STRICHARTZ INEQUALITIES AND THE NONLINEAR SCHRÖDINGER EQUATION ON COMPACT MANIFOLDS

by

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Abstract. — We prove Strichartz estimates with fractional loss of derivatives for the Schrödinger equation on any riemannian compact manifold. As a consequence we infer low regularity local well-posedness results in any dimension, as well as global existence results for the Cauchy problem of nonlinear Schrödinger equations on surfaces in the case of defocusing polynomial nonlinearities, and on three–manifolds in the case of cubic defocusing nonlinearities. We also discuss the optimality of these Strichartz estimates on spheres.

Résumé. — On démontre des inégalités de Strichartz avec pertes de dérivées pour l'équation de Schrödinger sur une variété compacte. On en déduit des résultats d'existence locale en toute dimension, ainsi que l'existence globale pour le problème de Cauchy sur les surfaces compactes dans le cas de non linéarités défocalisantes, et sur les variétés de dimension trois, pour les non linéarités cubiques défocalisantes. Nous discutons également l'optimalité de nos résultats dans le cas des sphères.

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2000 ***Mathematics Subject Classification.*** — 35Q55, 35BXX, 37K05, 37L50, 81Q20.
Key words and phrases. — Strichartz estimates, nonlinear Schrödinger, eigenfunctions, WKB method, dispersive equations.

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1. Introduction and statement of the results

For two decades, Strichartz inequalities have been a key tool in the study of nonlinear Schrödinger equations on the euclidean space (see e.g. Ginibre-Velo [**12, 13**], Kato [**21**], Yajima [**38**], Cazenave-Weissler [**9**]). Let us recall the general form of these inequalities. Given $v_0 \in L^2(\mathbb{R}^d)$, $d \geq 1$, denote by $v = v(t,x) \in C(\mathbb{R}, L^2(\mathbb{R}^d))$ the solution of the linear Schrödinger equation

$$i\partial_t v + \Delta v = 0, \quad v(0,x) = v_0(x). \tag{1.1}$$

Then

$$\|v\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \leq C\|v_0\|_{L^2(\mathbb{R}^d)} \tag{1.2}$$

where p, q are given by the scaling admissibility condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} \tag{1.3}$$

and satisfy moreover $p \geq 2$, $(p,q) \neq (2,\infty)$ (for the case $p = 2$, we refer to the paper by Keel-Tao [**22**]). It is now classical that the proof of (1.2) is a combination of functional analytic arguments and of the dispersion estimate

$$\|v(t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{|t|^{d/2}}\|v_0\|_{L^1(\mathbb{R}^d)}, \tag{1.4}$$

which can be easily derived from the explicit expression of v in terms of v_0.

If $\mathbb{R}^d$ is replaced by a compact riemannian manifold (M,g) any constant is solution of the free Schrödinger equation and therefore the above dispersion estimate trivially fails for large t. It is less obvious, but still true, that it also fails locally in time (see remark 2.6). However, in the special case where M is the flat torus $\mathbb{T}^d = (\mathbb{R}|2\pi\mathbb{Z})^d$, Bourgain [**3, 4**] (see also [**5**]) established inequalities related to (1.2) by means of the Fourier series representation of v. More precisely, observing that v is 2π periodic in time, an easy consequence of the results in [**3, 4**] is

$$\|v\|_{L^4(\mathbb{T}\times\mathbb{T}^d)} \leq C\|v_0\|_{H^s(\mathbb{T}^d)}, \quad s > \frac{d}{4} - \frac{1}{2}. \tag{1.5}$$

Of course, one infers estimates of the $L^p(\mathbb{T}\times\mathbb{T}^d)$, $p \geq 4$ norm of v by interpolating (1.5) with the Sobolev embedding $L^\infty(\mathbb{T}, H^{\frac{d}{2}+\varepsilon}(\mathbb{T}^d)) \subset L^\infty(\mathbb{T}\times\mathbb{T}^d)$. Moreover it can be proved that the loss s of derivatives in these estimates is essentially sharp (see also the proof of Theorem 4 below). By using these inequalities and techniques of conormal spaces adapted to harmonic analysis on the torus, Bourgain proves several results of local and global existence for the nonlinear Schrödinger equation on the torus.

Our first goal in this paper is to establish Strichartz inequalities on arbitrary compact riemannian manifolds without boundary[(1)], with a possible loss of derivatives. It turns out that this loss is half of the trivial one given by the Sobolev embedding. Specifically, our first result reads as follows.

Theorem 1. — *Let (M, g) be a riemannian compact manifold of dimension $d \geq 1$ and Δ be the Laplace Beltrami operator on M. Given p, q satisfying the scaling condition (1.3) and $p \geq 2$, $q < \infty$, the solution v of (1.1) satisfies for any finite time interval I,*

$$\|v\|_{L^p(I,L^q(M))} \leq C(I)\|v_0\|_{H^{1/p}(M)}. \tag{1.6}$$

Remark 1.1. —
1. *Theorem 1 can be easily generalized if one replaces Δ by any elliptic selfadjoint operator P of order m, assuming that the vertical Hessian of the principal symbol of P does not vanish outside the null section.*
2. *Other generalizations (abstract lower order perturbations, riemannian metrics on $\mathbb{R}^d$) are given in the appendix of this paper.*
3. *A natural open question is to obtain Strichartz estimates with a nontrivial loss of derivatives on compact manifolds with boundaries.*

Remark 1.2. — *After discussions with Daniel Tataru, we realized that the result of Theorem 1 can be deduced from from Theorem 4 in* [**31**]. *In* [**31**], *it is shown that*

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(1) All riemannian manifolds considered in the paper are without boundaries and that fact will not be explicitly mentioned.

local Strichartz norms of the solution of a variable coefficients Schrödinger equation are controlled by its $L^2H^{\frac{1}{2}}$ *space-time norm which clearly implies (1.6) for* $p=2$. *All other cases can be then covered by interpolation with the case* $p=\infty$ *which trivially follows from the* L^2 *boundedness of* $\exp(it\Delta)$. *We have however decided to keep our proof, since we believe that our analysis is less involved than the one presented in* [**31**]. *Moreover it is based on a different formalism, semiclassical calculus instead of FBI transform techniques.*

As a consequence of Theorem 1, we deduce a general result on nonlinear Schrödinger equations in dimension $d=2$.

Theorem 2. — *Let* (M,g) *be a riemannian compact surface, and let* G *be a polynomial with real coefficients. For every* $u_0 \in H^1(M)$, *there exists a unique maximal solution* $u \in C(I, H^1(M))$ *of the equation*

$$i\partial_t u + \Delta u = G'(|u|^2)u, \quad u(0,x) = u_0(x). \tag{1.7}$$

Moreover we have the following additional properties:

(i) *If* $\|u_0\|_{H^1(M)} \leq R$, *the length of* $I \cap \mathbb{R}_\pm$ *is bounded from below by a positive constant which only depends on* R.
(ii) *For any finite* p, $u \in L^p_{loc}(I, L^\infty(M))$.
(iii) *If* $G(r) \longrightarrow +\infty$ *as* $r \longrightarrow +\infty$, $I = \mathbb{R}$.
(iv) *If* $u_0 \in H^s(M)$ *for some* $s>1$, $u \in C(I, H^s(M))$. *In particular if* $u_0 \in C^\infty(M)$, $u \in C^\infty(I \times M)$.

Remark 1.3. —

1. *If* $G(r) \to +\infty$ *when* $r \to +\infty$, *the existence of global weak solutions in* H^1 *is easy. A consequence of our estimates is that such solutions satisfy property (ii) and therefore are strong and uniquely determined by their Cauchy data.*
2. *Theorem 2 and Remark 1.1 can be extended with very slight modifications to other kinds of nonlinearities (for instance* $G \in S^m(\mathbb{R})$, $G(r) = \lambda r^{\sigma+1}$*).*
3. *Observe that, if* $M = \mathbb{T}^2$, *Theorem 2 is a part of results by Bourgain quoted above. For general compact bidimensional domains, global existence for defocusing nonlinear Schrödinger equations was only known for cubic nonlinearities (deg(G) = 2), see Brézis-Gallouët* [**6**]. *Notice that even for such nonlinearities, the Brézis-Gallouët approach does not provide an existence theorem for strong solutions in the energy space* H^1.

In three space dimensions, our methods can be used to prove typically the following result.

Theorem 3. — *Let* $\dim(M) = 3$. *Then, for any* $u_0 \in H^1(M)$, *there exists a unique global weak solution* $u \in C_w(\mathbb{R}, H^1(M))$ *to the cubic defocusing nonlinear Schrödinger equation*

$$i\partial_t u + \Delta u = |u|^2 u, \qquad u(0,x) = u_0(x). \tag{1.8}$$

Moreover, $u \in C(\mathbb{R}, H^1(M))$ *with conservation of the energy, and if* $u_0 \in H^s(M)$, $s > 1$, *then* $u \in C(\mathbb{R}, H^s(M))$.
In particular, if $u_0 \in C^\infty(M)$, $u \in C^\infty(\mathbb{R} \times M)$.

Remark 1.4. — *Of course a similar result holds for more general nonlinearities with cubic or subcubic growth modulo some additional but inessential technicalities.*

Let us discuss briefly the main arguments for the above theorems. On (M, g) a natural generalization of Fourier series expansions is of course spectral decomposition of Laplace-Beltrami operator. However our knowledge of the spectrum and of the eigenfunctions of this operator on arbitrary riemannian manifolds is too poor for trying to adapt Bourgain's method at this level of generality. Rather, our proof of inequalities (1.6) relies on families of dispersive estimates on small time intervals depending on the size of the frequencies of the data. In the particular case of the torus, these estimates were proved by Vega [**35**] (see also [**23**]). Let us mention that the idea of splitting the time interval into small frequency dependent pieces also appeared in recent works by Bahouri–Chemin [**2**] and Tataru [**34**] on low regularity well–posedness of quasilinear wave equations. In the case of the Schrödinger equation, the strategy for proving such estimates is based on a simple time rescaling, first used by Lebeau [**24**] in the context of control theory (see also [**7**]), which transforms the equation into a semiclassical problem, for which parametrices can be constructed by WKB methods. These WKB constructions are to be related to those used by Kapitanski [**20**] in the case of wave equations. Due to the finite propagation speed, Kapitanski's estimates are the same as for the flat wave equation. On the other hand, in the case of the Schrödinger equation, a new phenomenon is produced, namely unavoidable losses of derivative in Strichartz inequalities have to be accepted (see Theorem 4 below for examples). This phenomenon is related to the existence of eigenfunctions, but the complete understanding of its mechanism is a delicate issue.

A combination of Theorem 1 with Sobolev estimates leads to crucial L^pL^∞ estimates, which yield easily Theorem 2 (see section 3.1 below for a general local well-posedness result in more singular Sobolev spaces). The proof of Theorem 3 is more intricate. Indeed, following the same method as in Theorem 2, the key here would be an L^2L^∞ estimate which would correspond to a gain of $\frac{1}{2}+\epsilon$ derivative with respect to Sobolev estimate combined with the H^1 estimate provided by the energy conservation law. This forces us to use Keel-Tao's endpoint estimate (1.6) with $p=2$, $q=6$ which corresponds exactly to a gain of $\frac{1}{2}$ derivative. In fact, in order to handle the nonlinear term, we rather use the local version of (1.6) on small time intervals depending on the size of the frequency. Summing up, we obtain a slightly weaker information than L^2L^∞. This information is used in two ways. Firstly, we get uniqueness of weak solutions in the energy space by means of an argument which goes back to Yudovitch [**37**] in the framework of two dimensional fluid mechanics. Secondly, in the case of smoother data, we derive a logarithmic estimate in the spirit of Wolibner [**36**] for 2d Euler equation and of Brézis-Gallouët [**6**] for 2d cubic nonlinear Schrödinger equation. Using these estimates we obtain regularity of weak solutions.

Let us now discuss the optimality of estimates (1.6). The results above are obtained by local considerations on M. A natural question is whether the global geometry of M plays a role in the Strichartz inequalities. In the present moment our knowledge on this issue is only partial. A first test is provided by taking eigenfunctions as initial data, so that the problem reduces to L^q estimates for L^2 normalized eigenfunctions. Using classical facts about spherical harmonics one infers that the endpoint estimates corresponding to $p=2$ are sharp on spheres (see section 4.2). On the other hand the work of Bourgain on tori shows that some intermediate inequalities, typically L^4 estimates, are not optimal. The aim of our fourth result is to prove L^4 estimates by imposing a global geometric assumption on M, which are optimal if $M=S^d$, $d\geq 3$.

