# The water-waves equations: from Zakharov to Euler

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**Abstract** Starting form the Zakharov/Craig-Sulem formulation of the water-waves equations, we prove that one can define a pressure term and hence obtain a solution of the classical Euler equations. It is proved that these results hold in rough domains, under minimal assumptions on the regularity to ensure, in terms of Sobolev spaces, that the solutions are $C^1$.

## 1 Introduction

We study the dynamics of an incompressible layer of inviscid liquid, having constant density, occupying a fluid domain with a free surface.

We begin by describing the fluid domain. Hereafter, $d \geq 1$, $t$ denotes the time variable and $x \in \mathbf{R}^d$ and $y \in \mathbf{R}$ denote the horizontal and vertical variables. We work in a fluid domain with free boundary of the form

$$\Omega = \{ (t,x,y) \in (0,T) \times \mathbf{R}^d \times \mathbf{R} : (x,y) \in \Omega(t) \},$$

where $\Omega(t)$ is the $d+1$-dimensional domain located between two hypersurfaces: a free surface denoted by $\Sigma(t)$ which will be supposed to be a graph and a fixed bottom $\Gamma$. For each time $t$, one has

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$$\Omega(t) = \{(x,y) \in \mathscr{O} : y < \eta(t,x)\},$$

where $\mathscr{O}$ is a given open connected domain and where $\eta$ is the free surface elevation. We denote by $\Sigma$ the free surface:

$$\Sigma = \{(t,x,y) : t \in (0,T), (x,y) \in \Sigma(t)\},$$

where $\Sigma(t) = \{(x,y) \in \mathbf{R}^d \times \mathbf{R} : y = \eta(t,x)\}$ and we set $\Gamma = \partial\Omega(t) \setminus \Sigma(t)$.

Notice that $\Gamma$ does not depend on time. Two classical examples are the case of infinite depth ($\mathscr{O} = \mathbf{R}^{d+1}$ so that $\Gamma = \emptyset$) and the case where the bottom is the graph of a function (this corresponds to the case $\mathscr{O} = \{(x,y) \in \mathbf{R}^d \times \mathbf{R} : y \geq b(x)\}$ for some given function $b$).

We introduce now a condition which ensures that, at time $t$, there exists a fixed strip separating the free surface from the bottom.

$$(H_t): \qquad \exists h > 0: \quad \Gamma \subset \{(x,y) \in \mathbf{R}^d \times \mathbf{R} : y < \eta(t,x) - h\}. \tag{1}$$

eta*

No regularity assumption will be made on the bottom $\Gamma$.

**The incompressible Euler equation with free surface**

Hereafter, we use the following notations

$$\nabla = (\partial_{x_i})_{1\leq i\leq d}, \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \sum_{1\leq i\leq d} \partial_{x_i}^2, \quad \Delta_{x,y} = \Delta + \partial_y^2.$$

The Eulerian velocity field $v\colon \Omega \to \mathbf{R}^{d+1}$ solves the incompressible Euler equation

$$\partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P = -g e_y, \quad \operatorname{div}_{x,y} v = 0 \quad \text{in } \Omega,$$

where $g$ is the acceleration of due to gravity ($g > 0$) and $P$ is the pressure. The problem is then given by three boundary conditions:

- a kinematic condition (which states that the free surface moves with the fluid)

$$\partial_t \eta = \sqrt{1 + |\nabla\eta|^2}\,(v \cdot n) \quad \text{on } \Sigma, \tag{2}$$

  where $n$ is the unitary exterior normal to $\Omega(t)$,
- a dynamic condition (that expresses a balance of forces across the free surface)

$$P = 0 \quad \text{on } \Sigma, \tag{3}$$

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- the ”solid wall” boundary condition at the bottom $\Gamma$

$$v \cdot \nu = 0, \tag{4}$$

where $\nu$ is the normal vector to $\Gamma$ whenever it exists. In the case of arbitrary bottom this condition will be implicit and contained in a variational formulation.

### The Zakharov/Craig-Sulem formulation

A popular form of the water-waves system is given by the Zakharov/Craig-Sulem formulation. This is an elegant formulation of the water-waves equations where all the unknowns are evaluated at the free surface only. Let us recall the derivation of this system.

Assume, furthermore, that the motion of the liquid is irrotational. The velocity field $v$ is therefore given by $v = \nabla_{x,y}\Phi$ for some velocity potential $\Phi\colon \Omega \to \mathbf{R}$ satisfying

$$\Delta_{x,y}\Phi = 0 \quad \text{in } \Omega, \qquad \partial_\nu \Phi = 0 \quad \text{on } \Gamma,$$

and the Bernoulli equation

$$\partial_t \Phi + \frac{1}{2}\left|\nabla_{x,y}\Phi\right|^2 + P + gy = 0 \quad \text{in } \Omega.$$

Following Zakharov [7], introduce the trace of the potential on the free surface:

$$\psi(t,x) = \Phi(t,x,\eta(t,x)).$$

Notice that since $\Phi$ is harmonic, $\eta$ and $\Psi$ fully determines $\Phi$. Craig and Sulem (see [3]) observe that one can form a system of two evolution equations for $\eta$ and $\psi$. To do so, they introduce the Dirichlet-Neumann operator $G(\eta)$ that relates $\psi$ to the normal derivative $\partial_n \Phi$ of the potential by

$$\begin{aligned}(G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\,\partial_n\Phi|_{y=\eta(t,x)} \\ &= (\partial_y\Phi)(t,x,\eta(t,x)) - \nabla_x\eta(t,x)\cdot(\nabla_x\Phi)(t,x,\eta(t,x)).\end{aligned}$$

(For the case with a rough bottom, we recall the precise construction later on). Directly from this definition, one has

$$\partial_t \eta = G(\eta)\psi. \tag{5}$$

It is proved in [3] (see also the computations in §3.5) that the condition $P = 0$ on the free surface implies that

$$\partial_t \psi + g\eta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{\left(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\right)^2}{1+|\nabla\eta|^2} = 0. \tag{6}$$

The system (5)–(6) is in Hamiltonian form (see [3, 7]), where the Hamiltonian is given by

$$\mathscr{H} = \frac{1}{2}\int_{\mathbf{R}^d} \psi G(\eta)\psi + g\eta^2\,dx.$$

The problem to be considered here is that of the equivalence of the previous two formulations of the water-waves problem. Assume that the Zakharov/Craig-Sulem system has been solved. Namely, assume that, for some $r > 1 + d/2$, $(\eta, \psi) \in C^0(I, H^r(\mathbf{R}^d) \times H^r(\mathbf{R}^d))$ solves (5)-(6). We would like to show that we have indeed solved the initial system of Euler equation with free boundary. In particular we have to define the pressure which does not appear in the above system (5)-(6). To do so, we recall from [1] that one can define a unique variational solution to the problem

$$\Delta_{x,y}\Phi = 0 \quad \text{in } \Omega, \qquad \Phi|_{\Sigma} = \Psi, \qquad \partial_\nu \Phi = 0 \quad \text{on } \Gamma.$$

Then we shall prove that the distribution $P \in \mathscr{D}'(\Omega)$ defined by

$$P := -\partial_t \Phi - gy - \frac{1}{2}\left|\nabla_{x,y}\Phi\right|^2$$

has a trace on $\Sigma$ which is equal to 0. This is not straightforward because we are working with solutions of low regularity and we consider general bottoms (namely no regularity assumption is assumed on the bottom). Indeed, the analysis would have been much easier for $r > 2 + d/2$ and a flat bottom.

