EXPONENTIAL DECAY FOR THE DAMPED WAVE EQUATION IN UNBOUNDED DOMAINS

by

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Abstract. — We study the decay of the semigroup generated by the damped wave equation in an unbounded domain. We first prove under the natural *geometric control condition* the exponential decay of the semigroup. Then we prove under a weaker condition the logarithmic decay of the solutions (assuming that the initial data are smoother). As corollaries, we obtain several extensions of previous results of stabilisation and control.

Résumé. — On étudie la décroissance du semi-groupe des ondes amorties dans un domaine non borné. Notre premier résultat est que, sous une hypothèse naturelle de contrôle géométrique, le semi-groupe décroît exponentiellement vite. On démontre ensuite sous une hypothèse plus faible la décroissance logarithmique des solutions associées à des données initiales plus régulières. On obtient en corollaire plusieurs généralisations de résultats de stabilisation et de contrôle.

1. Introduction

In this article we consider the damped wave equation. In the simplest case of constant coefficients Laplace operator, our main result takes the following form:

Theorem 1.1. — *Let $\gamma \in L^\infty(\mathbb{R}^d)$ be a non-negative damping. Assume that γ is a uniformly continuous function and that there exist $L, c > 0$ such that for any $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$,*

$$\int_{s=0}^{L} \gamma(x_0 + s\xi_0)dx \;\geq\; c \;>\; 0 \;.$$

Then, there exist M and $\lambda > 0$ such that any solution of

$$\partial^2_{tt} u + \gamma(x)\partial_t u = (\Delta - Id)u \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

satisfies

$$\|u(t)\|_{H^1(\mathbb{R}^d)} + \|\partial_t u(t)\|_{L^2(\mathbb{R}^d)} \;\leq\; Me^{-\lambda t}\Big(\|u(0)\|_{H^1(\mathbb{R}^d)} + \|\partial_t u(0)\|_{L^2(\mathbb{R}^d)}\Big) \;.$$

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1.1. The damped wave equation:— More precisely, our results concern a more general linear damped wave equation in $\mathbb{R}^d$, with $d \geq 1$:

$$(1.1) \qquad \left\{ \begin{array}{l} \partial^2_{tt}u(x,t) + \gamma(x)\partial_t u(x,t) = \mathrm{div}(K(x)\nabla u(x,t)) - u(x,t) \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \ , \\ (u,\partial_t u)(\cdot,0) = (u_0,u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \end{array} \right.$$

where $K \in \mathcal{C}^\infty(\mathbb{R}^d, \mathcal{M}_d(\mathbb{R}))$ is a smooth family of real symmetric matrices, which are uniformly positive in the sense that there exist two positive constants $K_{\inf}$ and $K_{\sup}$ such that

$$(1.2) \qquad \forall \xi \in \mathbb{R}^d \ , \ K_{\sup}|\xi|^2 \geq \xi^\intercal.K(x).\xi \geq K_{\inf}|\xi|^2 \ .$$

The damping coefficient $\gamma \in L^\infty(\mathbb{R}^d)$ is assumed to be a bounded and non-negative function. We set $X = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and

$$(1.3) \qquad A = \begin{pmatrix} 0 & Id \\ (\mathrm{div}(K(x)\nabla) - Id) & -\gamma(x) \end{pmatrix} \qquad D(A) = H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \ .$$

We equipped $H^1(\mathbb{R}^d)$ with the scalar product

$$(1.4) \qquad \langle u|v\rangle_{H^1} = \int_{\mathbb{R}^d} (\nabla u(x))^\intercal.K(x).(\overline{\nabla v(x)}) \ + \ u(x)\overline{v(x)}\, dx \ .$$

Obviously, this scalar product is equivalent to the classical one and direct computations show that it satisfies

$$\langle(\mathrm{div}(K(x)\nabla) - Id)u|v\rangle_{L^2} = -\langle u|v\rangle_{H^1} \ \text{ and } \ \mathrm{Re}(\langle AU|U\rangle_X) = -\int \gamma(x)|v(x)|^2\, dx$$

for any $U = (u,v) \in D(A)$. Then, one easily checks that A is a dissipative operator and therefore generates a semigroup e^{At} on X.

1.2. Exponential decay and Hamiltonian flow:— The main purpose of this paper is to investigate the exponential decay of the semigroup associated to (1.1): we ask whether there exist M and $\lambda > 0$ such that

$$(1.5) \qquad \forall t \geq 0 \ , \ |||e^{At}|||_{\mathcal{L}(X)} \leq Me^{-\lambda t} \ .$$

For the damped wave equation in a bounded domain and a continuous damping coefficient, it is well known that the exponential decay is equivalent to the fact that all the trajectories of the Hamiltonian flow intersect the support of the damping (see [**30**], [**3**], [**4**]) and [**6**]. More precisely, to the Laplacian operator with variable coefficients $\mathrm{div}(K(x)\nabla)$, we associate the symbol $g(x,\xi) = \xi^\intercal.K(x).\xi$ and the Hamiltonian flow $\varphi_t(x_0,\xi_0) = (x(t),\xi(t))$ defined on $\mathbb{R}^{2d}$ by

$$(1.6) \qquad \varphi_0(x_0,\xi_0) = (x_0,\xi_0) \ \text{ and } \ \partial_t\varphi_t(x,\xi) = (\partial_\xi g(x(t),\xi(t)), -\partial_x g(x(t),\xi(t)) \ .$$

We introduce the mean value of the damping along a ray a length T:

$$(1.7) \qquad \langle\gamma\rangle_T(x,\xi) = \frac{1}{T}\int_0^T \gamma(\varphi_t(x,\xi))dt$$

where we use the obvious notation $\gamma(x,\xi) := \gamma(x)$. We also introduce the set Σ of rays of speed one, that is

$$(1.8) \qquad \Sigma = \{(x,\xi) \in \mathbb{R}^{2d} \ , \ \xi^\intercal K(x)\xi = 1\} \ .$$

Some previous works:
If Ω is a bounded manifold, the uniform positivity of $\langle\gamma\rangle_T(x,\xi)$ in Σ for some $T > 0$ implies that the exponential decay (1.5) holds, as shown in the celebrated articles [**30**], [**3**] and [**4**] of Bardos,

Lebeau, Rauch and Taylor. The assumption that there exists $T > 0$ such that $\langle\gamma\rangle_T(x,\xi) > 0$ in Σ is called the *geometric control condition*. The article [**20**] underlines in addition the importance of the value of $\min_{(x,\xi)\in\Sigma}\langle\gamma\rangle_T(x,\xi)$ in order to control the rate of decay of the high frequencies.

