Fermionic semiclassical L^p estimates

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Workshop PDEs and Relativistic Quantum Mechanics
Nice
May 13$^{\mathrm{rd}}$, 2022.

Contents

Talk about $\underline{L^q \text{ estimates}}$ instead of L^p!

Outline

Protagonists :

- Quantum particles (**fermions**) moving in $\mathbb{R}^d$,

$$(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$$

- in force field derived from a potential $V : \mathbb{R}^d \to \mathbb{R}$.
  Crucial example: harmonic oscillator $V(x) = |x|^2 \rightsquigarrow$ **trapped** fermions!

Dean, Le Doussal, Majumdar, Schehr (2019): *Noninteracting fermions in a trap and random matrix theory*, Journal of Physics.

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Description of the discrete spectrum and the associated eigenfunctions of the Schrödinger operator $P_h = -h^2\Delta + V$ (on $L^2(\mathbb{R}^d)$) in the **semiclassical limit** *$h \to 0$?*

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  Trapping potentials s.t. $\lim_{|x| \to \infty} V(x) = +\infty$, yield a discrete spectrum.

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Description of the discrete spectrum and the associated eigenfunctions of the Schrödinger operator $P_h = -h^2\Delta + V$ (on $L^2(\mathbb{R}^d)$) in the ***semiclassical limit*** *$h \to 0$?*

Spectral properties

▷ **What about eigenfunctions?**

Spatial concentration of eigenfunctions of the Schrödinger operators with the associated eigenvalues in an interval I at the limit $h \to 0$?

1. One single particle:

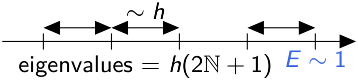
   concentration of $|u_h(x)|^2$?

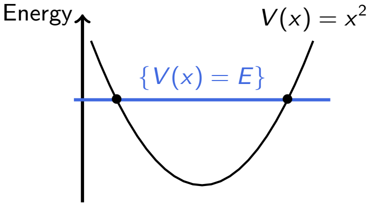
2. A family of free fermions:
   - description by N orthonormal functions $\{u_h^j\}_{1 \le j \le N}$
   - concentration of $\sum_{j=1}^{N} |u_h^j(x)|^2 =: \rho_h(x)$?

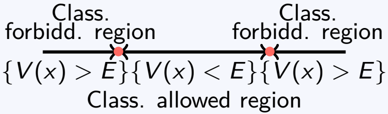
In your dreams: have an asymptotic pointwise description of the eigenfunctions.

Contents

The harmonic oscillator in 1d case







- turning points: $x \in \mathbb{R}$ such that $V(x) = E$ Main difficulty!

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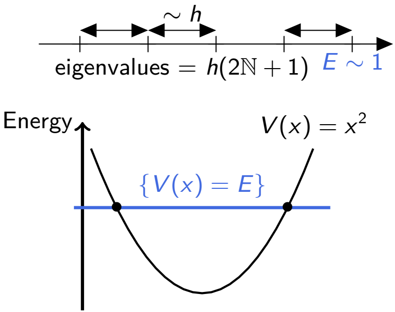

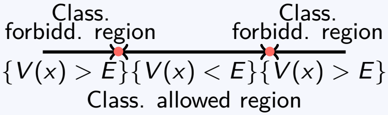

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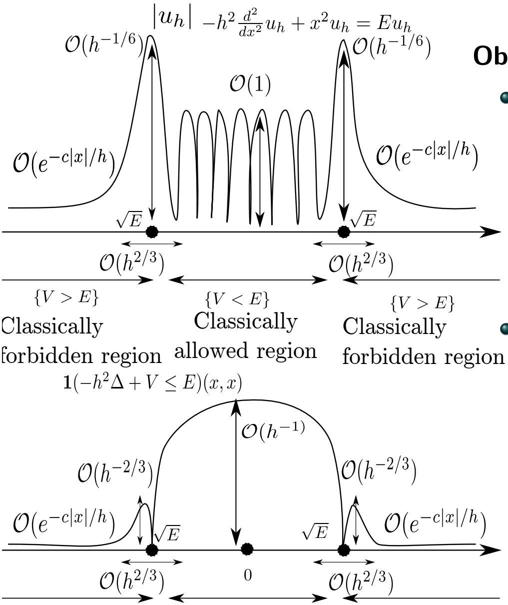
Pointwise description of the eigenfunctions u_h