Theorem 4. — *Assume (M,g) is a riemannian compact manifold of dimension $d\geq 2$, all of whose geodesics are closed with a common period. Then estimate (1.6) can be improved as*

$$\|v\|_{L^4(I\times M)} \leq C(I)\|v_0\|_{H^s(M)}, \quad s>s_0(d), \tag{1.9}$$

where $s_0(2)=\frac{1}{8}$, $s_0(d)=\frac{d}{4}-\frac{1}{2}$ for $d\geq 3$. Moreover, if $M=S^d$ is endowed with the canonical metric, then (1.9) is sharp in the sense that similar estimates fail with $s\leq s_0(d)$ if $d\geq 3$ (resp. $s<s_0(2)$ if $d=2$).

Remark 1.5. — *Notice that, using methods developed in the proof of Theorem 2, Theorem 4 gives local well-posedness of nonlinear Schrödinger equation with power* $m \leq 5$ *in space* H^s, $s > \frac{d}{2} - \frac{1}{2}$, $d \geq 3$, *which corresponds to a gain of* $\frac{1}{2}$ *derivative with respect to the energy approach applied in that context.*

For a general discussion on the geometric assumption on M in Theorem 4 we refer to the book by A. Besse [**1**].

The results of Theorem 4 should be discussed differently according to the dimension. If $d = 2$ the pair $(4, 4)$ is admissible and Theorem 4 provides a gain of $\frac{1}{8}$ derivative with respect to (1.6). If $d \geq 3$ the pair $(4, 4)$ is not admissible anymore and thus the comparison with Strichartz estimates should be done via Sobolev embeddings. It should be noticed that our estimates (1.9) are the same as L^4 estimates deduced from the Strichartz inequalities on $\mathbb{R}^d$ and Sobolev embeddings. Moreover they are also the same as estimates (1.5) on the torus for $d \geq 3$. Finally, observe that if $M = S^3$, Theorem 4 provides a fairly complete picture of Strichartz inequalities (see the end of section 4.3).

The proof of Theorem 4 combines several arguments : firstly, a theorem by Colin de Verdière [**11**] and Guillemin [**16**] provides a very precise localization of the spectrum of the Laplace-Beltrami operator on M. Secondly, general results by Sogge allow to handle the L^2–L^4 norm of spectral projectors of the Laplace operator associated to finite intervals of high frequencies. We conclude by using a number-theoretic argument close to the one used by Bourgain. The sharpness of estimate (1.9) on spheres is a consequence of the sharpness of Sogge's estimates on individual eigenfunctions combined with an argument based on Gauß sums which is also inspired from Bourgain's work.

We close this introduction by indicating briefly the structure of the paper. Theorem 1 is proved in section 2. In section 3, we infer a general local well-posedness result in singular Sobolev spaces which leads to Theorem 2. Then Theorem 3 is derived from endpoint Strichartz estimates and results of section 1. The optimality of estimates in Theorem 1 is discussed in section 4, where Theorem 4 is proven. Finally various extensions of our result are discussed in an appendix.

The results of this paper were announced in [**8**]. Notice that Theorem 3 has been improved with respect to the one in [**8**].

Acknowledgements. We thank H. Koch and S. Zelditch for discussions about the optimality of our results on spheres and Zoll manifolds, and E. Fouvry and R. De la Bretèche for discussions about Gauß sums. Part of this paper was written during the stays of the authors at the Erwin Schrödinger Institute in Vienna, and it is a pleasure to thank this institution for its hospitality.

2. General Strichartz inequalities

The goal of this section is to prove Theorem 1. As a first step we introduce spectral localizations which commute with the linear evolution. For that purpose we describe the effect of spectral localization relative to the Laplace-Beltrami operator in terms of Fourier analysis in a coordinate patch. Then we prove dispersion estimates for the spectrally localized Schrödinger evolution for small time intervals. Finally, we apply the abstract Keel-Tao $TT^\star$ result and we obtain Strichartz inequalities with loss of derivatives by summing up on the time intervals.

2.1. Semiclassical functional calculus. — Recall the usual pseudodifferential quantization rule,

$$a(x,D)u = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi$$

for every smooth symbol a and every $u \in C_0^\infty(\mathbb{R}^d)$ (see e.g. [**19**]).

Proposition 2.1. — *Let P be an elliptic selfadjoint differential operator of order $m > 0$ on a compact manifold M of dimension d. Let $\varphi \in C_0^\infty(\mathbb{R})$, $\kappa : U \subset \mathbb{R}^d \to V \subset M$ a coordinate patch, and $\chi_1, \chi_2 \in C_0^\infty(V)$ such that $\chi_2 = 1$ near the support of χ_1. There exists a sequence $(\psi_j)_{j\geq 0}$ of $C_0^\infty(U\times\mathbb{R}^d)$ such that, for every $N \in \mathbb{N}$, for every $h \in]0,1]$, for every $\sigma \in [0,N]$, for every $f \in C^\infty(M)$,*

$$\|\kappa^\star(\chi_1\varphi(h^mP)f) - \sum_{j=0}^{N-1} h^j\psi_j(x,hD)\kappa^\star(\chi_2 f)\|_{H^\sigma(\mathbb{R}^d)} \leq C_N h^{N-\sigma}\|f\|_{L^2(M)}.$$

Moreover $\psi_0(x,\xi) = \chi_1(\kappa(x))\varphi(p_m(x,\xi))$, where p_m denotes the principal symbol of P in κ, and every ψ_j is supported in $\{(x,\xi)\in U\times\mathbb{R}^d,\ \kappa(x)\in \mathrm{supp}(\chi_1),\ p_m(x,\xi)\in \mathrm{supp}(\varphi)\}$.

Proof of Proposition 2.1. This proof is essentially well known in the literature (see e.g. D. Robert [**26**]). Close results (where the semiclassical symbol $\varphi(h^m\lambda^m)$ is replaced by an element of $S^0(\mathbb{R})$) can be found for instance in Sogge [**29**] using

Hörmander's Fourier integral parametrix of $e^{itP^{1/m}}$ [**18**]. Our approach is to use Helffer-Sjöstrand's formula (see Helffer-Sjöstrand [**17**] or Davies [**10**]), namely

$$\varphi(h^mP) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\varphi}(z)(z-h^mP)^{-1}dL(z), \tag{2.1}$$

where $dL(z)$ denotes the Lebesgue measure on $\mathbb{C}$ and where $\tilde{\varphi}$ is an almost analytic extension of φ,

$$\tilde{\varphi}(z) = \left(\sum_{k=0}^{n} \frac{\varphi^{(k)}(\mathrm{Re}z)}{k!}(i\,\mathrm{Im}z)^k\right)\zeta(\mathrm{Im}z),$$

with $\zeta \in C_0^\infty(\mathbb{R})$, $\zeta = 1$ near zero. Notice that

$$|\bar{\partial}\tilde{\varphi}(z)| \leq C|\mathrm{Im}z|^{n+1}. \tag{2.2}$$

In order to use formula (2.1) we shall describe the resolvent $(z-h^mP)^{-1}$ using the pseudodifferential calculus, for which we refer to [**19**] or to [**26**] in the semi-classical setting. By the classical parametrix construction, one can find symbols $q_j = q_j(x,\xi,z)$ depending holomorphically on $z \notin \mathbb{R}$ such that, for every $N \geq 1$, for every $u \in C^\infty(M)$,

$$\left[\sum_{j=0}^{N-1} h^j q_j(x,hD,z)\right]\kappa^\star\left[\chi_2(z-h^mP)u\right] = \kappa^\star[\chi_1 u] + r_N(x,hD,z,h)\kappa^\star[\chi_2 u], \tag{2.3}$$

where

$$\left|\partial_x^\alpha\partial_\xi^\beta r_N(x,\xi,z,h)\right| \leq \frac{C_{N,\alpha,\beta}}{|\mathrm{Im}z|^{\gamma(N,\alpha,\beta)}}\ h^N\ (1+|\xi|)^{-N-|\beta|},$$

for some integer $\gamma(N,\alpha,\beta)$. More precisely, one has

$$q_0(x,\xi,z) = \frac{\chi_1(\kappa(x))}{z-p_m(x,\xi)}$$

and by induction on $j \geq 1$, the symbol q_j can be represented as the following finite sum,

$$q_j(x,\xi,z) = \sum_{k\geq 2}\frac{q_{jk}(x,\xi)}{(z-p_m(x,\xi))^k}$$

for some $q_{jk}(x,\xi)$ which are polynomials in ξ of degree less or equal to $(k-1)m$ and supported in the set $\{x \in U\ :\ \kappa(x) \in \mathrm{supp}(\chi_1)\}$. Applying formula (2.3) to

$u = (z - h^m P)^{-1} f$, we infer

$$\kappa^\star[\chi_1 (z - h^m P)^{-1} f] = \left(\sum_{j=0}^{N-1} h^j q_j(x, hD, z) \right) \kappa^\star[\chi_2 f] + R_N(z) f \tag{2.4}$$

with $R_N(z)f = -r_N(x, hD, z, h)\kappa^\star[\chi_2(z - h^m P)^{-1} f]$. For $\sigma \leq N$, the usual L^2 estimates for pseudodifferential operators gives

$$\|r_N(x, hD, z, h)\|_{L^2(\mathbb{R}^d) \to H^\sigma(\mathbb{R}^d)} \leq C_N \frac{h^{N-\sigma}}{|\mathrm{Im} z|^{\gamma_N}}$$

for some γ_N. As a consequence

$$\|R_N\|_{L^2(M) \to H^\sigma(\mathbb{R}^d)} \leq C_N h^{N-\sigma} |\mathrm{Im} z|^{-\gamma_N - 1}.$$

Plugging formula (2.4) into (2.1), we obtain

$$\kappa^\star[\chi_1 \varphi(h^m P) f] = \sum_{j=0}^{N-1} \psi_j(x, hD) \kappa^\star[\chi_2 f] + \tilde{R}_N f,$$

where

$$\tilde{R}_N f = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\varphi}(z) R_N(z) f \, dL(z),$$

and

$$\psi_j(x, \xi) = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\varphi}(z) q_j(x, \xi, z) dL(z).$$

Due to (2.2), taking $n \geq \gamma_N$, we get

$$\|\tilde{R}_N f\|_{H^\sigma} \leq C_N h^{N-\sigma} \|f\|_{L^2}$$

Taking into account the expressions for q_j and using Cauchy formula, we obtain the desired formula for ψ_0 and moreover for $j \geq 1$,

$$\psi_j = \sum_{k \geq 2} \frac{\varphi^{(k-1)}(p_m)}{(k-1)!} \; q_{jk}.$$

This completes the proof of Proposition 2.1. □

We now draw some consequences of Proposition 2.1 which will be useful in the sequel.

