Lectures VIASM: Convex geometry and algebraic geometry

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Contents

1 Introduction

This is part of a series of lectures and mini-course done at the VIASM in June 2026. The aim is to introduce to undergraduate and graduate certain aspect relating two a priori distinct areas, convex geometry and algebraic geometry. No knowledge of algebraic geometry is required (althought still appreciated) and rather only basic notions of algebra, measure theory are needed.

The connection between these two fields date back from the work of Gelfand, Kaveh, Khovanski, Teissier and was further developped by Okounkov, Lazarsfeld, Mustata. We will not present their work in full generality but rather some simplified versions of it, focusing to low dimension.

2 Overview, Hilbert-Samuel multiplicity I

First I will give a quick overview of the course. I have chosen to present certain results in algebraic geometry that are analogous to other problems in convex sets. Our analogy table is as follows

Commutative Algebra	Convex
Local ring A of dimension n	Cone $\mathbb{R}^n_+$
Primary Ideals $I, J \subset A$	Co-convex sets $K, L \subset \mathbb{R}^n_+$
Hilbert polynomial $\operatorname{len}(A/I^k)$	Co-volume $\operatorname{vol}(\mathbb{R}^n_+ \setminus tK)$
Hilbert-Samuel multiplicity $e(I)$	Co-volume : $\operatorname{coVol}(K) = \operatorname{vol}(\mathbb{R}^n_+ \setminus K)$.
Mixed multiplicities: e(I , . . . , I, J , . . . , J)	Mixed co-volume: $\operatorname{coVol}(K, \ldots, K, L, \ldots, L)$
Reversed Khovanski-Teissier inequalities	Reverse Alexandrov-Fenchel inequalities

(1)

Let us present the analog statements.

Theorem 2.1. *Given I, J two primary ideals in a local ring A. The following holds:*

(i) (Samuel) There exists a polynomial $P \in \mathbb{Q}[X]$ of degree n and an integer N such that for any $k \geqslant N$, one has:

$$\operatorname{len}_A(A/I^k) = P(k) = e(I)\frac{k^n}{n!} + l.o.t,$$

where $e(I)$ is called the Hilbert-Samuel multiplicity.

(ii) (Teissier) There exists $N \geqslant 0$ and a polynomial $Q \in \mathbb{Q}[X_1, X_2]$ such that for all $t_1, t_2 \geqslant N$,

$$e(I^{t_1} J^{t_2}) = Q(t_1, t_2) = \sum_{k=0}^{n} \binom{n}{k} e(I[k], J[n-k]) t_1^k t_2^{n-k} + l.o.t,$$

where $e(I[k], J[n-k]) = e(I, \ldots, I, J, \ldots, J)$ where I, J are repeated k times and $n-k$ times respectively.

(iii) (Rees, Teissier) For all $J_3, \ldots, J_n$ primary ideal of A, one has:

$$e(I, J, J_3, \ldots, J_n)^2 \leqslant e(I, I, J_3, \ldots, J_n) e(J, J, J_3, \ldots, J_n). \tag{2}$$

This inequality is what we refer as the reverse Khovanski-Teissier inequality.

The convex counterpart are given as follows.

Theorem 2.2. *Give K, L two co-convex sets in $\mathbb{R}^n_+$.*

(i) The function $(t_1, t_2) \mapsto \operatorname{vol}(\mathbb{R}^n_+ \setminus (t_1 K + t_2 L))$ is a homogeneous polynomial of degree n in t_1, t_2. We have thus:

$$\operatorname{coVol}(t_1 K + t_2 L) = \sum_{k \leqslant n} \binom{n}{k} t_1^k t_2^{n-k} \operatorname{coVol}(K[K], L[n-k]).$$

(ii) (Reverse Alexandrov-Fenchel inequalities) For any $L_3, \ldots, L_n$ co-convex sets. One has:

$$\operatorname{coVol}(K, L, L_3, \ldots, L_n)^2 \leqslant \operatorname{coVol}(K, K, L_3, \ldots, L_n) \operatorname{coVol}(L, L, L_3, \ldots, L_n)$$

3 Hilbert-Samuel multiplicity II

Theorem 3.1. *(see [Sam51, Théorème 9]) Let A be a local ring with maximal ideal $\mathfrak{m}$ (of Krull dimension d). Let I be a $\mathfrak{m}$-primary ideal. Then there exists an integer $N > 0$ and a polynomial Q of degree d such that the function*

$$k \mapsto \operatorname{len}_A(A/I^k) \tag{3}$$

coincides with Q for all $k \leqslant N$.

Take $A = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$.

We will follow the classical proof of Samuel, that does not require any algebraic geometry knowledge, but I will try to point out the corresponding geometric insight behind this construction.

We start with the proof of the homogeneous version of the theorem taken from Samuel [Sam51].

Theorem 3.2. *Let I be a homogeneous ideal in $A[X_1, \ldots, X_n]$ and denote by I_k the subset of I containing homogeneous polynomials of degree k (in the variables X_i). Then there exists a polynomial $P \in \mathbb{R}[X]$ of degree n and an integer N such that for all $k \geqslant N$, $P(k) = \operatorname{len}_A(I_k)$.*

Remark 3.3. When A is a field, the length is just the dimension of the corresponding vector space.

Proof. We proceed by induction on the Krull dimension of A. First assume that A is a field. This is Hilbert's theorem that we assume.

Let us prove the induction. Assume that A is artinian ring (i.e has finite length). If A is an integral domain, then it would be a field and we are reduced to the base case. Otherwise, A is not an integral domain, so there exists a divisor $\alpha \in A$ of zero. We set $\mathfrak{m}$ the ideal generated by α. The idea is to reduce the polynomials in I modulo $\mathfrak{m}$. Let us denote by M_k the homogeneous polynomials of degree k in $A[X_1, \ldots, X_n]$. Naturally M_k is an A-module, so are I_k and $\mathfrak{m}M_k$. In particular, $I_k + \mathfrak{m}M_k$ contains $\mathfrak{m}M_k$ as a A- submodule. Consider the morphism of A-module $\varphi \colon I_k \to I_k + \mathfrak{m}M_k/(\mathfrak{m}M_k)$. It is by construction surjective, and its kernel is exactly $I_k \cap \mathfrak{m}M_k$. In particular, there is an isomorphism of A-module from $I_k + \mathfrak{m}M_k/\mathfrak{m}M_k$ with $I_k/(I_k \cap \mathfrak{m}M_k)$. We thus get:

$$\operatorname{len}_A(I_k) = \operatorname{len}_A(I_k + \mathfrak{m}M_k/\mathfrak{m}M_k) + \operatorname{len}_A(\mathfrak{m}M_k \cap I_k) \tag{4}$$

(4) has to be understood as an exact sequence of A-modules, namely:

$$0 \to \mathfrak{m}M_k \cap I_k \to I_k \to I_k + \mathfrak{m}M_k/\mathfrak{m}M_k \to 0. \tag{5}$$