$$\left(-h^2 \frac{d^2}{dx^2} + x^2\right) u_h = E u_h$$

and of the spectral projector

$$\rho_h(x) = \mathbb{1}\left(-h^2 \frac{d^2}{dx^2} + x^2 \leq E\right)(x,x) = \sum_{j,\ \text{eigenvalues} \leq E} |u_h^j(x)|^2.$$

- Expression with **Hermite polynomials**.
- **BKW approximation** (Brillouin, Kramers, Wentzel).



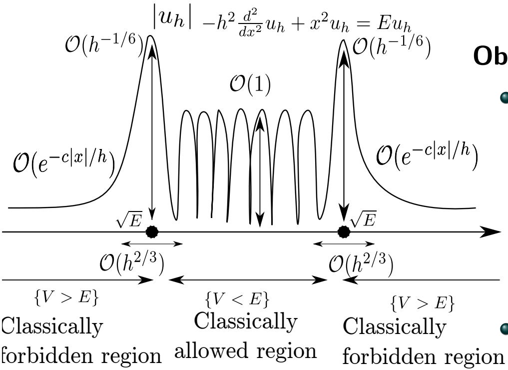
Observations:

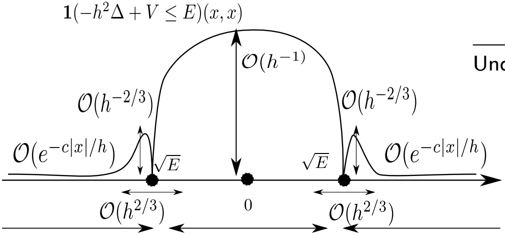
- one-body:

  concentration around turning points

- many-body:

  delocalization





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Understood by the pointwise Weyl law

$$\rho_h(x) \sim_{h\to 0} \frac{|B_{\mathbb{R}^d}(0,1)|}{(2\pi h)^d}(E - V(x))_+^{d/2},$$

for $V(x) = |x|^2$: Karadzhov (1989)

for more general trapping V

Deleporte, Lambert (2021)

Contents

Question: Spatial concentration of eigenfunctions of Schrödinger operators with the associated eigenvalues in an interval I at the limit $h \to 0$?

But the dreams don't always become true for more general potentials.

- ~~Pointwise expression~~
- Indicator : **asymptotic L^p estimates**

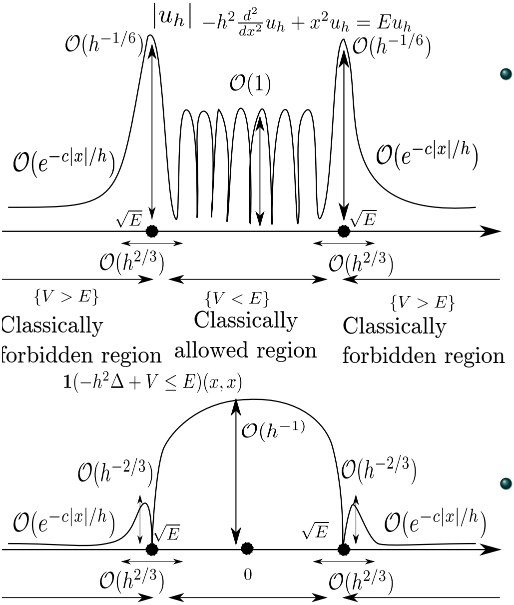
Hope : bounds of the form

$$\forall q \geq 2, \quad \|u_h\|_{L^q(\mathbb{R}^d)} \lesssim \log(1/h)^{-t} h^{-s} \|u_h\|_{L^2(\mathbb{R}^d)}$$

$$\|\rho_h\|_{L^{q/2}(\mathbb{R}^d)} = \|\sum_{1 \leq j \leq N} |u_h^j|^2\|_{L^{q/2}(\mathbb{R}^d)} \lesssim \log(1/h)^{-2t} h^{-2s} N^{1/\alpha}.$$

Have the smallest $s, t \in [0, +\infty]$ and the biggest $\alpha \in [0, +\infty]$.