Corollary 2.2. — *Under the conditions of Proposition 2.1, if $+\infty \geq r \geq q \geq 1$,*

$$\|\varphi(h^m P)\|_{L^q(M) \to L^r(M)} \leq C h^{d(\frac{1}{r} - \frac{1}{q})} \tag{2.5}$$

Proof. By duality and interpolation, it is enough to prove the statement in the two cases $q = r = +\infty$ and $q = 2, r = +\infty$. Taking N large enough so that $H^N(M) \subset L^\infty(M)$, we see from Proposition 2.1 that Corollary 2.2 will be a consequence of the analogous result for $\psi(x, hD)$, if $\psi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. This latter point trivially results from Schur's lemma. $\square$

Corollary 2.3. — *Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R}^*)$ such that*

$$\tilde{\varphi}(\lambda) + \sum_{k=1}^{\infty} \varphi(2^{-mk}\lambda) = 1, \quad \lambda \in \mathbb{R}. \tag{2.6}$$

Then for all $q \in [2, +\infty[$, we have

$$\|f\|_{L^q(M)} \lesssim C_q \left(\|\tilde{\varphi}(P)f\|_{L^q(M)} + \left(\sum_{k=1}^{\infty} \|\varphi(2^{-mk}P)f\|^2_{L^q(M)} \right)^{\frac{1}{2}} \right).$$

Proof. Of course, the above inequality holds for $q = 2$, by L^2 functional calculus and almost orthogonality. Hence it is enough to prove it for $q \in]2, +\infty[$ with an error term in the right hand side bounded by $C\|f\|_{L^2(M)}$. Moreover, by partition of unity, it is enough to estimate $\|\chi f\|_{L^q}$ where $\chi \in C^\infty(M)$ is supported in a coordinate patch κ. Let

$$1 = \tilde{\theta}(\xi) + \sum_{l=1}^{\infty} \theta(2^{-l}\xi)$$

be a dyadic partition of unity in $\mathbb{R}^d$, $\tilde{\theta} \in C_0^\infty(\mathbb{R}^d)$, $\theta \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. Applying the classical Littlewood-Paley estimate (see e.g. Stein [**32**, § 6, ex. 7.14]) combined with the Minkowski inequality to $g = \kappa^\star(\chi f)$, we get

$$\|g\|_{L^q(\mathbb{R}^d)} \leq \|\tilde{\theta}(D)g\|_{L^q(\mathbb{R}^d)} + \left(\sum_{l=1}^{\infty} \|\theta(2^{-l}D)g\|^2_{L^q(\mathbb{R}^d)} \right)^{\frac{1}{2}}. \tag{2.7}$$

Then we plug

$$g = \kappa^\star(\chi\tilde{\varphi}(P)f) + \sum_{k=1}^{\infty} \kappa^\star(\chi\varphi(2^{-mk}P)f)$$

into (2.7). Applying Proposition 2.1 with N so large that $H^{N-1} \subset L^q$, we obtain

$$\kappa^\star(\chi\varphi(2^{-mk}P)f) = \sum_{j=0}^{N-1} 2^{-kj}\psi_j(x, 2^{-k}D)\kappa^\star(\chi_2 f) + R_k,$$

$$\|R_k\|_{H^{N-1}(\mathbb{R}^d)} \leq C2^{-k}\|f\|_{L^2(M)}.$$

In view of the above estimate, the contribution of R_k to the right hand side of (2.7) is controlled by $\|f\|_{L^2}$. On the other hand, since

$$a|\xi|^m \leq |p_m(x,\xi)| \leq b|\xi|^m$$

for $x \in \operatorname{supp}(\chi)$, in view of the support properties of φ, we observe that $\psi_j(x,\xi)$ is supported in some fixed ring of $\mathbb{R}^d$. As a consequence, standard pseudodifferential estimates imply, if $|l-k|$ is larger than a fixed positive constant ν,

$$\|\theta(2^{-l}D)\psi_j(x, 2^{-k}D)\kappa^\star(\chi_2 f)\|_{L^q(\mathbb{R}^d)} \leq C2^{-l}\|\rho(2^{-k}D)\kappa^\star(\chi_2 f)\|_{L^2(\mathbb{R}^d)}$$

for some $\rho \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. Finally, (2.7) becomes

$$\|\kappa^\star(\chi f)\|_{L^q(\mathbb{R}^d)} \leq C\|\tilde{\varphi}(P)f\|_{L^q(M)} + \|\kappa^\star\chi\left(\sum_{k\leq\nu}\varphi(2^{-mk}P)f\right)\|_{L^q(\mathbb{R}^d)}$$

$$+\left(\sum_l \|\theta(2^{-l}D)\kappa^\star\chi\sum_{|l-k|\leq\nu}\varphi(2^{-mk}P)f\|^2_{L^q(\mathbb{R}^d)}\right)^{\frac{1}{2}} + C\|f\|_{L^2(M)},$$

which completes the proof. □

Corollary 2.4. — *Under the conditions of Proposition 2.1, there exists a function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that, for all $h \in]0,1]$, $\sigma > 0$, $N > 0$, $f \in C^\infty(M)$,*

$$\|(1-\psi(hD))\kappa^\star(\chi_1\varphi(h^mP)f)\|_{H^\sigma(\mathbb{R}^d)} \leq C_{\sigma,N}h^N\|f\|_{L^1(M)}. \tag{2.8}$$

Proof. Define $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi(\xi) = 1$ if $p_m(x,\xi) \in \operatorname{supp}(\varphi)$ for some $x \in \operatorname{supp}(\chi_1)$. Then Proposition 2.1 and standard pseudodifferential calculus imply the desired estimate with $\|f\|_{L^2}$ in place of $\|f\|_{L^1}$ in the right hand side.

Changing $\varphi(\lambda)$ into $\varphi(\lambda)(1+\lambda^2)^{s/2m}$ for some $s > \frac{d}{2}$, and f into $(1+h^{2m}P^2)^{-s/2m}f$, one can therefore estimate the left hand side of (2.8) by

$$C_{\sigma,N}h^N\|(1+h^{2m}P^2)^{-s/2m}f\|_{L^2(M)} \leq C_{\sigma,N}h^{N-s}\|f\|_{H^{-s}(M)} \leq \tilde{C}_{\sigma,N}h^{N-s}\|f\|_{L^1(M)}$$

which completes the proof, since N is arbitrary. □

2.2. A semiclassical dispersion estimate. —

Lemma 2.5. — *Let M be a compact riemannian manifold of dimension d. Let $\varphi \in C_0^\infty(\mathbb{R})$. There exists $\alpha > 0$ and $C > 0$ such that, for every $v_0 \in C^\infty(M)$, for every $h \in]0,1]$*

$$\|\mathrm{e}^{it\Delta}\varphi(h^2\Delta)v_0\|_{L^\infty(M)} \leq \frac{C}{|t|^{d/2}}\|v_0\|_{L^1(M)} \tag{2.9}$$

for every $t \in [-\alpha h, \alpha h]$.

Remark 2.6. — *Notice that the length of the time interval above is optimal, as shown by the following calculation. The kernel of $\mathrm{e}^{it\Delta}\varphi(h^2\Delta)$ is*

$$K_h(t,x,y) = \sum_{\lambda\in\sigma(\Delta)} \varphi(h^2\lambda)\mathrm{e}^{it\lambda}\Pi_\lambda(x,y) \tag{2.10}$$

where Π_λ is the kernel of the projector onto the eigenspace associated to λ. Therefore

$$\begin{aligned} \|\mathrm{e}^{it\Delta}\varphi(h^2\Delta)\|_{L^1(M)\to L^\infty(M)} &= \|K_h(t,\cdot,\cdot)\|_{L^\infty(M\times M)} \\ &\geq C\|K_h(t,\cdot,\cdot)\|_{L^2(M\times M)} \\ &\geq C\left(\sum_{\lambda\in\sigma(\Delta)} |\varphi(h^2\lambda)|^2\right)^{\frac{1}{2}} \geq \frac{C}{h^{\frac{d}{2}}} \end{aligned} \tag{2.11}$$

by the Weyl formula if φ is not identically zero. Therefore the estimate (2.9) *for some t implies $|t| \leq Ch$.*

Moreover, since $\varphi(h^2\Delta)$ is uniformly bounded on $L^\infty(M)$ by Corollary 2.2, estimate (2.11) *implies that for any t the operator $\mathrm{e}^{it\Delta}$ is not bounded from $L^1(M)$ to $L^\infty(M)$.*

Proof of Lemma 2.5. Notice that

$$w(s,\cdot) = \mathrm{e}^{ihs\Delta}\varphi(h^2\Delta)v_0$$

satisfies the semiclassical equation

$$ih\partial_s w + h^2\Delta w = 0, \quad w(0,x) = \varphi(h^2\Delta)v_0(x). \tag{2.12}$$

The following lemma provides an approximate solution of (2.12) in local coordinates. It is easy to see that it implies lemma 2.5 by using partitions of unity, Corollaries 2.2 and 2.4 and the boundedness of the propagator $\mathrm{e}^{it\Delta}$ on $H^\sigma(M)$.

Lemma 2.7. — *Let U_1 be an open ball in $\mathbb{R}^d$, endowed with a riemannian metric g. Let $U_2 \subset\subset U_1$ be an open ball, $\chi_0 \in C_0^\infty(U_2)$, $\psi \in C_0^\infty(\mathbb{R}^d)$. There exists $\alpha > 0$, $C > 0$ such that, for every $h \in]0,1]$, for every $w_0 \in C_0^\infty(U_1)$, one can find $\tilde{w} \in C^\infty([-\alpha,\alpha] \times U_2)$, compactly supported, satisfying*

$$ih\partial_s \tilde{w} + h^2 \Delta_g \tilde{w} = r, \quad \tilde{w}(0,x) = \chi_0(x)\psi(hD)w_0(x), \tag{2.13}$$

with estimates

$$\|\tilde{w}(s)\|_{L^\infty(U_2)} \leq \frac{C}{(h|s|)^{d/2}} \|w_0\|_{L^1(U_1)}, \quad |s| \leq \alpha \tag{2.14}$$

and

$$\|r\|_{L^\infty([-\alpha,\alpha],H^\sigma(\mathbb{R}^d))} \leq Ch\|w_0\|_{L^1(U_1)} \tag{2.15}$$

for some $\sigma > \frac{d}{2}$.

The proof of Lemma 2.7 is based on the usual WKB construction (see for instance Robert [**26**]). We seek $\tilde{w}$ as the following oscillatory integral

$$\tilde{w}(s,x) = \int_{\mathbb{R}^d} e^{\frac{i}{h}\phi(s,x,\xi)} a(s,x,\xi,h) \widehat{w_0}(\frac{\xi}{h}) \frac{d\xi}{(2\pi h)^d} \tag{2.16}$$

where

$$a(s,x,\xi,h) = \sum_{j=0}^{N} h^j a_j(s,x,\xi). \tag{2.17}$$

Here N is to be chosen large enough, $a_j \in C_0^\infty([-\alpha,\alpha] \times U_2 \times \mathbb{R}^d)$, with initial constraints

$$a_0(0,x,\xi) = \chi_0(x)\psi(\xi), \quad a_j(0,x,\xi) = 0, \quad \text{for } j \geq 1, \tag{2.18}$$

and $\phi \in C^\infty([-t_0,t_0] \times U_2 \times B, \mathbb{R})$, where B is a ball containing the support of ψ, with initial constraint $\phi(0,x,\xi) = x \cdot \xi$. Then the equations for ϕ, a_j are the eikonal equation

$$\partial_s \phi + \sum_{1 \leq i,j \leq d} g^{ij} \partial_i \phi \, \partial_j \phi = 0 \tag{2.19}$$

and the transport equations

$$\partial_s a_0 + 2g(\nabla_g \phi, \nabla_g a_0) + \Delta_g(\phi) a_0 = 0 \tag{2.20}$$

$$\partial_s a_j + 2g(\nabla_g \phi, \nabla_g a_j) + \Delta_g(\phi) a_j = -\Delta_g(a_{j-1}) \quad j \geq 1. \tag{2.21}$$

The Hamilton-Jacobi theory yields a solution ϕ of (2.19) on a small interval $[-\alpha, \alpha]$ for $\xi \in B$. Then, taking α smaller if necessary, equations (2.20) and (2.21) can be solved on $[-\alpha, \alpha]$ so that every a_j is supported in U_2. Then $\tilde{w}$ satisfies equation (2.13) with

$$r(s,x) = h^{N+2} \int_{\mathbb{R}^d} \mathrm{e}^{\frac{i}{h}\phi(s,x,\xi)} b(s,x,\xi) \widehat{w_0}(\frac{\xi}{h}) \frac{d\xi}{(2\pi h)^d}$$

where $b \in C_0^\infty([-\alpha, \alpha] \times U_2 \times B)$. By a crude estimate, we infer

$$\|r\|_{L^\infty([-\alpha,\alpha], H^\sigma(\mathbb{R}^d))} \leq C h^{N+2-d-\sigma} \|w_0\|_{L^1(U_1)},$$

which implies inequality (2.15) if N is large enough.

In order to prove the dispersion inequality (2.14), we write

$$\tilde{w}(s,x) = \int_{\mathbb{R}^d} k_h(s,x,y) w_0(y) dy,$$

with

$$k_h(s,x,y) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\frac{i}{h}(\phi(s,x,\xi) - y\cdot\xi)} a(s,x,\xi,h) d\xi. \tag{2.22}$$

In view of equation (2.19) and of the initial condition for ϕ, we have, for $(x,\xi) \in U_2 \times B$,

$$\phi''_{\xi_i \xi_j}(s,x,\xi) = -2sg^{i,j}(x) + O(s^2), \quad s \longrightarrow 0.$$

As a consequence, for $\alpha > 0$ small enough, the phase in the integral (2.22) has a unique nondegenerate critical point. Therefore the stationary phase formula can be applied, for all s such that $|s| \leq \alpha$. This yields the estimate

$$|k_h(s,x,y)| \leq \frac{C}{(h|s|)^{d/2}},$$

for $(x,y) \in U_2 \times U_1$, from which inequality (2.14) results. □

2.3. Strichartz inequalities on finite time intervals. — In this section we achieve the proof of Theorem 1. Of course, one may assume $I = [-1, 1]$.