In the case of an unbounded manifold, two situations have been investigated. First, some authors have considered the free wave equation (1.1) in an exterior domain (with $\gamma \equiv 0$ or $\gamma > 0$ only on a compact subset the exterior domain). They have shown that the local energy decays to zero in the sense that, under suitable assumptions, the energy of any solution escapes away from any compact set, see [**18**], [**26**] and [**2**] and the references therein. Secondly, several works have studied the damped wave equation in an unbounded manifold and with a non-linearity, but assuming that the damping satisfies $\gamma(x) \geq \alpha > 0$ outside a compact set, see [**33**], [**10**], [**9**] and [**16**].

Considering these previous works, it appears that one natural case has not been studied: the exponential decay of the semigroup e^{At} generated by the damped wave equation on a whole unbounded manifold, with the geometric control condition only, that is without assuming that $\gamma \geq \alpha > 0$ outside a compact set. To our knowledge, this case is surprisingly missing in the literature. The main purpose of this article is to settle this natural problem.

Main results:

We denote by $\mathcal{C}^k_b(\mathbb{R}^d)$ the set of functions in $\mathcal{C}^k(\mathbb{R}^d)$ which are bounded, as well as their k first derivatives. If $k = \infty$, the bound is not assumed to be uniform with respect to the derivatives. We recall that $\langle\gamma\rangle_T$ and Σ have been defined in (1.7) and (1.8). Our main result is as follows.

Theorem 1.2. — *We assume that the metric K belongs to $\mathcal{C}^\infty_b(\mathbb{R}^d, \mathcal{M}_d(\mathbb{R}))$ and that the bounded non-negative damping γ is uniformly continuous and satisfies*

$$(GCC) \qquad \text{there exist } T,\alpha > 0 \text{ such that } \langle\gamma\rangle_T(x,\xi) \geq \alpha > 0\ , \quad \text{for all } (x,\xi)\in\Sigma\ .$$

Then, the semigroup generated by the damped wave equation (1.1) *is exponentially decreasing that is that there exist M and $\lambda > 0$ such that*

$$\forall t \geq 0\ ,\ |||e^{At}|||_{\mathcal{L}(X)} \leq Me^{-\lambda t}\ . \tag{1.9}$$

Assume now that the geometric control condition (GCC) is violated but the damping is still efficient on a network of balls. The Lasalle invariance principle ensures that, for any initial data, the energy of the solution goes to 0 when $t \to +\infty$. Since the geometric control condition does not hold, it is classical that the convergence to 0 can be arbitrarily slow:

$$\forall T > 0, \quad \sup_{(u_0,u_1)\in H^1\times L^2} \frac{\|(u(T),\partial_t u(T))\|_{H^1\times L^2}}{\|(u_0,u_1)\|_{H^1\times L^2}} = 1.$$

Our second result extends to the non compact setting a similar result by Lebeau proved on compact manifolds [**20**] (see also [**21**]) and gives an upper bound for the rate of decay when the initial data are smoother (see [**22**, Definition 1.1 and Section 3.1] for a similar geometric setting developped independently).

Theorem 1.3. — *We assume that the metric K belongs to $\mathcal{C}_b^\infty(\mathbb{R}^d, \mathcal{M}_d(\mathbb{R}))$ and that $\gamma \in L^\infty(\mathbb{R}^d)$ satisfies*

$$(NCC) \qquad \text{there exist } L, r, a > 0 \text{ and a sequence } (x_n) \subset \mathbb{R}^d \text{ such that}$$
$$\gamma(x) \geq a > 0 \text{ on } \cup_n B(x_n, r) \text{ and } \forall x \in \mathbb{R}^d \ , \ d(x, \cup_n \{x_n\}) \leq L \ .$$

Then, for any $k > 0$, there exists $C_k > 0$ such that for any $(u_0, u_1) \in H^{k+1}(\mathbb{R}^d) \times H^k(\mathbb{R}^d)$,

$$\|(u(T), \partial_t u(T))\|_{H^1 \times L^2} \leq \frac{C_k}{\log(2+t)^k} \|(u_0, u_1)\|_{H^{k+1} \times H^k}.$$

Some extensions and applications:

i) A contradiction argument shows very easily that, as soon as the exponential decay holds for a damping coefficient $0 \leq \gamma$, it holds (with different constants, possibly worse) for any damping coefficient $\tilde{\gamma} \in L^\infty(\mathbb{R}^d)$ satisfying $\tilde{\gamma} \geq \gamma$ (see the arguments of the second step of Section 2). Consequently, Theorem 1.2 also holds for any damping $\gamma \in L^\infty(\mathbb{R}^d)$ for which there exists $\underline{\gamma}$ with $0 \leq \underline{\gamma} \leq \gamma$ satisfying (GCC) and being uniformly continuous. Notice that the existence of $\underline{\gamma}$ uniformly continuous satisfying (NCC) and $0 \leq \underline{\gamma} \leq \gamma$ is automatic in the case of Theorem 1.3. That is why, the uniform continuity can be omitted in its statement.

ii) Theorem 1.2 concerns solutions of (1.1) with finite energy. It is possible to consider solutions of (1.1) with infinite energy in the framework of uniformly local Sobolev spaces. The stabilisation in this case is a straightforward corollary of Theorem 1.2, see Section 6.

iii) The ideas of the proof of Theorems 1.2 and 1.3 may apply to other geometric situations. For example, if we consider an unbounded manifold without boundary as a cylinder instead of $\mathbb{R}^d$, then Theorems 1.2 and 1.3 will also hold with the obvious modifications of their statements.

iv) The smoothness assumptions on the coefficients $K(x)$ could be relaxed (probably up to $\mathcal{C}^2$, see [**5**]). To keep the paper short, we chose not to develop this issue here.

v) The exponential decay of the linear semigroup has important applications in the control theory and the study of dynamics for the wave equations. Some new results are obtained as corollaries of Theorem 1.2 as explained in Section 6.