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## 2 Low regularity Cauchy theory

Since we are interested in low regularity solutions, we begin by recalling the well-posedness results proved in [2]. These results clarify the Cauchy theory of the water waves equations as well in terms of regularity indexes for the initial conditions as for the smoothness of the bottom of the domain (namely no regularity assumption is assumed on the bottom).

Recall that the Zakharov/Craig-Sulem system reads

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{\left(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\right)^2}{1+|\nabla\eta|^2} = 0. \end{cases} \tag{7}$$

It is useful to introduce the vertical and horizontal components of the velocity,

$$B := (v_y)|_{y=\eta} = (\partial_y \Phi)|_{y=\eta}, \quad V := (v_x)|_{y=\eta} = (\nabla_x \Phi)|_{y=\eta}.$$

These can be defined in terms of $\eta$ and $\psi$ by means of the formulas

$$B = \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \qquad V = \nabla\psi - B\nabla\eta. \tag{8}$$

Also, recall that the Taylor coefficient $a = -\partial_y P|_\Sigma$ can be defined in terms of $\eta, V, B, \psi$ only (see §4.3.1 in [5]).

In [2] we proved the following results about low regularity solutions. We refer to the introduction of [2, 4] for references and a short historical survey of the background of this problem.

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**Theorem 1 ([2]).** *Let $d \geq 1$, $s > 1 + d/2$ and consider an initial data $(\eta_0, \psi_0)$ such that*

*$(i)$ $\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $\quad \psi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $\quad V_0 \in H^s(\mathbf{R}^d)$, $\quad B_0 \in H^s(\mathbf{R}^d)$,*

*$(ii)$ the condition $(H_0)$ in (1) holds initially for $t = 0$,*

*$(iii)$ there exists a positive constant $c$ such that, for all $x$ in $\mathbf{R}^d$, $a_0(x) \geq c$.*

*Then there exists $T > 0$ such that the Cauchy problem for (7) with initial data $(\eta_0, \psi_0)$ has a unique solution*

$$(\eta, \psi) \in C^0\big([0,T], H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d)\big),$$

*such that*

*1. $(V,B) \in C^0\big([0,T], H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)\big)$,*
*2. the condition $(H_t)$ in (1) holds for $t \in [0,T]$ with $h$ replaced by $h/2$,*
*3. $a(t,x) \geq c/2$, for all $(t,x)$ in $[0,T] \times \mathbf{R}^d$.*

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**Theorem 2 ([2]).** *Assume $\Gamma = \emptyset$. Let $d = 2$, $s > 1 + \frac{d}{2} - \frac{1}{12}$ and consider an initial data $(\eta_0, \psi_0)$ such that*

$$\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad \psi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad V_0 \in H^s(\mathbf{R}^d), \quad B_0 \in H^s(\mathbf{R}^d).$$

*Then there exists $T > 0$ such that the Cauchy problem for (7) with initial data $(\eta_0, \psi_0)$ has a solution $(\eta, \psi)$ such that*

$$(\eta, \psi, V, B) \in C^0\big([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)\big).$$

*Remark 1.* $(i)$ For the sake of simplicity we stated Theorem 2 in dimension $d = 2$ (recall that $d$ is the dimension of the interface). One can prove such a result in any dimension $d \geq 2$, the number $1/12$ being replaced by an index depending on $d$.

$(ii)$ Notice that in infinite depth ($\Gamma = \emptyset$) the Taylor condition (which is assumption $(iii)$ in Theorem 1) is always satisfied as proved by Wu ([6]).

Now having solved the system (7) in $(\eta, \psi)$ we have to show that we have indeed solved the initial system in $(\eta, v)$. This is the purpose of the following section.

There is one point that should be emphasized concerning the regularity. Below we consider solutions $(\eta, \psi)$ of (7) such that

$$(\eta, \psi) \in C^0\big([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d)),$$

with the only assumption that $s > \frac{1}{2} + \frac{d}{2}$ (and the assumption that there exists $h > 0$ such that the condition $(H_t)$ in (1) holds for $t \in [0,T]$). Consequently, the result proved in this note apply to the settings considered in the above theorems.

# 3 From Zakharov to Euler

## 3.1 The variational theory

In this paragraph the time is fixed so we will skip it and work in a fixed domain $\Omega$ whose top boundary $\Sigma$ is Lipschitz i.e $\eta \in W^{1,\infty}(\mathbf{R}^d)$.

We recall here the variational theory, developed in [1], allowing us to solve the following problem in the case of arbitrary bottom,

$$\Delta \Phi = 0 \quad \text{in } \Omega, \quad \Phi|_\Sigma = \psi, \quad \frac{\partial \Phi}{\partial \nu}|_\Gamma = 0. \tag{9}$$

Notice that $\Omega$ is not necessarily bounded below. We proceed as follows.

Denote by $\mathscr{D}$ the space of functions $u \in C^\infty(\Omega)$ such that $\nabla_{x,y} u \in L^2(\Omega)$ and let $\mathscr{D}_0$ be the subspace of functions $u \in \mathscr{D}$ such that $u$ vanishes near the top boundary $\Sigma$.

**Lemma 1 (see Prop 2.2 in [1]).** *There exist a positive weight $g \in L^\infty_{loc}(\Omega)$ equal to 1 near the top boundary $\Sigma$ of $\Omega$ and $C > 0$ such that for all $u \in \mathscr{D}_0$*

$$\iint_\Omega g(x,y)|u(x,y)|^2 dxdy \leq C \iint_\Omega |\nabla_{x,y} u(x,y)|^2 dxdy. \tag{10}$$

Using this lemma one can prove the following result.

**Proposition 1 (see page 422 in [1]).** *Denote by $H^{1,0}(\Omega)$ the space of functions $u$ on $\Omega$ such that there exists a sequence $(u_n) \subset \mathscr{D}_0$ such that*

$$\nabla_{x,y} u_n \to \nabla_{x,y} u \quad \text{in } L^2(\Omega), \quad u_n \to u \quad \text{in } L^2(\Omega, g dxdy),$$

*endowed with the scalar product*

$$(u,v)_{H^{1,0}(\Omega)} = (\nabla_x u, \nabla_x v)_{L^2(\Omega)} + (\partial_y u, \partial_y v)_{L^2(\Omega)}.$$

*Then $H^{1,0}(\Omega)$ is a Hilbert space and (10) holds for $u \in H^{1,0}(\Omega)$.*

Let $\psi \in H^{\frac{1}{2}}(\mathbf{R}^d)$. One can construct (see below after (19)) $\underline{\psi} \in H^1(\Omega)$ such that

$$\operatorname{supp} \underline{\psi} \subset \{(x,y) : \eta(t,x) - h \leq y \leq \eta(x)\}, \quad \underline{\psi}|_\Sigma = \psi.$$

Using Proposition 1 we deduce that there exists a unique $u \in H^{1,0}(\Omega)$ such that, for all $\theta \in H^{1,0}(\Omega)$,

$$\iint_\Omega \nabla_{x,y} u(x,y) \cdot \nabla_{x,y} \theta(x,y) dxdy = -\iint_\Omega \nabla_{x,y} \underline{\psi}(x,y) \cdot \nabla_{x,y} \theta(x,y) dxdy.$$

Then to solve the problem (9) we set $\Phi = u + \underline{\psi}$.

