Control and stabilization of Schrödinger and wave linear equations on tori

Understanding the influence of the ambient geometry on the behaviour of solutions to PDE's

Nicolas Burq[1]

Université Paris-Sud, Laboratoire de Mathématiques d'Orsay
UMR 8628 du CNRS

HYP 2016 Aachen, aug. 2nd, 2016

Based on a series of works with M. Zworski, M.Hitrik, M.Zworski-J.Bourgain, C. Zuily and P. Gérard

[1]Work supported in parts by ANR projects NOSEVOL-11-BS01-019-01 and ANAÉ-13-BS01-0010-03.

Control for Schrödinger equations

Control problem on a manifold or a domain M: Fix $T > 0$ and an open set $\omega \subset M$, $a \in L^\infty$ supported in ω. For $u_0, v_0 \in L^2(\mathbb{T}^2)$, does there exists a control $g \in L^2(0, T) \times \omega$ such that the solution to

$$(i\partial_t + \Delta)u = g(x,t)1_{(0,T)}a(x), \qquad u\,|_{t=0} = u_0 (+\text{bdry conditions})$$

satisfies

$$u\,|_{t=T} = v_0$$

Control of waves

Consider the wave equation on a Riemanian manifold M_g, $a \in L^\infty(M), a \geq 0,\ T > 0$

$$(\partial_t^2 - \Delta)u = f \times 1_{(0,T)} \times a(x), \quad (u\mid_{t=0}, \partial_t u\mid_{t=0}) = (u_0, u_1)$$

Given $(u_0, u_1) \in \mathcal{H}^1 = H^1(M) \times L^2(M)$ initial data and $(v_0, v_1) \in \mathcal{H}^1$ target data in energy space, can we choose f in suitable space such that

$$(u\mid_{t=T}, \partial_t u\mid_{t=T}) = (v_0, v_1)?$$

Natural space for f is $L^2((0,T) \times M)$. If answer yes: exact controlability

Stabilization for waves

$$(\partial_t^2 - \Delta + a(x)\partial_t)u = 0,$$
$$(u\mid_{t=0}, \partial_t u\mid_{t=0}) = (u_0, u_1) \in H^1 \times L^2 = \mathcal{H}^1$$

The natural energy is decaying ($a \geq 0$)

$$E(u)(t) = \int_M |\nabla_x u|^2 + |\partial_t u|^2 dx, \frac{d}{dt}E(t) = \int_M -a(x)|\partial_t u|^2 dx$$

Question: speed of decay of $E(u)(t)$?

- The energy of all solutions tend to 0 iff there exists no non trivial stationary equilibrium, i.e.
  $-\Delta e = \lambda^2 e, a \times e = 0 \Rightarrow e = 0.$
- Semi-group property: If there exists a uniform rate $f(t)$,

$$\forall (u_0, u_1) \in \mathcal{H}^1, E(u)(t) \leq f(t)E(u)(0), \lim_{t \to +\infty} f(t) = 0,$$

then can choose $f(t) = Ce^{-ct}$ (uniform) stabilization.

Observation and HUM duality imply equivalence

- There exists a rate $f(t)$ such that $\lim_{t\to+\infty} f(t) = 0$ and

$$\forall (u_0, u_1) \in H^1(M) \times L^2(M), E(u)(t) \leq f(t)E(u)(0).$$

  (and then can choose $f(t) = Ce^{-ct}$)
- $\exists T > 0, c > 0; \forall (u_0, u_1) \in H^1(M) \times L^2(M)$, if u is the solution to the damped wave equation, then

$$E(u)(0) \leq C \int_0^T \int_M 2a(x)|\partial_t u|^2 dxdt.$$

- $\exists T > 0, c > 0; \forall (u_0, u_1) \in H^1(M) \times L^2(M)$, if u is the solution to the undamped wave equation then

$$E(u)(0) \leq C \int_0^T \int_M 2a(x)|\partial_t u|^2 dxdt.$$

- The wave equation is exactly controlable in time T

Observation and HUM duality imply equivalence

- $\exists T > 0, C > 0; \forall (u_0) \in L^2(M)$, if u is the solution to the Schrödinger equation, then

$$\|u_0\|_{L^2}^2 \leq C \int_0^T \int_M a(x)|u|^2(x,t)\,dxdt.$$

- The Schrödinger equation is exactly controlable in time T

The geometric control assumption for waves

$(a \in C^0(M), T)$ controls geometrically (M, g) if every geodesic starting from any point $x_0 \in M$ in any direction ξ_0, $\gamma_{(x_0,\xi_0)}(s)$, encounters $\{a > 0\}$ in time smaller that T

Theorem (Rauch-Taylor, Bardos-Lebeau-Rauch 88', N.B- P.G.)