Remarks:

i) The simplest applications of Theorem 1.2 are the periodic frameworks satisfying the geometric control condition, see for example Figure 1.a). To our knowledge, the exponential decay of the semigroup was not known in this simple case (notice that one cannot directly use the framework of the torus since the initial data (u_0, u_1) are not periodic).

ii) The proof of Theorem 1.2 follows the lines of the proofs of the results on compact manifolds (see [**3**], [**4**], [**34**]...). It also uses classical properties of pseudo-differential calculus (see e.g. [**1**], [**24**] or [**19**]). The main point in the analysis is to be careful when using the classical arguments to deal with the infinity (in space). In particular this forbids the use of tools as the defect measure, which only yields informations on a compact subset of the domain. As usual, the proof of the stabilisation stated in Theorem 1.2 splits into two parts. The first is the control of the high frequencies, where we fully use the geometric control condition (GCC). This part is contained in Section 3, where we have to return to the semiclassical analysis behind the classical defect measure arguments. The second part is the control of the low frequencies by using a Carleman estimate as shown in Section 4. In this section, we do not use (GCC) but the weaker hypothesis (NCC), which is a uniform control of the damping on a network of balls stated in Theorem 1.3.

iii) In a first version of this article by the second author alone, it was shown that Theorem 1.2 can be obtained in dimension one by multipliers techniques following the ideas of [**23**] and [**32**]. In some simple geometrical situation in higher dimension, the multipliers techniques should also apply. The interest of this kind of proofs is to provide explicit constants M and λ, but the geometrical assumptions cannot be as general as the ones of the main result of this paper, except in dimension one.

iv) Of course, our theorem also hold when the operator $\mathrm{div}(K(x)\nabla) - Id$ is replaced by the operator $\mathrm{div}(K(x)\nabla) - \varepsilon Id$ with $\varepsilon > 0$. However, when $\varepsilon = 0$, that is when the right-hand-side is not a negative operator but only a non-positive one, it is known that one cannot hope an exponential decay of the solutions. Indeed, it has been established since a long time (see [**25**]) that the solutions of $\partial^2_{tt}u + \partial_t u = \Delta u$ in $\mathbb{R}^d$ asymptotically behave as the ones of $\partial_t v = \Delta v$ (see for example [**27**], [**29**] and the references therein). It is shown in [**8**] that if u is solution of $\partial^2_{tt}u + \partial_t u = \Delta u$ in $\mathbb{R}^d$, with initial data $(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and v is solution of $\partial_t v = \Delta v$ in $\mathbb{R}^d$, with initial data $u_0 + u_1$, then $\|(u - v)(t)\|_{H^1} \leq C/t$. In particular, u is generally decaying not faster than $C/t^{d/4}$ for $d \leq 3$.

v) The uniform continuity assumption on γ in Theorem 1.2 ensures that it can be regularised into a smooth damping coefficient $\underline{\gamma}$ satisfying $\underline{\gamma} \leq \gamma$ and belonging to $\mathcal{C}^\infty_b(\mathbb{R}^d, \mathbb{R})$. In particular, the fact that the derivative of $\underline{\gamma}$ can be taken uniformly bounded will be important in our proof order to apply the pseudo-differential calculus (notice that these uniform bounds would also be required if we used the multipliers techniques, at least for the first derivative). Of course, in the usual compact case, this assumption is automatically satisfied. In Figures 1.b) and 1.c), we show examples where all the Hypotheses of Theorem 1.2 apply, if one neglects the regularity hypothesis. In these cases, it would be natural to expect the exponential decay of the semigroup, but this is still an open problem. Notice that the simple requirement that γ belongs to L^∞ is not sufficient to define properly the mean value $\langle\gamma\rangle_T(x,\xi)$ everywhere. This could be a hint that the regularisation assumption is not just a technical one.

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2. Proof of Theorem 1.2

In this section, we outline the proof of Theorem 1.2. The real technical parts of its proof will be detailed in Sections 3 and 4.

There exist several ways to obtain the exponential decay (1.9) of the semigroup e^{At}. The most classical one is to argue by contradiction to establish the observation inequality $E(v(0)) \leq C\int_0^T \gamma|\partial_t v|^2 dt$ for some $T > 0$ and any solution v of the free wave equation (see for example [**14**] for the relation between this observation estimate and the exponential decay of the damped semigroup). A less usual method consists in uniformly estimating the resolvent $(A - \lambda Id)^{-1}$ on the imaginary axis (see for example chapter 5 of [**34**]). We use here this last method as a direct corollary of the results of [**12**], [**28**] and [**15**].