*Remark 2.* As for the usual Neumann problem the meaning of the third condition in (9) is included in the definition of the space $H^{1,0}(\Omega)$. It can be written as in (9) if the bottom $\Gamma$ is sufficiently smooth.

Let us assume that the Zakharov system (7) has been solved on $I=(0,T)$, which means that we have found

$$(\eta,\psi)\in C^0(\bar I, H^{s+\frac12}(\mathbf{R}^d)\times H^{s+\frac12}(\mathbf{R}^d)),$$

with $s>\frac12+\frac d2$ solution of the system

$$\begin{cases}\partial_t\eta = G(\eta)\psi,\\ \partial\psi = -g\eta-\dfrac12|\nabla\psi|^2+\dfrac12\dfrac{(\nabla\psi\cdot\nabla\eta+G(\eta)\psi)^2}{1+|\nabla\eta|^2}.\end{cases} \tag{11}$$

We set

$$B=\frac{\nabla\psi\cdot\nabla\eta+G(\eta)\psi}{1+|\nabla\eta|^2},\quad V=\nabla\psi-B\nabla\eta. \tag{12}$$

Then $(B,V)\in C^0(I,H^{s-\frac12}(\mathbf{R}^d)\times H^{s-\frac12}(\mathbf{R}^d))$ with $(s-\frac12>\frac d2)$

We would like to show that we have indeed solved the initial system of Euler equation with free boundary. In particular we have to define the pressure which does not appear in the above system. We proceed in several steps.

## 3.2 Straightenning the free boundary

First of all if condition $(H_t)$ is satisfied on $I$, for $T$ small enough, one can find $\eta_*\in L^\infty(\mathbf{R}^d)$ independent of $t$ such that

$$\begin{cases}(i)\quad \nabla_x\eta_*\in H^\infty(\mathbf{R}^d),\quad \|\nabla_x\eta_*\|_{L^\infty(\mathbf{R}^d)}\le C\|\eta\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d))},\\ (ii)\quad \eta(t,x)-h\le\eta_*(x)\le\eta(t,x)-\dfrac h2,\quad \forall(t,x)\in I\times\mathbf{R}^d,\\ (iii)\quad \Gamma\subset\{(x,y)\in\mathcal{O}: y<\eta_*(x)\}.\end{cases} \tag{13}$$

Indeed using the first equation in (11) we have

$$\begin{aligned}\|\eta(t,\cdot)-\eta_0\|_{L^\infty(\mathbf{R}^d)}&\le\int_0^t\|G(\eta)\psi(\sigma,\cdot)\|_{H^{s-\frac12}(\mathbf{R}^d)}\,d\sigma\\ &\le TC\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d)\times H^{s+\frac12}(\mathbf{R}^d))}\big).\end{aligned}$$

Therefore taking $T$ small enough we make $\|\eta(t,\cdot)-\eta_0\|_{L^\infty(\mathbf{R}^d)}$ as small as we want. Then we take $\eta_*(x)=-\frac{2h}{3}+e^{-\delta|D_x|}\eta_0$ and writing

$$\eta_*(x) = -\frac{2h}{3} + \eta(t,x) - (\eta(t,x) - \eta_0(x)) + (e^{-\delta|D_x|}\eta_0 - \eta_0(x)),$$

we obtain (13).

In what follows we shall set

$$\begin{cases} \Omega_1(t) = \{(x,y) : x \in \mathbf{R}^d, \eta_*(x) < y < \eta(t,x)\}, \\ \Omega_1 = \{(t,x,y) : t \in I, (x,y) \in \Omega_1(t)\}, \\ \Omega_2 = \{(x,y) \in \mathscr{O} : y < \eta_*(x)\}, \end{cases} \tag{14}$$

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and

$$\tilde{\Omega}_1 = \{(x,z) : x \in \mathbf{R}^d, z \in (-1,0)\}. \tag{15}$$

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Following Lannes ([4]), for $t \in I$ consider the map $(x,z) \mapsto (x,\rho(t,x,z))$ from $\tilde{\Omega}_1$ to $\mathbf{R}^{d+1}$ defined by

$$\rho(t,x,z) = (1+z)e^{\delta z|D_x|}\eta(t,x) - z\eta_*(x) \tag{16}$$

diffeo

where $\delta$ is chosen such that

$$\delta\|\eta\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d))} := \delta_0 << 1.$$

Notice that since $s > \frac{1}{2} + \frac{d}{2}$, taking $\delta$ small enough and using (13) $(i),(ii)$, we obtain the estimates

$$\begin{aligned} &(i) \quad \partial_z\rho(t,x,z) \geq \frac{h}{3}, \quad \forall (t,x,z) \in I \times \tilde{\Omega}_1, \\ &(ii) \quad \|\nabla_x\rho\|_{L^\infty(I\times\tilde{\Omega}_1)} \leq C(1 + \|\eta\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d))}). \end{aligned} \tag{17}$$

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It follows from (17) $(i)$ that the map $(t,x,z) \mapsto (t,x,\rho(t,x,z))$ is a $C^1$-diffeomorphism from $I \times \tilde{\Omega}_1$ to $\Omega_1$.

We denote by $\kappa$ the inverse map of $\rho$:

$$(t,x,z) \in I \times \tilde{\Omega}_1,\ (t,x,\rho(t,x,z)) = (t,x,y) \Leftrightarrow (t,x,z) = (t,x,\kappa(t,x,y)),\ (t,x,y) \in \Omega_1.$$

### 3.3 The Dirichlet-Neumann operator

Let $\Phi$ be the variational solution described above (with fixed $t$) of the problem

$$\begin{cases} \Delta_{x,y}\Phi = 0 \quad \text{in } \Omega(t), \\ \Phi|_{\Sigma(t)} = \psi(t,\cdot), \\ \partial_\nu\Phi|_\Gamma = 0. \end{cases} \tag{18}$$

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Let us recall that

$$\Phi = u + \underline{\psi} \tag{19}$$
u

where $u \in H^{1,0}(\Omega(t))$ and $\underline{\psi}$ is an extension of $\psi$ to $\Omega(t)$.

Here is a construction of $\underline{\psi}$. Let $\chi \in C^\infty(\mathbf{R}), \chi(a) = 0$ if $a \le -1, \chi(a) = 1$ if $a \ge -\frac{1}{2}$. Let $\underline{\tilde{\psi}}(t,x,z) = \chi(z)e^{z|D_x|}\psi(t,x)$ for $z \le 0$. It is classical that $\underline{\tilde{\psi}} \in L^\infty(I, H^1(\tilde{\Omega}_1))$ if $\psi \in L^\infty(I, H^{\frac{1}{2}}(\mathbf{R}^d))$ and

$$\|\underline{\tilde{\psi}}\|_{L^\infty(I,H^1(\tilde{\Omega}_1))} \le C\|\psi\|_{L^\infty(I,H^{\frac{1}{2}}(\mathbf{R}^d))}.$$

Then we set

$$\underline{\psi}(t,x,y) = \underline{\tilde{\psi}}\big(t,x,\kappa(t,x,y)\big). \tag{20}$$
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Since $\eta \in C^0(I, W^{1,\infty}(\mathbf{R}^d))$ we have $\underline{\psi}(t,\cdot) \in H^1(\Omega(t))$, $\underline{\psi}|_{\Sigma(t)} = \psi$ and

$$\|\underline{\psi}(t,\cdot)\|_{H^1(\Omega(t))} \le C\big(\|\eta\|_{L^\infty(I,W^{1,\infty}(\mathbf{R}^d))}\big)\|\psi\|_{L^\infty(I,H^{\frac{1}{2}}(\mathbf{R}^d))}.$$

Then we define the Dirichlet-Neumann operator by

$$\begin{aligned} G(\eta)\psi(t,x) &= \sqrt{1+|\nabla\eta|^2}\,\partial_n\Phi|_\Sigma \\ &= (\partial_y\Phi)(x,\eta(t,x)) - \nabla_x\eta(t,x)\cdot(\nabla_x\Phi)(t,x,\eta(t,x)). \end{aligned} \tag{21}$$

It has been shown in ABZ3 [2] (see §3) that $G(\eta)\psi$ is well defined in $C^0(\bar{I}, H^{-\frac{1}{2}}(\mathbf{R}^d))$ if $\eta \in C^0(\bar{I}, W^{1,\infty}(\mathbf{R}^d))$ and $\psi \in C^0(\bar{I}, H^{\frac{1}{2}}(\mathbf{R}^d))$.

