An example of singular dynamics for the nonlinear Schrödinger equation on bounded domains

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1. Introduction and statement of the results

Let Ω be bounded subset of $\mathbb{R}^2$. The purpose of this note is to stress the influence of the geometry of Ω on the dynamics of the nonlinear Schrödinger equation,

$$\left\{\begin{array}{l} i\partial_t u + \Delta u + |u|^2 u = 0, \quad t \in \mathbb{R},\ x \in \Omega, \\ u|_{\mathbb{R}\times\partial\Omega} = 0. \end{array}\right. \tag{1.1}$$

In view of the formal conservation laws,

$$\left\{\begin{array}{l} \|u(t)\|_{L^2}^2 = cst, \\ \\ \|\nabla_x u(t)\|_{L^2}^2 - \frac{1}{2}\|u(t)\|_{L^4}^4 = cst, \end{array}\right. \tag{1.2}$$

a natural space for (1.1) is of course the Sobolev space $H_0^1(\Omega)$. First we recall the classical existence results in this phase space.

Theorem 1. (Brézis-Gallouet [**6**], Vladimirov [**11**], Ogawa-Ozawa [**10**])
For every $u_0 \in H_0^1(\Omega)$, there exists a unique maximal solution $u \in C(I, H_0^1(\Omega))$ with $u(0,x) = u_0(x)$. Moreover u satisfies the conservation laws (1.2) on I, and, if I has a finite extremity a, then

$$\|\nabla_x u(t)\|_{L^2} \to +\infty$$

as $t \in I$ tends to a.
Moreover, if Ω is regular of class C^2 and $u_0 \in H^s(\overline{\Omega})$ for some $s \in]1,2]$, then $u \in C(I, H^s(\overline{\Omega}))$.
Finally, there exists $\delta_0 > 0$ such that if $\|u_0\|_{L^2} < \delta_0$, then $I = \mathbb{R}$.

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Notice that the optimal value of δ_0 is known and does not depend on Ω. If Q is the positive radial solution on $\mathbb{R}^2$ of the equation

$$\Delta Q + Q^3 = Q$$

one can prove that $\delta_0 = \|Q\|_{L^2}$. Indeed, that $\delta_0 \geq \|Q\|_{L^2(\mathbb{R}^2)}$ is a consequence of a refinement of the Gagliardo-Nirenberg inequality due to M. Weinstein [**13**]. Moreover, the existence on Ω of blow up solutions u such that $\|u(t)\|_{L^2(\Omega)} = \|Q\|_{L^2(\mathbb{R}^2)}$ was recently proved by the authors in [**4**].

Here we shall study another feature of dynamics, namely the possibility of extending the flow map to more singular data. More precisely, for every $s \in [0,1]$, let us denote by $H^s_D(\Omega)$ the domain of the operator $(-\Delta_D)^{\frac{s}{2}}$ on $L^2(\Omega)$, where $-\Delta_D$ is the Dirichlet Laplacian. Of course, $H^0_D = L^2(\Omega)$, $H^1_D(\Omega) = H^1_0(\Omega)$ and, if Ω is smooth enough and $s < \frac{1}{2}$, $H^s_D(\Omega) = H^s(\overline{\Omega})$ (see [**9**]).
We address the following problem :

For which values of $s \in [0,1]$ the flow map, $u_0 \longmapsto u$ is uniformly continuous on bounded subsets B of $H^s_D(\Omega)$, with values in $C([-T,T], H^s_D(\Omega))$, at least for $T = T(B)$ small enough ?

Denote by $\mathcal{P}_s$ this property. $\mathcal{P}_s$ means local wellposedness of (1.1) on $H^s_D(\Omega)$, including stability of the flow map with respect to possible oscillations and concentration effects of the data. Let us briefly recall what is known in the literature about this problem.

First of all, we only addressed the validity of $\mathcal{P}_s$ for $s \leq 1$, since $\mathcal{P}_s$ trivially holds for $s \in]1,2]$ (and Ω smooth) because of the Sobolev embedding $H^s(\overline{\Omega}) \subset L^\infty(\Omega)$.

The analogue of property $\mathcal{P}_s$ on $\mathbb{R}^2$ is well known. Indeed, for $s > 0$, it is a classical consequence of Strichartz inequalities for the linear Schrödinger evolution on $\mathbb{R}^2$, as was shown in Cazenave-Weissler [**7**].