The last term corresponds to polynomials in I_k whose coefficients are reduced modulo $\mathfrak{m}$ (geometrically one restricts the functions in I_k to the di $\mathfrak{m}$)

We first describe the first term in the sum. Observe that $I_k + \mathfrak{m}M_k/\mathfrak{m}M_k$ is an A-module, which is annihilated by all elements of $\mathfrak{m}$. In particular, $I_k + \mathfrak{m}M_k/\mathfrak{m}M_k$ descends to an $A/\mathfrak{m}$ module. More precisely, $I_k + \mathfrak{m}M_k/\mathfrak{m}M_k$ is a submodule of polynomials in $A/\mathfrak{m}[X_1, \ldots, X_n]$ of degree k. By the induction hypothesis, $\operatorname{len}_{A/\mathfrak{m}} I_k + \mathfrak{m}M_k/\mathfrak{m}M_k$ is a polynomial for k large enough. Now any chain of A-module annihilated by $\mathfrak{m}$ yields a sequence of $A/\mathfrak{m}$ module. Conversely, any chain $B_1 \subset B_2 \subset \ldots \subset B_p$ of $A/\mathfrak{m}$-submodule yields a sequence of A submodule of the form $B_i + \mathfrak{m}M_k/\mathfrak{m}M_k$. This shows that $\operatorname{len}_A(I_k + \mathfrak{m}M_k/\mathfrak{m}M_k)$ is a polynomial in k for k large enough.

Let us now describe the A-modules $\mathfrak{m}M_k \cap I_k$. We consider $B = Ann(\mathfrak{m}) = \{b \in A \mid bx = 0 \forall x \in \mathfrak{m}\}$. By definition, $\mathfrak{m}M_k \cap I_k$ is annihilated by B hence defines an A/B module. One checks that B is a non-trivial ideal by construction of $\mathfrak{m}$. . As a result, the length of A/B is strictly smaller than the length of A. We now analyze the structure of $\mathfrak{m}M_k \cap I_k$. Let α be a generator of the ideal $\mathfrak{m}$. Then the multiplication by α in A and in $A[X_1 \ldots, X_n]$ induces some isomorphism between A/B and $\mathfrak{m}A$ and $A/B[X_1, \ldots, X_n]$ with $\mathfrak{m}A[X_1, \ldots, X_n]$. The same holds if one intersects with degree k homogeneous polynomials of degree k intersected with I. (in fact A/B is a field) The induction can then be applied to $A/B[X_1, \ldots, X_n] \cap I_k$, and we obtain a polynomial R in k. One then concludes as the sum of two polynomials is a polymial. ☐

This theorem will have striking consequences. Before stating the result, we will use the following notions in commutative algebra.

Definition 3.4. *A ring A is called local if the set of all non-invertible element form an ideal $\mathfrak{m}$.*

Exercice: Check that the ideal $\mathfrak{m}$ is then a maximal ideal in such local ring.

Example 3.5. The main example we will use is the following $A = \mathbb{C}[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ (the localization with respect to the ideal $(x_1, \ldots, x_n)$).

Definition 3.6. *Given a local ring A with maximal ideal $\mathfrak{m}$, another ideal $\mathfrak{p}$ is called $\mathfrak{m}$-primary if $\mathfrak{m}^n \subset \mathfrak{p} \subset \mathfrak{m}$ for some integer n.*

We can now state the main result which is due to Samuel [Sam51, Théorème 9].

Theorem 3.7. *Let A be a local ring (noetherian) with maximal ideal $\mathfrak{m}$. Then for any $\mathfrak{m}$-primary ideal $\mathfrak{p}$, there exists a polynomial $P \in \mathbb{Z}[X]$ and N such that for all $n \geqslant N$, $P(n) = \mathrm{len}_A(A/\mathfrak{p}^n)$.*

The proof relies on a beautiful trick that I will present below.

3.1 The normal cone ring

Given a $\mathfrak{m}$-primary ideal $\mathfrak{p}$. We construct the "normal cone ring" as the following graded ring:

$$F = \oplus_{n=0}^{\infty} \mathfrak{p}^n / \mathfrak{p}^{n+1}. \tag{6}$$

Remark 3.8. This terminology fits with the modern formulation in algebraic geometry. Namely a prime ideal of the ring F correspond to point in the normal cone of $\mathfrak{p}$ in the underlying space $Spec(A)$ (see [Ful98, Appendix B.6.1]) while the homogeneous ideals in it correspond to the points on the exceptional divisors (see [Ful98, Appendix B.6.3]). More intuitively, if $\mathfrak{p}$ correspond to the ideal given by a say a point or dimension k-plane in $\mathbb{C}^n$ near 0, then the normal cone is the affine cone over the normal bundle of the k-plane inside $\mathbb{C}^n$, the set of points on the k-plane with the data of a vector going normally out of the k-plane. The ring F is obtained after a change of variables from $\mathbb{C}^n$

The following statement is simple but crucial.

Proposition 3.9. *The following properties hold:*

(i) F is a graded $A/\mathfrak{p}$-module and has a ring structure (induced by the multiplication in A)

(ii) There is an isomorphism between F and a quotient ofthe graded ring $(A/\mathfrak{p})[X_1, \ldots, X_k]$ where k is the number of generators of $\mathfrak{p}$.

Proof. We prove assertion (i). Each graded piece $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ has a structure of A-module, annihilated by $\mathfrak{p}$ so descends to a $A/\mathfrak{p}$-module. The ring structure comes as follows. Assume that $\bar{x} \in \mathfrak{p}^n/\mathfrak{p}^{n+1}$ and $\bar{y} \in \mathfrak{p}^m/\mathfrak{p}^{m+1}$ where $x, y \in \mathfrak{p}^n, \mathfrak{p}^m$ respectively. Then $xy \in \mathfrak{p}^{n+p}$ by definition, and the class $\overline{xy} \in \mathfrak{p}^{m+n}/\mathfrak{p}^{m+n+1}$ does not depend on the choice of representative of $\bar{x}, \bar{y}$.

Let us prove the second assertion. Choose some generators $(x_1, \ldots, x_k) \in A^k$ of $\mathfrak{p}$. By definition, every graded piece is generated by homogeneous monomials in $\bar{x}_1, \ldots, \bar{x}_k$. Consider the morphism $\varphi\colon A[X_1, \ldots, X_k] \mapsto F$ given by $P \mapsto P(\bar{x}_1, \ldots, \bar{x}_k)$. This morphism is a morphism of A-module which is by definition surjective (since every homogeneous piece surjects on the corresponding graded piece). However, the kernel of φ consists of $\mathfrak{p}A[X_1, \ldots, X_k]$ as well as other relations elements and so we obtain a first the quotient is isomorphic to $A/\mathfrak{p}[X_1, \ldots, X_k]$ and quotient by the relations in $\mathfrak{p}$, as required. ☐

We can now prove the main result.