## *3.4 Preliminaries*

Recall that we have set

$$\Omega(t) = \{(x,y) \in \mathscr{O} : y < \eta(t,x)\}, \quad \Omega = \{(t,x,y) : t \in I, (x,y) \in \Omega(t)\}. \tag{22}$$
omega

For a function $f \in L^1_{loc}(\Omega)$ if $\partial_t f$ denotes its derivative in the sense of distributions we have

$$\langle \partial_t f, \varphi \rangle = \lim_{\varepsilon\to 0}\Big\langle \frac{f(\cdot+\varepsilon,\cdot,\cdot) - f(\cdot,\cdot,\cdot)}{\varepsilon}, \varphi \Big\rangle, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{23}$$
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This point should be clarified due to the particular form of the set $\Omega$ since we have to show that if $(t,x,y) \in \operatorname{supp}\varphi = K$ then $(t+\varepsilon,x,y) \in \Omega$ for $\varepsilon$ sufficiently small independently of the point $(t,x,y)$. This is true. Indeed if $(t,x,y) \in K$ there exists a fixed $\delta > 0$ (depending only on $K,\eta$) such that $y \le \eta(t,x) - \delta$. Since by Zaharov (11)

$$|\eta(t+\varepsilon,x) - \eta(t,x)| \le \varepsilon\|G(\eta)\psi\|_{L^\infty(I\times\mathbf{R}^d)} \le \varepsilon C$$

where $C = C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))}\big)$, we have if $\varepsilon < \frac{\delta}{C}$,

$$y-\eta(t+\varepsilon,x) = y-\eta(t,x)+\eta(t,x)-\eta(t+\varepsilon,x) \leq -\delta+\varepsilon C < 0.$$

Notice that since $\eta \in C^0(\bar{I},H^{s+\frac{1}{2}}(\mathbf{R}^d)), \partial_t\eta = G(\eta)\psi \in C^0(\bar{I},H^{s-\frac{1}{2}}(\mathbf{R}^d))$ and $s > \frac{1}{2}+\frac{d}{2}$ we have $\rho \in C^1(I\times\tilde{\Omega}_1)$.

If $f(t,\cdot)$ is a function defined on $\Omega_1(t)$ we shall denote by $\tilde{f}$ its image by the diffeomorphism $(t,x,z)\mapsto(t,x,\rho(t,x,z))$. Thus we have

$$\tilde{f}(t,x,z) = f(t,x,\rho(t,x,z)) \Leftrightarrow f(t,x,y) = \tilde{f}\big(t,x,\kappa(t,x,y)\big). \tag{24}$$

image

Formally we have the following equalities for $(x,y) = (x,\rho(t,x,z)) \in \Omega_1(t)$ and $\nabla = \nabla_x$

$$\begin{cases} \partial_y f(t,x,y) = \dfrac{1}{\partial_z\rho}\partial_z\tilde{f}(t,x,z) \Leftrightarrow \partial_z\tilde{f}(t,x,z) = \partial_z\rho(t,x,\kappa(t,x,y))\partial_y f(t,x,y), \\ \nabla f(t,x,y) = \big(\nabla\tilde{f} - \dfrac{\nabla\rho}{\partial_z\rho}\,\partial_z\tilde{f}\big)(t,x,z) \Leftrightarrow \nabla\tilde{f}(t,x,z) = \big(\nabla f + \nabla\rho\;\partial_y f\big)(t,x,y), \\ \partial_t f(t,x,y) = \big(\partial_t\tilde{f} + \partial_t\kappa(t,x,y)\partial_z\tilde{f}\big)(t,x,\kappa(t,x,y)). \end{cases} \tag{25}$$

dt

We shall set in what follows

$$\Lambda_1 = \frac{1}{\partial_z\rho}\partial_z, \quad \Lambda_2 = \nabla_x - \frac{\nabla_x\rho}{\partial_z\rho}\partial_z \tag{26}$$

lambda

Eventually recall that if $u$ is the function defined by (19) we have

$$\iint_{\Omega(t)} \nabla_{x,y}u(t,x,y)\cdot\nabla_{x,y}\theta(x,y)dxdy = -\iint_{\Omega(t)} \nabla_{x,y}\underline{\psi}(t,x,y)\cdot\nabla_{x,y}\theta(x,y)dxdy \tag{27}$$

egvar

for all $\theta \in H^{1,0}(\Omega(t))$ which implies that for $t\in I$,

$$\|\nabla_{x,y}u(t,\cdot)\|_{L^2(\Omega(t))} \leq C(\|\eta\|_{L^\infty(I,W^{1,\infty}(\mathbf{R}^d))})\|\psi\|_{L^\infty(I,H^{\frac{1}{2}}(\mathbf{R}^d))}. \tag{28}$$

est

Let $u$ be defined by (19). Since $(\eta,\psi)\in C^0(\bar{I},H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))$ the elliptic regularity theorem proved in [2] (see Theorem 3.16), shows that,

$$\partial_z\tilde{u}, \nabla_x\tilde{u} \in C^0_z([-1,0],H^{s-\frac{1}{2}}(\mathbf{R}^d)) \subset C^0([-1,0]\times\mathbf{R}^d),$$

since $s-\frac{1}{2} > \frac{d}{2}$.

It follows from (25) that $\partial_y u$ and $\nabla_x u$ have a trace on $\Sigma$ and

$$\partial_y u|_\Sigma = \frac{1}{\partial_z\rho(t,x,0)}\partial_z\tilde{u}(t,x,0), \quad \nabla_x u|_\Sigma = \big(\nabla_x\tilde{u} - \frac{\nabla_x\eta}{\partial_z\rho(t,x,0)}\partial_z\tilde{u}\big)(t,x,0).$$

Since $\tilde{u}(t,x,0) = 0$ it follows that

$$\nabla_x u|_\Sigma + (\nabla_x \eta)\partial_y u|_\Sigma = 0$$

from which we deduce, since $\Phi = u + \underline{\psi}$,

$$\nabla_x \Phi|_\Sigma + (\nabla_x \eta)\partial_y \Phi|_\Sigma = \nabla_x \psi. \tag{29}$$

On the other hand one has

$$G(\eta)\psi = \big(\partial_y \Phi - \nabla_x \eta \cdot \nabla_x \Phi\big)|_\Sigma. \tag{30}$$

It follows from (29), (30) and (12) that we have

$$\nabla_x \Phi|_\Sigma = V, \quad \partial_y \Phi|_\Sigma = B. \tag{31}$$

### 3.5 The regularity results

We shall need the following results.