2.3.1. **Step 1**. *Strichartz estimates for spectrally localized data.*— We first recall the functional-analytic result which allows to transform a dispersion estimate into a family of Strichartz estimates.

Proposition 2.8 **(Keel-Tao [22]).** — *Let $(X, \mathcal{S}, \mu)$ be a σ-finite measured space, and $U : \mathbb{R} \to B(L^2(X, \mathcal{S}, \mu))$ be a weakly measurable map satisfying, for some A, $\sigma > 0$,*

(i)

$$\|U(t)\|_{L^2\to L^2} \leq A, \quad t \in \mathbb{R},$$

(ii)

$$\|U(t_1)U(t_2)^\star f\|_{L^\infty} \leq \frac{A}{|t_1 - t_2|^\sigma}\|f\|_{L^1}, \quad t_1, t_2 \in \mathbb{R}.$$

Then for every pair $p, q \in [1, \infty]$ satisfying

$$\frac{2}{p} + \frac{2\sigma}{q} = \sigma, \quad p \geq 2, \quad (p, q) \neq (2, \infty), \tag{2.23}$$

one has

$$\left(\int_{\mathbb{R}} \|U(t)f\|_{L^q}^p dt\right)^{\frac{1}{p}} \leq B\|f\|_{L^2}.$$

Moreover, B only depends on A, σ, p, q.

For $h \in]0, 1]$, $\varphi \in C_0^\infty(\mathbb{R})$, let us apply the above proposition to

$$U(t) = \mathbb{1}_J(t)\mathrm{e}^{it\Delta}\varphi(h^2\Delta)$$

on an interval J of length $\leq \alpha h$, where α is given by Lemma 2.5. In view of this lemma we conclude

Proposition 2.9. — *Under the assumptions of Theorem 1, for any $\varphi \in C_0^\infty(\mathbb{R})$ there exists $C > 0$ such that for any $h \in]0, 1]$ and any interval J of length $|J| \leq \alpha h$,*

$$\left(\int_J \|e^{it\Delta}\varphi(h^2\Delta)v_0\|_{L^q}^p dt\right)^{\frac{1}{p}} \leq C\|v_0\|_{L^2}.$$

Writing $[-1, 1]$ as a union of N intervals J_k of length $\leq \alpha h$ with $N \lesssim \frac{1}{h}$, we obtain finally

$$\begin{aligned}\int_{-1}^{1} \|\mathrm{e}^{it\Delta}\varphi(h^2\Delta)v_0\|_{L^q}^p dt &\leq \sum_k \int_{J_k} \|\mathrm{e}^{it\Delta}\varphi(h^2\Delta)v_0\|_{L^q}^p dt \\ &\leq NC\|v_0\|_{L^2}^p,\end{aligned}$$

therefore

$$\left(\int_{-1}^{1} \|\mathrm{e}^{it\Delta}\varphi(h^2\Delta)v_0\|_{L^q}^p dt\right)^{\frac{1}{p}} \leq \frac{\tilde{C}}{h^{1/p}}\|v_0\|_{L^2}. \tag{2.24}$$

Notice that $\|v_0\|_{L^2}$ can be replaced in the right side of (2.24) by $\|\varphi(h^2\Delta)v_0\|_{L^2}$ (first replace φ by $\varphi^\star$ in (2.24) where $\varphi^\star \in C_0^\infty$ satisfies $\varphi\varphi^\star = \varphi^\star$ and then apply the estimate to $\varphi(h^2\Delta)v_0$ instead of v_0).

2.3.2. **Step 2.** *Strichartz estimates.—* Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$, $\varphi \in C_0^\infty((\mathbb{R}) \setminus \{0\})$ such that

$$\tilde{\varphi}(\lambda) + \sum_{k=1}^{\infty} \varphi(2^{-2k}\lambda) = 1.$$

Apply Corollary 2.3 with $P = \Delta$ to $f = \mathrm{e}^{it\Delta} v_0 = v(t)$ and take the L^p norm for $t \in [-1, 1]$. We obtain

$$\|v\|_{L^p([-1,1],L^q)} \leq C\|v_0\|_{L^2} + \left\| \left(\sum_{k=1}^{\infty} \|\mathrm{e}^{it\Delta} \varphi(2^{-2k}\Delta) v_0\|_{L^q}^2 \right)^{1/2} \right\|_{L^p[-1,1]},$$

which by the Minkowski inequality and estimate (2.24), leads to

$$\begin{aligned} \|v\|_{L^p([-1,1],L^q)} &\leq C\|v_0\|_{L^2} + C \left(\sum_{k=1}^{\infty} 2^{2k/p} \|\varphi(2^{-2k}\Delta) v_0\|_{L^2}^2 \right)^{1/2} \\ &\leq C\|v_0\|_{H^{1/p}}. \end{aligned}$$

This completes the proof of Theorem 1. □

As usual for Strichartz estimates, we infer estimates for the nonhomogeneous equation from estimates for the homogeneous one.

Corollary 2.10. — *If p, q satisfy $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $p \geq 2$ and $q < \infty$,*

$$\left\| \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau \right\|_{L^p([0,T],L^q(M))} \leq C_T \|f\|_{L^1([0,T],H^{\frac{1}{p}}(M))}. \tag{2.25}$$

Proof. The proof is straightforward through the following Minkowski inequality: The left hand side reads

$$I = \left\| \int_0^T F_\tau d\tau \right\|_{L^p([0,T],L^q(M))}, \qquad F_\tau(t) = \mathbb{1}_{\tau \leq t}\, \mathrm{e}^{i(t-\tau)\Delta} f(\tau), \tag{2.26}$$

hence

$$\begin{aligned} I &\leq \int_0^T \|F_\tau\|_{L^p([0,T];L^q(M))} d\tau \\ &\leq \int_0^T \|\mathrm{e}^{i(t-\tau)\Delta} f(\tau)\|_{L^p([0,T];L^q(M))} d\tau \leq C \int_0^T \|f(\tau)\|_{H^{\frac{1}{p}}} d\tau. \end{aligned} \tag{2.27}$$

□

3. Nonlinear Schrödinger equations on compact manifolds

In this section we discuss existence results for the initial value problem

$$i\partial_t u + \Delta u = F(u), \quad u(0,x) = u_0(x) \tag{3.1}$$

with $t \in \mathbb{R}, x \in M$, $\dim(M) \geq 2$, and where F is a (nonlinear) polynomial, $F(0) = 0$. Our first result is local in time and holds for all such polynomials F. In sections 3.2 and 3.3, our global results use conservation laws, hence the specific structure $F(u) = G'(|u|^2)u$ with real valued G.

3.1. A local well-posedness result in singular Sobolev spaces. —

Proposition 3.1. — *Let β be the degree of F and let*

$$s > \frac{d}{2} - \frac{1}{\max(\beta-1, 2)}.$$

For every $u_0 \in H^s(M)$, there exists $T > 0$ and a unique solution

$$u \in C([-T,T], H^s(M)) \cap L^p([-T,T], L^\infty(M))$$

for some $p > \beta - 1$. Moreover

(i) *If $\|u_0\|_{H^s}$ is bounded, then T is bounded from below by a positive constant.*
(ii) *If $u_0 \in H^r$ for some $r > s$, $u \in C([-T,T], H^r(M))$.*

Proof. We proceed classically using a contraction principle in a suitable Banach space. In view of (ii) we may assume $s < \frac{d}{2}$. Select $p > \max(\beta - 1, 2)$ such that $s > \frac{d}{2} - \frac{1}{p}$, and consider

$$Y_T = C([-T,T], H^s(M)) \cap L^p([-T,T], W^{\sigma,q}(M))$$

where q is given by $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, and $\sigma = s - \frac{1}{p} > \frac{d}{q}$. We set

$$\|u\|_{Y_T} = \max_{|t|\leq T} \|u(t)\|_{H^s} + \|(1-\Delta)^{\sigma/2} u\|_{L^p([-T,T],L^q)}.$$

Observe that $Y_T \subset L^p([-T,T], L^\infty)$ by the Sobolev embedding. In view of the Duhamel formula, the first part of the proposition is proved if we show that the functional

$$\Phi(u)(t) = \mathrm{e}^{it\Delta} u_0 - i\int_0^t \mathrm{e}^{i(t-\tau)\Delta}(F(u(\tau)))d\tau$$

is a contraction on some ball of Y_T centered at the origin, if T is small enough. Using Theorem 1, Corollary 2.10 and standard nonlinear estimates, we get

$$\begin{aligned}\|\Phi(u)\|_{Y_T} &\leq C\left(\|u_0\|_{H^s} + \int_{-T}^{T} \|F(u(\tau))\|_{H^s} d\tau\right)\\ &\leq C\left(\|u_0\|_{H^s} + \int_{-T}^{T} (1 + \|u(\tau)\|_{L^\infty}^{\beta-1})\|u(\tau)\|_{H^s} d\tau\right)\\ &\leq C\left(\|u_0\|_{H^s} + T^\gamma(1 + \|u\|_{L^p(L^\infty)}^{\beta-1})\|u\|_{L^\infty(H^s)}\right).\end{aligned}$$

with $\gamma = 1 - \frac{\beta-1}{p} > 0$. Thus

$$\|\Phi(u)\|_{Y_T} \leq C\left(\|u_0\|_{H^s} + T^\gamma(\|u\|_{Y_T} + \|u\|_{Y_T}^\beta).\right) \tag{3.2}$$

Similarly,

$$\|\Phi(u) - \Phi(v)\|_{Y_T} \leq CT^\gamma(1 + \|u\|_{Y_T} + \|v\|_{Y_T})^{\beta-1}\|u - v\|_{Y_T} \tag{3.3}$$

and

$$\|\Phi(u) - \Phi(v)\|_{L^\infty(H^r)} \leq C_r T^\gamma(1 + \|u\|_{Y_T} + \|v\|_{Y_T})^{\beta-1}\|u - v\|_{L^\infty(H^r)}. \tag{3.4}$$

The existence part of the proposition and (i) are then consequence of estimates (3.2) and (3.3), while (ii) is a consequence of estimate (3.4). It remains to prove the uniqueness. Given u, $\tilde{u} \in C([-T,T], H^s(M)) \cap L^p([-T,T], L^\infty(M))$ two solutions of (3.1) one gets similarly to above

$$\sup_{|t|\leq T} \|u(t) - \tilde{u}(t)\|_{H^s(M)} \lesssim CT^\gamma B^{\beta-1} \sup_{|t|\leq T} \|u(t) - \tilde{u}(t)\|_{H^s(M)}, \tag{3.5}$$

if

$$1 + \|u\|_{L^p([-T,T],L^\infty(M))} + \|v\|_{L^p([-T,T],L^\infty(M))} \leq B.$$

This completes the proof of Proposition 3.1 □

Remark 3.2. — *Using (3.5), it is easy to show that the flow map $u_0 \mapsto u$ is Lipschitz continuous from bounded subsets of $H^s(M)$ to $C([-T,T], H^s(M))$.*

3.2. The case of surfaces. — We now come to the proof of Theorem 2, so the dimension d of M is assumed to be 2. Observe that, for every $p < \infty$, $\frac{d}{2} - \frac{1}{p} = 1 - \frac{1}{p} < 1$, hence existence and uniqueness of a maximal solution $u \in C(I, H^1) \cap \bigcap_{p<\infty} L^p_{loc}(I, L^\infty)$ as well as estimates (i) on the length of $I \cap \mathbb{R}_\pm$ and regularity (iv), are direct consequences of Proposition 3.1 above. If u is smooth, we have classically the following conservation laws,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{3.6}$$

and

$$\int_M |\nabla u(t,x)|^2 dx + \int_M G(|u(t,x)|^2) dx = cst \tag{3.7}$$

and one can easily show that these conservation laws are still valid if $u_0 \in H^1$, by approximation. As a consequence, if $G(r) \longrightarrow +\infty$ as $r \longrightarrow +\infty$, the H^1 norm of $u(t)$ remains bounded for all time, and $I = \mathbb{R}$ in view of property (i).