On bounded subsets, very few is known about $\mathcal{P}_s$, with the notable exception of a square. Indeed, if $\Omega =]0,1[^2$, then $H^1_0(\Omega)$ can be identified to the space of $u \in H^1((\mathbb{R}/2\mathbb{Z})^2)$ satisfying

$$u(\varepsilon_1 x_1, \varepsilon_2 x_2) = \varepsilon_1 \varepsilon_2 u(x_1, x_2), \quad \varepsilon_j \in \{-1, 1\}. \tag{1.3}$$

Since condition (1.3) is clearly invariant through the NLS flow, $\mathcal{P}_s$ for every $s \in]0,1]$ is a consequence of local wellposedness results of Bourgain [**2**] for the nonlinear Schrödinger equation on $H^s(\mathbb{T}^2)$, $s \in]0,1]$. This result is in strong contrast with our main observation, which concerns the case of a disc.

THEOREM 2. *Assume Ω is the unit disc and $s \in [0, \frac{1}{3}[$. Then, for any ball B centered at 0 in $H^s(\overline{\Omega}) = H^s_D(\Omega)$, for any $T > 0$, the flow map*

$$u_0 \in B \cap H^1_0(\Omega) \longmapsto u \in C([0,T], H^s(\overline{\Omega}))$$

is not uniformly continuous.

The key of Theorem 2 is a nonlinear ansatz for u if u_0 is an eigenfunction of the Dirichlet Laplacian on the disc, with a large eigenvalue. Recall that such eigenfunctions read, in polar coordinates (r, θ) associated to $x \in \Omega$,

$$f_{n,k}(x) = J_n(z_{n,k}\, r)\, e^{in\theta} \tag{1.4}$$

where $n \in \mathbb{N}$, J_n is the Bessel function of index n, and $(z_{n,k})_{k\geq 1}$ is the increasing sequence of positive zeros of J_n. The corresponding eigenvalue of $-\Delta_D$ is $\lambda = z^2_{n,k}$. We shall chose u_0 to be proportional to $f_{n,1}$ and set

$$\varphi_n = n^{\frac{2}{3}-s} f_{n,1} \tag{1.5}$$

so that, in view of the asymptotic behavior of J_n (see section 2), $\|\varphi_n\|_{H^s} \approx 1$. An important quantity is then

$$\omega_n = \frac{\|\varphi_n\|^4_{L^4}}{\|\varphi_n\|^2_{L^2}} \approx n^{\frac{2}{3}-2s} \longrightarrow +\infty \tag{1.6}$$

is $s \in [0, \frac{1}{3}[$. Then one can prove

THEOREM 3. *Assume $u_0 = \kappa\varphi_n$ for some constant $\kappa > 0$ and $s \in [0, \frac{1}{3}[$. Then, for n large enough, the solution of (1.1) is defined for every time and satisfies, locally uniformly in κ and $t \in \mathbb{R}$,*

$$u(t) = \kappa e^{-it(z^2_{n,1} - \kappa^2\omega_n)}\, \varphi_n + \mathcal{O}(n^{-4s}), \tag{1.7}$$

in $H^s(\Omega)$, if $s \in [0, \frac{1}{3}[$ and $n \gg 1$. In the particular case $s = 0$, the above formula can be modified into

$$u(t) = \kappa e^{-it(z^2_{n,1} - \kappa^2\omega_n - 3\kappa^4\theta_n)}\, \varphi_n + o(1) \tag{1.8}$$

in $L^2(\Omega)$, $n \gg 1$, where

$$\theta_n = \frac{((-\Delta_D - z^2_{n,1})^{-1} r_n \mid r_n)_{L^2}}{\|\varphi_n\|^2_{L^2}}$$

and r_n is the projection of $|\varphi_n|^2\varphi_n$ orthogonally to φ_n. One can show that (θ_n) is bounded.

Let us make some brief comments about Theorem 3. Of course, we have

$$e^{it\Delta_D}\varphi_n = e^{-itz_{n,1}^2}\varphi_n,$$

hence formulae (1.7), (1.8) show how the nonlinear term in (1.1) modifies the dynamics. It is somewhat surprising that this modification is only affected by a phase shift, at the first order if $n \to \infty$. As we shall see, this simple but nontrivial modification will allow us to prove Theorem 2 very easily.