**Lemma 2.** *Let $u$ be defined by (19) and $\tilde{u}$ by (24). Then*

$$\partial_t \nabla_{x,z} \tilde{u} \in L^\infty(I, L^2(\tilde{\Omega}_1)), \quad \partial_t \tilde{u} \in L^\infty(I, L^2(\tilde{\Omega}_1)), \quad \nabla_{x,y} \partial_t u \in L^\infty(I, L^2(\Omega_2)).$$

**Lemma 3.** *In the sense of distributions on $\Omega_1$ we have the chain rule*

$$\partial_t u(t,x,y) = \partial_t \tilde{u}(t,x,\kappa(t,x,y)) + \partial_t \kappa(t,x,y)\partial_z \tilde{u}(t,x,\kappa(t,x,y)).$$

These lemmas are proved in the next paragraph.

Assume for a moment these results proved. Then a classical interpolation Theorem shows that for almost all $t \in I$ we have $\partial_t \tilde{u}(t,\cdot) \in C^0_z([-1,0], H^{\frac{1}{2}}(\mathbf{R}^d))$ so $\partial_t \tilde{u}(t,\cdot)$ has a trace in $H^{\frac{1}{2}}(\mathbf{R}^d)$ on $z = 0$ which is equal to zero since $\tilde{u}(t,x,0) = 0$. Moreover it follows from Lemma 3 and (25) that

$$\partial_t \tilde{u}(t,x,z) = \partial_t u(t,x,\rho(t,x,z)) + \partial_t \rho(t,x,z)\,\partial_y u(t,x,\rho(t,x,z)).$$

It follows that the right hand side has a trace on $z = 0$. Since $\partial_y u|_\Sigma = B - \partial_y \underline{\psi}|_\Sigma$ we deduce that $\partial_t u$ has a trace on $\Sigma$ and using the fact that $\partial_t \rho(t,x,0) = G(\eta)\psi$ we have

$$\partial_t \Phi|_\Sigma = \partial_t u|_\Sigma + \partial_t \underline{\psi}|_\Sigma = -G(\eta)\psi \cdot (B - \partial_y \underline{\psi}|_\Sigma) + \partial_t \underline{\psi}|_\Sigma.$$

Now by an elementary computation on sees that

$$\partial_t \underline{\psi}|_\Sigma - G(\eta)\psi \cdot \partial_y \underline{\psi}|_\Sigma = \partial_t \psi.$$

Therefore,

$$\partial_t \Phi|_\Sigma = \partial_t \psi - G(\eta)\psi \cdot B. \tag{32}$$

Now we introduce the pressure $P \in \mathscr{D}'(\Omega)$ by setting

$$P = -\partial_t \Phi - gy - \frac{1}{2}|\nabla_{x,y}\Phi|^2. \tag{33}$$

It follows then from (32) and (31) that $P$ has a trace on $\Sigma$. We claim that

$$P|_{\Sigma} = 0. \tag{34}$$

Indeed according to (31) and (32) we have

$$P|_{\Sigma} = -\partial_t \psi + BG(\eta)\psi \cdot - g\eta - \frac{1}{2}(|V|^2 + B^2).$$

Now using System (11), (12) and the fact that $G(\eta)\psi = (1+|\nabla\eta|^2)B - \nabla\psi\cdot\nabla\eta$ we can write

$$\begin{aligned}
-\partial_t\psi + BG(\eta)\psi &= g\eta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}(1+|\nabla\eta|^2)B^2 + (1+|\nabla\eta|^2)B^2 - B\nabla\psi\cdot\nabla\eta\\
&= g\eta + \frac{1}{2}|\nabla\psi - B\nabla\eta|^2 - \frac{1}{2}B^2|\nabla\eta|^2 + \frac{1}{2}(1+|\nabla\eta|^2)B^2\\
&= g\eta + \frac{1}{2}(|V|^2 + B^2)
\end{aligned}$$

which proves (34).

Now (33) and (34) show that $v = \nabla_{x,y}\Phi$ solves the Euler system. Moreover we have trivially div $v = 0$, curl $v = 0$ and $\partial_t \eta = \sqrt{1+|\nabla\eta|^2}\, v\cdot n$ on the surface $\Sigma$.

Therefore we are left with the proof of Lemma 2 and Lemma 3.

## 3.6 *Proof of the Lemmas*

### 3.6.1 Proof of Lemma 2

Recall (see (16) and (26)) that we have set

$$\begin{cases}
\rho(t,x,z) = (1+z)e^{\delta z|D_x|}\eta(t,x) - z\eta_*(x),\\
\Lambda_1(t) = \dfrac{1}{\partial_z\rho(t,\cdot)}\partial_z, \quad \Lambda_2(t) = \nabla_x - \dfrac{\nabla_x\rho(t,\cdot)}{\partial_z\rho(t,\cdot)}\partial_z
\end{cases} \tag{35}$$

We shall prove that

$$\begin{cases}
\Lambda_j(t)\partial_t\tilde{u} \in L^\infty\big(I, L^2(\tilde{\Omega}_1, \partial_z\rho(t,\cdot)dxdz)\big),\; j = 1,2,\\
\nabla_{x,y}\partial_t u \in L^\infty(I, L^2(\Omega_2)).
\end{cases} \tag{36}$$

This will imply our claim according to (17) and $\nabla_x\partial_t\tilde{u} = \Lambda_2\tilde{u} - (\nabla_x\rho)\Lambda_1\tilde{u}$.

Now we fix $t_0 \in I$, we take $\varepsilon \in \mathbf{R}\setminus\{0\}$ small enough and we set for $t \in I$

$$\begin{cases} F_1(t) = \iint_{\Omega_1(t)} \nabla_{x,y} u(t,x,y) \cdot \nabla_{x,y} \theta(x,y) dxdy \\ F_2(t) = \iint_{\Omega_2} \nabla_{x,y} u(t,x,y) \cdot \nabla_{x,y} \theta(x,y) dxdy \\ H_1(t) = -\iint_{\Omega_1(t)} \nabla_{x,y} \underline{\psi}(t,x,y) \cdot \nabla_{x,y} \theta(x,y) dxdy \\ H_2(t) = -\iint_{\Omega_2} \nabla_{x,y} \underline{\psi}(t,x,y) \cdot \nabla_{x,y} \theta(x,y) dxdy \end{cases} \tag{37}$$

FH

where

$$\theta(x,y) = \frac{u(t_0+\varepsilon,x,y) - u(t_0,x,y)}{\varepsilon}. \tag{38}$$

theta

It follows from (27) [eqvar] that for all $t \in I$ we have

$$F_1(t) + F_2(t) = H_1(t) + H_2(t). \tag{39}$$

F1+F2

Let us set for $j = 1,2$

$$J_\varepsilon^j(t_0) =: \frac{F_j(t_0+\varepsilon) - F_j(t_0)}{\varepsilon}, \quad \mathscr{J}_\varepsilon^j(t_0) =: \frac{H_j(t_0+\varepsilon) - H_j(t_0)}{\varepsilon}.$$