Finally, in order to complete the proof of Theorem 2, we only need to check property (ii) for solutions $u \in L^\infty_{\rm loc}(I, H^1)$, i.e. to check that $u \in \cap_{p<\infty} L^p_{\rm loc}(I; L^\infty)$. Indeed by Sobolev embedding, $u \in L^\infty_{\rm loc}(I, L^r)$ for all $r < \infty$, hence $G'(|u|^2)u \in L^\infty_{\rm loc}(I, W^{1,q})$ for all $q < 2$, therefore $G'(|u|^2)u \in L^\infty_{\rm loc}(I, H^s)$ for all $s < 1$.

By Corollary 2.10 and Theorem 1 and from Duhamel formula

$$u(t) = \mathrm{e}^{it\Delta} u_0 - i \int_0^t \mathrm{e}^{i(t-\tau)\Delta} (G'(|u(\tau)|^2) u(\tau)) d\tau$$

we infer that $u \in L^p_{loc}(I, W^{s-\frac{1}{p},q})$, $s < 1$ for all p, q satisfying

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad q < \infty.$$

But $s - \frac{1}{p} - \frac{2}{q} = s - \frac{1}{2} - \frac{1}{q} > 0$ if s is close to 1 since $q > 2$, thus Sobolev estimate completes the proof. Notice that the latter argument applies to weak solutions as well, which therefore satisfy property (ii) and are uniquely determined by their Cauchy data. □

3.3. The case of three-manifolds.

— Firstly we recall the non-homogeneous version of Keel-Tao's result [**22**]. If $r \in [1, +\infty]$ we denote by $\overline{r}$ the Hölder conjugate exponent of r.

Proposition 3.3. — *Under the assumptions of proposition 2.8, one has*

$$\left(\int_{\mathbb{R}} \| \int_{-\infty}^t U(t)U(\tau)^* f(\tau) d\tau \|_{L^q}^p dt \right)^{\frac{1}{p}} \leq C \|f\|_{L^{\overline{r}}(\mathbb{R}, L^{\overline{s}})} \tag{3.8}$$

for all pairs $(p,q), (r,s)$ *satisfying condition* (2.23).

In view of Lemma 2.5, we infer the following complement to proposition 2.9

Lemma 3.4. — *Under the assumptions of Theorem 1 and for $d = 3$, for any $\varphi \in C_0^\infty(\mathbb{R})$, there exists $C > 0$ such that for any $h \in]0,1]$, f supported in J*

$$\left(\int_J \| \int_{-\infty}^t \mathrm{e}^{i(t-\tau)\Delta}\varphi(h^2\Delta)f(\tau)d\tau\|_{L^6}^2 dt\right)^{\frac{1}{2}} \leq C\|\varphi(h^2\Delta)f\|_{L^2(J,L^{\frac{6}{5}})} \tag{3.9}$$

if $|J| \leq \frac{\alpha}{2}h$.

Indeed, it is enough to apply proposition 3.3 to $U(t) = \mathrm{e}^{it\Delta}\varphi(h^2\Delta)\mathbb{1}_{J-J}(t)$. □

Remark 3.5. — *Let us make a brief comment about the choice of the L^2L^6 estimate above. In three space dimensions, to take advantage of the energy conservation for estimating the nonlinear terms, one needs to gain at least one half of a derivative with respect to the Sobolev embedding $H^{\frac{3}{2}+\varepsilon} \subset L^\infty$. In view of Theorem 1, this constraint forces us to use the endpoint Strichartz estimate.*

We can now proceed to the main step of our proof.

Lemma 3.6. — *For any $\varphi \in C_0^\infty(\mathbb{R}\setminus\{0\})$, $0 < h < 1$, there exists $C > 0$ such that for any $a < b$ with $b - a \leq 1$ and $u \in C([a,b],H^1)$ a solution of*

$$i\partial_t u + \Delta u = G'(|u|^2)u \tag{3.10}$$

where $G \in S^m$, $m < \frac{5}{2}$, one has

$$\|\varphi(h^2\Delta)u\|_{L^2([a,b],L^6)} \leq Ch^{\frac{1}{2}}\|\varphi(h^2\Delta)u\|_{L^2([a,b],H^1)} + Ch^{\frac{1}{2}+\varepsilon}\Lambda(\|u\|_{L^\infty([a,b],H^1)}) \tag{3.11}$$

for some $\varepsilon > 0$, where $\Lambda : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is some nondecreasing function.

Proof. As will appear later on in the proof of Theorem 3, the specific structure of the term $\|\varphi(h^2\Delta)u\|_{L^2([a,b],H^1)}$ in estimate (3.11) is crucial, since it belongs to ℓ^2 for $h = 2^{-k}$. We shall get this term by reducing equation (3.10) to equations on small intervals of length h with null Cauchy data.

Let $0 < h < 1$ and $J = [c - \frac{\alpha h}{8}, c + \frac{\alpha h}{8}] \subset [a + \frac{\alpha h}{8}, b - \frac{\alpha h}{8}]$ be an interval of length $\frac{\alpha h}{4}$. Denote by $J' = J + [-\frac{\alpha h}{8}, \frac{\alpha h}{8}]$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 1$ on $[-\frac{1}{8}, \frac{1}{8}]$ and $\mathrm{supp}(\chi) \subset [-\frac{1}{4}, \frac{1}{4}]$. Set $\chi_h(t) = \chi(\frac{t-c}{\alpha h})$. Then $v_h(t) = \chi_h(t)\varphi(h^2\Delta)u(t)$ is solution of

$$i\partial_t v_h + \Delta v_h = i(\chi_h)'(t)\varphi(h^2\Delta)u + \chi_h\varphi(h^2\Delta)f \tag{3.12}$$

where $f = G'(|u|^2)u$. Observe that $|\nabla f| \leq C\langle u\rangle^{2m-2}|\nabla u|$, where $\langle\lambda\rangle = (1+\lambda^2)^{1/2}$, hence

$$\|f\|_{L^\infty([a,b];W^{1,r})} \leq C\Lambda(\|u\|_{L^\infty([a,b],H^1)}) \tag{3.13}$$

with $\frac{1}{r} = \frac{2m-2}{6} + \frac{1}{2}$, therefore $r > 1$.

As a consequence (with $(\frac{5}{6} - \frac{1}{r})_- = \min(0, \frac{5}{6} - \frac{1}{r})$),

$$\begin{aligned} \|\chi_h \varphi(h^2\Delta) f\|_{L^2(J', L^{\frac{6}{5}})} &\leq C h^{\frac{1}{2}} \|\chi_h \varphi(h^2\Delta) f\|_{L^\infty(J', L^{\frac{6}{5}})} \\ &\leq C h^{\frac{1}{2} + 3(\frac{5}{6} - \frac{1}{r})_-} \|\chi_h \varphi(h^2\Delta) f\|_{L^\infty(J', L^r)} \\ &\leq C h^{\frac{1}{2} + 3(\frac{5}{6} - \frac{1}{r})_- + 1} \Lambda(\|u\|_{L^\infty([a,b], H^1)}) \\ &\leq C h^{1+\varepsilon} \Lambda(\|u\|_{L^\infty([a,b], H^1)}). \end{aligned} \tag{3.14}$$

for some $\varepsilon > 0$, where we have made an extensive use of corollary 2.2.

According to equation (3.12), proposition 2.9 and lemma 3.4 imply

$$\begin{aligned} \|v_h\|_{L^2(J, L^6)} &\leq \|(\chi_h)' \varphi(h^2\Delta) u\|_{L^1(J', L^2)} + \|\chi_h \varphi(h^2\Delta) f\|_{L^2(J', L^{\frac{6}{5}})} \\ &\leq C h^{\frac{1}{2}} \|(\chi_h)' \varphi(h^2\Delta) u\|_{L^2(J', L^2)} + C h^{1+\varepsilon} \Lambda(\|u\|_{L^\infty([a,b], H^1)}). \end{aligned} \tag{3.15}$$

By Corollary 2.2 (and using that $|(\chi_h)'| \leq C/h$), we conclude

$$\|\varphi(h^2\Delta) u\|_{L^2(J, L^6)} \leq h^{\frac{1}{2}} \|\varphi(h^2\Delta) u\|_{L^2(J', H^1)} + C h^{1+\varepsilon} \Lambda(\|u\|_{L^\infty([a,b], H^1)}). \tag{3.16}$$

Now we write

$$[a, b] = [a, \tilde{a}] \cup \cup_{k=1}^{N} J_k \cup [\tilde{b}, b], \tag{3.17}$$

where intervals J_k have length $\frac{\alpha h}{4}$ and disjoint interiors, $J'_k \subset [a, b]$, and

$$\tilde{a} - a \leq \alpha h, \qquad b - \tilde{b} \leq \alpha h, \qquad N \leq \frac{4(b-a)}{\alpha h}.$$

Since

$$(i\partial_t + \Delta)\varphi(h^2\Delta) u = \varphi(h^2\Delta) f$$

we obtain similarly as above

$$\begin{aligned} \|\varphi(h^2\Delta) u\|_{L^2([a, \tilde{a}], L^6)} &\leq C h \|\varphi(h^2\Delta) u(a)\|_{H^1} + C h^{1+\varepsilon} \Lambda(\|u\|_{L^\infty([a,b], H^1)}). \\ \|\varphi(h^2\Delta) u\|_{L^2([\tilde{b}, b], L^6)} &\leq C h \|\varphi(h^2\Delta) u(b)\|_{H^1} + C h^{1+\varepsilon} \Lambda(\|u\|_{L^\infty([a,b], H^1)}). \end{aligned} \tag{3.18}$$

Summing the squares of the latter estimates and of estimates (3.16), we obtain finally

$$\begin{aligned} \|\varphi(h^2\Delta) u\|^2_{L^2([a,b], L^6)} \leq{}& C h^2 \|\varphi(h^2\Delta) u(a)\|^2_{H^1} + C h^2 \|\varphi(h^2\Delta) u(b)\|^2_{H^1} \\ &+ C h \|\varphi(h^2\Delta) u\|^2_{L^2([a,b], H^1)} + C h^{1+2\varepsilon} \Lambda(\|u\|_{L^\infty([a,b], H^1)}) \end{aligned} \tag{3.19}$$

which completes the proof of lemma 3.6. ☐

Now we turn to the proof of Theorem 3. We begin with global existence of smooth classical solutions $u \in C(\mathbb{R}, H^s)$, for s large enough, to the equation

$$i\partial_t u + \Delta u = |u|^2 u. \tag{3.20}$$

Since this equation is locally well-posed in H^s, it is clearly enough to prove that $\|u\|_{L^\infty([0,T],H^s)}$ remains bounded as T remains bounded.

First we take advantage of the usual conservation laws,

$$\begin{aligned} \|u(t)\|_{L^2} &= \|u_0\|_{L^2}, \\ \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\|u(t)\|_{L^4}^4 &= E_0. \end{aligned} \tag{3.21}$$

to conclude that $\|u\|_{L^\infty([0,T];H^1)} \leq C$ uniformly with respect to T.

Next we use lemma 3.6 in the following way. We write

$$u = \tilde{\varphi}(\Delta)u + \sum_{k=0}^{\infty} \varphi(2^{-2k}\Delta)u$$

with $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$, $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, hence, for every l,

$$\begin{aligned} \|u\|_{L^2([0,T],L^\infty)} &\leq \|\tilde{\varphi}(\Delta)u\|_{L^2([0,T],L^\infty)} \\ &+ \sum_{k \leq l} \|\varphi(2^{-2k}\Delta)u\|_{L^2([0,T],L^\infty)} + \sum_{k > l} \|\varphi(2^{-2k}\Delta)u\|_{L^2([0,T],L^\infty)}. \end{aligned} \tag{3.22}$$

By Corollary 2.2 we have

$$\|\tilde{\varphi}(\Delta)u\|_{L^2([0,T],L^\infty)} \leq CT^{\frac{1}{2}}\|u\|_{L^\infty([0,T],H^1)} \leq \tilde{C} \tag{3.23}$$

if $T \leq 1$, and, in view of Lemma 3.6,

$$\|\varphi(2^{-2k}\Delta)u\|_{L^2([0,T],L^\infty)} \leq C(\|\varphi(2^{-2k}\Delta)u\|_{L^2([0,T],H^1)} + 2^{-k\varepsilon}). \tag{3.24}$$

We apply this estimate and Schwarz inequality to the first sum in (3.22), while in the second sum we write

$$\|\varphi(2^{-2k}\Delta)u\|_{L^2([0,T],L^\infty)} \leq C(2^{-\beta k}\|\varphi(2^{-2k}\Delta)u\|_{L^2([0,T],H^s)} + 2^{-k\varepsilon}) \tag{3.25}$$

with $\beta = s - 1 > 0$. We obtain finally, for every $l \geq 0$,

$$\|u\|_{L^2([0,T],L^\infty)} \leq C\left(\sqrt{Tl} + 2^{-\beta l}\|u\|_{L^2([0,T],H^s)}\right)$$

thus, by choosing $l = D\log(2 + \|u\|_{L^2([0,T],H^s)})$ with D large enough, we conclude

$$\|u\|_{L^2([0,T],L^\infty)} \leq C\left[(T\log(2 + \|u\|_{L^2([0,T],H^s)}))^{\frac{1}{2}} + 1\right] \tag{3.26}$$

At this stage it is worth to observe, for future reference, that estimate (3.26) holds for all smooth solutions as soon as $s > 1$.