The rest of the paper is organized as follows : in section 2, we show how Theorem 3 implies Theorem 2, and we give a complete proof of Theorem 2. In section 3, we discuss various generalizations of these results. In particular, we sketch a generalization to higher dimensions which, roughly speaking , implies illposedness, in the energy space $H_0^1(\Omega)$, of any nonlinear Schrödinger equation on the d-dimensional unit ball for $d \geq 5$.

We close this introduction by mentioning that the results contained in this paper are slight improvements of Theorem 2 in [**4**].

2. Instability on the unit disc

2.1. We begin with proving that Theorem 3 implies Theorem 2. Choose $\kappa > 0$, and κ_n a sequence such that $\kappa_n \to \kappa$ as $n \to +\infty$ so slowly that

$$(\kappa_n - \kappa)\omega_n \longrightarrow +\infty, \quad \text{as } n \text{ tends to } +\infty. \tag{2.1}$$

Let u_n, $\tilde{u}_n$ be the solutions of (1.1) with $u_n(0) = \kappa\varphi_n$, $\tilde{u}_n(0) = \kappa_n\varphi_n$. Since $\kappa_n \to \kappa$ and $\|\varphi_n\|_{H^s} \approx 1$, it is clear that

$$\|u_n(0) - \tilde{u}_n(0)\|_{H^s} \longrightarrow 0, \quad \text{as } n \text{ tends to } +\infty.$$

Moreover, by Theorem 3, we have, if $s \in]0, \frac{1}{3}[$,

$$\|u_n(t) - \tilde{u}_n(t)\|_{H^s} = 2\kappa \left|\sin\left(\frac{(\kappa_n^2 - \kappa^2)\omega_n t}{2}\right)\right| \|\varphi_n\|_{H^s} + o(1)$$

By (2.1), $(\kappa_n^2 - \kappa^2)\omega_n \longrightarrow +\infty$ as $n \to +\infty$, hence $\|u_n(t) - \tilde{u}_n(t)\|_{H^s}$ cannot tend to zero uniformly on any time interval $[0, T]$, $T > 0$.

2.2. Proof of Theorem 3. First we collect some basic properties about Bessel functions, see e.g. Watson [**12**]. For $n \in \mathbb{N}$, recall that

$$J_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin\theta - in\theta}\,d\theta.$$

Using this formula, one can prove easily the following asymptotics for $n \gg 1$ (see e.g. [4]),

$$|J_n(n\rho)| \le c_k n^{-k} \text{ for } \rho \ge 0,\ 1-\rho \gtrsim n^{-\gamma},\ \gamma < \frac{1}{2} \tag{2.2}$$

and

$$J_n(n\rho) = \left(\frac{2}{n\rho}\right)^{\frac{1}{3}} \mathrm{Ai}\left(n^{\frac{2}{3}}(1-\rho)\left(\frac{2}{\rho}\right)^{\frac{1}{3}}\right) + \mathcal{O}(n^{1-3\gamma+\varepsilon}) \tag{2.3}$$

for $1-\beta_1 n^{-\gamma} \le \rho \le 1+\beta_2 n^{-\frac{2}{3}}$, $\gamma < \frac{1}{2}$. Here Ai denotes the usual Airy function, which is exponentially decreasing for $x \to +\infty$, and satisfies $\mathrm{Ai}(x) \sim c|x|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|x|^{\frac{3}{2}}\right)$, if $x \to -\infty$. Moreover, the first zeroes of J_n satisfy

$$z_{n,1} = n + \alpha n^{\frac{1}{3}} + \mathcal{O}(n^{-\frac{1}{3}}), \quad z_{n,2} \ge z_{n,1} + Cn^{\frac{1}{3}} \tag{2.4}$$

for some $\alpha > 0$, $C > 0$.

As a consequence of (2.2), (2.3), (2.4), we infer the following behavior of the L^p norms of $f_{n,1}$,

$$\|f_{n,1}\|_{L^p} \approx n^{-\frac{2}{3p}-\frac{1}{3}}, \quad p \in [2, +\infty]. \tag{2.5}$$

In particular, $\varphi_n = n^{\frac{2}{3}-s} f_{n,1}$ is essentially normalized in H^s, as announced in the introduction.