We begin by estimating $J_\varepsilon^2(t_0)$. Since $\Omega_2$ does not depend on $t$ we have

$$J_\varepsilon^2(t_0) = \iint_{\Omega_2} \frac{\nabla_{x,y} u(t_0+\varepsilon,x,y) - \nabla_{x,y} u(t_0,x,y)}{\varepsilon} \nabla_{x,y} \theta(x,y) dxdy$$

therefore

$$J_\varepsilon^2(t_0) = \left\| \frac{\nabla_{x,y} u(t_0+\varepsilon,x,y) - \nabla_{x,y} u(t_0,x,y)}{\varepsilon} \right\|_{L^2(\Omega_2)}^2. \tag{40}$$

estJ2

It remains to estimate $J_\varepsilon^1(t_0)$. With the notation used in (35) [lambdaj] we have,

$$\Lambda_j(t_0+\varepsilon) - \Lambda_j(t_0) = \beta_{j,\varepsilon}(t_0,x,z)\partial_z \tag{41}$$

champ

and we have

estbeta

**Lemma 4.**

$$\sup_{t_0 \in I} \iint_{\tilde{\Omega}_1} |\beta_{j,\varepsilon}(t_0,x,z)|^2 dxdz \leq \varepsilon^2 C\big(\|(\eta,\psi)\|_{L^\infty(I, H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d))}\big).$$

*Proof.* The most delicate term to deal with is

$$(1) =: \frac{\nabla_x \rho}{\partial_z \rho}(t_0+\varepsilon,x,z) - \frac{\nabla_x \rho}{\partial_z \rho}(t_0,x,z) = \varepsilon \int_0^1 \partial_t \Big(\frac{\nabla_x \rho}{\partial_z \rho}\Big)(t_0+\varepsilon\lambda,x,z) d\lambda.$$

We have

$$\partial_t \Big(\frac{\nabla_x \rho}{\partial_z \rho}\Big) = \frac{\nabla_x \partial_t \rho}{\partial_z \rho} - \frac{(\partial_z \partial_t \rho) \nabla_x \rho}{(\partial_z \rho)^2}.$$

First of all we have $\partial_z \rho \geq \frac{h}{3}$. Now since $s - \frac{1}{2} > \frac{d}{2} \geq \frac{1}{2}$, we can write

$$\begin{aligned}
\|\nabla_x \partial_t \rho(t,\cdot)\|_{L^2(\tilde{\Omega}_1)} &\leq 2\|e^{\delta z|D_x|} G(\eta)\psi(t,\cdot)\|_{L^2((-1,0),H^1(\mathbf{R}^d))}\\
&\leq C\|G(\eta)\psi(t,\cdot)\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} \leq C\|G(\eta)\psi(t,\cdot)\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)}\\
&\leq C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))}\big).
\end{aligned} \tag{42}$$

nablarho

On the other hand we have

$$\begin{aligned}
\|\nabla_x \rho(t,\cdot)\|_{L^\infty(\tilde{\Omega}_1)} &\leq C\|e^{\delta z|D_x|}\nabla_x \eta(x,\cdot)\|_{L^\infty((-1,0),H^{s-\frac{1}{2}}(\mathbf{R}^d)} + \|\nabla_x \eta_*\|_{L^\infty(\mathbf{R}^d)}\\
&\leq C'\|\eta(t,\cdot)\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} + \|\nabla_x \eta_*\|_{L^\infty(\mathbf{R}^d)} \leq C"\|\eta(t,\cdot)\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}
\end{aligned}$$

by (13). Eventually since

$$\partial_z \partial_t \rho = e^{\delta z|D_x|} G(\eta)\psi + (1+z)\delta e^{\delta z|D_x|}|D_x| G(\eta)\psi,$$

we have as in (42)

$$\|\partial_z \partial_t \rho(t,\cdot)\|_{L^2(\tilde{\Omega}_1)} \leq C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))}\big). \tag{43}$$

estrho2

Then the Lemma follows.

Now after making the change of variables defined in (16) we obtain

$$\begin{aligned}
J^1_\varepsilon(t_0) = \frac{1}{\varepsilon}\sum_{j=1}^{2}\iint_{\tilde{\Omega}_1} \big[&\Lambda_j(t_0+\varepsilon)\tilde{u}(t_0+\varepsilon,x,z)\Lambda_j(t_0+\varepsilon)\tilde{\theta}(x,z)\partial_z\rho(t_0+\varepsilon,x,z)\\
&-\Lambda_j(t_0)\tilde{u}(t_0,x,z)\Lambda_j(t_0)\tilde{\theta}(x,z)\partial_z\rho(t_0,x,z)\big]\,dxdz =: \sum_{j=1}^{2} K_{j,\varepsilon}(t_0).
\end{aligned} \tag{44}$$

J=sumK

Thus we can write for $j = 1,2$,

$$\begin{aligned}
K_{j,\varepsilon}(t_0) &= \sum_{k=1}^{4}\iint_{\tilde{\Omega}_1} A^k_{j,\varepsilon}(t_0,x,z)\,dxdz,\\
A^1_{j,\varepsilon}(t_0,\cdot) &= \Lambda_j(t_0)\Big[\frac{\tilde{u}(t_0+\varepsilon,\cdot)-\tilde{u}(t_0,\cdot)}{\varepsilon}\Big]\Lambda_j(t_0)\tilde{\theta}(\cdot)\partial_z\rho(t_0,\cdot),\\
A^2_{j,\varepsilon}(t_0,\cdot) &= \Big[\frac{\Lambda_j(t_0+\varepsilon)-\Lambda_j(t_0)}{\varepsilon}\Big]\tilde{u}(t_0,\cdot)\Lambda_j(t_0)\tilde{\theta}(\cdot)\partial_z\rho(t_0,\cdot),\\
A^3_{j,\varepsilon}(t_0,\cdot) &= \Lambda_j(t_0+\varepsilon)\tilde{u}(t_0+\varepsilon,\cdot)\Big[\frac{\Lambda_j(t_0+\varepsilon)-\Lambda_j(t_0)}{\varepsilon}\Big]\tilde{\theta}(\cdot)\partial_z\rho(t_0,\cdot),\\
A^4_{j,\varepsilon}(t_0,\cdot) &= \Lambda_j(t_0+\varepsilon)\tilde{u}(t_0+\varepsilon,\cdot)\Lambda_j(t_0+\varepsilon)\tilde{\theta}(\cdot)\Big[\frac{\partial_z\rho(t_0+\varepsilon,\cdot)-\partial_z\rho(t_0,\cdot)}{\varepsilon}\Big].
\end{aligned} \tag{45}$$

J1=

In what follows to simplify the notations we shall set $X = (x,z) \in \tilde{\Omega}_1$.