Back to the $1d$ harmonic oscillator.



- Concentration for an individual eigenfunction

$$\|u_h\|_{L^q(||x|^2-E|\leq h^{2/3})} \lesssim h^{-s(q)} \|u\|_{L^2(\mathbb{R})}$$

with

$$s(q) = \begin{cases} 0 & \text{if } 2 \leq q \leq 4, \\ \frac{1}{6} - \frac{2}{3q} & \text{if } q \geq 4. \end{cases}$$

- Delocalization for the spectral projector :

$$\|\rho_h\|_{L^{q/2}(\mathbb{R})} \lesssim h^{-1} \quad \forall q \geq 2.$$

Review of the results.

Outline	Compact manifolds (without potential)	Euclidian space $\mathbb{R}^d$ (with confining potentials)
One body	Sogge (1986) **Q** A lot of improvements	Koch-Tataru (2005) : harmonic oscillator Koch-Tataru-Zworski (2007) **Q**
Many-body	Frank-Sabin (2017) **SC**	Nguyen (2022) **Q** **SC**

Notations:

- $p(x, \xi) = |\xi|^2 + V(x)$, $P_h = -h^2\Delta + V$,
- Energy $E \in \mathbb{R}$,
- Spectral projector $\Pi_h := \mathbb{1}\left(P_h \in [E - h, E + h]\right)$.

Our hypothesis: Potential $V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ with polynomial growth.

Where?

- For u_h a quasimode of P_h: $(P_h - E)u_h = \mathcal{O}_{L^2}(h)$.
- In a **spectral cluster** of P_h: linear combinations of P_h's' eigenfunctions associated with eigenvalues in an intervals of length $\sim h$.

Density matrices formalism

We indeed prove that

$$(\star)\qquad \left\| \sum_{j=1}^{N_h} \nu_j |u_j|^2 \right\|_{L^{q/2}(\mathbb{R}^d)} \le C h^{-2s(q,d)} \underbrace{\|\nu\|_{l^{\alpha(q,d)}}}_{:=\left(\sum_{1\le j\le N_h}|\nu_j|^\alpha\right)^{1/\alpha}} , \quad \forall \{\nu_j\}_j \subset \mathbb{R}_+$$

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which is the same as

$$(\star') \qquad \|\rho_\gamma\|_{L^{q/2}(\mathbb{R}^d)} \leq C h^{-2s(q,d)} \|\gamma\|_{\mathfrak{S}^{\alpha(q,d)}(L^2(\mathbb{R}^d))}$$

where γ is the compact operator on $L^2(\mathbb{R}^d)$ defined by $\gamma = \sum_{1 \leq j \leq N_h} \nu_j \langle u_j, \cdot \rangle u_j$, ρ_γ is its density and $\|\cdot\|_{\mathfrak{S}^\alpha}(L^2(\mathbb{R}^d))$ is the Schatten norm

$$\|\gamma\|_{\mathfrak{S}^\alpha} := \left(\mathrm{Tr}_{L^2_x} \left((\gamma^* \gamma)^{\alpha/2} \right) \right)^{2/\alpha}.$$

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$$\|\gamma\|_{\mathfrak{S}^\alpha} := \left(\mathrm{Tr}_{L^2_x}\left((\gamma^*\gamma)^{\alpha/2}\right)\right)^{2/\alpha}.$$

Examples:

- if $\gamma \geq 0$, $\|\gamma\|_{\mathfrak{S}^1} = \mathrm{tr}_{L^2}(\gamma)$
- $\|\cdot\|_{\mathfrak{S}^2} =$ Hilbert-Schmidt norm
- $\|\gamma\|_{\mathfrak{S}^\infty} = \|\gamma\|_{L^2\to L^2}$.