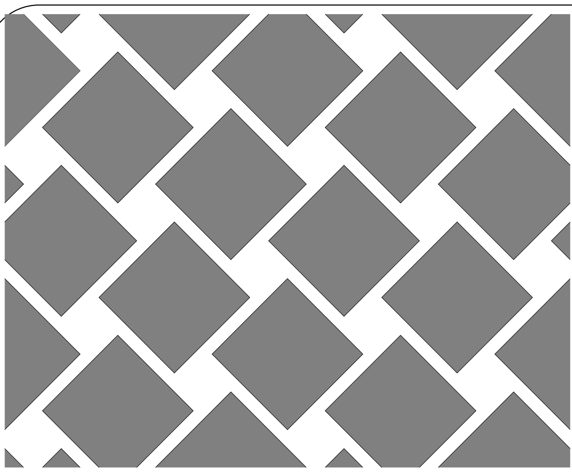

Figure 1.a): A periodic two-dimensional example for which Theorem 1.2 holds: the semigroup generated by the corresponding damped wave equation is decaying exponentially fast.

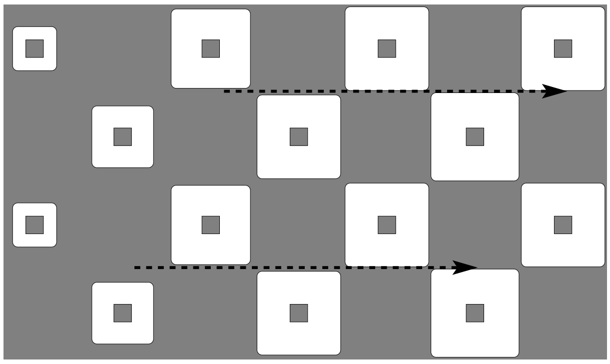

Figure 1.b): A two-dimensional quasi-periodic exam[le where only the regularisation condition in Hypothesis i) is not satisfied. Theorem 1.2 fails to apply because for any uniformly continuous damping $\tilde{\gamma}$ satisfying $\tilde{\gamma} \leq \gamma$, the infimum of the mean value $\langle\tilde{\gamma}\rangle$ is equal to 0.

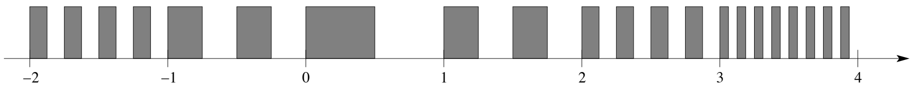

Figure 1.c): A one-dimensional example where the mean value of the damping $\langle\gamma\rangle_1(x,\xi)$ is uniformly positive in Σ, but where Theorem 1.2 does not apply since there is no uniformly continuous regularisation $\tilde{\gamma}$ with mean value $\langle\tilde{\gamma}\rangle_T$ uniformly positive for some T and with a derivative uniformly bounded in $\mathbb{R}$.

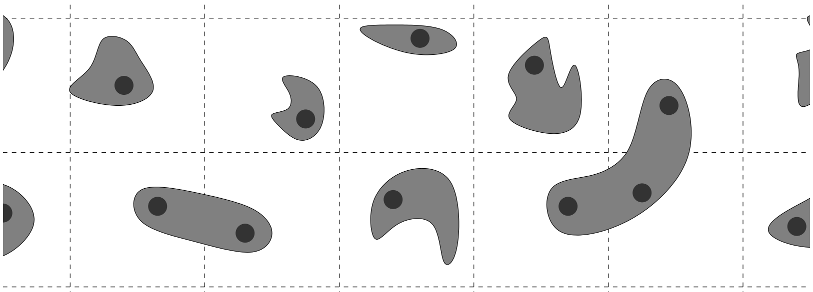

Figure 1.d): An example where (NCC) holds but (GCC) does not. A network of balls where the damping is effective is in dark grey. In this case, the exponential decay of Theorem 1.2 fails but the logarithmic decay of Theorem 1.3 holds.

FIGURE 1. *Discussion on some examples of damping. In the two-dimensional situations, the damping is equal to* 1 *on the grey regions and equal to* 0 *in the other regions. In the one-dimensional situation, the figure represents the graph of the damping. In both cases, the metric is assumed to be flat, i.e.* $K(x) = Id$.

• *First step: a characterisation in terms of resolvent estimates.*
To study the exponential decay, we use here the characterisation given by Theorem 3 of [**15**].

***Theorem 2.1* (Gearhart-Prüss-Huang).** — *Let e^{At} be a $\mathcal{C}^0-$semigroup in a Hilbert space X and assume that there exists a positive constant $M>0$ such that $|||e^{At}|||\leq M$ for all $t\geq 0$. Then e^{At} is exponentially stable if and only if $i\mathbb{R}\subset\rho(A)$ and*

$$\sup_{\mu\in\mathbb{R}} |||(A-i\mu Id)^{-1}|||_{\mathcal{L}(X)} \;<\; +\infty \;. \tag{2.1}$$

Since the linear operator A associated to the damped wave equation is dissipative, we have $|||e^{At}|||\leq 1$ for all $t\geq 0$. To prove Theorem 1.2, it remains to show that (2.1) holds. We argue by contradiction and assume that there exist two sequences $(U_n)=(u_n,v_n)\subset D(A)=H^2(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$ and $(\mu_n)\subset\mathbb{R}$ such that

$$\|U_n\|_X^2=\|u_n\|_{H^1}^2+\|v_n\|_{L^2}^2=1 \;\text{ and }\; (A-i\mu_n)U_n \xrightarrow[n\longrightarrow+\infty]{} 0 \;\text{ in } X\;. \tag{2.2}$$

Notice that, here, u_n and v_n are complex valued functions.