$a \in C^0(M)$ *geometric control is equivalent to observability (and hence control and stabilization) for wave equations.* $a \in L^\infty(M)$ *Strong Geometric Control is sufficient for observability which implies Weak Geometric Control.*

$$\begin{gathered}\exists T, c > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T), \exists \delta > 0; \\ a \geq c \text{ a.e. on } B(\gamma_{\rho_0}(s), \delta).\end{gathered} \tag{SGCC}$$

$$\exists T > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T); \gamma_{\rho_0}(s) \in \operatorname{supp}(a) \tag{WGCC}$$

$\operatorname{supp}(a)$ is the support (in the distributional sense) of a,

The geometric control assumption

Theorem (Lebeau 92')

$a \in C^0(M)$ geometric control is sufficient for observability (and hence control) for Schrödinger. $a \in L^\infty(M)$ Strong Geometric Control is sufficient for observability.

The geometric control assumption

Theorem (Lebeau 92')

$a \in C^0(M)$ geometric control is sufficient *for observability (and hence control) for Schrödinger. $a \in L^\infty(M)$ Strong Geometric Control is* sufficient *for observability.*

Yes No

Some examples on tori

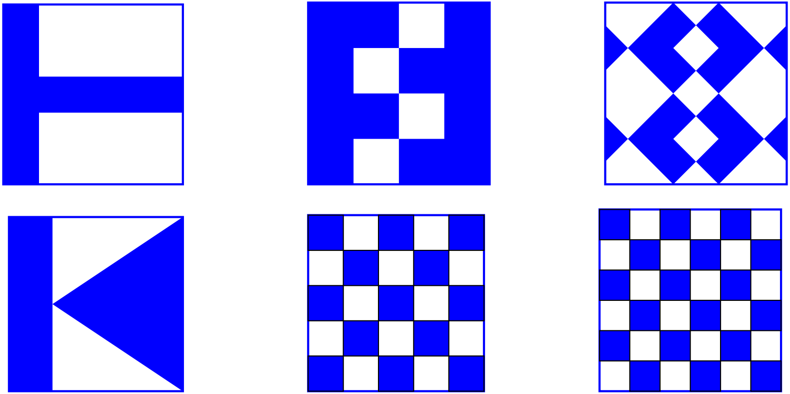


Figure: Checkerboards: the damping a is equal to 1 in the blue region, 0 elsewhere. The geodesics are (periodized) straight lines. The first example satisfies (SGCC) while all others satisfy (WGCC) but not (SGCC)

Going beyond the Strong Geometric Control condition

- For Schrödinger (GCC) Sufficient, not necessary. Give some examples and study their stability
- For wave equations, understand the difference between (SGCC) and (WGCC)

Jaffard's result

Jaffard's theorem states that Schrödinger equation is exactly controlable by any (non empty) space-time open set :

Theorem (Jaffard, 1990)

for any $T > 0$ *and any (non trivial) open set* $\omega \subset \mathbb{T}^2$, *for any* $u_0 \in L^2(\mathbb{T}^2)$, *there exists a control* $g \in L^2(0,T) \times \omega$ *such that the solution to*

$$(i\partial_t + \Delta)u = g(x,t)1_{(0,T)\times\omega}, \qquad u\mid_{t=0} = u_0$$

satisfies $u\mid_{t>T} \equiv 0$

- Stable with respect to dimension (Komornik 90' true on any rational torus)
- Not stable by perturbations of the metric (false on spheres)
- Probably not stable by first order perturbations (Wunsch 2010')
- Stable by zeroth' order perturbations

Control for Schrödinger equations

Theorem (J. Bourgain, N. B., M. Zworski 2012)

Let $\mathbb{T}^2 = (0, 2\pi) \times (0, a), a > 0$ *be any torus. Assume that* $V \in L^2(\mathbb{T}^2)$*. Then, for any* $T > 0$ *and any* $\omega \subset \mathbb{T}^2$*, there exists* $C > 0$ *such that for any* $u_0 \in L^2(\mathbb{T}^2)$*,*

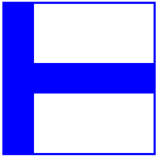
$$\|u_0\|_{L^2}^2 \leq C \int_0^T \|e^{it(\Delta+V)} u_0\|_{L^2(\omega)}^2 dt$$