Proof of Theorem 3.1. Take the graded ring F associated with $\mathfrak{p}$. It is isomorphic to a quotient of the graded ring $A/\mathfrak{p}[X_1, \ldots, X_k]$. We want to compute the length of $A/\mathfrak{p}^n$. This is the same as computing the length of $\sum_{1 \leqslant k \leqslant n-1} \mathrm{len}_A(\mathfrak{p}^k/\mathfrak{p}^{k+1}) + \mathrm{len}_A(A/\mathfrak{p}) = \mathrm{len}_{A/\mathfrak{p}}(A/\mathfrak{p}) + \sum_{1 \leqslant k \leqslant n-1} \mathrm{len}_{A/\mathfrak{p}}(\mathfrak{p}^k/\mathfrak{p}^{k+1})$. By Theorem 3.2, each graded piece has a length $Q(k)$ where A is a polynomial of degree $\dim(A) - 1$. This is a polynomial of degree $\dim(A)$. ☐

3.2 Twisted version of the normal cone and stabilization phenomena

Our goal is to understand the ring $A/\mathfrak{p}^k\mathfrak{q}^l$ in order to prove the following result.

Theorem 3.10. *[?,] Given a local ring A and any two $\mathfrak{m}$-primary ideal $\mathfrak{p}, \mathfrak{q}$. Then there exists $N \geqslant 0$ and a polynomial $Q \in \mathbb{Q}[X, Y]$ such that for any $k, l \geqslant N$, $Q(k, l) = \mathrm{len}_A(A/(\mathfrak{p}^k\mathfrak{q}^l))$.*

To that end, Teissier introduces a twisted version of the normal cone construction or blow-up construction in a general finite type graded module M

Namely, given two $\mathfrak{m}$-primary ideals $\mathfrak{p}, \mathfrak{q}$, one considers the ring:

$$G = \oplus_{k,l} \frac{\mathfrak{p}^k\mathfrak{q}^l M}{\mathfrak{p}^{k+1}\mathfrak{q}^{\mathfrak{l}} M} \tag{7}$$

and one consider a weight 1 for elements in $\mathfrak{p}$ while we put a weight 0 on $\mathfrak{q}$. We introduce the graded ring A':

$$A' = \oplus_{k,l} \frac{\mathfrak{p}^k\mathfrak{q}^l}{\mathfrak{p}^{k+1}\mathfrak{q}^{\mathfrak{l}}} \tag{8}$$

Similarly, we see that A' has the structure of graded $A/\mathfrak{p}$ module and we also have an isomorphism between a quotient of $A/\mathfrak{p}[X_1, \ldots, X_k]$ with A', so it is noetherian. While G is a module of finite type over A'.

We will use the following notation.

Definition 3.11. *Given a submodule N inside an A-module M and an ideal $\mathfrak{p}$, then $(N\colon \mathfrak{p})_M = \{x \in M \mid \mathfrak{p}x \subset N\}$.*

In order to make the induction work, we need to proceed as in the proof of Theorem 3.2 as well as in the proof of Theorem 3.1. We will not use an arbitrary minimal ideal, but one contained in $\mathfrak{p}$ and will need to work with arbitrary A-modules instead of A as we will pass to some quotient after the first step of the induction. The first step is the following:

Lemma 3.12. *Given a noetherian A-module M. Consider* $a \in \mathfrak{p} \setminus \mathfrak{p}^2$ *and let* $\mathfrak{a} = (a)$*. Then the following sequence is exact:*

$$\begin{tikzcd} 0 \arrow[r] & \dfrac{(\mathfrak{p}^k\mathfrak{q}^l M : \mathfrak{a})_M}{\mathfrak{p}^{k-1}\mathfrak{q}^l M} \arrow[r] & \dfrac{M}{\mathfrak{p}^{k-1}\mathfrak{q}^l M} \arrow[r, "\times a"] & \dfrac{M}{\mathfrak{p}^k\mathfrak{q}^l M} \arrow[r] & \dfrac{M}{(a)M + \mathfrak{p}^{k+1}\mathfrak{q}^l M} \arrow[r] & 0 \end{tikzcd}$$

Our aim is to find an appropriate $\mathfrak{a}$ so that the part $(\mathfrak{p}^{k+1}\mathfrak{q}^l : \mathfrak{a})_A = \mathfrak{p}^k\mathfrak{q}^l$ for k large enough. This will be the result of Noetherianity.

Lemma 3.13. *Let* $a \in \mathfrak{p} \setminus \mathfrak{p}^2$*. View* $\bar{a}$ *the element in G. Then if* $\bar{a}$ *is non-trivial, then there exists N such that for any* $k \geqslant N$ *and any l,*

$$(\mathfrak{p}^{k+1}\mathfrak{q}^l : a)_M \cap \mathfrak{p}^N\mathfrak{q}^l M = \mathfrak{p}^k\mathfrak{q}^l M. \tag{9}$$

Remark 3.14. The element a and its ideal play the role of a divisor whose support contains the support of $\mathfrak{p}$.

Proof. G is a graded $A/\mathfrak{p}$-module which is also filtered by $\mathfrak{a}$. The multiplication by $\bar{a}$ induces a morphism in G and we claim that for k large enough, the image stabilizes. G is of finite type since it is a finite module over A' Assume by contradiction that $\mathfrak{a}\mathfrak{p}^{k-1}\mathfrak{l} \neq \mathfrak{p}^k\mathfrak{q}^l$ for infinitely many k and l. Let us consider the submodules $K = (0 : \bar{a})_G$ then K is a submodule and for each n, K_n the n-th graded piece is a submodule of G_n. Observe that K is noetherian. But this gives infinitely many submodules that are non-trivial, which is impossible. In particular, for n large enough $K_n = 0$. Writing K_n explicitly gives:

$$K_n = \{\bar{x} \in \mathfrak{p}^n\mathfrak{q}^l M/\mathfrak{p}^{n+1}\mathfrak{q}^l M \,|ax \in \mathfrak{p}^{n+2}\mathfrak{q}^l M\} = \mathfrak{p}^n\mathfrak{q}^l M \cap (p^{n+2}q^l M : a)_M/\mathfrak{p}^{n+1}\mathfrak{q}^l M \tag{10}$$

So this gives that $\mathfrak{p}^n \cap \mathfrak{q}^l M \cap (p^{n+2}q^l M : a)_M = \mathfrak{p}^{n+1}\mathfrak{q}^l$ for n large enough, as required. $\square$

The next part is also crucial and concerns the control of these kernels asymptotically.