• *Second step: replacing γ by a smooth damping.*
We recall that $H^1(\mathbb{R}^d)$ is equipped with the convenient scalar product (1.4). Let us denote the operator $\mathrm{div}(K(x)\nabla)$ by Δ_K. We have

$$(A-i\mu_n Id)U_n=\begin{pmatrix} v_n-i\mu_n u_n\\ (\Delta_K-Id)u_n-\gamma(x)v_n-i\mu_n v_n\end{pmatrix}$$

and

$$\mathrm{Re}(\langle (A-i\mu_n)U_n|U_n\rangle_X)=-\int\gamma(x)|v_n(x)|^2\,dx\;.$$

Thus, (2.2) implies that $\int\gamma(x)|v_n(x)|^2\,dx$ goes to zero. Therefore, we can replace γ by any smooth damping $\underline{\gamma}$ satisfying $0\leq\underline{\gamma}\leq\gamma$ without changing (2.2). Let us show that we can construct such a damping $\underline{\gamma}\in\mathcal{C}_b^\infty(\mathbb{R}^d)$. First choose $\theta=\max(0,\gamma-\varepsilon)$. For a small enough $\varepsilon>0$, the damping θ still satisfies that its mean value $\langle\theta\rangle_T(x,\xi)$ is uniformly bounded away from 0. Moreover, since γ is uniformly continuous, the support of θ stays at a uniform distance $\delta>0$ of the set where γ vanishes. Now, mollify θ into $\theta*\rho_\delta$ where ρ_δ is a $\mathcal{C}^\infty$ regularisation kernel with support in $B(0,\delta)$. We obtain a smooth damping $\underline{\gamma}$ with a support included in the one of γ. Thus, one can use this new damping without changing (2.2). Moreover, this new damping $\underline{\gamma}$ belongs to $\mathcal{C}_b^\infty(\mathbb{R}^d)$, which ensures that the multiplication by $\underline{\gamma}$ is a pseudo-differential operator of order 0. In the remaining part of this proof, to simplify the notations, we will assume that γ itself belongs to $\mathcal{C}_b^\infty(\mathbb{R}^d)$.

• *Third step: separation between high and low frequencies.*
We now work with a smooth damping γ with bounded derivatives. To obtain a contradiction from (2.2), we consider two cases.

→ **High frequencies:** assume that $|\mu_n|$ goes to $+\infty$. Since A is a real operator, by symmetry, we can assume that $\mu_n>0$ and we set $h_n=1/\mu_n$. We have to show that one cannot have $\|U_n\|_X=1$ and $(A-i/h_n)U_n\longrightarrow 0$. This will be shown in Section 3 by semiclassical pseudo-differential arguments, using the geometric control condition of Theorem 1.2.

→ **Low frequencies:** assume that (μ_n) has a bounded subsequence. Then, up to extracting a subsequence, one can assume that (μ_n) converges to a real number μ. Then (2.2) is equivalent to have a sequence (U_n) with $\|U_n\|_X=1$ and $(A-i\mu)U_n\longrightarrow 0$. In Section 4, we will show that this is not possible by using a global Carleman estimate. In this part, it is in fact sufficient to replace the geometric control condition by the network control condition (NCC) stated in Theorem 1.3. A similar argument was developped independently by Le Rousseau and Moyano for the study of the Kolmogorov equation.

Since Sections 3 and 4 provide a contradiction in both cases, Theorem 2.1 yields the proof of Theorem 1.2.

3. Proof of Theorem 1.2: high frequencies

The purpose of this section is to obtain a contradiction from the existence of sequences (U_n) with $\|U_n\|_X = 1$ and (h_n) with $h_n \to 0$ satisfying $(A - i/h_n)U_n \longrightarrow 0$. To simplify the notations, we may forget the index n for the remaining part of this section and set $U_n = U_h = (u_h, v_h)$. We have

$$\left\{ \begin{array}{l} v_h - \frac{i}{h} u_h = o_{H^1}(1) \\ (\Delta_K - id)u_h - \gamma(x)v_h - \frac{i}{h} v_h = o_{L^2}(1) \end{array} \right.$$

and thus

$$\left\{ \begin{array}{l} v_h - \frac{i}{h} u_h = o_{H^1}(1) \\ h^2(\Delta_K - Id)u_h - ih\gamma(x)u_h + u_h = o_{L^2}(h^2) + o_{H^1}(h) \end{array} \right. \tag{3.1}$$

To obtain a contradiction between (3.1) and Hypothesis (GCC) of Theorem 1.2, we will use the semiclassical microlocal analysis and follow the ideas of the chapter 5 of [**34**]. Notice that the usual way to deal with high frequencies is to use semiclassical defect measures (see for example [**34**]). However, this is not possible in our case since we work in an unbounded domain and the semiclassical defect measure will only tell us what happens in compact subsets.

Lemma 3.1. — *Assume that the operator $P_h = h^2(\Delta_K - Id) - ih\gamma(x) + Id$ has a resolvent in $L^2(\mathbb{R}^d)$ satisfying*

$$\|(P_h)^{-1} f\|_{L^2} \leq \frac{C}{h} \|f\|_{L^2} \ . \tag{3.2}$$

Then P_h has a resolvent in $H^1(\mathbb{R}^d)$ also satisfying

$$\|(P_h)^{-1} f\|_{H^1} \leq \frac{C}{h} \|f\|_{H^1} \ .$$

Proof: We argue by contradiction. Assume that there exists a sequence (u_h) with $\|u_h\|_{H^1} = 1$ and $P_h u_h = o_{H^1}(h)$. Multiplying by $\overline{u_h}$ and integrating, we get that

$$-h^2 \|u_h\|_{H^1}^2 - ih \int_{\mathbb{R}^d} \gamma(x)|u_h|^2 + \|u_h\|_{L^2}^2 = o(h)\|u_h\|_{L^2} \ .$$

Taking the real part and solving the equation in $\|u_h\|_{L^2}$, we get

$$\|u_h\|_{L^2} = \frac{1}{2}\left(o(h) + \sqrt{o(h^2) + 4h^2\|u_h\|_{H^1}^2} \right) \sim h \ .$$