First of all, using (38) and the lower bound $\partial_z \rho(t_0, X) \geq \frac{h}{3}$, we obtain

$$\iint_{\tilde{\Omega}_1} A^1_{j,\varepsilon}(t_0, X)dX \geq \frac{h}{3} \left\| \Lambda_j(t_0) \Big[ \frac{\tilde{u}(t_0+\varepsilon,\cdot) - \tilde{u}(t_0,\cdot)}{\varepsilon} \Big] \right\|^2_{L^2(\tilde{\Omega}_1)}. \tag{46}$$

A1

Now it follows from (41) that

$$\left| \iint_{\tilde{\Omega}_1} A^2_{j,\varepsilon}(t_0, X)dX \right| \leq \sup_{t\in I} \| \frac{\beta_\varepsilon}{\varepsilon} \|_{L^2(\tilde{\Omega}_1)} \sup_{t\in I} \| \partial_z \tilde{u}(t,\cdot) \|_{L^\infty_z(-1,0,L^\infty(\mathbf{R}^d))} \| \Lambda_j(t_0)\tilde{\theta} \|_{L^2(\tilde{\Omega}_1)}.$$

Since $s - \frac{1}{2} > \frac{d}{2}$ the elliptic regularity theorem shows that

$$\begin{aligned} \sup_{t\in I} \| \partial_z \tilde{u}(t,\cdot) \|_{L^\infty_z(-1,0,L^\infty(\mathbf{R}^d))} &\leq \sup_{t\in I} \| \partial_z \tilde{u}(t,\cdot) \|_{L^\infty_z(-1,0,H^{s-\frac{1}{2}}(\mathbf{R}^d))} \\ &\leq C\big( \|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))} \big) \end{aligned} \tag{47}$$

regtheo

Using Lemma 4 we deduce that

$$\left| \iint_{\tilde{\Omega}_1} A^2_{j,\varepsilon}(t_0, X)dX \right| \leq C\big( \|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))} \big) \| \Lambda_j(t_0)\tilde{\theta} \|_{L^2(\tilde{\Omega}_1)}. \tag{48}$$

A2

Now write

$$\begin{aligned} &\iint_{\tilde{\Omega}_1} A^3_{j,\varepsilon}(t_0, X)dX = \\ &\qquad = \iint_{\tilde{\Omega}_1} \Lambda_j(t_0+\varepsilon)\tilde{u}(t_0+\varepsilon, X)\beta_\varepsilon(t_0+\varepsilon, X)\partial_z\tilde{\theta}(t_0,X)\partial_z\rho(t_0,X)dX. \end{aligned}$$

By elliptic regularity $\Lambda_j(t)\tilde{u}$ is bounded in $L^\infty_{t,x,z}$ by a constant depending only on $\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))}$. Therefore we can write

$$\left| \iint_{\tilde{\Omega}_1} A^3_{j,\varepsilon}(t_0, X)dX \right| \leq C\big( \|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))} \big) \| \partial_z\tilde{\theta} \|_{L^2(\tilde{\Omega}_1)}. \tag{49}$$

A3

Eventually since

$$\frac{\partial_z\rho(t_0+\varepsilon,x,z) - \partial_z\rho(t_0,x,z)}{\varepsilon} = \int_0^1 \partial_t\partial_z\rho(t_0+\lambda\varepsilon,x,z)d\lambda,$$

we find using (47) and (43)

$$\left| \iint_{\tilde{\Omega}_1} A^4_{j,\varepsilon}(t_0, X)dX \right| \leq C\big( \|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))} \big) \| \partial_z\tilde{\theta} \|_{L^2(\tilde{\Omega}_1)}. \tag{50}$$

A4

Let us consider now the right hand side of (39). Due to the presence of the cut-off in the expression of $\underline{\psi}$ we have $H_2(t) = 0$. Now we consider

$$\mathscr{J}_\varepsilon^1 = \frac{H_1(t_0+\varepsilon) - H_1(t_0)}{\varepsilon}. \tag{51}$$

estdH

We make the change of variable $(x,z) \to (x, \rho(t_0,x,z))$ in the integral and we decompose the new integral as in (44), (45). This gives, with $X = (x,z)$,

$$\mathscr{J}_\varepsilon^1 = \sum_{j=1}^{2} \mathscr{K}_{j,\varepsilon}(t_0), \quad \mathscr{K}_{j,\varepsilon}(t_0) = \sum_{k=1}^{4} \iint_{\tilde{\Omega}_1} \mathscr{A}_{j,\varepsilon}^k(t_0,X)\,dX,$$

where $\mathscr{A}_{j,\varepsilon}^k$ has the same form as $-A_{j,\varepsilon}^k$ in (45) except the fact that $\tilde{u}$ is replaced by $\underline{\tilde{\psi}}$. Recall that $\underline{\tilde{\psi}}(t,x,z) = \chi(z) e^{z|D_x|}\psi(t,x)$. Now we have

$$\begin{aligned}
\|\Lambda_j \partial_t \underline{\tilde{\psi}}\|_{L^\infty(I,L^2(\tilde{\Omega}_1))} &\le C\big(\|\eta\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d))}\big)\|\partial_t \underline{\tilde{\psi}}\|_{L^\infty(I,L^2_z((-1,0),H^1(\mathbf{R}^d)))}\\
&\le C\big(\|\eta\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d))}\big)\|\partial_t \psi\|_{L^\infty(I,H^{\frac12}(\mathbf{R}^d))}\\
&\le C\big(\|\eta\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d))}\big)\|\partial_t \psi\|_{L^\infty(I,H^{s-\frac12}(\mathbf{R}^d))}
\end{aligned}$$

since $s - \frac12 \ge \frac12$. Using the equation (11) on $\psi$, and the fact that $H^{s-\frac12}(\mathbf{R}^d)$ is an algebra we obtain

$$\|\Lambda_j \partial_t \underline{\tilde{\psi}}\|_{L^\infty(I,L^2(\tilde{\Omega}_1))} \le C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d))\times H^{s+\frac12}(\mathbf{R}^d))}\big).$$

It follows that we have

$$\left|\iint_{\tilde{\Omega}_1} \mathscr{A}_{j,\varepsilon}^1(t_0,X)\,dX\right| \le C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d)\times H^{s+\frac12}(\mathbf{R}^d))}\big)\|\Lambda_j(t_0)\tilde{\theta}\|_{L^2(\tilde{\Omega}_1)}. \tag{52}$$

estAA1

Now since

$$\begin{aligned}
\|\Lambda_j(t_0)\underline{\tilde{\psi}}(t,\cdot)\|_{L^2_z((-1,0),L^\infty(\mathbf{R}^d))} &\le C(\|\eta\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d))})\|\underline{\tilde{\psi}}(t_0,\cdot)\|_{L^2_z((-1,0),H^{\frac{d}{2}+\varepsilon}(\mathbf{R}^d))}\\
&\le C(\|\eta\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d))})\|\psi\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d))}
\end{aligned}$$

we can use the same estimates as in (48), (49), (50) to bound the terms $\mathscr{A}_{j,\varepsilon}^k$ for $k = 2,3,4$. We obtain finally

$$\left|\frac{H_1(t_0+\varepsilon)-H_1(t_0)}{\varepsilon}\right| \le C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d)\times H^{s+\frac12}(\mathbf{R}^d))}\big)\sum_{j=1}^{2}\|\Lambda_j(t_0)\tilde{\theta}\|_{L^2(\tilde{\Omega}_1)}. \tag{53}$$

H

Summing up using (40), (45), (46), (48),(49),(50), (53) we find that setting

$$U_\varepsilon(t_0,\cdot) = \frac{u(t_0+\varepsilon,\cdot)-u(t_0,\cdot)}{\varepsilon}, \quad \tilde{U}_\varepsilon(t_0,\cdot) = \frac{\tilde{u}(t_0+\varepsilon,\cdot)-\tilde{u}(t_0,\cdot)}{\varepsilon}$$

the quantity

$$\sum_{j=1}^{2} \left\| \Lambda_j(t_0)\tilde{U}_\varepsilon(t_0,\cdot)\right\|_{L^2(\tilde{\Omega}_1)} + \left\|\nabla_{x,y}U_\varepsilon(t_0,\cdot)\right\|_{L^2(\Omega_2)} \tag{54}$$

is uniformly bounded (with respect to $t_0$ and $\varepsilon$) by a constant depending only on $\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d)\times H^{s+\frac12}(\mathbf{R}^d))}$.