Main results

Let $\varepsilon > 0$.
Then, for all $2 \leq q \leq \infty$ and all bounded operator self-adjoint $\gamma \geq 0$ on $L^2(\mathbb{R}^d)$:

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Then, for all $2 \le q \le \infty$ and all bounded operator self-adjoint $\gamma \ge 0$ on $L^2(\mathbb{R}^d)$:
(the more general statement)

$$\left\| \rho_{\Pi_h \gamma \Pi_h} \right\|_{L^{q/2}(\mathbb{R}^d)} \lesssim h^{-2 s_{gene}(q,d)} \log(1/h)^{2 t_{gene}(q,d)} \left\| \gamma \right\|_{\mathfrak{S}^{\alpha_{gene}(q,d)}} .$$

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More precisely:

- In the classically forbidden region

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(x \in \mathbb{R}^d : V(x) - E > \varepsilon)} = \mathcal{O}(h^\infty) \|\gamma\|_{\mathfrak{S}^\infty} .$$

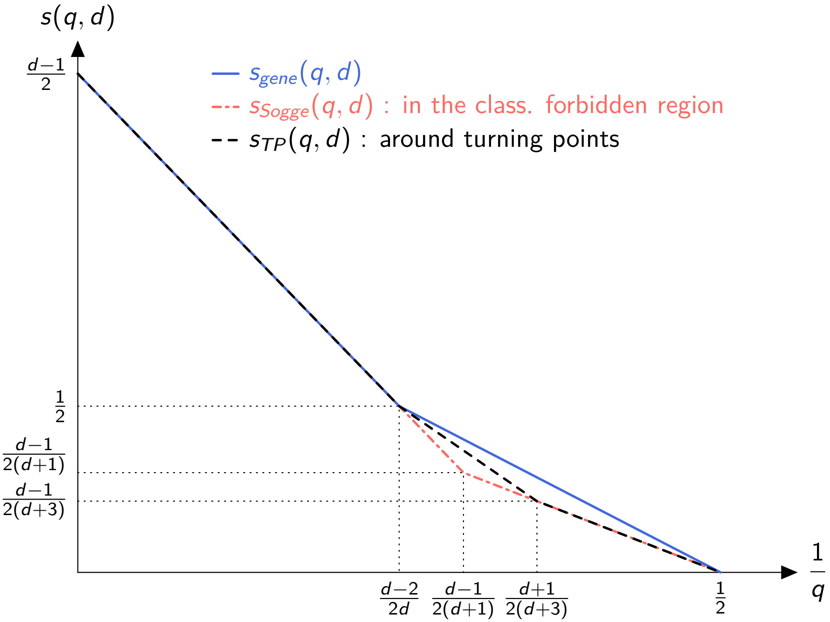
- In the classically allowed region

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(x \in \mathbb{R}^d : V(x) - E < -\varepsilon)} \lesssim h^{-2s_{Sogge}(q,d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{Sogge}(q,d)}} .$$

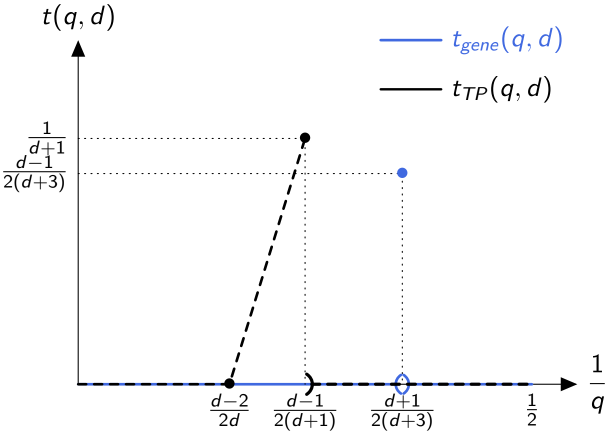
- Near the turning points: if $V(x) = E \implies \nabla_x V(x) \neq 0$