We come to the proof of the ansatz (1.7) for $s > 0$. Since $\|\varphi_n\|_{L^2} \approx n^{-s}$, Theorem 1 implies, for n large enough, that u exists globally in time. The nonlinear frequency ω_n arises when testing the nonlinearity on the linear solution,

$$|\varphi_n|^2\varphi_n = \omega_n\varphi_n + r_n, \tag{2.6}$$

where $r_n \perp \varphi_n$. We then decompose $u(t)$ in the same way,

$$u(t) = \kappa e^{-it(z_{n,1}^2 - \kappa^2\omega_n)}\left[\gamma(t)\varphi_n + q(t)\right]$$

where $q(t) \perp \varphi_n$, and $\gamma(t) \in \mathbb{C}$. Notice that here and in the sequel, the dependence of $\gamma(t)$ and $q(t)$ on n is not explicitly mentioned. By projecting the equation (1.1) on φ_n, we get the following evolution equation for γ,

$$i\dot{\gamma} - \kappa^2\omega_n(1-|\gamma|^2)\gamma = -\kappa^2[2\zeta|\gamma|^2 + \bar{\zeta}\gamma^2 + 2\varepsilon\gamma + \eta\bar{\gamma} + \sigma], \tag{2.7}$$

where

$$\zeta(t) = \frac{(q(t)|r_n)}{\|\varphi_n\|_{L^2}^2}, \quad \varepsilon(t) = \frac{\int_\Omega |\varphi_n|^2|q(t)|^2}{\|\varphi_n\|_{L^2}^2}, \quad \eta(t) = \frac{\int_\Omega q(t)^2\bar{\varphi}_n^2}{\|\varphi_n\|_{L^2}^2}, \quad \sigma(t) = \frac{\int_\Omega |q(t)|^2 q(t)\bar{\varphi}_n}{\|\varphi_n\|_{L^2}^2}.$$

Moreover, the conservation laws (1.2) read

$$\begin{cases} |\gamma(t)|^2\|\varphi_n\|_{L^2}^2 + \|q(t)\|_{L^2}^2 = \|\varphi_n\|_{L^2}^2 \\ \\ |\gamma(t)|^2\|\nabla\varphi_n\|_{L^2}^2 + \|\nabla q(t)\|_{L^2}^2 - \frac{\kappa^2}{2}\|\gamma(t)\varphi_n + q(t)\|_{L^4}^4 = \|\nabla\varphi_n\|_{L^2}^2 - \frac{\kappa^2}{2}\|\varphi_n\|_{L^4}^4. \end{cases}$$

Substracting $z_{n,1}^2$ times the first conservation law from the second one, we get

$$\|\nabla q(t)\|_{L^2}^2 - z_{n,1}^2\|q(t)\|_{L^2}^2 = \frac{\kappa^2}{2}\left[\|\gamma(t)\varphi_n + q(t)\|_{L^4}^4 - \|\varphi_n\|_{L^4}^4\right] \tag{2.8}$$

The key information for estimating $q(t)$ lies in the following lemma.

Lemma 2.1. *The decomposition of $q(t)$ on eigenfunctions of $-\Delta_D$ reads*

$$q(t) = \sum_{k=2}^{+\infty} c_k(t) f_{n,k}.$$

Proof of lemma 2.1. For any $\alpha \in \mathbb{R}$, let R_α be the rotation of angle α in $\mathbb{R}^2$. Notice that $u(0,\cdot) = \kappa\varphi_n$ satisfies, for every $\alpha \in \mathbb{R}$,

$$u(0, R_\alpha(x)) = e^{in\alpha}u(0,x).$$

On the other hand, equation (1.1) is invariant through the change of variables $x \mapsto R_\alpha(x)$ and through $u \mapsto e^{in\alpha}u$. As a consequence of uniqueness in Theorem 1, we conclude, for every $t \in \mathbb{R}$, for every $\alpha \in \mathbb{R}$,

$$u(t, R_\alpha(x)) = e^{in\alpha}u(t,x). \tag{2.9}$$

Expanding $u(t)$ into eigenfunctions of $-\Delta_D$,

$$u(t) = \sum_{m,k} v_{m,k}(t) f_{m,k}$$

we infer from (2.9) that $v_{n,k}(t) = 0$ if $m \neq n$. The lemma is a consequence of this information combined with the fact that $q(t) \perp f_{n,1}$ by definition. $\square$