Lemma 3.15. *For k, l large enough, one has:*

$$(\mathfrak{p}^k\mathfrak{q}^l M : a)_M/\mathfrak{p}^{k-1}\mathfrak{q}^l M = (0 : a)_M \tag{11}$$

Proof. We shall apply first the Artin-Rees Lemma twice, recall that it says,

Lemma 3.16. *If* $N \subset M$ *is a submodule of M which is finitely generated A-module where A is noetherian. Let I be an ideal, then there exists N such that for all* $k \geqslant N$, $I^kM \cap N = I^{k-N}(I^NM \cap N)$

We apply for N the submodule aM and to the ideal $\mathfrak{p}, \mathfrak{q}$. We obtain the existence of two integers N_1, N_2 such that for all $k, l \geqslant N_1, N_2$

$$\mathfrak{p}^k\mathfrak{q}^l M \cap aM = \mathfrak{p}^{k-N_1}\mathfrak{q}^{l-N_2}(\mathfrak{p}^{N_1}\mathfrak{q}^{N_2}M \cap aM) \tag{12}$$

Now take $m \in (\mathfrak{p}^k\mathfrak{q}^l M : a)_M$, then by definition $am \in \mathfrak{p}^k\mathfrak{q}^l M$. But by the previous equation, we can write $am = am'$ with $m' \in \mathfrak{p}^{k-N_1}\mathfrak{q}^l M$. Choosing k large enough gives $m' \in \mathfrak{p}^N\mathfrak{q}^l M$ where N is the integer given in the stabilization lemma. We get that the difference $m - m'$ belongs to $(0 : a)_M$ while m' belongs to $\mathfrak{p}^{N_1}\mathfrak{q}^l$. This shows that $(\mathfrak{p}^k\mathfrak{q}^l M : a)_M = (0 : a)_M + \mathfrak{p}^N\mathfrak{q}^l M$

We can replace the quotient by the quotient by N_1 by the previous lemma and using the second isomorphism theorem (i.e $X + N/N \simeq X/X \cap N$) and the fact that $((A/N) : a) \simeq (A : a) + N/N$:

$$\frac{(\mathfrak{p}^k\mathfrak{q}^l M : a)_M}{\mathfrak{p}^{k-1}\mathfrak{q}^l M} = \frac{(\mathfrak{p}^k\mathfrak{q}^l M : a)_M + \mathfrak{p}^N\mathfrak{q}^l M}{\mathfrak{p}^N\mathfrak{q}^l M} = \frac{(O : a)_M}{\mathfrak{p}^N\mathfrak{q}^l M \cap (0 : a)_M} \tag{13}$$

For l large enough, the length of the last quotient only depends on N_1 (because M is finitely generated). Note that the submodule $\mathfrak{p}^N\mathfrak{q}^l M$ decrease toward zero as N, l grow. So choosing them large enough from the beginning, this quotient is thus $(0 : a)_M$. $\square$

3.3 Polynomial behavior of the Hilbert-Samuel multiplicity

We now reproduce the argument due to Teissier, that justifies the definition of the mixed multiplicity:

Proof of Theorem 3.10. Apply the exact sequence and pass to length and we obtain a recursive formula involving $f_M(k+1,l) - f_M(k,l) = len_A((0:a)_M) + f_{M/aM}(k+1,l)$

$\square$

4 Mixed covolumes in convex geometry

We reset the whole discussion and will focus namely on special subset in the cone $\mathbb{R}_+^n$ for $n \geqslant 1$. Although our first definitions will be fairly general, we will later focus all the discussion in the case where $n = 2$ where many phenomenon can be noticed but for which many integral formulas are explicit and do not require too much analysis.

4.1 Basic definition and operations

We start with a cone $\mathcal{C} = \mathbb{R}_+^n$ with $n \geqslant 1$ and we will focus on convex subset $A \subset \mathcal{C}$. A subset A is called $\mathcal{C}$-coconvex if A is convex and the complement $\mathcal{C} \setminus A$ is compact. In the rest of the lecture, since $\mathcal{C}$ will be fixed once and for all, we will say that A is coconvex if A is $\mathcal{C}$-coconvex.

Given two coconvex sets A, B, we denote by $+$ the Minkowski sum :

$$A + B = \{x + y \mid x \in A, y \in B\}. \tag{14}$$

We also introduce the scaling of A by $\lambda \in \mathbb{R}_+$, defined as :

$$\lambda A = \{\lambda x \mid x \in A\}. \tag{15}$$

One should check that the following are satisfied.

Properties 4.1. One has:

(i) For all $A, B \subset \mathcal{C}$ with A, B coconvex, $A + B$ is also coconvex.

(ii) For all $\lambda \in \mathbb{R}_+$ and all coconvex $A \subset \mathcal{C}$, λA is also coconvex.

We now give examples, some of combinatorial nature while others will be more analytic

Example 4.2. (Polytope) Take a finite number of points $p_1, \ldots, p_m \in \mathcal{C}$ with $m \geqslant n$ and assume that the first n elements p_i belong to the each of the coordinate axii. Then the subset:

$$P = \left\{ \sum_{i=1}^{m} \lambda_i p_i \mid \sum_{i=1}^{m} \lambda_i \geqslant 1 \right\} \tag{16}$$

is coconvex.

In the previous example, the boundary is polyhedral. We give an example where the boundary is smooth.

Example 4.3. (the "co-Ball") Consider the unit ball at the point $(1, 1, \ldots, 1) \in \mathcal{C}$ and consider the subset B containing all the radial half-line contained in $\mathcal{C}$ starting from the boundary of the unit ball. Then B is coconvex.

Now if we focus in dimension 2, there is a way to define all these convex subsets.

Example 4.4. If $n = 2$, given any convex function $f\colon [0,a] \to \mathbb{R}_+$ such that $f(a) = 0$ with $a > 0$, the subset

$$A = \{(tx, tf(x)) \mid x \in [0,a], t \geqslant 1\} \tag{17}$$

is coconvex.

4.2 Covolumes and mixed covolumes

We endow the cone $\mathcal{C}$ with the Lebesgue measure and the main statement is the following.

Theorem 4.5. *K, L two co-convex sets in $\mathbb{R}^n_+$. Then $(t_1, t_2) \mapsto \operatorname{coVol}(t_1K + t_2L)$ is a polynomial.*

As an application, we shall write this expression as:

$$\operatorname{coVol}(t_1K + t_2L) = \sum_{k \leqslant n} \binom{n}{k} \operatorname{coVol}(K[k], L[n-k]) t_1^k t_2^{n-k} + l.o.t \tag{18}$$

For the proof of this statement, we will infer it from Alexandrov's theorem (see [**?**, XXX])

4.3 Surface area measure

In this section, we will define the surface area measure for general polytopes, but for general co-convex sets in dimension 2.

Definition 4.6. *Given a co-convex set K whose boundary is polyhedral. The surface area measure is the measures S such that:*

$$S = \sum_{v \in N(S)} Leb_{n-1}(F(v))\delta_v. \tag{19}$$

where $N(S)$ is the subset of outer normal vectors to S and where $F(v)$ is the corresponding face whose normal vector is v.

Proposition 4.7. *(Give as exercice) Let $K \subset \mathbb{R}^2_+$ be a co-convex set defined as the epigraph of a smooth strictly convex function $f\colon [0, a] \to \mathbb{R}$. Then almost everywhere on $[0, a]$, the measures S is determined by the formula*

$$N_* dS = \sqrt{1 + f'(x)^2} dx, \tag{20}$$

where $N\colon [0, a] \to \mathbb{R}$ is the Gauss map.

