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Bouncing Ball Modes and Quantum Chaos*

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Abstract. Quantum ergodicity for classically chaotic systems has been studied extensively both theoretically and experimentally in mathematics and physics. Despite this long tradition we are able to prove a new rigorous result using only elementary calculus. In the case of the famous Bunimovich stadium shown in Figure 1, we prove that the wave functions have to spread to any neighborhood of the wings.

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The quantum/classical correspondence is a puzzling issue that has been with us since the advent of quantum mechanics a hundred years ago. Many aspects of it go back to the Newton–Huyghens debate over the wave vs. corpuscular theories of light.

The simplest description of a wave comes from solving the Helmholtz equation:

$$(-\Delta - \lambda^2)u = 0\,, \quad \Delta = \partial_x^2 + \partial_y^2\,, \quad (x,y) \in \Omega\,, \quad u\restriction_{\partial\Omega} = 0\,.$$

Here we put our wave inside a two-dimensional region Ω. In classical wave mechanics the limit $\lambda \to \infty$ is described using geometrical optics where the waves propagate along straight lines reflecting in the boundary $\partial\Omega$. Roughly speaking, we expect something similar in the classical/quantum correspondence, with the Helmholtz equation replaced by its quantum mechanical version, the Schrödinger equation. For many fascinating illustrations of this we refer the reader to the web art gallery of the Harvard physicist Eric Heller [11].

Many researchers on different aspects of semiclassical analysis have been interested in the correspondence of solutions to the equation above and the classical geometry of balls bouncing from the walls of Ω: Bäcker, Cvitanović, Eckhardt, Gaspard, Heller, Sridhar in physics, and Colin de Verdière, Melrose, Sjöstrand, Zelditch in mathematics, to mention a few (see [1] for references to the physics literature, and [17] for mathematics).

Billiard tables for which the motion is chaotic are a particularly interesting model to study. One of the most famous is the Bunimovich billiard table shown in Figure 1.

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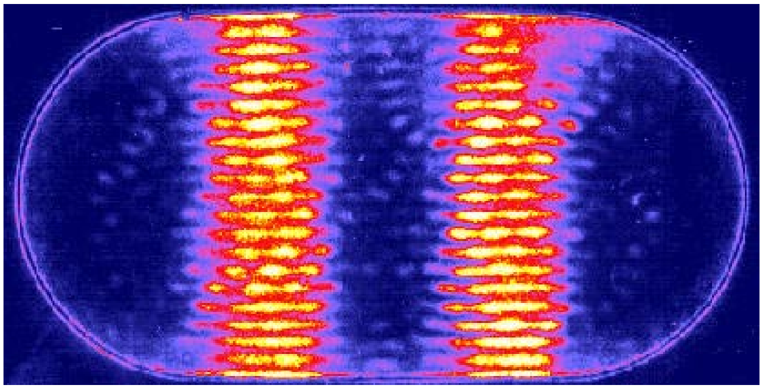

Fig. 1 *An experimental image of bouncing ball modes in a Bunimovich stadium cavity—see* [6] *and http://www.bath.ac.uk/∼pyscmd/acoustics. The bouncing ball modes spread all the way to the boundaries of the rectangular part of the billiard, which is consistent with the results of this paper.*

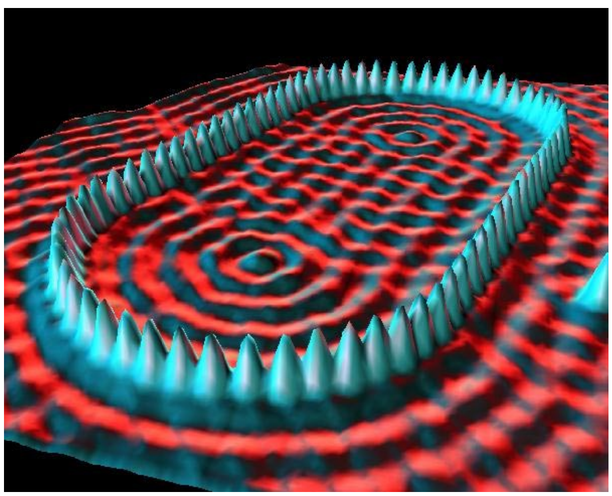

Fig. 2 *Quantum corral in the shape of the Bunimovich stadium. Image reproduced by permission of IBM Research, Almaden Research Center. Unauthorized use not permitted.*

By adding two circular "wings" to a rectangular table, the motion of a reflecting billiard ball becomes chaotic, or more precisely, hyperbolic, in the sense that changes in initial conditions lead to exponentially large changes in motion as time goes on. It is also ergodic in the sense that any flow-invariant subset of the phase space has either full measure or zero measure.