In view of lemma (2.1), the left hand side of (2.8) can be estimated from below as

$$\begin{aligned} \|\nabla q(t)\|_{L^2}^2 - z_{n,1}^2\|q(t)\|_{L^2}^2 &\geq \sum_{k\geq 2}(z_{n,k}^2 - z_{n,1}^2)|c_k(t)|^2\|f_{n,k}\|_{L^2}^2 \\ &\geq Cn^{\frac{1}{3}}\|q(t)\|^2_{H_D^{\frac{1}{2}}} \end{aligned} \tag{2.10}$$

because of the second inequality in (2.4). On the other hand, the right hand side of (2.8) is estimated by

$$\|\gamma(t)\varphi_n + q(t)\|_{L^4}^4 - \|\varphi_n\|_{L^4}^4 \leq \\ (|\gamma(t)|^4 - 1)\|\varphi_n\|_{L^4}^4 + C|\gamma(t)|^3 \int_\Omega |\varphi_n|^3 |q(t)| + C\|q(t)\|_{L^4}^4. \tag{2.11}$$

Coming back to the first conservation law, we observe that $|\gamma(t)| \leq 1$. We obtain, from (2.10) and (2.11),

$$n^{\frac{1}{3}}\|q(t)\|^2_{H^{\frac{1}{2}}_D} \leq \frac{C\kappa^2}{2}\left(\int_\Omega |\varphi_n|^3|q(t)| + \|q(t)\|_{L^4}^4\right).$$

By the Sobolev embedding, we have

$$\|q(t)\|_{L^4} \lesssim \|q(t)\|_{H^{\frac{1}{2}}_D}.$$

Moreover

$$\int \Omega|\varphi_n|^3|q(t)| \leq \|\varphi_n\|^3_{L^6}\|q(t)\|_{L^2} \lesssim n^{\frac{1}{6}-3s}\|q(t)\|_{H^{\frac{1}{2}}_D},$$

in view of the estimate on $\|f_{n,1}\|_{L^6}$ and by lemma 2.1. We conclude,

$$n^{\frac{1}{3}}\|q(t)\|_{H^{\frac{1}{2}}_D} \lesssim n^{\frac{1}{6}-3s} + \|q(t)\|^3_{H^{\frac{1}{2}}_D}.$$

Since $q(0) = 0$, a bootstrap argument leads to

$$\|q(t)\|_{H^{\frac{1}{2}}_D} \lesssim n^{-\frac{1}{6}-3s} \tag{2.12}$$

for n large enough. As a consequence,

$$\|q(t)\|_{L^2} \lesssim n^{-\frac{2}{3}-3s} \tag{2.13}$$

and, by interpolation,

$$\|q(t)\|_{H^s} \lesssim n^{-\frac{2}{3}-2s}. \tag{2.14}$$

Now we come back to the evolution equation for γ. We can use (2.12), (2.13), to estimate the coefficients of this equation,

$$|\zeta(t)| \lesssim n^{-4s}, \quad |\eta(t)| \leq \varepsilon(t) \lesssim n^{-\frac{2}{3}-6s}, \quad |\sigma(t)| \lesssim n^{-\frac{2}{3}-8s}.$$

Since, by the first conservation law,

$$1 - |\gamma(t)|^2 = \frac{\|q(t)\|^2_{L^2}}{\|\varphi_n\|^2_{L^2}} \lesssim n^{-\frac{4}{3}-4s},$$

the evolution equation for $\gamma(t)$ reads

$$\dot{\gamma}(t) = \mathcal{O}(n^{-4s}), \quad \gamma(0) = 1,$$

which completes the proof of (1.7).
Let us now discuss the case $s = 0$. A priori, since $\|u(0)\|_{L^2}$ is not small, $u(t)$ is defined only on some open time interval I. However, for $t \in I$, the estimates on $q(t)$ and $\gamma(t)$ can still be deduced from conservation laws, so that (2.12), (2.13), (2.14) hold for $s = 0$. Coming back to (2.8), we deduce a control of the H^1 norm,

$$\|\nabla q(t)\|_{L^2} \lesssim n^{\frac{1}{3}} \tag{2.15}$$

which, combined with $|\gamma(t)| \lesssim 1$, implies global existence of $u(t)$ in view of Theorem 1. It remains to study the evolution of γ. The only not small coefficient in this equation is $\zeta(t)$, so that

$$\begin{cases} i\dot{\gamma}(t) = -2\kappa^2\zeta(t) - \kappa^2\bar{\zeta}(t)\gamma^2(t) + o(1) \\ \gamma(0) = 1. \end{cases}$$