Proof. The derivative on the left and right coincide at every point except countably many. This means that the Gauss map is also almost everywhere defined. □

4.4 Support function

Let K be a cocompact convex set in $\mathbb{R}^2_+$. We will define the following function:

Definition 4.8. *The support function of K with respect to a base point $p \in \mathbb{R}^n$ is defined as:*

$$h_{K,p}(u) = \sup_{x \in K} \langle u, x - p \rangle, \tag{21}$$

where $u \in S^1 \cap R^2_-$.

When p is the origin, we will denote by h_K the support function.

In the rest of the lectures, we will denote by Ω the subset $S^1 \cap \mathbb{R}^2_-$.

Our aim is to prove the following formula from integral geometry, which is due to Schneider [**?**]

Theorem 4.9. *Let K be a cocompact convex set in $\mathbb{R}^2_+$. Then one has:*

$$\operatorname{coVol}(K) = -\frac{1}{2}\int_\Omega h_K(u)dS_K(u) \tag{22}$$

Before treating the general case, we will prove this formula when K is a convex set with strictly convex $\mathcal{C}^2$ boundary. Its proof relies on beautiful aspect of differential geometry that we will use later on.

Proof of Theorem 4.9 in the smooth case. We proceed in several steps.

Step 1 (Parametrizing normal vectors, i.e the Gauss map) The boundary of K is the graph of a strictly convex function $f\colon [0,+\infty] \to \mathbb{R}$. We also assume that f is zero on $[a,+\infty[$. The function $x \mapsto \gamma(x) = (x, f(x))$ gives a $\mathcal{C}^2$-smooth parametrization of the boundary. The unit normal vector at each point $(x, f(x))$ is given by

$$n(x) = \frac{\begin{pmatrix} f'(x) \\ -1 \end{pmatrix}}{\sqrt{1+f'(x)^2}} \in \Omega \tag{23}$$

Since $x \mapsto f'(x)$ is a diffeomorphism, this Gauss map is also a local diffeomorphism.

Step 2 (Rewrite the support function) Since $u(x)$ is normal to ∂K at $(x, f(x))$ and K is convex, it is contained in the half plane $H = \{y | \langle n(x), y\rangle \leqslant \langle n(x), \gamma(x)\rangle\}$. In particular, the supremum is realized exactly at $\gamma(x)$, so that we have:

$$h_K(n(x)) = \langle \gamma(x), n(x)\rangle. \tag{24}$$

Combining with the previous formula together with the fact that $dS_K(n(x)) = \sqrt{1+f'(x)^2}dx$, we obtain:

$$\begin{aligned}\int h_K(u)dS_K(u) &= \int_x \langle (f'(x), -1), (x, f(x))\rangle = \int_{[0,+\infty[} (xf'(x) - f(x))dx \\ &= -\int_{\mathbb{R}_+} f(x)dx + [xf]_0^{+\infty} - \int_{\mathbb{R}_+} f(x)dx = -2\int_{\mathbb{R}_+} f(x)dx,\end{aligned}$$

where we have used an integration by part. This gives the appropriate formula. □

We now give later a more general proof which will be valid in higher dimension.

5 Mixed co-volumes and the three body inequality

5.1 Surface area measure from support functions in $\mathbb{R}^2$

So far, we have defined two distinct objects, support functions and surface area measures. Our aim is to connect these two notions, and the connection comes from a differential operator. This is where magical differential geometry comes.

Proposition 5.1. *Given any co-convex set K in $\mathbb{R}^2_+$ with smooth boundary. Then one has:*

$$dS(v) = (h_K(v) + \Delta h_K(v))dv, \tag{25}$$

where dv is the Lebesgue measure on the circle.

Proof. Let us push forward by the Gauss map N, we get:

$$N_* dS(x) = \sqrt{1 + (f'(x))^2} dx \tag{26}$$

Differentiating the relation

$$f'(x) = -\cot\theta$$

gives

$$f''(x)\, dx = \csc^2\theta\, d\theta.$$

Therefore

$$dx = \frac{\csc^2\theta}{f''(x)}\, d\theta.$$

Substituting into the arc length formula,

$$ds = \sqrt{1 + f'(x)^2} \frac{\csc^2\theta}{f''(x)}\, d\theta.$$

Using

$$1 + f'(x)^2 = 1 + \cot^2\theta = \csc^2\theta,$$

we obtain

$$ds = \frac{\csc^3\theta}{f''(x(\theta))}\, d\theta.$$

Since $\sin\theta < 0$, this may also be written as

$$ds = \frac{1}{|\sin\theta|^3\, f''(x(\theta))}\, d\theta.$$

We now express the other hand of the equality We now compute $h + h''$ directly. Since

$$h(\theta) = x(\theta)\cos\theta + f(x(\theta))\sin\theta,$$

differentiate with respect to θ:

$$h'(\theta) = x'(\theta)\cos\theta - x(\theta)\sin\theta + f'(x)x'(\theta)\sin\theta + f(x)\cos\theta.$$

Using the critical point relation

$$\cos\theta + f'(x)\sin\theta = 0,$$

the terms involving $x'(\theta)$ cancel, and therefore

$$h'(\theta) = -x(\theta)\sin\theta + f(x(\theta))\cos\theta.$$

Differentiate again:

$$\begin{aligned} h''(\theta) = &-x'(\theta)\sin\theta - x(\theta)\cos\theta \\ &+ f'(x)x'(\theta)\cos\theta - f(x)\sin\theta. \end{aligned}$$

Now add $h(\theta)$:

$$h''(\theta) + h(\theta) = -x'(\theta)\sin\theta + f'(x)x'(\theta)\cos\theta.$$

Factor out $x'(\theta)$:

$$h'' + h = x'(\theta)\big(-\sin\theta + f'(x)\cos\theta\big).$$

Using

$$f'(x) = -\cot\theta,$$

we obtain

$$\begin{aligned}-\sin\theta + f'(x)\cos\theta &= -\sin\theta - \frac{\cos^2\theta}{\sin\theta}\\ &= -\frac{\sin^2\theta+\cos^2\theta}{\sin\theta}\\ &= -\frac{1}{\sin\theta}.\end{aligned}$$

Hence

$$h'' + h = -\frac{x'(\theta)}{\sin\theta}.$$

It remains to compute $x'(\theta)$. Differentiate

$$f'(x(\theta)) = -\cot\theta :$$

$$f''(x)x'(\theta) = \csc^2\theta.$$

Therefore

$$x'(\theta) = \frac{\csc^2\theta}{f''(x)}.$$

Substituting,

$$h''(\theta) + h(\theta) = -\frac{1}{\sin\theta}\frac{\csc^2\theta}{f''(x)}.$$

Since $\sin\theta < 0$,

$$-\frac{1}{\sin\theta} = \frac{1}{|\sin\theta|},$$

and thus

$$\boxed{h''(\theta) + h(\theta) = \frac{1}{|\sin\theta|^3 f''(x(\theta))},}$$

as required. ☐

Corollary 5.2. *Given K, L two co-convex sets in $\mathbb{R}^2_+$, one has $S_{K+L} = S_K + S_L$ where S_K, S_L are the surface area measures of K, L respectively.*

Proof. One sees that $h_{K+L} = h_K + h_L$, so that applying (22) gives the required formula. ☐

We also get:

Corollary 5.3. *The function $(K, L) \mapsto \mathrm{coVol}(K, L)$ is multi-linear with respect to the Minkowski sum in each factor.*

We will deduce out of the previous Proposition the famous reverse Alexandrov Fenchel inequality, in convex geometry, which was proved in the works of Kaveh, Khovanskii [KK14]. However, the scheme of the proof dates back to the work of Alexandrov.