$V \in L^2(\mathbb{T}^2)$ replaced by

- $V = 0$, rational torus, Jaffard, (1990), lacunary series, Kahane results, harmonic analysis
- $V = 0$, $\mathbb{T}^d$, rational torus Komornik (2005) lacunary series, Kahane results, harmonic analysis
- $\{0\}$ any torus, N.B. M. Zworski (2004), geometric, microlocal proof
- $C^0(\mathbb{T}^2)$ any torus, N.B. M. Zworski (2011)
- $\{V \in L^\infty(\mathbb{T}^d), V$ continuous except on a set of Lebesgue measure $0\}$, rational torus?, Anantharaman-Macia (2011)

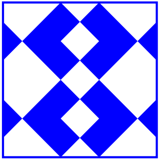
Stabilization for wave equations: the result

Theorem (Does Stabilization holds? $a = 1$ in blue region 0 otherwise)



YES 80'
(Taylor-Rauch)

YES
(NB-PG 16)



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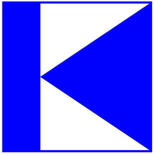

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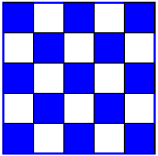

YES
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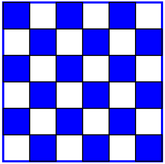

NO
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Another geometric condition

When the manifold is a two dimensional torus and the damping a is a linear combination of characteristic functions of rectangles, i.e. there exists N rectangles (or polygons), $R_j, j = 1, \dots N$ (disjoint and non necessarily vertical), and $0 < a_j, j = 1, \dots, N$ such that

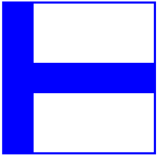
$$a(x) = \sum_{j=1}^{N} a_j 1_{x \in R_j}, \tag{1}$$

then

Theorem (NB–P. Gérard 16)

Stabilization holds for the waves on $\mathbb{T}^2$ iff there exists $T > 0$ such that all geodesics (straight lines) of length T either encounters the interior of one of the rectangles or follows for some time one of the sides of a rectangle R_{j_1} on the left and for some time one of the sides of another (possibly the same) rectangle R_{j_2} on the right.

Stabilization for wave equations: the result



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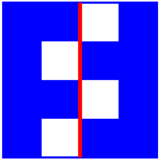

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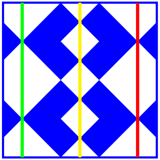

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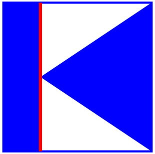

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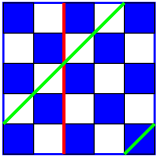
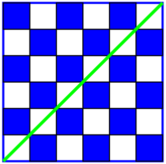


YES



NO

Reduction to resolvent estimate (time Fourier analysis)

Proposition

Stabilization (or control) for waves is equivalent to a resolvent estimate:

$$\exists C > 0; \forall \tau \in \mathbb{R}, \|(-\Delta - \tau^2 + 2ia(x)\tau)^{-1}\|_{\mathcal{L}(L^2)} \leq C(1 + |\tau|)^{-1}$$

$$\Leftrightarrow (-\Delta - \tau^2 + 2ia(x)\tau)u = f \Rightarrow \|u\|_{L^2} \leq C(1 + |\tau|)^{-1}\|f\|_{L^2}$$

and to stationary observation estimate

$$(-\Delta - \tau^2)v = g \Rightarrow \|v\|_{L^2} \leq C(1 + |\tau|)^{-1}\|g\|_{L^2} + C\|\sqrt{a}(x)v\|_{L^2}$$

Control for Schrödinger is equivalent to stationary observation estimate

$$(-\Delta - \lambda)v = g \Rightarrow \|v\|_{L^2} \leq C\|g\|_{L^2} + C\|\sqrt{a}(x)v\|_{L^2}$$

A remark: Jaffard's result implies a weaker estimate, after N.B-M.Hitrik and N. Anantharaman-M.Leautaud

A priori estimate: $(-\Delta - \tau^2 + ia(x)\tau)u = f$, multiply by $\overline{u}$, integrate on $\mathbb{T}^d$ get (for any non trivial $a \geq 0$)

$$\int_{\mathbb{T}^d} |\nabla_x u|^2 - \tau^2 |u|^2 + i\tau a(x)|u|^2 dx = \int_M f\overline{u}.$$

$$\Rightarrow |\tau| \int_M a(x)|u|^2 dx = \operatorname{Im}\big(\int_M f\overline{u}dx\big) \leq \|f\|_{L^2}\|u\|_{L^2}.$$