We introduce the operator $\nabla_K = (-\Delta_K + Id)^{1/2}$. It is the particular case $h = 1$ of the semiclassical operator $(-h^2\Delta_K + Id)^{1/2}$, which has for principal symbol $\sqrt{\xi^\intercal . K(x).\xi + 1}$ (see Section A in Appendix for a brief recall about pseudo-differential semiclassical calculus). Obviously, it commutes with any polynomial of Δ_K. Moreover, applying i) of Corollary A.2 of the Appendix, with $h = 1$ fixed, we get that the commutator $[\nabla_K, \gamma(x)\cdot]$ is bounded in $L^2(\mathbb{R}^d)$. Thus, since $u_h = \mathcal{O}_{L^2}(h)$,

$$\nabla_K(P_h u_h) = P_h(\nabla_K u_h) - ih[\nabla_K, \gamma(x)]u_h = P_h(\nabla_K u_h) + \mathcal{O}_{L^2}(h^2) \ .$$

Since $P_h u_h = o_{H^1}(h)$, we obtain that $P_h(\nabla_K u_h) = o_{L^2}(h)$. Using the assumption on the resolvent of P_h, we obtain that $\nabla_K u_h$ goes to 0 in $L^2(\mathbb{R}^d)$ when h goes to 0. However, $\|\nabla_K u_h\|_{L^2}$ is equivalent to $\|u_h\|_{H^1}$ and we obtain a contradiction with the assumption $\|u_h\|_{H^1} = 1$. □

Proposition 3.2. — *If the operator* $P_h = h^2(\Delta_K - Id) - ih\gamma(x) + Id$ *has a resolvent in* $L^2(\mathbb{R}^d)$ *satisfying* (3.2)*, then* (3.1) *cannot hold.*

Proof: We argue by contradiction again. Assume that P_h satisfies (3.2) and assume that there exists $U_h = (u_h, v_h)$ with $\|U_h\|_X = 1$ such that (3.1) holds. As in the beginning of the proof of Lemma 3.1, multiplying the second equation of (3.1) by $\overline{u}_h$, integrating, taking the real part and solving the equation of second degree in $\|u_h\|_{L^2}$, we get that

$$\|u_h\|_{L^2} = \frac{1}{2}\left(o(h) + \sqrt{o(h^2) + 4h^2\|u_h\|_{H^1}^2}\right) .$$

Due to the first equation of (3.1) and since $\|U_h\|_X = 1$, we must have $\|u_h\|_{H^1} \sim 1/\sqrt{2}$, $\|u_h\|_{L^2} \sim h/\sqrt{2}$ and $\|v_h\|_{L^2} \sim 1/\sqrt{2}$.

We introduce $w_h = P_h^{-1}(f_h)$ where f_h is the term $o_{H^1}(h)$ in the second equation of (3.1). By assumption and by Lemma 3.1, we have that $w_h = o_{H^1}(1)$. By the same straightforward computation than the one just above, we also have $w_h = o_{L^2}(h)$. Then, $u_h - w_h$ solves $P_h(u_h - w_h) = o_{L^2}(h^2)$ and by assumption, we get that $u_h - w_h = o_{L^2}(h)$ and thus that $u_h = o_{L^2}(h)$. This is a contradiction with the fact that $\|u_h\|_{L^2} \sim h/\sqrt{2}$, which was proved above. □

Due to Proposition 3.2, to obtain a contradiction from (3.1), it remains to show the L^2-resolvent estimate (3.2). Obtaining this estimate is the central argument for controlling the high frequencies. Here, we will use pseudo-differential calculus and we will see the importance of Hypothesis (GCC) of Theorem 1.2. The remaining part of this section is thus devoted to the proof of the following result.

Proposition 3.3. — *The operator*

$$P_h = h^2(\Delta_K - Id) - ih\gamma(x) + Id$$

has a resolvent in $L^2(\mathbb{R}^d)$ *satisfying*

$$\|(P_h)^{-1}f\|_{L^2} \leq \frac{C}{h}\|f\|_{L^2} .$$

Proof: As usual, we argue by contradiction and assume that there exists a sequence (h_n) going to zero and functions $(u_n) \subset H^2(\mathbb{R}^d)$ such that $\|u_n\|_{L^2} = 1$ and $P_{h_n}u_n = o_{L^2}(h_n)$. Once again, we may forget the indices and assume that $\|u_h\|_{L^2} = 1$ and

$$h^2(\Delta_K - Id)u_h - ih\gamma(x)u_h + u_h = o_{L^2}(h) . \tag{3.3}$$

In what follows, we will use the notations and the results of the pseudo-differential semiclassical calculus recalled in Section A. Our proof follows the lines of Chapter 5 of [**34**], omitting the notion of defect measures, which is not convenient in the case of unbounded domains.

• *First step:* u_h *is concentrating along the radial speeds* $\xi^\intercal K(x)\xi = 1/h^2$.
First notice that the main part of P_h is $h^2\Delta_K + Id$ in the sense that $(h^2\Delta_K + Id)u_h = o_{L^2}(1)$. As explained in Appendix, up to an error term $\mathcal{O}_{L^2}(h^2)$, this main part is a pseudo-differential

semiclassical operator $\mathrm{Op}_h(-\xi^\intercal K(x)\xi+1)$. Let $\chi(x,\xi)\in\mathcal{C}^\infty(\mathbb{R}^d,\mathbb{R}_+)$ be a smooth cutting function which is equal to one in a neighbourhood of the sphere $\Sigma=\{(x,\xi),\xi^\intercal K(x)\xi=1\}$ and equal to 0 outside the annulus $1/2K_{\sup}\leq|\xi|\leq 2/K_{\inf}$. Also assume that χ and its derivatives are bounded, which implies that $\chi(x,\xi)$ is a symbol of order 0. We claim that u_h is concentrating on the microlocal set $\{(x,\xi),\xi^\intercal K(x)\xi=1/h^2\}$ in the sense that $\langle\mathrm{Op}_h(1-\chi(x,\xi))u_h|u_h\rangle_{L^2}$ goes to 0 when h goes to 0.