5.2 Some Fourier analysis and a three body inequality

We will now apply some little Fourier analysis to recover a convex analog of the Ruggiero-Gignac inequality [GR21], in a special case.

We state this analog theorem.

Theorem 5.4. *Let A, B, C be three co-convex sets in $\mathbb{R}^2_+$. Then one has:*

$$\mathrm{coVol}(C, C)\,\mathrm{coVol}(A, B) \geqslant \mathrm{coVol}(A, C)\,\mathrm{coVol}(B, C). \tag{27}$$

The proof of this result will be proved in the next lectures if one relies on Ruggiero-Gignac's algebraic result. Our aim is to prove this result when C is the "co-ball" in Example 4.3. Before that we will need one more result. The support functions are supported on Ω and vanish at the boundary points. We extend then on the whole circle by zero so that one obtains some functions defined on the whole circle. Viewed as periodic functions on $\mathbb{R}$ and given a coconvex set A, we can write the Fourier transform of h_A as follows:

$$h_A(\theta) = a_0(A) + \frac{1}{2}\sum_{n\geqslant 1} a_n(A)\cos(n\theta) + b_n(A)\sin(n\theta), \tag{28}$$

where $a_i(A), b_i(A)$ are real numbers.

We derive the first consequence of Proposition 5.1.

Corollary 5.5. *Let $a_n(A), b_n(A)$ be the Fourier coefficients of h_A where A is a co-convex set. Then the function*

$$\theta \mapsto \sum_{n>1} a_n(A)(1-n^2)\cos(n\theta) + b_n(A)(1-n^2)\sin(n\theta) \tag{29}$$

is non-negative.

Proof. We rely on the fact that the surface area measure is positive. Given any co-convex set A, by Proposition 5.1 together with the Fourier decomposition, we get for $v = -(\cos(\theta), \sin(\theta))$,

$$dS(v) = a_0(A) + \sum_{n>1} a_n(A)(1-n^2)\cos(n\theta) + b_n(A)(1-n^2)\sin(n\theta)d\theta \tag{30}$$

is a non-negative measure. This implies that the density is non-negative, hence for any $\theta \in \mathbb{R}$:

$$a_0(A) + \sum_{n>1} a_n(A)(1-n^2)\cos(n\theta) + b_n(A)(1-n^2)\sin(n\theta) \geqslant 0. \tag{31}$$

However, the average $a_0(A) \leqslant 0$ since $h_A \leqslant 0$. This gives

$$a_0(A) + \sum_{n>1} a_n(A)(1-n^2)\cos(n\theta) + b_n(A)(1-n^2)\sin(n\theta) \geqslant -a_0(A) \geqslant 0, \tag{32}$$

as required. ☐

We thus introduce the operator $\mathcal{S}$ which takes a support function and associates the surface area measure. We set for any function h on the circle:

$$\mathcal{S}(h) = (\mathrm{Id} - \Delta)h. \tag{33}$$

Theorem 5.6. *Theorem 5.4 holds when C is the coball.*

Proof. Let us consider the difference $D = \mathrm{coVol}(A,B) - \dfrac{\mathrm{coVol}(A,C)\,\mathrm{coVol}(B,C)}{\mathrm{coVol}(C,C)}$. Writing as an integral allows one to consider as difference of functions:

$$D = -\langle h_A - \frac{\mathrm{coVol}(A,C)}{\mathrm{coVol}(C,C)} h_C, \mathcal{S}(h_B)\rangle = -\langle \mathcal{S}\left(h_A - \frac{\mathrm{coVol}(A,C)}{\mathrm{coVol}(C,C)} h_C\right), h_B\rangle.$$

Now the point is that the function given by $\mathcal{S}(h_A - \dfrac{\mathrm{coVol}(A,C)}{\mathrm{coVol}(C,C)} h_C)$ is exactly the function whose Fourier decomposition is given in Corollary 5.5. Indeed, this relies on the following observation (Exercice), that $h_C = 1 - \cos(\theta) - \sin(\theta)$. So $\mathcal{S}(h_C) = 1$ and the difference coincide with the substraction of the average of h_A, which kills the constant coefficient in the Fourier decomposition. This shows that $D \geqslant 0$, as required. ☐

6 Khovanskii theorem and Newton polygons

6.1 Kaveh-Khovanski's main theorem

In this lecture, the aim is to connect ideals with co-convex sets. We will focus on particular ideals. GIven $A = \mathbb{C}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$, we focus on primary ideals $\mathfrak{p}$ generated by monomials.

We describe the following subset:

$$\Gamma(\mathfrak{p}) = \{\alpha \in \mathbb{N}^n \ |x^\alpha \in \mathfrak{p}\} \tag{34}$$

Observe that $\Gamma(\mathfrak{p}\mathfrak{q}) = \Gamma(\mathfrak{p}) + \Gamma(\mathfrak{q})$ We then define the convex hull $C(\mathfrak{p})$ as the convex hull of all points of $\Gamma(\mathfrak{p})$.

This provides a dictionnary between ideals and co-convex sets.

Lemma 6.1. *Let $\mathfrak{p}$ be a monomial ideal. The following properties hold:*

(i) $\Gamma(\mathfrak{p})$ *is an additive semi-group in* $\mathbb{Z}^n$*.*

(ii) If $\mathfrak{p}$ is $\mathfrak{m}$-primary if and only if $C(\mathfrak{p})$ is co-convex.

The following fundamental result is taken from Kaveh-Khovanski's paper [KK14, Proposition 7.9].

Lemma 6.2. *One has* $\mathrm{len}_A(A/\mathfrak{p}) = \# \mathbb{N}^n \backslash (\Gamma(\mathfrak{p}) \cup \{0\})$.

Proof. Step 1: We show that $\mathrm{len}_A(A/\mathfrak{p}) = \dim_{\mathbb{C}}(A/\mathfrak{p})$. Given any maximal sequence of ideals I_j strictly increasing, observe that this gives a sequence of subrings such that the quotient I_{k+1}/I_k is as small as possible, and isomorphic to $\mathbb{C}$. This shows that the length of $A/\mathfrak{p}$ equals the dimension over $\mathbb{C}$ of $A/\mathfrak{p}$.