Notice that $|\zeta(t)| \lesssim 1$, hence $|\dot{\gamma}(t)| \lesssim 1$, and therefore $\gamma(t)$ is equicontinuous in time. As a consequence, we only need to know the behavior of ζ in the weak $\star$ topology of $L^\infty(\mathbb{R})$. Let us look at the evolution for $q(t)$, which reads

$$\begin{aligned} &i\dot{q} + (\Delta + z_{n,1}^2 - \kappa^2\omega_n)q = \\ &\qquad - \kappa^2|\gamma|^2\gamma\, r_n - \kappa^2\Pi\left[2|\gamma|^2|\varphi|^2 q + \gamma^2\varphi^2\bar{q} + 2\gamma\varphi|q|^2 + \bar{\gamma}\bar{\varphi}q^2 + |q|^2 q\right], \end{aligned}$$

where Π is the projector onto the orthogonal subspace of φ. In view of the estimates on q and γ, this equation can be simplified as

$$i\dot{q}(t) + (\Delta + z_{n,1}^2)q(t) = -\kappa^2\gamma(t)\, r_n + f(t)$$

where $\|f(t)\|_{L^2} \lesssim 1$ uniformly in t. Let $A = -\Delta - z_{n,1}^2$. Notice that, on the space spanned by $f_{n,k}$, $k \geq 2$, $A \geq z_{n,2}^2 - z_{n,1}^2 \gtrsim n^{\frac{4}{3}}$. From the equation on q, we infer,

$$q(t) = iA^{-1}\dot{q}(t) + \kappa^2\gamma(t)A^{-1}r_n - A^{-1}f(t)$$

hence

$$(q(t)\,|\,r_n) = i\frac{d}{dt}(A^{-1}q(t)\,|\,r_n) + \kappa^2\gamma(t)(A^{-1}r_n\,|\,r_n) - (A^{-1}f(t)\,|\,r_n)$$

Notice that $(A^{-1}q(t)\,|\,r_n) \lesssim n^{-\frac{4}{3}}$ and $(A^{-1}f(t)\,|\,r_n) \lesssim n^{-\frac{2}{3}}$. As a consequence, in $L^\infty_{w\star}$,

$$\zeta(t) = \kappa^2\theta_n\ \gamma(t) + o(1)$$

with

$$\theta_n = \frac{(A^{-1}r_n\,|\,r_n)}{\|\varphi_n\|_{L^2}^2} = \mathcal{O}(1).$$

Coming back to the equation in γ, we have, in $L^\infty_{w^\star}$,

$$i\dot\gamma(t) = -3\kappa^4\theta_n\ \gamma(t) + o(1),$$

thus

$$\gamma(t) = e^{3it\kappa^4\theta_n} + o(1)$$

locally uniformly in t. This completes the proof of Theorem 3. $\square$

3. Open questions and Generalizations to higher dimensions

3.1. Open questions about improvements of Theorems 2 and 3. A very natural question about Theorem 2 is of course the optimality of $s = \frac{1}{3}$ as a well-posedness threshold on the disc. In other words, does property $\mathcal{P}_s$ hold on the disc for $s \in]\frac{1}{3}, 1]$? At this stage, we cannot give any answer to this interesting question. Notice however that $s = \frac{1}{3}$ has some universal feature on the disc, since it is essentially not sensitive to the strength of the nonlinearity : indeed , by similar arguments, Theorem 2 still holds if $|u|^2u$ is replaced in (1.1) by $(1+|u|^2)^{\alpha/2}u$, for every $\alpha \in]0, 2]$.

Another series of open questions concerns the specific dynamics described in Theorem 3. First, if $s \in]0, \frac{1}{3}[$, it should be observed that our proof yields long time asymptotics : since $|\dot\gamma(t)| \lesssim n^{-4s}$, $u(t)$ is approximated by the first term in the right hand side of (1.7) as soon as $|t| \ll n^{4s}$. A natural question would be to improve the time interval of validity of this approximation. In particular, does the approximation hold uniformly in time ?