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(x \in \mathbb{R}^d : |V(x) - E| < \varepsilon)} \lesssim h^{-2s_{TP}(q,d)} \log(1/h)^{2t_{TP}(q,d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{TP}}} .$$

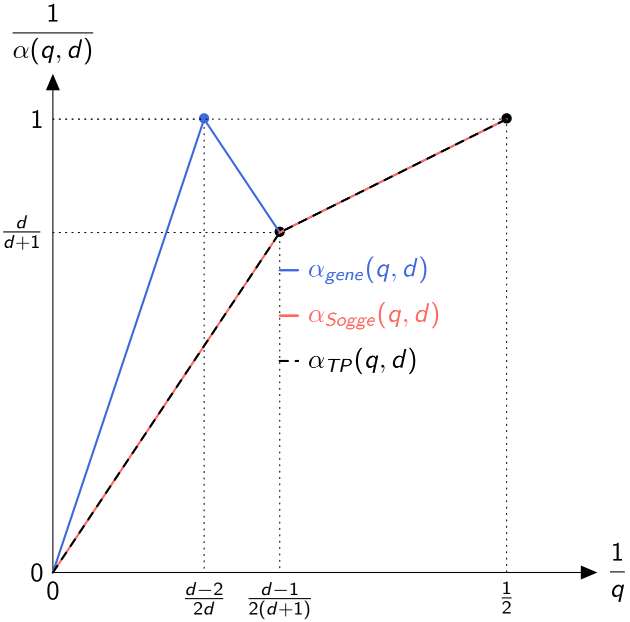
N, Fermionic semiclassical Lp estimates, submitted 2022.



Have the smallest $s(q, d)$.



Have the smallest $t(q,d)$.



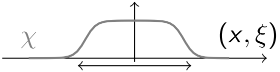
Have the biggest $\alpha(q,d)$.

Contents

Main steps of the proof

(cond) = **(ellip)**, **(gene)**, **(Sogge)** or **(TP)** for $p_E(x,\xi) = |\xi|^2 + V(x) - E$.

✂ Cut the phase space $\mathbb{R}^d \times \mathbb{R}^d$ into several regions by **microlocalization**.



1. **Microlocalized estimates**:
   for (x_0,ξ_0) satisfying **(cond)** and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ supported in its neighborhood

$$\|\rho_{\chi^w\gamma\chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \lesssim h^{-2s_{\mathrm{cond}}(q,d)} \left(\|\gamma\|_{\mathfrak{S}^{\alpha_{\mathrm{cond}}(q,d)}} + \frac{1}{h^2} \|(P_h - E)\gamma(P_h - E)\|_{\mathfrak{S}^{\alpha_{\mathrm{cond}}(q,d)}} \right)$$

2. Deduce with **localization in space**:
   $\Omega \subset \mathbb{R}^d$ and χ s.t. each point of $(\Omega \times \mathbb{R}^d) \cap \operatorname{supp}\chi$ satisfies **(cond)**

$$\|\rho_{\chi^w\gamma\chi^w}\|_{L^{q/2}(\Omega)} \lesssim h^{-2s_{\mathrm{cond}}(q,d)} \left(\|\gamma\|_{\mathfrak{S}^{\alpha_{\mathrm{cond}}(q,d)}} + \frac{1}{h^2} \|(P_h - E)\gamma(P_h - E)\|_{\mathfrak{S}^{\alpha_{\mathrm{cond}}(q,d)}} \right).$$

3. Application to spectrals clusters.

$$\Pi_h = \text{"}\chi^w + \mathcal{O}_{\mathscr{S}' \to \mathscr{S}}(h^\infty)\text{"}.$$

Sub-results

Theorem (Microlocalized estimates)

Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ *satisfying the condition* **(cond)** *for* p *and* E*.*
Then, for $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *supported in a neighborhood of* (x_0, ξ_0)

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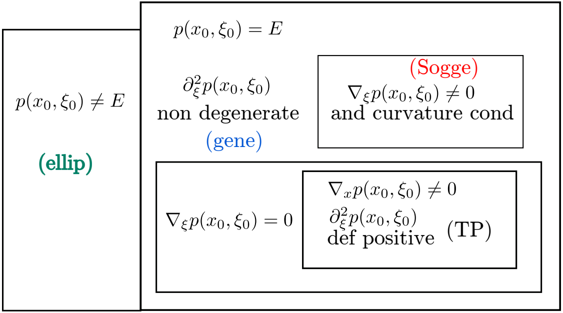
One-body estimates : [Koch-Tataru-Zworski, Semiclassical Lp estimates, 2007]

Many-body estimates: [N, Fermionic semiclassical Lp estimates, submitted 2022].