Step 2: We finish the proof of the result. The dimension over $\mathbb{C}$ is the same as the dimension of a basis over $\mathbb{C}$ of $A/\mathfrak{p}$ which consists of all monomials x^β with $\beta \notin \Gamma(\mathfrak{p})$. This gives the required equality. ☐

Remark 6.3. The previous lemma shows how this problem in commutative algebra is deeply connected to another hard problem, that is counting integral points on a set. Fortunately, we are only interested in a first asymptotic and not a precise counting, which is much more subtle and which has not any connection to convex geometry any more.

Theorem 6.4. *(Kaveh-Khovanski) If $\mathfrak{p}$ is a monomial ideal in A, then* $e(\mathfrak{p}) = (n!)\,\mathrm{coVol}(C(\mathfrak{p}))$.

Proof. Choose k large enough, then by the previous lemma, one has:

$$\frac{\mathrm{len}_A(A/\mathfrak{p}^k)}{k^n} = \frac{\#\mathbb{N}^n \setminus \Gamma(\mathfrak{p}^k)}{k^n}.$$

Let us show the following, that $C(\mathfrak{p}\mathfrak{p}) = 2C(\mathfrak{p})$. Indeed, take two boundary points $u, v \in \Gamma(\mathfrak{p})$. Then $(u + v)/2$ is inside of $C(\mathfrak{p})$ by convexity. This shows an inclusion. More generally, for all $k \geqslant 0$, we get $C(\mathfrak{p}^k) \subset kC(\mathfrak{p})$. In fact there is an equality $kC(\mathfrak{p}) = C(\mathfrak{p}^k)$. We shall compare the number of integral points to the Lebesgue measure, we use the old method due to Gauss. Let Q the unit cube in $\mathbb{R}^n_+$. We shall over cover by unit tiles of $\mathbb{R}^n_+ \backslash C(\mathfrak{p}^k)$:

$$\mathbb{R}^n_+ \backslash C(\mathfrak{p}^k) \subset \cup_{u \in \#\mathbb{N}^n \backslash \Gamma(\mathfrak{p}^{\mathfrak{k}})}(u + Q) \tag{35}$$

So taking the volume yields:

$$\mathrm{coVol}(kC(\mathfrak{p})) \leqslant \mathrm{coVol}(C(\mathfrak{p}^k)) \leqslant \sum_{u \in \#\mathbb{N}^n \backslash \Gamma(\mathfrak{p}^k)} \mathrm{vol}(u + Q) = \mathrm{len}_A(A/\mathfrak{p}^k), \tag{36}$$

where we have used that $\mathbb{R}_+^n \setminus kC(\mathfrak{p}) \subset \mathbb{R}_+^n \setminus C(\mathfrak{p}^k)$. Dividing by k^n and multiplying by $(n!)$ yields and taking the limit as $k \to +\infty$ gives:

$$(n!)\operatorname{coVol}(C(\mathfrak{p})) \leqslant e(\mathfrak{p}). \tag{37}$$

We now prove the converse inequality. We use the following inequality:

$$\cup_{u\in\mathbb{N}^{*,n}\setminus\Gamma(\mathfrak{p}^k)}(u-Q) \subset \mathbb{R}_+^n \setminus C(\mathfrak{p}^k) \tag{38}$$

Taking the volume yields:

$$\operatorname{len}_A(A/\mathfrak{p}^k) - O(k^{n-1}) = \sum_{u\in\mathbb{N}\setminus\Gamma(\mathfrak{p}^k)} \operatorname{vol}(u-Q) - O(k^{n-1}) \leqslant \operatorname{coVol}(C(\mathfrak{p}^k)) = k^n \operatorname{coVol}(C(\mathfrak{p})). \tag{39}$$

Dividing by k^n and multiplying yields the other bound.

□

6.2 Application in convex geometry

We reproduce the following arguments due to Kaveh-Khovanski in convex geometry.

The arrow that the reverse Khovanski-Teissier inequality implies the reverse Alexandrov Fenchel inequality.

Similarly, we prove the three body inequality of Ruggiero-Gignac.

Theorem 6.5. *(Ruggiero-Gignac) Given any three primary ideal I, J, K in a local ring of length 2 the following holds:*

$$e(K,K)e(I,J) \geqslant e(I,K)e(J,K). \tag{40}$$

Corollary 6.6. *The three body inequality holds for any convex body.*

Proof. It is clear if A, B, C are rational polytopes. Otherwise for any co-convex set A, for each k, we consider the monomial ideal I_k such that $\Gamma(I_k)$ coincides with the integral points in $kA \cap \mathbb{N}^n$. If $x^\alpha \in I_k$ and $x^\beta \in I_m$, then this means that $\alpha \in kA$ and $\beta \in mA$. In particular, the sum is a lattice point $\alpha + \beta \in kA + mA = (k+m)A$. This shows that $I_k \cdot I_m \subset I_{m+k}$. We get a graded sequence of ideals (I_k). The same argument as in the previous proof shows that $\operatorname{coVol}(C(I_k))/k^n$ converges to $\operatorname{coVol}(A)$. □

Conjecture 6.7. *Can one prove the following inequality*

$$e(K,K,J_3,\dots,J_n)e(I,J,J_3,\dots,J_n) \geqslant e(I,K,J_3,\dots,J_n)e(J,K,J_3,\dots,J_n). \tag{41}$$

for any $\mathfrak{m}$-primary ideals I, J, K, J_i.

Conjecture 6.8. *Can one prove the analog with convex bodies ?*

7 A glimpse in higher dimension

I will give some background on mixed volumes. It is the sister of the covolume theory, but with more formulas.

Definition 7.1. *A convex body is a compact convex subset in $\mathbb{R}^n$.*

Theorem 7.2. *(Alexandrov) Given any convex bodies K, L, the volume function behaves polynomially:*

$$(t_1,t_2) \mapsto \operatorname{vol}(t_1K + t_2L) \tag{42}$$

is a polynomial in t_1, t_2 of degree n.

We can expand the formula into:

$$\mathrm{vol}(t_1K+t_2L)=\sum_{k=0}^{n}\binom{n}{k}t_1^kt_2^l\,\mathrm{vol}(K[k],L[n-k]). \tag{43}$$

We will focus on special formulas. Similarly to the co-convex case, we can define the surface area measure by the same formula for polytopes and also support functions. One then has:

Proposition 7.3. *One has for any convex body K,*

$$\mathrm{vol}(K)=\frac{1}{n}\int_{S^{n-1}}h_KdS_K. \tag{44}$$

Proof. Assume that the origin is inside the convex body K. This follows from a pyramid formula. Consider a face at distance d oriented along the normal vector v whose area is $a(v)$. Then the volume of the cone based at O with base given by the face $F(v)$ is given by:

$$vol(P(v))=\int_{t=0}^{1}t^{n-1}da(v)dt=\frac{1}{n}da(v). \tag{45}$$

If the origin is not in the convex set K, we can use a translate of the support function $h_{K,O'}$. The difference is of the form $\int\langle v,O'\rangle dS_K(v)$ and we shall use the fact that for a polytope, the sum $\int vdS_K(v)=0$. □

Proposition 7.4. *(see [?, XXX]) Given K,L two convex bodies in $\mathbb{R}^n$, one has:*

$$\mathrm{vol}(K,L[n-1])=\frac{1}{n}\int_{S^{n-1}}h_KdS_L. \tag{46}$$

We will not prove this formula, but rather use it in the following situation.