As a model for studying quantum phenomena in chaotic systems this billiard table has become popular in experimental physics. A genuinely quantum example is shown in Figure 2; it comes from the scanning tunneling microscope work of Crommie, Lutz, and Eigler [7].

The basic mathematical result in the theory of quantum chaos is the so-called Shnirelman theorem.

Shnirelman Theorem. *Suppose that the billiard flow on a bounded domain, Ω, with boundary, $\partial\Omega$, is ergodic. Let u_j be the sequence of normalized eigenfunctions of the Dirichlet (or Neumann) Laplacian,*

$$-\Delta u_j = \lambda_j^2 u_j\,, \quad u_j\restriction_{\partial\Omega} = 0\,, \quad \int_\Omega |u_j(x)|^2 dx = 1\,. \tag{1}$$

Then there exists a sequence $\{j_k\}_{k=1}^\infty \subset \mathbb{N}$ of density 1*, that is,* $\lim_{k\to\infty} j_k/k = 1$*, such that for any open subset V of Ω,*

$$\lim_{k\to\infty} \int_V |u_{j_k}(x)|^2 dx = \frac{\mathrm{Area}(V)}{\mathrm{Area}(\Omega)}\,. \tag{2}$$

This means that for almost all eigenfunctions there cannot be any concentration: they have to be uniformly spread out over the billiard table. The integral of the square of the eigenfunction over V is interpreted as the probability of finding the quantum state in V. A stronger version of the theorem gives a phase space version of this statement.

The Shnirelman theorem for billiards was first proved for convex billiards (in particular, the Bunimovich billiard) by Gérard and Leichtnam [10], and for arbitrary manifolds with piecewise smooth boundaries by Zelditch and Zworski [18]. We refer to these papers and to [1], [9], [17] for history and pointers to the literature.

One question still mysterious to mathematicians and physicists alike is whether the quantum states of a classically ergodic system (in our case, solutions of the Helmholtz equation for an ergodic billiard) can concentrate on the highly unstable closed orbits of the classical flow, or on some invariant tori formed by such orbits. The Shnirelman theorem allows the possibility of such concentration on sequences of density zero.

A system is called *quantum unique ergodic* if there is no such concentration—see [17] and references given there. In particular, quantum unique ergodicity means that (2) holds for the *full* sequence of eigenfunctions, that is,

$$\lim_{j\to\infty} \int_V |u_j(x)|^2 dx = \frac{\mathrm{Area}(V)}{\mathrm{Area}(\Omega)}\,. \tag{3}$$

In the arithmetic[1] case investigated and popularized by Sarnak [16], spectacular advances have been recently achieved by Lindenstrauss [13]. On the other hand, for the quantization of the *Arnold cat map*, results showing the *lack* of quantum unique ergodicity were produced by Bonechi and De Bièvre [5] and Faure, Nonnenmacher, and De Bièvre [8].

Here we prove an elementary but striking result which shows that no sequence of eigenfunctions of the Bunimovich stadium can concentrate on a single bouncing ball orbit, or indeed on any smaller phase space set than the union of the bouncing ball orbits of the entire rectangle. It follows from adapting the first author's earlier work in control theory. Although motivated by the more general aspects of [2], we give a simple self-contained proof. Without estimates, which will be given in Theorem′ below, it can be stated as follows.

[1] That is, for billiards given by arithmetic surfaces with hyperbolic metrics, the motion is given by the geodesic flow, and the quantum Hamiltonian is the Laplace–Beltrami operator.

THEOREM. *Let Ω be the Bunimovich billiard table $\Omega = R \cup W$, where R is the rectangular part and W the circular wings. Let V be any open neighborhood of the closure of W in Ω. With the notation of* (1) *we have*

$$\liminf_{j\to\infty} \int_V |u_j(x)|^2 dx > \frac{1}{C_V}\,, \tag{4}$$

where $C_V > 0$ depends only on V.

In particular, the result says that single bouncing ball orbits (that is, orbits following an interval perpendicular to the horizontal straight boundaries) cannot produce localized waves. Our result allows concentration on the full invariant set of all vertical orbits over R, which is consistent with the existing physical literature, both numerical and experimental; see [1] and [6]. In fact, it is expected that there exists a density zero subsequence, $\{p_k\}_{k=1}^{\infty}$, for which

$$\lim_{k\to\infty} \int_R |u_{p_k}(x)|^2 dx = 1 > \frac{\mathrm{Area}(R)}{\mathrm{Area}(\Omega)}\,.$$

Establishing this would constitute a major breakthrough.