Then we come back to equation (3.20) and apply usual Sobolev estimates

$$\|u\|_{L^\infty([0,T],H^s)} \leq \|u_0\|_{H^s} + C\int_0^T \|u(t)\|_{L^\infty}^2 \|u(t)\|_{H^s} dt$$

so that, by Gronwall's lemma,

$$\begin{aligned} \|u\|_{L^\infty([0,T],H^s)} &\leq \|u_0\|_{H^s} \exp\left(C\|u\|_{L^2([0,T],L^\infty)}^2\right) \\ &\leq C\|u_0\|_{H^s} \exp\left(CT\log(2 + \|u\|_{L^\infty([0,T],H^s)})\right). \end{aligned}$$

If $CT \leq \frac{1}{2}$, we conclude

$$\|u\|_{L^\infty([0,T],H^s)} \leq C\left(\|u_0\|_{H^s} + \|u_0\|_{H^s}^2\right). \tag{3.27}$$

Applying this estimate successively on small time intervals, we conclude easily that $\|u\|_{L^\infty([0,T],H^s)}$ remains bounded if T remains bounded, which completes the proof for classical (smooth) solutions. If $u_0 \in H^s$ for some smaller $s > 1$, we regularize the data so that we obtain a family (u^ε) of smooth solutions. The above estimates applied to (u^ε) imply that u^ε is bounded in $L^\infty_{loc}(\mathbb{R}, H^s)$ and in $L^2_{loc}(\mathbb{R}, L^\infty)$. Passing to the limit in the equation, we obtain a solution u to our Cauchy problem which belongs to $L^\infty_{loc}(\mathbb{R}, H^s) \cap L^2_{loc}(\mathbb{R}, L^\infty)$, therefore $u \in C(\mathbb{R}, H^s)$ by using the Duhamel formula.

Finally, we turn to weak solutions in the energy space. The global existence of such solutions is easily obtained by standard compactness arguments from a sequence (u_n) of smooth solutions with Cauchy data converging to u_0 in H^1. Moreover, passing to the limit in the energy conservation law, gives for every time t,

$$\|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\|u(t)\|_{L^4}^4 \leq \|\nabla u_0\|_{L^2}^2 + \frac{1}{2}\|u_0\|_{L^4}^4.$$

If we prove uniqueness for the Cauchy problem, the above inequality becomes an equality by reversing time, leading to conservation of energy for u. As a consequence for any sequence (t_n) converging to t, the sequence $(u_n(t_n))$ converges strongly in H^1 to $u(t)$. Thus (u_n) converges strongly in H^1 to u, uniformly on any compact time interval.

It remains to prove the uniqueness of weak solutions in $L^\infty(H^1)$. Coming back to estimate (3.11) of lemma 3.6, we have that, by corollary 2.2, for every $p \in [6, \infty[$, for every $T \leq 1$,

$$\|\varphi(h^2\Delta)u\|_{L^2([0,T],L^p)} \leq C\left(h^{\frac{3}{p}}\|\varphi(h^2\Delta)u\|_{L^2([0,T],H^1)} + h^\varepsilon\right).$$

Summing on $h = 2^{-k}$ and applying the Cauchy-Schwarz inequality, we infer

$$\|u\|_{L^2([0,T],L^p)} \leq C\left(\sqrt{Tp}+1\right) \tag{3.28}$$

Given two such solutions u, $\tilde{u}$ with the same Cauchy data, we introduce $g(t) = \|u(t) - \tilde{u}(t)\|^2_{L^2(M)}$ and compute

$$\begin{aligned} \frac{dg}{dt} &= 2\,\mathrm{Im}\left(|u(t)|^2u(t) - |\tilde{u}(t)|^2\tilde{u}(t), u(t) - \tilde{u}(t)\right)_{L^2(M)} \\ &\leq C\int_M (|u(t)|^2 + |\tilde{u}(t)|^2)|u(t,x) - \tilde{u}(t,x)|^2 dx \end{aligned}$$

Following Yudovitch [**37**], we estimate the right hand side as

$$\frac{dg}{dt} \leq C\left(\|u(t)\|^2_{L^{2p}} + \|\tilde{u}(t)\|^2_{L^{2p}}\right)\|u(t) - \tilde{u}(t)\|^2_{L^{2\bar{p}}}$$

where $p < \infty$ is large, and $\bar{p} = \frac{p}{p-1}$. Interpolating the $L^{2\bar{p}}$ norm between the L^2 norm and the L^6 norm (which is bounded by the Sobolev embedding), we get

$$\frac{dg}{dt} \leq C\left(\|u(t)\|^2_{L^{2p}} + \|\tilde{u}(t)\|^2_{L^{2p}}\right)[g(t)]^{1-\frac{3}{2p}}$$

Integrating this differential inequality with $g(0) = 0$, and taking advantage of (3.28) for u and $\tilde{u}$, we obtain

$$p\,[g(t)]^{\frac{3}{2p}} \leq C(pt+1)$$

thus

$$g(t) \leq \left(C(t + \frac{1}{p})\right)^{\frac{2p}{3}}$$

which goes to 0 as p tends to infinity if $t > 0$ is small enough. This completes the proof of Theorem 3. $\square$

Remark 3.7. — *Let us make some remarks on the continuity properties of the flow map in Theorem 3. Using the arguments in the compactness proof above, one easily shows that this map is continuous on $H^1(M)$. However, the uniform continuity on bounded subsets of $H^1(M)$ is an open problem.*

4. On the optimality of Strichartz estimates

4.1. Sogge's estimates. — For all $k \geq 1$, denote by Π_k the spectral projector

$$\Pi_k = \mathbb{1}_{-\Delta\in[k^2,(k+1)^2]},$$

where Δ is the Laplace-Beltrami operator on a compact riemannian d-manifold M, $d \geq 2$. The following estimate is proved in [**28**] (see also [**29**]).

Proposition 4.1. —

$$\|\Pi_k\|_{L^2(M)\to L^q(M)} \leq Ck^{s(q)} \tag{4.1}$$

where

$$s(q) = \begin{cases} \frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right), & \text{if} \quad 2 \leq q \leq \frac{2(d+1)}{d-1}, \\ \frac{d-1}{2}-\frac{d}{q}, & \text{if} \quad \frac{2(d+1)}{d-1} \leq q \leq \infty. \end{cases}$$

Moreover, these estimates are sharp.

In particular, the above estimates give bounds on the growth of L^q norms of eigenfunctions normalized in L^2. However the sharpness of these bounds is a much more difficult problem (see Sogge-Zelditch [**30**] for recent results in this direction), except in specific geometries. For instance, if $M = S^d$, one can find for all $q \geq 2$, a sequence (h_k) of spherical harmonics, such that h_k has degree k and

$$\|h_k\|_{L^q} \approx c(q)k^{s(q)}\|h_k\|_{L^2}, \quad k \longrightarrow \infty \tag{4.2}$$

where $s(q)$ is the exponent in estimate (4.1) (see [**27**]).

4.2. Optimality of endpoint Strichartz estimates on spheres.

— If $d \geq 3$, endpoint Strichartz estimate of Theorem 1 reads

$$\|v\|_{L^2(I,L^q(M))} \leq C_I\|v_0\|_{H^{1/2}} \quad q = 2^\star = \frac{2d}{d-2}. \tag{4.3}$$

On the other hand, according to Proposition 4.1 $s(2^\star) = \frac{1}{2}$. But if $M = S^d$ and $v_0 = h_k$, $v(t) = \mathrm{e}^{-itk(k+d-1)}h_k$. Thus

$$\|v\|_{L^2(I,L^{2^\star}(M))} = \|h_k\|_{L^{2^\star}} \approx \|h_k\|_{H^{1/2}}$$

according to (4.2). As a consequence the endpoint Strichartz estimates of Theorem 1 are optimal on S^d for $d \geq 3$. In the case $d = 2$, letting p tend to 2, the argument gives the optimality of the L^pL^q estimate up to the loss of ε derivatives.

4.3. Improved Strichartz estimates for specific geometries. — In this section and the next one, we prove Theorem 4. Observe that $s_0(d)$ in this theorem is $s(4)$ in Proposition 4.1. It remains to prove estimate (1.9). By a dilation of the metric we may assume that the geodesics of M have 2π as a common period. Our starting point is the following result (see Guillemin [**16**], Colin de Verdière [**11**]).

Proposition 4.2. — *If the geodesics of M are 2π periodic, there exists $\alpha \in \mathbb{N}$ and $A > 0$ such that the spectrum of $-\Delta$ is contained in $\cup_{k=1}^{\infty} I_k$, where $I_k = [(k+\frac{\alpha}{4})^2 - A, (k+\frac{\alpha}{4})^2 + A]$.*

For $k \geq 1$, we denote by P_k the spectral projector $\mathbb{1}_{-\Delta \in I_k}$. Note that P_k satisfies the same estimates (4.1) as Π_k. Now we introduce the conormal spaces used by Bourgain in this context ([**3**], see also Ginibre [**14**] for a survey). For $f = f(t,x) \in C_0^\infty(\mathbb{R} \times M)$ set

$$\|f\|_{X^{s,b}} = \|\langle i\partial_t + \Delta\rangle^b f\|_{L^2(\mathbb{R},H^s)} \sim \left(\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \langle \tau + \mu_k\rangle^{2b} \|\widehat{P_k f}(\tau,\cdot)\|_{H^s}^2 d\tau\right)^{\frac{1}{2}}$$

where $\mu_k = (k+\frac{\alpha}{4})^2$, and $\widehat{P_k f}(\tau,x)$ denotes the Fourier transform of $P_k f(t,x)$ with respect to t. The connection to the linear Schrödinger equation is given by the following lemma.

Lemma 4.3. — *If $v(t) = \mathrm{e}^{it\Delta} v_0$ and $\chi \in C_0^\infty(\mathbb{R})$, then*

$$\|\chi v\|_{X^{s,b}} \leq C_{s,b} \|v_0\|_{H^s}$$

for all $s, b > 0$.

Proof. Denote by p_λ the spectral projector on the eigenspace associated to λ. We have

$$\|\widehat{P_k(\chi v)}(\tau)\|_{H^s}^2 = \sum_{\lambda \in I_k} |\hat{\chi}(\tau+\lambda)|^2 \|p_\lambda v_0\|_{H^s}^2$$

and, for $\lambda \in I_k$

$$|\hat{\chi}(\tau+\lambda)| \leq C_N \langle \tau+\lambda\rangle^{-N} \leq \tilde{C}_N \langle \tau+\mu_k\rangle^{-N}$$

for all N. Therefore

$$\|\chi v\|_{X^{s,b}}^2 \leq C \sum_k \sum_{\lambda \in I_k} \|p_\lambda v_0\|_{H^s}^2 \leq \tilde{C} \|v_0\|_{H^s}^2.$$

$\square$

In view of Lemma 4.3, estimate (1.9) in Theorem 4 is a consequence of the following result.

Lemma 4.4. — *If $f \in C_0^\infty(\mathbb{R} \times M)$,*

$$\|f\|_{L^4(\mathbb{R}\times M)} \leq C_{s,b}\|f\|_{X^{s,b}}$$

for all $s > s_0(d)$, $b > \frac{1}{2}$.