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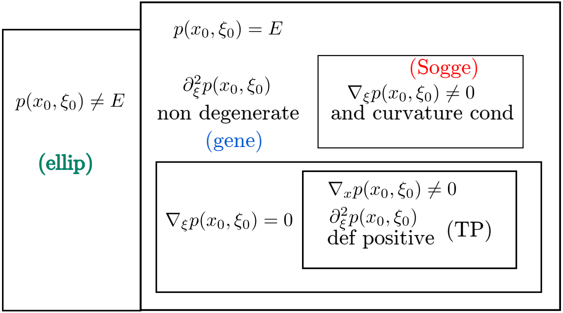
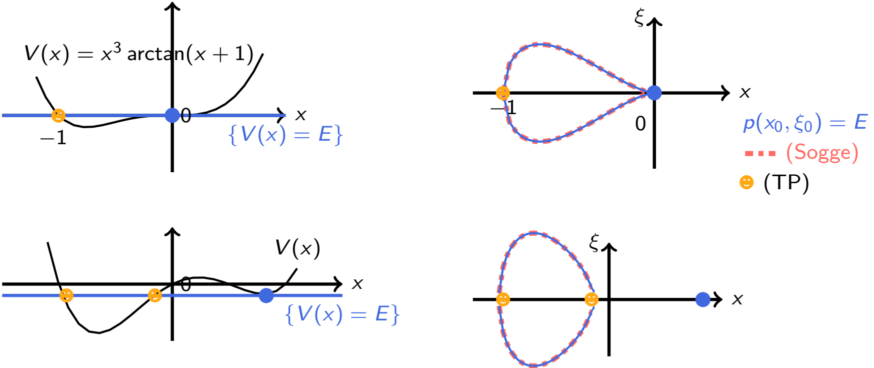


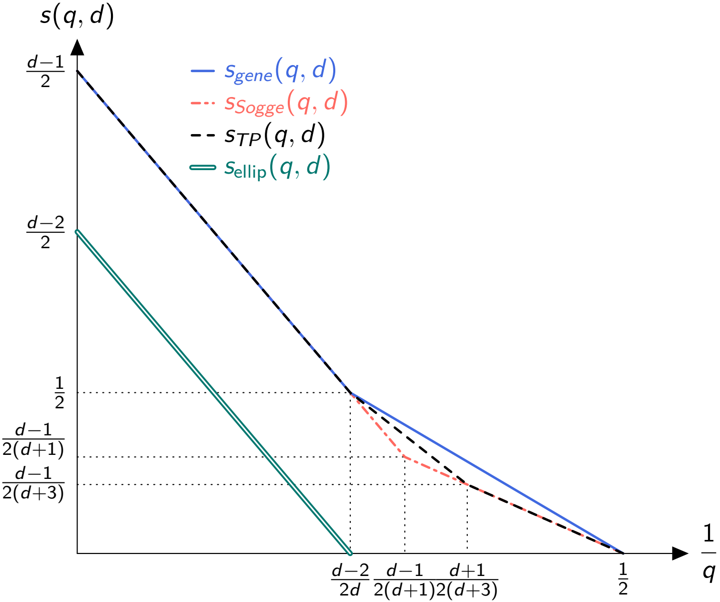
Crucial ingredient: Strichartz estimates.

(gene) $p(x_0,\xi_0)=E$ and $\partial_\xi^2 p(x_0,\xi_0)$ non degenerate:
$|\xi_0|^2+V(x_0)=E$.