Since $\tilde{u}(t,x,0)=0$, using (54) and the Poincaré inequality in $\tilde{\Omega}_1$ we find that

$$\left\|\tilde{U}_\varepsilon(t_0,\cdot)\right\|_{L^2(\tilde{\Omega}_1)} \leq C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac12}(\mathbf{R}^d)\times H^{s+\frac12}(\mathbf{R}^d))}\big). \tag{55}$$

It follows that we can extract a subsequence such that

$$\begin{cases} \big(\Lambda_j(t_0)\tilde{U}_{\varepsilon_k}\big) \text{ converges in the weak-star topology of } L^\infty(I,L^2(\tilde{\Omega}_1)),\\ \big(U_{\varepsilon_k}\big) \text{ converges in the weak-star topology of } L^\infty(I,L^2(\tilde{\Omega}_1)).\\ \big(\nabla_{x,y}U_{\varepsilon_k}\big) \text{ converges in the weak-star topology of } L^\infty(I,L^2(\Omega_2)).\end{cases}$$

But these sequences converge in $\mathscr{D}'(I\times\tilde{\Omega}_1)$ (resp.$\mathscr{D}'(I\times\Omega_2)$) to $\Lambda_j(t_0)\partial_t\tilde{u},\partial_t\tilde{u}$ (resp.$\nabla_{x,y}\partial_t u$), Then $\Lambda_j(t_0)\partial_t\tilde{u}\in L^\infty(I,L^2(\tilde{\Omega}_1)),\partial_t\tilde{u}\in L^\infty(I,L^2(\tilde{\Omega}_1))$ and $\nabla_{x,y}\partial_t u\in L^\infty(I,L^2(\Omega_2))$ which completes the proof of (36).

#### 3.6.2 Proof of Lemma 3

Let $\varphi\in C_0^\infty(\Omega_1)$ and set

$$\begin{aligned} &v_\varepsilon(t,x,y)=\frac{1}{\varepsilon}[\tilde{u}(t+\varepsilon,x,\kappa(t+\varepsilon,x,y))-\tilde{u}(t+\varepsilon,x,\kappa(t,x,y))],\\ &w_\varepsilon(t,x,y)=\frac{1}{\varepsilon}[\tilde{u}(t+\varepsilon,x,\kappa(t,x,y))-\tilde{u}(t,x,\kappa(t,x,y))],\\ &J_\varepsilon=\iint_{\Omega_1} v_\varepsilon(t,x,y)\varphi(t,x,y)dtdxdy,\quad K_\varepsilon=\iint_{\Omega_1} w_\varepsilon(t,x,y)\varphi(t,x,y)dtdxdy,\\ &I_\varepsilon=J_\varepsilon+K_\varepsilon.\end{aligned} \tag{56}$$

Let us consider first $K_\varepsilon$. In the integral in $y$ we make the change of variable $\kappa(t,x,y)=z\Leftrightarrow y=\rho(t,x,z)$. Then setting $\tilde{\varphi}(t,x,z)=\varphi(t,x,\rho(t,x,z))$ and $X=(x,z)\in\tilde{\Omega}_1$ we obtain

$$K_\varepsilon=\iint_I\int_{\tilde{\Omega}_1}\frac{\tilde{u}(t+\varepsilon,X)-\tilde{u}(t,X)}{\varepsilon}\tilde{\varphi}(t,X)\partial_z\rho(t,X)dtdX.$$

Since $\rho\in C^1(I\times\tilde{\Omega}_1)$ we have $\tilde{\varphi}\cdot\partial_z\rho\in C_0^0(I\times\tilde{\Omega}_1)$. Now we know that the sequence $\tilde{U}_\varepsilon=\frac{\tilde{u}(\cdot+\varepsilon,\cdot)-\tilde{u}(\cdot,\cdot)}{\varepsilon}$ converges in $\mathscr{D}'(I\times\tilde{\Omega}_1)$ to $\partial_t\tilde{u}$. We use this fact, we approximate $\tilde{\varphi}\cdot\partial_z\rho$ by a sequence in $C_0^\infty(I\times\tilde{\Omega}_1)$ and we use (55) to deduce that

$$\lim_{\varepsilon\to 0} K_\varepsilon = \int_I \iint_{\tilde{\Omega}_1} \partial_t \tilde{u}(t,X)\tilde{\varphi}(t,X)\partial_z\rho(t,X)\,dtdX.$$

Coming back to the $(t,x,y)$ variables we obtain

$$\lim_{\varepsilon\to 0} K_\varepsilon = \iint_{\Omega_1} \partial_t \tilde{u}(t,x,\kappa(t,x,y))\varphi(t,x,y)\,dtdxdy. \tag{57}$$

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Let us look now to $J_\varepsilon$. We cut it into two integrals; in the first we set $\kappa(t+\varepsilon,x,y)=z$ in the second we set $\kappa(t,x,y)=z$. With $X=(x,z)\in\tilde{\Omega}_1$ we obtain

$$J_\varepsilon = \frac{1}{\varepsilon}\int_I \iint_{\tilde{\Omega}_1} \tilde{u}(t+\varepsilon,X)\Big(\int_0^1 \frac{d}{d\sigma}\big\{\varphi(t,x,\rho(t+\varepsilon\sigma,X))\partial_z\rho(t+\varepsilon\sigma,X)\big\}d\sigma\Big)dtdX.$$

Differentiating with respect to $\sigma$ we see easily that

$$J_\varepsilon = \int_I \iint_{\tilde{\Omega}_1} \tilde{u}(t+\varepsilon,X)\frac{\partial}{\partial z}\Big(\int_0^1 \partial_t\rho(t+\varepsilon\sigma,X)\varphi(t,x,\rho(t+\varepsilon\sigma,X))d\sigma\Big)dtdX.$$

Since $\tilde{u}$ is continuous in $t$ with values in $L^2(\tilde{\Omega}_1), \partial_t\rho$ is continous in $(t,x,z)$ and $\varphi\in C_0^\infty$ we can pass to the limit and we obtain

$$\lim_{\varepsilon\to 0} J_\varepsilon = \int_I \iint_{\tilde{\Omega}_1} \tilde{u}(t,X)\frac{\partial}{\partial z}\Big(\partial_t\rho(t,X)\varphi(t,x,\rho(t,X))\Big)dtdX.$$

Now we can integrate by parts. Since, thanks to $\varphi$, we have compact support in $z$ we obtain

$$\lim_{\varepsilon\to 0} J_\varepsilon = -\int_I \iint_{\tilde{\Omega}_1} \partial_z\tilde{u}(t,X)\partial_t\rho(t,X)\varphi(t,x,\rho(t,X))\,dtdX.$$

Now since

$$\partial_t\rho(t,X) = -\partial_t\kappa(t,x,y)\partial_z\rho(t,x,z)$$

setting in the integral in $z$, $\rho(t,X)=y$ we obtain

$$\lim_{\varepsilon\to 0} J_\varepsilon = \iint_{\Omega_1} \partial_z\tilde{u}(t,x,\kappa(t,x,y))\partial_t\kappa(t,x,y)\varphi(t,x,y)\,dtdxdy. \tag{58}$$

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Then Lemma 3 follows from (56), (57) and (58).

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