In [2] we show a stronger result than (4), namely, that the neighborhood of the wings can be replaced by any neighborhood of the vertical intervals between the wings and the rectangular part. The proof of that predictable (to experts) improvement is, however, no longer elementary.

The proof of the Theorem depends on the following unpublished result of the first author (see [2] for detailed references and background material).

PROPOSITION. *Let $\Delta = \partial_x^2 + \partial_y^2$ be the Laplace operator on the rectangle $R = [0,1]_x \times [0,a]_y$. Then for any open $\omega \subset R$ of the form $\omega = \omega_x \times [0,a]_y$, there exists C such that for any solution of*

$$(-\Delta - \lambda^2)u = f + \partial_x g \quad \text{on } R\,, \quad u\restriction_{\partial R} = 0\,,$$

with an arbitrary $\lambda \geq 0$, we have

$$\int_R |u(x,y)|^2 dxdy \leq C\left(\int_R (|f(x,y)|^2 + |g(x,y)|^2)dxdy + \int_\omega |u(x,y)|^2 dxdy\right). \tag{5}$$

Here, f and g are assumed to be square integrable.

Proof. We decompose u and $f + \partial_x g$ in terms of the basis of $L^2([0,a])$ formed by the Dirichlet eigenfunctions $e_k(y) = \sqrt{2/a}\sin(2k\pi y/a)$,

$$u(x,y) = \sum_k e_k(y)u_k(x), \qquad f(x,y) + \partial_x g(x,y) = \sum_k e_k(y)(f_k(x) + \partial_x g_k(x))\,. \tag{6}$$

We then get the following equations for u_k, f_k:

$$-\left(\partial_x^2 + z\right)u_k = f_k + \partial_x g_k, \qquad u_k(0) = u_k(1) = 0, \quad z = \lambda^2 - (2k\pi/a)^2\,.$$

It is now easy to see that

$$\int_0^1 |u_k(x)|^2 dx \leq C\left(\int_0^1 (|f_k(x)|^2 + |g_k(x)|^2)dx + \int_{\omega_x} |u_k(x)|^2 dx\right), \tag{7}$$

where C is independent of z. In fact, let us first assume that $\omega_x = (0,\delta)$, $\delta > 0$, and $z = \lambda_1^2$, with $\mathrm{Im}\lambda_1 \leq 1$. We then choose $\chi \in \mathcal{C}_c^\infty([0,1])$ identically zero near 0 and identically one on $[\delta/2, 1]$. Then

$$\left(\partial_x^2 + \lambda_1^2\right)(\chi u_k) = F_k\,, \quad F_k = -\left(\chi(f_k + \partial_x g_k) + 2\partial_x\chi\partial_x u_k + \partial_x^2\chi u_k\right).$$

We can now use the explicit solution given by

$$\chi(x)u_k(x) = \frac{1}{\lambda_1}\int_0^x \sin(\lambda_1(x-y))F_k(y)dy.$$

All the terms with $\partial_x g_k$ and $\partial_x u_k$ can be converted to g_k and u_k by integration by parts (with boundary terms 0 at both ends). Due to the λ_1^{-1} factor that produces no loss, the estimate follows. The argument is symmetric under the $x \mapsto -x$ change, so we can place our control interval anywhere.

It remains to discuss the case $z \leq -1 < 0$. Then the estimate (7) follows from integration by parts (where now we do not need ω_x):

$$\begin{aligned}\int_0^1 \left(f(x)\overline{u(x)} - g(x)\overline{\partial_x u(x)}\right)dx &= \int_0^1 (f(x) + \partial_x g(x))\overline{u(x)}dx\\ = \int_0^1 (-\partial_x^2 - z)u(x)\overline{u(x)}dx &= \int_0^1 \left(|\partial_x u(x)|^2 + |z||u(x)|^2\right)dx.\end{aligned}$$

But the Cauchy–Schwarz theorem shows that the left-hand side is bounded from above by

$$\left(\int_0^1 \left(|f(x)|^2 + |g(x)|^2\right)dx\right)^{\frac12}\left(\int_0^1 \left(|u(x)|^2 + |\partial_x u(x)|^2\right)dx\right)^{\frac12}.$$

Since $|z| > 1 > 0$, (7) follows from elementary inequalities (see [2, Lemma 4.1] for a general microlocal argument). We then sum the estimate in k to obtain the proposition. ◻

We can now present a more precise version of the theorem. For a yet finer version we refer the reader to [2, Theorem 3′ and Figure 5].