Finally, other problems rely on the optimality of bounds on the remainder terms in (1.7). Let us mention that, in the close framework of equation (1.1) on S^2, V. Banica [**1**] gave the exact description of these remainder terms, both parallelly to φ_n and orthogonally to φ_n. Of course, all these questions become much more difficult in the case $s = 0$, where a refined analysis of nonlinear resonances seems necessary to get a better estimate. Moreover, it would be interesting to decide whether formula (1.8) holds for some $s < 0$.

3.2. The role of the geometry. An important problem is of course the extension of Theorem 2 and 3 to other subsets of $\mathbb{R}^2$. Indeed, strictly convex domains yield the same kind of concentration effects on the boundary for convenient sequences of approximate eigenfunctions (or quasi-modes), as shown by Lazutkin (see e.g. [**8**]).

However, the difficulty in the extension of our proof lies in the "ground state effect" displayed in lemma 2.1, which we are unable to generalize to such a situation. For the same reasons, let us mention that, even on the disc, we cannot prove an analogue of Theorem 3 if $f_{n,1}$ is replaced in the Cauchy data by $f_{n,2}$, which enjoys the same concentration properties.

3.3. Generalization to higher dimensions. If Ω is a bounded open set of $\mathbb{R}^d$, $d \geq 3$, no global existence result such as Theorem 1 is known on $H_0^1(\Omega)$ for any equation of the kind

$$i\partial_t u + \Delta u + (1+|u|^2)^{\frac{\alpha}{2}} u = 0, \tag{3.1}$$

for $\alpha > 0$. The only global existence result at our disposal concerns weak solutions. In this context, it is possible to extend Theorem 2 as follows.

THEOREM 4. *Assume $\Omega = B^d$ is the d-dimensional unit ball, $d \geq 3$. There exists $\alpha_0 = \alpha_0(d) > 0$ such that, for any $\alpha \in]0, \alpha_0[$, for any $s \in [0, \frac{1}{3} + \frac{d-2}{4}[$, $s \leq 1$, one can find two sequences u_n, $\tilde{u}_n \in C_w(\mathbb{R}, H_0^1(\Omega))$ of weak solutions of (3.1) such that*

$$\lim_{n\to\infty} \|u_n(0) - \tilde{u}_n(0)\|_{H_D^s} = 0,$$

with $\|u_n(0)\|_{H_D^s} + \|\tilde{u}_n(0)\|_{H_D^s} \lesssim 1$, but such that, for every $T > 0$, the quantity

$$\sup_{t\in[0,T]} \|u_n(t) - \tilde{u}_n(t)\|_{H_D^s}$$

does not tend to zero as n tends to infinity.

Notice that, for $d \geq 5$, the only restriction on s is $s \leq 1$, and Theorem 4 implies illposedness in the energy space $H_0^1(\Omega)$ for any $\alpha \in]0, \alpha_0[$. This is in contrast with wellposedness results for the nonlinear Schrödinger equation on $\mathbb{R}^d$ for $d = 5, 6$ (see e.g. [**5**]), and suggests that global solvability for (3.1) in these dimensions is a very intricate problem.

Let us sketch the proof of Theorem 4. The eigenfunctions of $-\Delta_D$ on B_d can be described as

$$f(x) = \frac{1}{|x|^{\frac{d-2}{2}}} J_{n+\frac{d-2}{2}}(z_n|x|) H_n\left(\frac{x}{|x|}\right), \tag{3.2}$$

where J_ν is the Bessel function, z_n is a positive zero of $J_{n+\frac{d-2}{2}}$, and H_n is a spherical harmonic of degree n. The corresponding eigenvalue of $-\Delta_D$ is then z_n^2. We choose our initial data to be proportional to f, where z_n is the first positive zero of $J_{n+\frac{d-2}{2}}$, and

$$H_n(y) = (y_1 + iy_2)^n.$$

More precisely we set

$$u(0,x) = \kappa\varphi(x) = \kappa n^\beta f(x)$$

where $\beta = \frac{2}{3} + \frac{d-2}{4} - s$, which yields $\|u(0)\|_{H_D^s} \approx 1$. The nonlinear frequency is then

$$\omega(\kappa) = \frac{1}{\|\varphi\|_{L^2}^2} \int_\Omega (1+\kappa^2|\varphi(x)|^2)^{\frac{\alpha}{2}} |\varphi(x)|^2 dx \approx \kappa^\alpha n^{\alpha(\frac{1}{3}+\frac{d-2}{4}-s)},$$

which justifies the choice $s < \frac{1}{3} + \frac{d-2}{4}$. The choice of z_n and H_n is justified by the following generalization of lemma 2.1.