Proof. Using dyadic decomposition it is enough to prove that

$$\|f\|_{L^4(\mathbb{R}\times M)} \leq C_{s,b}N^s\|f\|_{X^{0,b}}$$

if $P_k f = 0$ unless $N \leq k \leq 2N$(see Corollary 2.3). Then

$$\begin{aligned}
\|f\|^4_{L^4(\mathbb{R}\times M)} &= \|f^2\|^2_{L^2(\mathbb{R}\times M)} \\
&= \Big\| \sum_{N\leq k_1,k_2\leq 2N} P_{k_1}f\, P_{k_2}f \Big\|^2_{L^2(\mathbb{R}\times M)} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} d\tau \Big\| \sum_{N\leq k_1,k_2\leq 2N} (\widehat{P_{k_1}f\,P_{k_2}}f)(\tau)\Big\|^2_{L^2(M)}.
\end{aligned}$$

But

$$\begin{aligned}
\Big\|\sum_{k_1,k_2}(\widehat{P_{k_1}f\,P_{k_2}}f)(\tau)\Big\|_{L^2} &= \Big\|\int_{\mathbb{R}} d\sigma \sum_{k_1,k_2} P_{k_1}\hat{f}(\sigma)P_{k_2}\hat{f}(\tau-\sigma)\Big\|_{L^2} \\
&\leq \int_{\mathbb{R}}\sum_{k_1,k_2}\|P_{k_1}\hat{f}(\sigma)\,P_{k_2}\hat{f}(\tau-\sigma)\|_{L^2}d\sigma \\
&\leq CN^{2s_0(d)}\int_{\mathbb{R}}\sum_{k_1,k_2}\|P_{k_1}\hat{f}(\sigma)\|_{L^2}\|P_{k_2}\hat{f}(\tau-\sigma)\|_{L^2}d\sigma
\end{aligned}$$

in view of Hölder's inequality and Sogge's estimate (4.1).

Using Cauchy-Schwarz inequality in σ, k_1, k_2, we infer

$$\begin{aligned}
(4.4)\quad \|f\|^4_{L^4(\mathbb{R}\times M)} \leq CN^{4s_0(d)}\int_{\mathbb{R}} d\tau \left(\int_{\mathbb{R}} d\sigma\sum_{k_1,k_2}\langle\sigma+\mu_{k_1}\rangle^{-2b}\langle\tau-\sigma+\mu_{k_2}\rangle^{-2b}\right) \\
\times\left(\int_{\mathbb{R}} d\sigma\sum_{k_1,k_2}\langle\sigma+\mu_{k_1}\rangle^{2b}\|P_{k_1}\hat{f}(\sigma)\|^2_{L^2}\langle\tau-\sigma+\mu_{k_2}\rangle^{2b}\|P_{k_2}\hat{f}(\tau-\sigma)\|^2_{L^2}\right).
\end{aligned}$$

Therefore Lemma 4.4 will be proved by the following estimate

$$\sup_{\tau\in\mathbb{R}}\int_{\mathbb{R}} d\sigma \sum_{N\leq k_1,k_2\leq 2N} \langle\sigma+\mu_{k_1}\rangle^{-2b}\langle\tau-\sigma+\mu_{k_2}\rangle^{-2b} = O(N^\epsilon) \tag{4.5}$$

for all $\epsilon>0$, $b>\frac{1}{2}$. Observe that the left hand side of (4.5) is controlled by

$$\begin{aligned}\sup_{\tau\in\mathbb{R}} \sum_{N\leq k_1,k_2\leq 2N} & \langle\tau+\mu_{k_1}+\mu_{k_2}\rangle^{-2b} \\ & \leq C\sup_{\tau\in\mathbb{R}}\sum_{N^2\leq n\leq 200N^2}\langle 16\tau+n\rangle^{-2b} \\ & \qquad \#\{(k_1,k_2)\in\mathbb{N}\times\mathbb{N},\ n=(4k_1+\alpha)^2+(4k_2+\alpha)^2\} \\ & \leq C\sup_{\tau\in\mathbb{R}}\sum_{N^2\leq n\leq 200N^2}\langle 16\tau+n\rangle^{-2b}\ r_2(n)\end{aligned} \tag{4.6}$$

where $r_2(n)=\#\{(p,q)\in\mathbb{N}\times\mathbb{N},\ n=p^2+q^2\}=O(n^\epsilon)$ for all $\epsilon>0$ (see e.g. [**15**]). This completes the proof of estimate (1.9). □

Remark 4.5. — *It is worth noticing that on S^3 we dispose with a set of optimal Strichartz inequalities for admissible pairs, up to a loss of ε derivative for the intermediate cases. Let (p,q) be an admissible pair in 3d, i.e. $\frac{2}{p}+\frac{3}{q}=\frac{3}{2}$ with $p\geq 2$. Then an interpolation between the results of Theorem 1 and Theorem 4 yields the estimate*

$$\|\exp(it\Delta_{S^3})v_0\|_{L^p(I,L^q(S^3))}\leq C(I)\|v_0\|_{H^s(S^3)}, \tag{4.7}$$

provided $s>s(q)=\max(\frac{1}{2}-\frac{1}{q},1-\frac{3}{q})$. Indeed, applying Hölder inequality in time together with Theorem 4, one easily obtains

$$\|\exp(it\Delta_{S^3})v_0\|_{L^{\frac{8}{3}}(I,L^4(S^3))}\leq C(I)\|v_0\|_{H^s(S^3)},\quad s>\frac{1}{4}. \tag{4.8}$$

Interpolating between (4.8) and the L^2 conservation yields (4.7) for $q\in[2,4]$ while an interpolation between (4.8) and the 3d endpoint result in Theorem 1 gives (4.7) for $q\in[4,6]$. Remark also that starting from inequality (1.9) *rather than* (4.8) *one even obtains* (4.7) *with p replaced by $\tilde{p}$ given by*

$$\begin{cases}\frac{1}{\tilde{p}}+\frac{3}{q} & =1\ \text{if}\ 4\leq q\leq 6\\ \frac{1}{\tilde{p}}+\frac{1}{q} & =\frac{1}{2}\ \text{if}\ 2\leq q\leq 4.\end{cases} \tag{4.9}$$

Finally, the estimate (4.7) fails for $s < s(q)$ by taking $v_0 = h_k$ as in (4.2) and letting k tend to infinity.

4.4. Optimality in Theorem 4. — If $d = 2$, the optimality stated in Theorem 4 is guaranteed by (4.2). Now suppose $d \geq 3$, and proceed by contradiction. Assume we have the following inequality for $v(t) = \mathrm{e}^{it\Delta} v_0$,

$$\|v\|_{L^4(I\times S^d)} \leq C(I)\|v_0\|_{H^s(S^d)}, \quad s = s_0(d) = \frac{d}{4} - \frac{1}{2}. \tag{4.10}$$

Let $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\}$. By Corollary 2.2, we infer, for every $N > 1$,

$$\|\varphi(N^{-2}\Delta)v\|_{L^4(I,L^\infty(S^d))} \leq C(I)N^{\frac{d-1}{2}}\|\varphi(N^{-2}\Delta)v_0\|_{L^2(S^d)}. \tag{4.11}$$

Let $x_0 \in S^d$ and h_n be a sequence of zonal spherical harmonics of degree n (see e.g. Stein-Weiss [**33**]) such that

$$\|h_n\|_{L^2} = 1, \quad h_n(x_0) = \|h_n\|_{L^\infty} \approx n^{\frac{d-1}{2}} \ .$$

Apply the above inequality to

$$v_N(t,x) = \sum_n \varphi(-\frac{n(n+d-1)}{N^2})\mathrm{e}^{-itn(n+d-1)}\frac{h_n(x)}{h_n(x_0)} \ .$$

We obtain

$$\|\sum_n \varphi(-\frac{n(n+d-1)}{N^2})\mathrm{e}^{-itn(n+d-1)}\|_{L^4(I)} \leq C\sqrt{N}. \tag{4.12}$$

We are therefore reduced to disprove inequality (4.12), which can be classically done by use of Gauß sums. For the sake of completeness, we give a proof below.

We begin with the following estimate related to Gauß sums, which is rather crude but will be sufficient for our purpose.

Lemma 4.6. — *Let $\Psi \in C^1(\mathbb{R})$, compactly supported, $N \in \mathbb{N}$, p be a prime odd number, $a \in \{1, \dots, p-1\}$, $b \in \mathbb{Z}$. Then*

$$\left| \Big|\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{2i\pi}{p}(an^2+bn)}\Psi(\frac{n}{N})\Big| - \frac{N}{\sqrt{p}}\Big|\int_{\mathbb{R}}\Psi(y)dy\Big| \right| \leq p\int_{\mathbb{R}}|\Psi'(y)|dy \tag{4.13}$$

Proof. Set $e(x) = \mathrm{e}^{-2i\pi x}$. Then

$$\sum_{n\in\mathbb{Z}} e(\frac{an^2+bn}{p})\Psi(\frac{n}{N}) = \sum_{r=0}^{p-1} e(\frac{ar^2+br}{p})S_r \tag{4.14}$$

with

$$S_r = \sum_{q\in\mathbb{Z}} \Psi(\frac{pq+r}{N}) \tag{4.15}$$

Observe that for every r,
(4.16)

$$\int_{\mathbb{R}} \Psi(x)dx = \sum_{q\in\mathbb{Z}} \int_{pq/N}^{p(q+1)/N} \Psi\left(y+\frac{r}{N}\right) dy$$
$$= \sum_{q\in\mathbb{Z}} \frac{p}{N}\Psi\left(\frac{pq+r}{N}\right) + \sum_{q\in\mathbb{Z}} \int_{pq/N}^{p(q+1)/N} \left(\int_{pq/N}^{y} \Psi'\left(t+\frac{r}{N}\right) dt\right) dy$$

so that

$$\left|S_r - \frac{N}{p}\int_{\mathbb{R}} \Psi(y)dy\right| \leq \int_{-\infty}^{+\infty} |\Psi'(y)|dy \tag{4.17}$$

and therefore

$$\left|\sum_{r=0}^{p-1} e(\frac{ar^2+br}{p})S_r - \frac{N}{p}\int_{\mathbb{R}} \Psi(y)dy \sum_{r=0}^{p-1} e(\frac{ar^2+br}{p})\right| \leq p\int_{\mathbb{R}} |\Psi'(y)|dy \tag{4.18}$$

This completes the proof in view of the classical Gauß formula

$$\left|\sum_{r=0}^{p-1} e(\frac{ar^2+br}{p})\right| = \sqrt{p}. \tag{4.19}$$

□

Assume $t = \frac{2\pi a}{p} + \frac{s}{N^2}$ with $|s| \leq \delta$ small enough. Apply Lemma 4.6 to

$$\Psi_N(y) = \varphi(-y(y+\frac{d-1}{N}))\mathrm{e}^{-isy(y+\frac{d-1}{N})} .$$

We get

$$\left||f_N(t)| - \frac{N}{\sqrt{p}}\left|\int_{\mathbb{R}} \Psi_N(y)\, dy\right|\right| \leq Cp,$$

where

$$f_N(t) = \sum_n \varphi(-\frac{n(n+d-1)}{N^2})\mathrm{e}^{-itn(n+d-1)}.$$

Observe that

$$\int_{\mathbb{R}} \Psi_N(y)\, dy \to \int_{\mathbb{R}} \varphi(-y^2)\, dy$$

as N tends to infinity and as s tends to 0. By choosing φ such that this limit is not zero, we obtain

Lemma 4.7. — *There exists $\delta > 0$ such that for all N, all odd prime number p such that $p \ll N^{2/3}$, all $a \in \{1, \dots, p-1\}$,*

$$|f_N(t)| \geq \frac{N}{2\sqrt{p}} \left| \int_{-\infty}^{\infty} \varphi(-y^2) dy \right| \tag{4.20}$$

if

$$\left| t - \frac{2\pi a}{p} \right| \leq \frac{\delta}{N^2}. \tag{4.21}$$

We now complete the proof of optimality in estimate (1.9) of Theorem 4. We may assume $I = [0, 2\pi]$. It is easy to estimate $\|f_N\|_{L^4(\mathbb{T})}$ from below using Lemma 4.7. Observe that the sets

$$\Omega_{a,p} = \left\{ t \in [0, 2\pi], |t - \frac{2\pi a}{p}| \leq \frac{\delta}{N^2}, \right\} \tag{4.22}$$

are pairwise disjoint as p varies in the set of prime numbers such that $2 < p \ll N^{2/3}$ and $a \in \{1, \dots, p-1\}$. As a consequence

$$\begin{aligned} \|f_N\|^4_{L^4(\mathbb{T})} &\geq \sum_{2<p\ll N^{2/3}} \sum_{a=1}^{p-1} \|f_N\|^4_{L^4(\Omega_{a,p})} \\ &\geq c \sum_{2<p\ll N^{2/3}} p \times \frac{N^4}{p^2} \times \frac{1}{N^2} \\ &\geq c\, N^2 \sum_{2<p\ll N^{2/3}} \frac{1}{p} \end{aligned} \tag{4.23}$$

which disproves inequality (4.12) since $\sum_{2<p\ll N^{2/3}} \frac{1}{p}$ is unbounded as N tends to the infinity.