Proposition 7.5. *For any two co-convex sets K,L, one has:*

$$\mathrm{coVol}(K,L[n-1])=-\frac{1}{n}\int_{\Omega}h_KdS_L, \tag{47}$$

where $\Omega=S^{n-1}\cap\mathbb{R}^n_-$.

Proof. By scaling, we can assume that the complement of the K,L in $\mathbb{R}^n_+$ is contained in the unit cube Q. Take $\tilde{K},\tilde{L}$ the convex bodies $K\cap Q$ and $L\cap Q$ respectively. In order to estimate the covolume of t_1K+t_2L, we observe that $t_1\tilde{K}\subset t_1Q$ and that $t_2\tilde{L}\subset t_2Q$ so the inclusion $t_1\tilde{K}+t_2\tilde{L}\subset(t_1+t_2)Q$ holds. In particular, we get:

$$\mathrm{coVol}(t_1K+t_2L)=\mathrm{vol}((t_1+t_2)Q)-\mathrm{vol}(t_1\tilde{K}+t_2\tilde{L})=(t_1+t_2)^n-\sum_{k=0}^{n}\binom{n}{k}t_1^kt_2^{n-k}\,\mathrm{vol}(\tilde{K}[k],\tilde{L}[n-k]). \tag{48}$$

Identifying the coefficient in $t_1t_2^{n-1}$ in both sides and using the previous formula on mixed volumes gives:

$$\mathrm{coVol}(K,L[n-1])=1-\frac{1}{n}\int_{S^{n-1}}h_{\tilde{K}}dS_{\tilde{L}}=1-\frac{1}{n}\int_{\Omega}h_KdS_L-\frac{1}{n}\int_{S^{n-1}\setminus\Omega}h_{\tilde{K}}dS_{\tilde{L}}.$$

By construction, the measure $S_{\tilde{L}}$ restricted to the complement of Ω has only atoms at vectors of the canonical basis. More precisely $(S_{\tilde{L}})_{|S^{n-1}\setminus\Omega}=\sum_{i=1}^{n}\delta_{e_i}$. Moreover, one has also $h_{\tilde{K}}(e_i)=1$ for all i. Hence the last term is equal to 1 and we get:

$$\mathrm{coVol}(K,L[n-1])=-\frac{1}{n}\int_{\Omega}h_KdS_L, \tag{49}$$

as required. □

8 Examples and further discussions

8.1 Examples

Using Khovanski's theorem, we can thus compute explicitly all the mixed covolumes in dimension 2 associated with monomial ideals.

Here are a few examples one can compute explicitly.

Proposition 8.1. *Given any monomial* $\mathfrak{p}$ *generated by* x^{α_i} *with* $\alpha_i \in \mathbb{N}^n$ *which is supposed* $\mathfrak{m}$*-primary. One has:*

$$e(\mathfrak{p}, \mathfrak{m}[n-1]) = (n-1)! \min_{i \in I} |\alpha_i|, \tag{50}$$

where $|\alpha_i|$ *denotes the sum of the weights. In particular,* $e(\cdot, \mathfrak{m}[n-1])$ *computes up to a scalar the order of vanishing of an ideal.*

Proof. Observe that the polygon induced by $\mathfrak{m}$ has normal $w = (-1, \ldots, -1)$. The area of this face is $\sqrt{n}$. So we get:

$$e(\mathfrak{p}, \mathfrak{m}[n-1]) = -(n!)\frac{1}{n} \max_i(\langle w, \alpha_i \rangle) = (n-1)! \min_i |\alpha_i| \tag{51}$$

□

8.2 An aparte on ideals in dimension 2

One modern way to prove in dimension 2 the Teissier inequalities can be proved via some formula that can be found in Lazarsfeld's book [Laz04]. We will need some notions of complex geometry.

Given an ideal I supported on $\mathbb{C}^2$ at zero. There exists a blow-up of $\mathbb{P}^2(\mathbb{C})$ and a surface S_I and a birational morphism $\pi_I : S_I \to \mathbb{P}^2$ such that $\pi^* I$ is given by a divisor Z_I (curve in the surface S_I determined by a unique function locally). Similarly if J is another such ideal, we can also find a common resolution of both ideals, so that we get a surface with Z_I, Z_J.

The following formula relates the Hilbert-Samuel multiplicity with the intersection of these curves. Namely:

$$-(Z_I \cdot Z_J) = e(I, J) \tag{52}$$

This intersection product can be viewed as a geometric intersection counted with multiplicity. In fact, since Z_I and Z_J are curves, they correspond to classes in $H^{1,1}(S)$ can the intersection product corresponds to the cup-product.

Here is what we can use.

Theorem 8.2. *(Hodge index theorem) The intersection form* $(\ \cdot\ )$ *has signature* $(1, \dim H^{1,1}(S) - 1)$ *on* S*. Moreover, it is negative definite on the components that are exceptional.*

Corollary 8.3. *One has:*

$$e(I, J)^2 \leqslant e(I, I) e(J, J). \tag{53}$$

Proof. Now since the curves Z_I are pulled back of the equation that are supported over a point, we deduce that Z_I, Z_J are exceptional, hence the intersection form on the span of Z_I, Z_J is negative definite by Hodge index theorem.

We can thus apply the Cauchy-Schwartz inequality:

$$e(I, J)^2 = (Z_I \cdot Z_J)^2 \leqslant (Z_I \cdot Z_I)(Z_J \cdot Z_J) = e(J, J) e(I, I). \tag{54}$$

This gives the required conclusion. - □

8.3 A pseudo-distance

We state the following result due to Gignac-Ruggiero [GR21] which is a geometric reinterpretation of the Teissier inequalities.

Theorem 8.4. *The function* $d(I,J) = \log \frac{e(I,I)e(J,J)}{e(I,J)^2}$ *yields a pseudo-distance in the space of primary ideals with* $e(I) = e(J) = 1$.

Proof. Let us view why this satisfies the triangular inequality. $d(I,K) = \log \frac{e(I,I)e(K,K)}{e(I,K)^2}$ and the sum $d(I,J) + d(J,K) = \log(\frac{e(I,I)e(J,J)^2 e(K,K)}{e(I,J)^2 e(J,K)^2})$

But by the Ruggiero-Ginac inequality, we have:

$$\frac{1}{e(I,K)} \leqslant \frac{e(J,J)}{e(J,I)e(J,K)}, \tag{55}$$

This gives:

$$d(I,K) \leqslant \log(\frac{e(I,I)e(K,K)e(J,J)^2}{e(J,I)^2 e(J,K)^2}) = d(I,J) + d(J,K), \tag{56}$$

as required. □

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