(Sogge) $p(x_0,\xi_0)=E \implies \nabla_\xi p(x_0,\xi_0)\neq 0$ + locally curvature condition:
$|\xi_0|^2+V(x_0)=E$ and $\xi_0\neq 0$.

(TP) $p(x_0,\xi_0)=E,\ \nabla_\xi p(x_0,\xi_0)=0,\ \nabla_x p(x_0,\xi_0)\neq 0$ and $\partial_\xi^2 p(x_0,\xi_0)$ pos. def.:
$\xi_0=0,\ V(x_0)=E$ and $\nabla_x V(x_0)\neq 0$.





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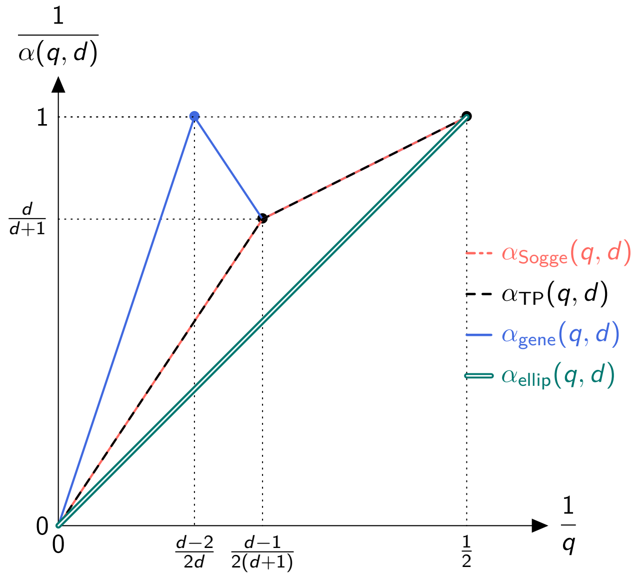

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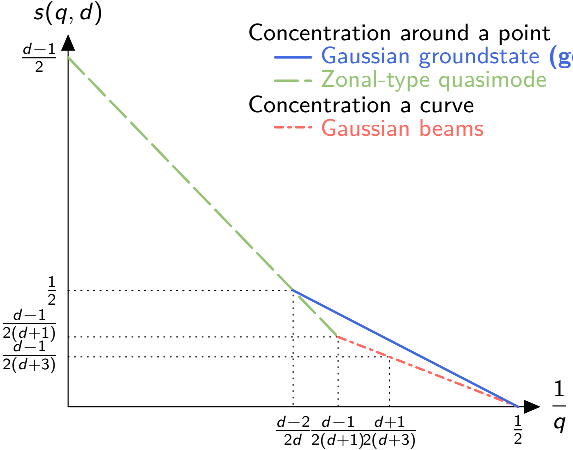
Functions u_h which saturate the inequalities?

$$\|u_h\|_{L^q} \gtrsim h^{-s(q,d)} \left(\|u_h\|_{L^2} + \frac{1}{h} \|(P_h - E)u_h\|_{L^2} \right).$$



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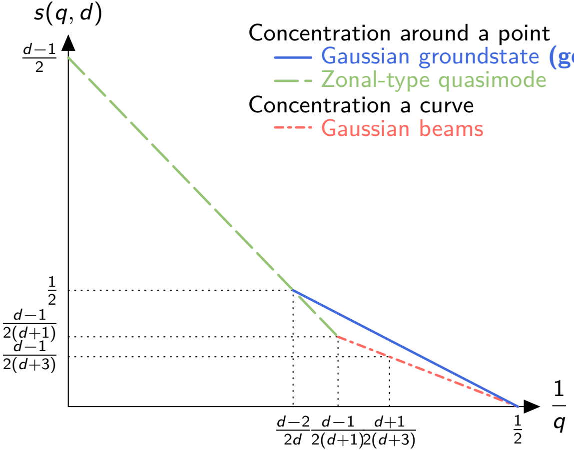