Appendix A
Generalizations

In this last section, we collect some generalizations of theorem 1 and sketch their proofs.

A.1. Metrics on $\mathbb{R}^d$. — The first generalization is concerned with the Laplace operator associated to a riemannian metric, g, on $\mathbb{R}^d$ satisfying the following uniform bounds

$$\begin{aligned} &\exists 0 < m < M;\ \forall x \in \mathbb{R}^d,\ m\mathrm{Id} \leq g(x) \leq M\mathrm{Id} \\ &\forall \alpha \in \mathbb{N}^d,\ \exists C_\alpha;\ \forall x \in \mathbb{R}^d,\ |\partial^\alpha g(x)| \leq C_\alpha \end{aligned} \tag{A.1}$$

In the case where g is a *compactly supported perturbation* of the standard metric and if every geodesic of the metric g goes out of any compact set after some time (i.e. g is *non-trapping*), then it is known that the corresponding linear Schrödinger equation satisfies the same Strichartz estimates as in $\mathbb{R}^d$ with the standard metric (see Staffilani and Tataru [**31**]). Here we can prove the following (weaker) statement, without any geometric assumption on g.

Theorem 5. — *Assume that* (A.1) *holds. Then the solution to the Schrödinger equation*

$$(i\partial_t + \Delta_g)v = 0, \qquad v(0,x) = v_0(x) \tag{A.2}$$

satisfies the same estimates as in Theorem 1,

$$\|v\|_{L^p(I,L^q(\mathbb{R}^d))} \leq C(I)\|v_0\|_{H^{1/p}(\mathbb{R}^d)} \tag{A.3}$$

for all (p,q) *such that* $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $p \geq 2$, $q < \infty$, *and all finite intervals* I.

Sketch of the proof of Theorem 5. We follow essentially the same lines as in the proof of Theorem 1, with the following variant, due to the different status of the L^p spaces on $\mathbb{R}^d$. First we state the following version of proposition 2.1, the proof of which being similar. We denote by C_b^∞ the space of C^∞ functions with bounded derivatives at any order.

Proposition A.1. — *Let P be an elliptic selfadjoint operator of order $m > 0$ on $\mathbb{R}^d$. Assume that the symbol p and the principal symbol p_m satisfy*

$$\begin{aligned} &\forall \alpha,\beta \in \mathbb{N}^d, \exists C_{\alpha,\beta}; \forall (x,\xi) \in \mathbb{R}^{2d}, |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}\langle \xi \rangle^{m-|\beta|} \\ &\exists C > 0; \forall (x,\xi) \in \mathbb{R}^{2d}, |p_m(x,\xi)| \geq C|\xi|^m. \end{aligned} \tag{A.4}$$

Given $\varphi \in C_0^\infty(\mathbb{R})$ there exists a sequence $(\Psi_j) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for any $N \in \mathbb{N}$, any $h \in]0,1]$ and any $f \in C_0^\infty(\mathbb{R}^d)$

$$\|\varphi(h^m P)f - \sum_{j=0}^{N-1} h^j \Psi_j(x, hD_x)f\|_{H^\sigma} \leq Ch^{N-\sigma}\|f\|_{L^2}. \tag{A.5}$$

Moreover $\Psi_0(x,\xi) = \varphi(p_m(x,\xi))$ *and, for every* $j \geq 1$, Ψ_j *is a linear combination of* $\varphi^{(k)}(p_m)$ *with coefficients in* $C_b^\infty(\mathbb{R}^{2d})$.

Corollary A.2. — *Given* $\Psi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, *there exists* $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ *such that for all* $h \in]0,1]$, $\sigma > 0$, $N > 0$, $f \in C_0^\infty(\mathbb{R}^d)$,

$$\|\Psi(hD)(1-\varphi(h^m P))f\|_{H^\sigma(\mathbb{R}^d)} \leq C_{\sigma,N} h^N \|f\|_{L^2(\mathbb{R}^d)} \tag{A.6}$$

Choose $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that $\varphi = 1$ near the following compact subset of $\mathbb{R} \setminus \{0\}$:

$$K = \{p_m(x,\xi), \xi \in \operatorname{supp}(\Psi), x \in \mathbb{R}^d\} \tag{A.7}$$

so that by Proposition A.1, for $j \geq 1$ and $|\alpha| \geq 0$,

$$\partial_\xi^\alpha \Psi(\xi) \partial_x^\alpha (1 - \Psi_0(x,\xi)) \equiv 0, \qquad \partial_\xi^\alpha \Psi(\xi) \partial_x^\alpha \Psi_j(x,\xi) \equiv 0, \tag{A.8}$$

and Corollary A.2 follows. □

Next by the same WKB method we prove the following analogue of Lemma 2.5.

Lemma A.3. — *Given* $\Psi \in C_0^\infty(\mathbb{R}^d)$, *there exists* $\alpha > 0$, $C > 0$ *such that, for every* $v_0 \in C_0^\infty(\mathbb{R}^d)$ *and every* $h \in]0,1]$, $t \in J = [-\alpha h, \alpha h]$,

$$\|e^{it\Delta_g}\Psi(hD)v_0\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{|t|^{d/2}} \|v_0\|_{L^1(\mathbb{R}^d)} \tag{A.9}$$

Remark A.4. — *Remark that although the WKB method is used in Lemma 2.5* locally *with respect to the* x *variable, here, according to assumptions* (A.1)*, the solutions to eikonal and transport equations are defined* globally *on* $[-\alpha,\alpha] \times \mathbb{R}^d$ *for* α *small enough, which gives Lemma A.3.*

Applying Proposition 2.8 to $U(t) = \mathbb{1}_J(t)\Psi(hD)e^{it\Delta_g}$, we obtain

$$\left(\int_J \|\Psi(hD)e^{it\Delta_g} v_0\|_{L^q}^p dt \right)^{1/p} \leq C\|v_0\|_{L^2}. \tag{A.10}$$

If we assume $\Psi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ and introduce $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ given by Corollary A.2, we obtain

$$\left(\int_J \|\Psi(hD)e^{it\Delta_g} v_0\|_{L^q}^p \right)^{1/p} \leq C\|\varphi(h^2\Delta_g)v_0\|_{L^2} + C_N h^N \|v_0\|_{L^2}. \tag{A.11}$$

Then the rest of the proof is similar to what was done in section 2, by summing on intervals J and on frequencies by the usual Littlewood-Paley estimate.

A.2. Abstract lower order terms. — Our second generalization deals with abstract perturbations, which typically cover potentials associated to N-body problems.

Theorem 6. — *Let V be an unbounded symmetric operator on $L^2(\mathbb{R}^d)$ with domain $D(V) = H^1(\mathbb{R}^d)$ and satisfying*

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{H^1(\mathbb{R}^d)} \tag{A.12}$$

Under the conditions of Theorem 5, the solution to the linear Schrödinger equation

$$(i\partial_t + \Delta_g + V)u = 0, u(0,x) = u_0(x) \tag{A.13}$$

satisfies the same estimates as in Theorem 1,

$$\|u\|_{L^p(I,L^q(\mathbb{R}^d))} \leq C(I)\|u_0\|_{H^{1/p}(\mathbb{R}^d)} \tag{A.14}$$

for all (p,q) such that $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $p \geq 2$, $q < \infty$, and all finite intervals I.

Sketch of the proof. Let $A = \Delta_g + V$. From the Kato-Rellich lemma (see e.g. Reed-Simon [**25**]), A is self-adjoint with domain $D(A) = H^2(\mathbb{R}^d)$.

Moreover for any $\varphi \in C_0^\infty(\mathbb{R})$, any $0 \leq r \leq 1$, $0 \leq s \leq 1$,

$$\|\varphi(h^2A) - \varphi(h^2\Delta_g)\|_{H^r \to H^s} \leq Ch^{1-s+r} \tag{A.15}$$

Indeed, from Helffer-Sjöstrand's formula (2.1) applied to A and to Δ_g,

$$\begin{aligned}\varphi(h^2A) - \varphi(h^2\Delta_g) &= -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\tilde{\varphi}(z)\left((z-h^2A)^{-1} - (z-h^2\Delta_g)^{-1}\right)dxdy \\ &= -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\tilde{\varphi}(z)(z-h^2A)^{-1}h^2V(z-h^2\Delta_g)^{-1}dxdy\end{aligned} \tag{A.16}$$

and (A.15) follows from the assumptions on V and from the standard estimates

$$\begin{aligned}\|(z-h^2A)^{-1}\|_{L^2\to H^s} &\leq \frac{C}{h^s|\mathrm{Im} z|} \\ \|(z-h^2\Delta_g)^{-1}\|_{H^r\to H^1} &\leq \frac{C}{h^{1-r}|\mathrm{Im} z|}.\end{aligned} \tag{A.17}$$

As a second step, we are going to take advantage of the conservation of the L^2 norm

$$\|\mathrm{e}^{itA}u_0\|_{L^2} = \|u_0\|_{L^2} \tag{A.18}$$

to handle the perturbation as a source term: write the solution of equation (A.13) as

$$u(t) = \mathrm{e}^{it\Delta_g}u_0 - \int_0^t \mathrm{e}^{it(t-\tau)\Delta_g}(Vu(\tau))d\tau. \tag{A.19}$$

Using estimates (A.11) we obtain, if $J = [0, \alpha h]$,

$$\|\Psi(hD)u\|_{L^p(J,L^q)} \leq C\|u_0\| + C\|\varphi(h^2\Delta_g)(Vu)\|_{L^1(J,L^2)} + C_N h^N \|Vu\|_{L^1(J,L^2)} \tag{A.20}$$

Since φ is compactly supported,

$$\|\varphi(h^2\Delta_g)(Vf)\|_{L^2} \leq \frac{1}{h}\|\varphi(h^2\Delta_g)(Vf)\|_{H^{-1}} \leq \frac{C}{h}\|f\|_{L^2} \tag{A.21}$$

since V is bounded from L^2 to H^{-1} by duality.

Taking advantage of the length of J we conclude

$$\|\Psi(hD)u\|_{L^p(J,L^q)} \leq C\|u_0\| + C_N h^N \|u_0\|_{H^1} \tag{A.22}$$

Next we apply the above estimate to $\varphi(h^2A)u_0$ in place of u_0. Then u is replaced by $\varphi(h^2A)u$ and since

$$\|\varphi(h^2A)u_0\|_{H^1} \leq \frac{C}{h}\|\varphi(h^2A)u_0\|_{L^2} \tag{A.23}$$

we obtain

$$\|\Psi(hD)\varphi(h^2A)u\|_{L^p(J,L^q)} \leq C\|\varphi(h^2A)u_0\|_{L^2}. \tag{A.24}$$

Next we sum on intervals J of length $|J| \leq \alpha h$ to get an estimate on intervals of fixed length

$$\|\Psi(hD)\varphi(h^2A)u\|_{L^p(I,L^q)} \leq Ch^{-1/p}\|\varphi(h^2A)u_0\|_{L^2} \leq C\|\varphi(h^2A)u_0\|_{H^{1/p}}. \tag{A.25}$$

It remains to sum on frequencies. Observe that

$$\Psi(hD) = \Psi(hD)\varphi(h^2A) + \Psi(hD)(1-\varphi(h^2\Delta_g)) + \Psi(hD)(\varphi(h^2\Delta_g) - \varphi(h^2A)) \tag{A.26}$$

Applying Corollary A.2 and estimate (A.15) with $s = \frac{2}{p} = d(\frac{1}{2} - \frac{1}{q})$ and $r = \frac{1}{p}$ we conclude

$$\|\Psi(hD)u\|_{L^p(I,L^q)} \leq C\|\varphi(h^2A)u_0\|_{H^{1/p}} + Ch^{1-\frac{1}{p}}\|u_0\|_{H^{1/p}} \tag{A.27}$$

Therefore Theorem 6 follows from the usual Littlewood-Paley estimate. □

Remark A.5. — *Note that the nonlinear arguments involved in the proof of Theorem 2 and Theorem 3 can be easily adapted to the framework of this appendix.*

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