To prove this claim, we introduce another smooth cutting function θ which is equal to 1 in a neighbourhood of the sphere $\Sigma=\{(x,\xi),\xi^\intercal K(x)\xi=1\}$ and equal to 0 in the support of $1-\chi$. The symbol $a(x,\xi)=-\xi^\intercal K(x)\xi+1+i\theta$ is of order 2 and uniformly bounded away from 0. By Corollary A.3 in Appendix, the symbol $b(x,\xi)=\frac{1}{a(x,\xi)}$ is of order -2 and satisfies

$$\mathrm{Op}_h(a)\circ\mathrm{Op}_h(b)=Id+\mathcal{O}_{L^2}(h)\ \text{ and }\ \mathrm{Op}_h(b)\circ\mathrm{Op}_h(a)=Id+\mathcal{O}_{L^2}(h)\ .$$

Thus,

$$\langle\mathrm{Op}_h(1-\chi)u_h|u_h\rangle_{L^2}=\langle\mathrm{Op}_h(1-\chi)\circ\mathrm{Op}_h(b)\circ\mathrm{Op}_h(a)u_h|u_h\rangle_{L^2}+\mathcal{O}(h)\ .$$

On the other hand, $\mathrm{Op}_h(a)=\mathrm{Op}_h(-\xi^\intercal K(x)\xi+1)+i\mathrm{Op}_h(\theta)$ and thus $\mathrm{Op}_h(a)u_h=o_{L^2}(1)+i\mathrm{Op}_h(\theta)u_h$. Since $1-\chi$ and b are of order 0 or less, their corresponding operators are bounded in $L^2(\mathbb{R}^d)$, uniformly with respect to h and

$$\langle\mathrm{Op}_h(1-\chi)u_h|u_h\rangle_{L^2}=i\langle\mathrm{Op}_h(1-\chi)\mathrm{Op}_h(b)\mathrm{Op}_h(\theta)u_h|u_h\rangle_{L^2}+o(1)\ .$$

Now, it remains to apply Proposition A.1 in Appendix to see that, since $1-\chi$ and θ have disjoint supports,

$$\mathrm{Op}_h(1-\chi)\mathrm{Op}_h(b)\mathrm{Op}_h(\theta)u_h=\mathrm{Op}_h((1-\chi)b\theta)+o_{L^2}(1)=o_{L^2}(1)\ .$$

This shows that

$$\langle\mathrm{Op}_h(1-\chi)u_h|u_h\rangle_{L^2}\xrightarrow[h\longrightarrow 0]{}0\ .$$

• *Second step: using the geometric control condition of Theorem 1.2.*
First notice that

$$P_h=\mathrm{Op}_h(-\xi^\intercal K(x)\xi+1)-ih\mathrm{Op}_h(\gamma(x))+\mathcal{O}_{L^2\to L^2}(h^2)$$

and that we may assume that γ is smooth and bounded and so that it is a symbol of order 0 (see Section 2). Let $a(x,\xi)$ be a symbol of order 0. By Corollary A.2 in Appendix, the commutator of P_h and $\mathrm{Op}_h(a)$ is

$$[\mathrm{Op}_h(a),P_h]=-ih\mathrm{Op}_h\big(\{\xi^\intercal K(x)\xi,a(x,\xi)\}\big)+\mathcal{O}_{L^2\to L^2}(h^2)\ .$$

On the other hand, since $P_hu_h=o_{L^2}(h)$,

$$\begin{aligned}\langle[\mathrm{Op}_h(a),P_h]u_h|u_h\rangle_{L^2}&=\langle\mathrm{Op}_h(a)P_hu_h|u_h\rangle_{L^2}-\langle P_h\mathrm{Op}_h(a)u_h|u_h\rangle_{L^2}\\&=o(h)-\langle\mathrm{Op}_h(a)u_h|P_h^*u_h\rangle_{L^2}\\&=-\langle\mathrm{Op}_h(a)u_h|(P_h+2ih\gamma(x))u_h\rangle_{L^2}+o(h)\\&=2ih\langle\mathrm{Op}_h(a)u_h|\gamma(x)u_h\rangle_{L^2}+o(h)\\&=2ih\langle\gamma(x)\mathrm{Op}_h(a)u_h|u_h\rangle_{L^2}+o(h)\\&=2ih\langle\mathrm{Op}_h(a\gamma)u_h|u_h\rangle_{L^2}+o(h)\end{aligned}$$

Thus, setting $g(x,\xi)=\xi^\intercal K(x)\xi$, we obtain that

$$\langle\mathrm{Op}_h(2a\gamma+\{g,a\})u_h|u_h\rangle_{L^2}\xrightarrow[h\longrightarrow 0]{}0\ . \tag{3.4}$$

Due to Corollary A.3 of Appendix, we will get a contradiction with $\|u_h\|_{L^2} = 1$ if we find a such that $2a\gamma + \{g, a\}$ is uniformly bounded away from zero. Assume that $a(x, \xi)$ is constant equal to 1 for large ξ, then $2a\gamma + \{g, a\}$ is a symbol of order 0. Moreover, the first step of this proof shows that modifying $2a\gamma + \{g, a\}$ away from the sphere $\Sigma = \{(x, \xi), \xi^\intercal K(x)\xi = 1\}$ has no influence on (3.4). Thus, it is sufficient to exhibit a symbol a such that $2a\gamma + \{g, a\}$ is uniformly bounded and stay uniformly away from zero on Σ.