Theorem′. *Consider Ω, the Bunimovich stadium constructed from a rectangle R and wings W. Let V be any open neighborhood of $\overline{W}$ in Ω. There exists a constant C depending only on Ω and V such that for any solution of the equation*

$$(-\Delta - \lambda^2)v = F\,, \quad u\restriction_{\partial\Omega} = 0\,, \quad \lambda \geq 0,$$

we have

$$\int_\Omega |v(x,y)|^2 dxdy \leq C\left(\int_\Omega |F(x,y)|^2 dxdy + \int_V |v(x,y)|^2 dxdy\right).$$

We apply the second theorem with $F = 0$ to obtain the first one.

Proof. Let us take x, y as the coordinates on the stadium, so that x is the horizontal direction, y the vertical direction, and the internal rectangle is $[0,1]_x \times [0,a]_y$. Let us then consider u and F satisfying $(-\Delta - \lambda^2)u = F$, $u = 0$ on the boundary of the stadium, and $\chi(x) \in \mathcal{C}_c^\infty(0,1)$ equal to 1 on $[\varepsilon, 1-\varepsilon]$. Then $\chi(x)u(x,y)$ is the solution of

$$(-\Delta - \lambda^2)\chi u = \chi F + [\Delta, \chi]u \;\text{ in } R$$

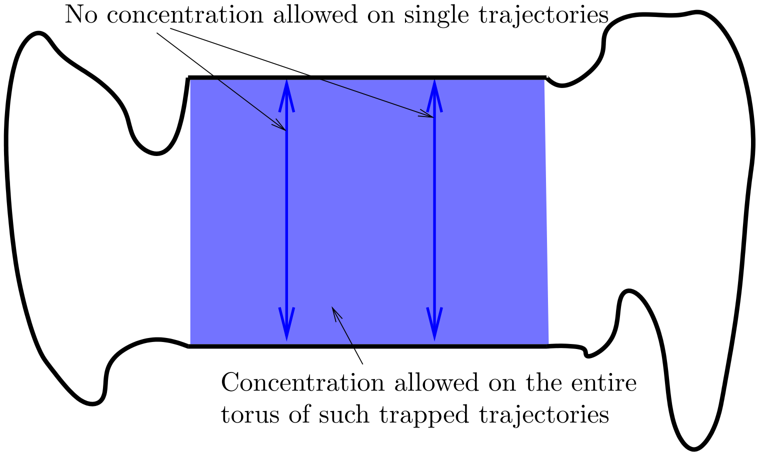

Fig. 3 *A partially rectangular billiard.*

with Dirichlet boundary conditions on ∂R. Since $[\Delta, \chi]u = 2\partial_x(\chi' u) - \chi'' u$ we can apply the proposition with $f = \chi F - \chi'' u$ and $g = 2\chi' u$ to obtain

$$\int_R |\chi(x)u(x,y)|^2 dxdy \leq C \left(\int_R |\chi(x)f(x,y)|^2 dxdy + \int_{\omega_\varepsilon} |u(x,y)|^2 dxdy \right),$$

where ω_ε is a neighborhood of the support of $\partial_x \chi$. Since we can choose it to be contained in $R \cap V$, the theorem follows. $\square$

The reader might notice that the structure of the wings did not play any role in the argument, and hence the result holds for any billiard $\Omega = R \cup W$ as long as $\partial\Omega$ contains a pair of parallel sides of R; see Figure 3. In any such billiard it is easy to construct a weak quasi-mode that is an approximation to an eigenstate. Again let $R = [0,1]_x \times [0,a]_y$ with the sides parallel to the x axis contained in the boundary of $\Omega = R \cup W$. Consider $\phi \in \mathcal{C}_c^\infty((0,1))$, a smooth one-dimensional function vanishing near the boundaries of $[0,1]$, with the property $\int_0^1 |\phi(x)|^2 dx = 1$. Then put

$$v_k(x,y) = e_k(y)\phi(x)\,, \quad e_k(y) = \sqrt{\frac{2}{a}} \sin\left(\frac{2k\pi y}{a}\right),$$

so that

$$(-\Delta - (2\pi k/a)^2)v_k = -e_k(y)\phi''(x) = \mathcal{O}(1)\,, \quad k \to \infty\,. \tag{8}$$

A strong quasi-mode would be defined by (8) with $\mathcal{O}(1)$ replaced by $\mathcal{O}(k^{-N})$, $N \gg 1$. Theorem$'$ shows, however, that the trivial construction of v_k is the best possible: if $\mathcal{O}(1)$ in (8) were replaced by $o(1)$, as $k \to \infty$, we would obtain a contradiction with the estimate (5) once we took ω outside the support of ϕ. The same argument works in the settings of [9] and [17], where similar weak quasi-modes were considered, since the rectangle can be replaced by a torus.

In [3] we show how similar methods imply the control theory result of Jaffard [12] and some new results for Sinai billiards. However, they require slightly more advanced tools.

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