Apply Schrödinger estimate to

$$(-\Delta - \tau^2)u = f - ia(x)\tau u \Rightarrow \|u\|_{L^2} \leq C(\|f - ia(x)\tau u\|_{L^2} + \|au\|_{L^2}),$$

Gives

$$\|u\|_{L^2} \leq C\|f\|_{L^2} + |\tau|^{1/2}\|f\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2} \Rightarrow \|u\|_{L^2} \leq C(1 + |\tau|)\|f\|_{L^2}.$$

Estimates for $(-\Delta - \tau^2 + 2ia(x)\tau)u = f$,

- Argue by contradiction. If wave estimate not true then there exists sequences $\tau_n \to +\infty$, u_n, f_n, $\quad (\Delta - \tau_n^2)u_n = f_n$, and

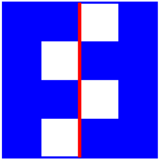
$$1 = \|u_n\|_{L^2} > n(\frac{1}{1+|\tau_n|}\|f_n\|_{L^2} + \|au_n\|_{L^2}) \qquad (2)$$

- $a \not\equiv 0$. Low frequency regime follows from uniqueness of solutions to elliptic second order operators (Carleman estimates):

$$(-\Delta - \tau^2)u = 0 \to \|u\|_{L^2} \leq C_\tau \|ua\|_{L^2}$$

- High frequency regime, describe how mass concentrate in phase space $(X, \Xi) \in \mathbb{T}^2 \times \mathbb{R}^2$ (position and momentum). Natural scales: $X \sim 1$, $\Xi \sim \tau_n$.
- From (2) deduce that no mass accumulate in (the interior of $\{a > 0\}$) and that the mass accumulate along geodesics (propagation properties of microlocal defect measures)
- Get contradiction from the Geometric control property.

Second microlocalization



$$(-\Delta - \tau_n^2)u_n = f_n,$$
$$\|f_n\|_{L^2} = o(|\tau_n|),$$
$$\|au_n\|_{L^2} = o(1)$$
$$\|u_n\|_{L^2} = 1$$

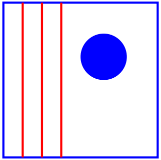
- We know that the total mass of u_n is (asymptotically when $n \to +\infty$) concentrated on the red geodesic. We also know that in some regions there is no mass on the right and on some other regions no mass on the left along this geodesic
- Develop second microlocalization to understand at finer scales how the mass can concentrate on the red geodesic $\{x = 1/2, y \in [0,1) \in \mathbb{T}^2\}$ from left or right. Scales:

$$y \sim 1, \eta \sim \tau_n, x - 1/2 \sim \tau_n^{-\alpha}, \xi \sim \tau_n^{1-\alpha}.$$

Describe concentration at these scales and conclude by contradiction.

A geometric proof of Jaffard's estimate N.B-M.Zworski02', see also Anantharaman-Macia 11'

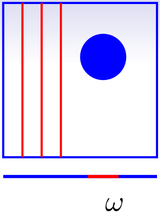
$a = 1$ in blue circle



$$(-\Delta - \tau_n^2)u_n = f_n,$$
$$\|f_n\|_{L^2} = o(1),$$
$$\|au_n\|_{L^2} = o(1)$$
$$\|u_n\|_{L^2} = 1$$

- We know that the total mass of u_n is (asymptotically when $n \to +\infty$) concentrated on geodesics which do not encounter the blue region, hence on periodic geodesics (non periodic geodesics are dense and enter the interior of the blue region). Say one of the red vertical ones
- Write

$$u_n = \sum_k e^{iky} u_{n,k}(x), \qquad -\partial_x^2 - (\tau_n^2 - k^2))u_{n,k} = f_{n,k}.$$



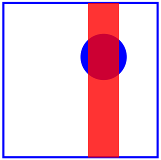
$$(-\partial_x^2 - \tau_n^2 + k^2) u_{n,k} = f_{n,k},$$

The 1-d red region, $\omega \subset (0,1)$ satisfies the SGCC. We deduce

$$\|u_{n,k}\|^2_{L^2(0,1)} \leq C(\|f_{n.k}\|^2_{L^2(0,1)} + \|u_{n,k} 1_\omega\|^2_{L^2(0,1)}),$$

and summing in k

$$\|u_n\|^2_{L^2} \leq C(\|f_n\|^2_{L^2} + \|u_n 1_{\omega \times (0,1)}\|^2_{L^2}) \leq o(1) + C\|u_n 1_\omega\|^2_{L^2}),$$



Propagating again vertically gives the contradiction.