LEMMA 3.1. *Let $R_\alpha \in SO(d)$ be the rotation of angle α in the (y_1, y_2) plane. Let $v \in L^2(\Omega)$ such that, for every $\alpha \in \mathbb{R}$,*

$$v(R_\alpha(x)) = e^{in\alpha} v(x). \tag{3.3}$$

Then the expansion of v as sum of eigenfunctions only contains terms (3.2) with $m \geq n$. Moreover, if $m = n$, $H_n(y) = (y_1 + iy_2)^n$.

The proof of lemma 3.1 is an easy consequence of a similar result on S^{d-1} (see [**3**]). As a consequence, if q satisfies (3.3) and is orthogonal to φ, then the only eigenfunctions arising in the expansion of q correspond to eigenvalues $\lambda = z^2$, where z is either a higher zero of $J_{n+\frac{d-2}{2}}$, or a zero of $J_{m+\frac{d-2}{2}}$ with $m \geq n+1$. In both cases, $z - z_n \geq C > 0$ (see Watson [**13**]) hence $z^2 - z_n^2 \gtrsim z$ and

$$\|\nabla q(t)\|_{L^2}^2 - z_n^2 \|q(t)\|_{L^2}^2 \gtrsim \|q(t)\|^2_{H_D^{\frac{1}{2}}}$$

if $n \to \infty$.
Let us conclude this section by indicating the construction of the sequences (u_n), $(\tilde{u}_n)$. As before they will correspond to $u_n(0) = \kappa\varphi_n$, $\tilde{u}_n(0) = \kappa_n \varphi_n$, where $\kappa > 0$, $\kappa_n \to \kappa$ slowly. The construction of a weak solution u with such Cauchy data is classical. Given $\lambda > 0$, we solve

$$\begin{cases} i\partial_t u^\lambda + \Delta u^\lambda + f^\lambda(u^\lambda) = 0, \\ u^\lambda(0) = u(0), \\ u^\lambda|_{\mathbb{R}\times\partial\Omega} = 0 \end{cases}$$

with $f^\lambda(w) = \chi\left(\frac{|w|^2}{\lambda}\right)(1+|w|^2)^{\frac{\alpha}{2}} w$ and $\chi \in C_0^\infty(\mathbb{R})$ is 1 on $[0,1]$. Since f^λ is globally Lipschitz, the global existence of $u^\lambda \in C(\mathbb{R}, H_0^1(\Omega) \cap H^2(\Omega))$ is easily obtained, as well as L^2 and energy conservation laws. Assuming $\alpha < \frac{4}{d}$, these two conservation laws imply a bound on u^λ in $L^\infty(\mathbb{R}, H_0^1(\Omega))$, thanks to Gagliardo-Nirenberg inequality. Letting $\lambda \to +\infty$, we get a weak solution u of (3.1), satisfying the L^2 conservation law (by Rellich theorem) and the inequality

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 - \frac{2}{\alpha+2} \int_\Omega (1 + |u(t,x)|^2)^{\frac{\alpha+2}{2}} dx \leq \\ \|\nabla u(0)\|_{L^2}^2 - \frac{2}{\alpha+2} \int_\Omega (1 + |u(0,x)|^2)^{\frac{\alpha+2}{2}} dx. \end{aligned} \tag{3.4}$$

Moreover since property (3.3) is satisfied by $u(0)$, it extends to $u^\lambda(t)$ and $u(t)$. Then, by similar arguments to section 2, we prove, if $u(0) = \kappa\varphi$,

$$u(t) = \kappa e^{-it(z_n^2 - \omega(\kappa))}\varphi + o(1)$$

in $H^s_D(\Omega)$, locally uniformly in t and κ. The smallness assumption on α merely guarantees the extension of the bootstrap argument for estimating the orthogonal part $q(t)$ from the L^2 conservation law and inequality (3.4). □

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