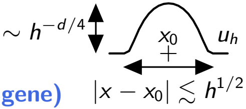

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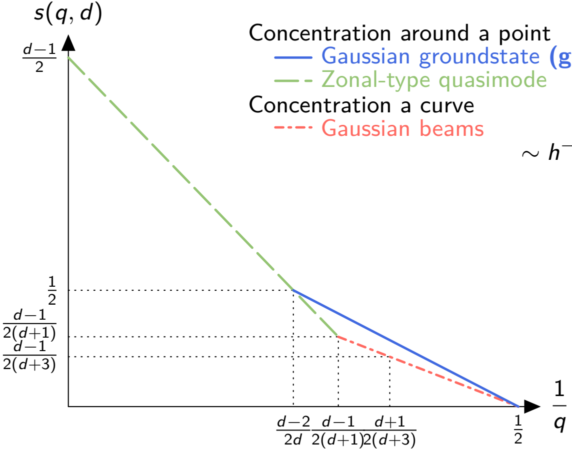

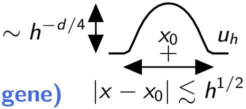

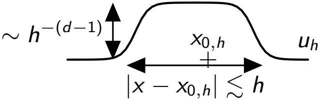

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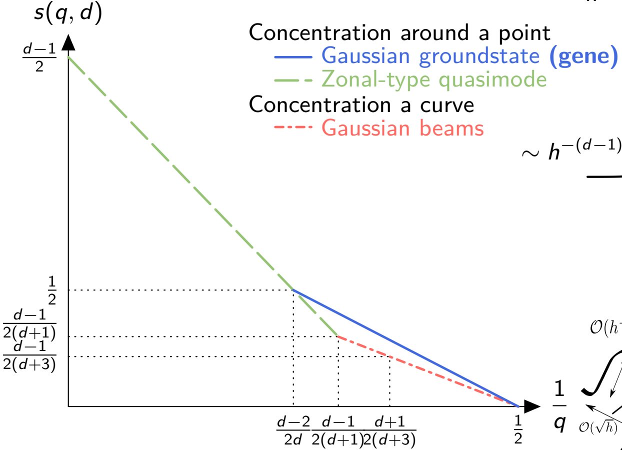

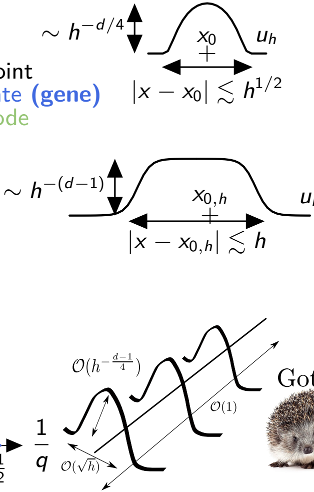

Got it!

Question: What functions u_h and operators γ_h saturate the inequalities?

- ☑ Optimality for individual function?
- ☐ Optimality of orthonormal families of eigenfunctions?

$$\left\|\rho_{\Pi_h \gamma_h \Pi_h}\right\|_{L^{q/2}(\Omega_{V,E})} \gtrsim h^{-2s(q,d)} \left\|\gamma_h\right\|_{\mathfrak{S}^{\alpha(q,d)}} .$$

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Got it!

Optimality of **(Sogge)** with γ_h the spectral projector.

Lemma (In the class. allowed region)

Let $E > \min V$ *such that for any* λ *in a compact neighborhood of* E

$$\lim_{\varepsilon \to 0} \left| \{ x \in \mathbb{R}^d \ : \ |V(x) - \lambda| \le \varepsilon \} \right| = 0.$$

Then, there exist $\{E_{h_n}\}_{n \to \infty}$ *in a compact neighborhood of* E *on* $(\min V, \infty)$ *and* $\varepsilon > 0$

$$\forall 2 \le q \le \infty \quad \left\| \rho_{\Pi_{h_n}} \right\|_{L^{q/2}\left(\{ x \in \mathbb{R}^d \ : \ V(x) \le E_{h_n} - \varepsilon \} \right)} \gtrsim h_n^{-(d-1)}.$$

Have the optimality for **(gene)** and **(TP)** .

Thank you very much for your attention! ☼