appeared in Theorem 80 and Example 81.

As observed in [93], a given Argyres-Douglas theory can be realized in several ways. Since the map Φ is well-defined, whenever this happens, it means that we have an isomorphisms between W -algebras. We believe that such a phenomenon essentially reflects that the level is collapsing, provided that one of the W -algebras in an affine vertex algebra. Actually, from the geometry of nilpotent Slodowy slices, it is sometimes possible to predict isomorphisms between non-trivial W -algebras. For example, we conjecture that

$$\mathcal{W}_{-7+7/3}(\mathfrak{sl}_7, f) \cong \mathcal{W}_{-4+4/3}(\mathfrak{sl}_4, f'),$$

where f belongs to the nilpotent orbit of \mathfrak{sl}_7 associated with the partition $(3, 2^2)$ and f' belongs to the nilpotent orbit of \mathfrak{sl}_4 associated with the partition (2^2) . We have checked that the central charges, the amplitudes, the asymptotic growths, and of course the associated varieties coincide, but we are not able to conclude for the moment since our Theorem 79 does not apply.

Unfortunately, our understanding of the Argyres-Douglas theory is limited for the moment, but it strongly motivates our investigations on collapsing levels and their variants.

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