Let us recall that φ_t is the Hamiltonian flow associated to g and that T is a time such that the mean value $\langle\gamma\rangle_T(x, \xi) = \frac{1}{T}\int_0^T \gamma(\varphi_t(x, \xi))dt$ is uniformly bounded away from 0 away from Σ, according to Assumption (GCC) of Theorem 1.2. We choose $a(x, \xi) = e^{c(x,\xi)}$ with

$$c(x, \xi) = \frac{2}{T}\int_0^T (T - t)\gamma(\varphi_t(x, \xi))\, dt = \frac{2}{T}\int_0^T \int_0^t \gamma(\varphi_s(x, \xi))\, ds\, dt \ .$$

By definition of the Hamiltonian flow, for any function $f \in \mathcal{C}^1(\mathbb{R}^{2d}, \mathbb{R})$, we have

$$\{g, f\}(x, \xi) = \partial_\tau f(\varphi_\tau(x, \xi))_{|\tau=0} \ .$$

Since

$$\begin{aligned} c(\varphi_\tau(x, \xi)) &= \frac{2}{T}\int_0^T (T - t)\gamma(\varphi_{t+\tau}(x, \xi))\, dt \\ &= \frac{2}{T}\int_\tau^{T+\tau} (T - t + \tau)\gamma(\varphi_t(x, \xi))\, dt \end{aligned}$$

we get that

$$\{g, c\}(x, \xi) = \frac{2}{T}\int_0^T \gamma(\varphi_t(x, \xi))\, dt - 2\gamma(x, \xi) = 2\langle\gamma\rangle_T(x, \xi) - 2\gamma(x, \xi) \ .$$

Thus, we have

$$2a\gamma + \{g, a\} = 2e^{c(x,\xi)}\langle\gamma\rangle_T(x, \xi) \ .$$

By assumption (GCC) of Theorem 1.2 and since $c \geq 0$, there exists $\alpha > 0$ such that, for all $(x, \xi) \in \Sigma$, $2a\gamma + \{g, a\} \geq \alpha > 0$. As explained above, we can neglect any (x, ξ) away from Σ and this yields that

$$\langle \mathrm{Op}_h(2a\gamma + \{g, a\})u_h|u_h\rangle_{L^2} \sim \langle \mathrm{Op}_h\left(2e^{c(x,\xi)}\langle\gamma\rangle_T(x, \xi)\right) u_h|u_h\rangle_{L^2} \geq 2\alpha\|u_h\|^2_{L^2} \ ,$$

which contradicts (3.4) since $\|u_h\|_{L^2} = 1$. □

4. Proof of Theorem 1.2: low frequencies

We now have to deal with the low frequencies to finish the proof of Theorem 1.2. This is done by using Carleman estimates. Notice that the same tool will provide the logarithmic decay of Theorem 1.3 (see Section 5).

In this section, we fix first a real number μ and we assume that there is a sequence (U_n) with $\|U_n\|_X = 1$ and $(A - i\mu)U_n \longrightarrow 0$, that is that $U_n = (u_n, v_n)$ satisfies $v_n = i\mu u_n + o_{H^1}(1)$ and

(4.1) $$(\Delta_K - Id)u_n - i\mu\gamma(x)u_n + \mu^2 u_n = o_{L^2}(1) \ .$$

We work with $\gamma \in \mathcal{C}_b^\infty(\mathbb{R}^d)$ satisfying the geometric control condition (GCC) of Theorem 1.2 (see Section 2). This condition yields that, for any $(x, \xi) \in \Sigma$, the ray of length T contains a point y

such that $\gamma(y) \geq \alpha/T$. Since the metric K is uniformly bounded, we can find a sequence $(x_n) \subset \mathbb{R}^d$ such that $\gamma(x_n) \geq \alpha/T$ and any point of $\mathbb{R}^d$ is at bounded distance L of the set $\cup_n \{x_n\}$. Since γ is uniformly continuous, we can find $r, a > 0$ such that $\gamma(x) \geq a > 0$ on $\cup_n B(x_n, r)$. This is the control by a network of balls (NCC) stated in Theorem 1.3, which will be a sufficient condition to control the low frequencies in this section. We will denote by ω the set $\omega = \cup_n B(x_n, r)$.

As a first basic computation, we can multiply (4.1) by $\overline{u}_n$ and integrate. Taking real and imaginary parts, we obtain

$$\|u_n\|_{H^1}^2 = \mu^2 \|u_n\|_{L^2}^2 + o(1) \quad \text{and} \quad \int_{\mathbb{R}^d} \gamma(x) |u_n(x)|^2 \, dx \xrightarrow[n \longrightarrow \infty]{} 0 \ . \tag{4.2}$$

Also notice that $\|v_n\|_{L^2} = \mu \|u_n\|_{L^2} + o(1)$. Thus, if $\mu = 0$ we obtain a contradiction between $\|U_n\|_X = 1$ and $\|U_n\|_X \sim \mu \|u_n\|_{L^2} + o(1)$. Assume from now on that $\mu \neq 0$, we get that $\|U_n\|_X$ is equivalent to $\|u_n\|_{L^2}$ and, up to a renormalisation, we can assume that (u_n) satisfies $\|u_n\|_{L^2} = 1$.

Now, we can multiply (4.1) by $\Delta_K \overline{u}_n$ and integrate. The real part of the result shows that $\|\Delta_K u_n\|^2 = \mathcal{O}(1) + \langle o_{L^2}(1) | \Delta_K u_n \rangle_{L^2}$, which implies that $\|\Delta_K u_n\|$ is bounded. Then, considering the imaginary part, we get that

$$\int_{\mathbb{R}^d} \gamma(x) |\nabla u_n(x)|^2 \, dx \xrightarrow[n \longrightarrow \infty]{} 0 \ . \tag{4.3}$$

• *First step: using Hörmander sub-ellipticity argument.*
We set $P = \Delta_K - i\mu\gamma(x) + (\mu^2 - 1) Id$ and

$$Q_h = h^2 e^{\varphi/h} (-\Delta_K + (1 - \mu^2) Id) e^{-\varphi/h} \ .$$

We follow the classical arguments (see for example [**19**]). We have

$$Q_h u = -h^2 \Delta_K u + 2h \nabla\varphi^\intercal K(x) \nabla u + u(-\nabla\varphi^\intercal K(x) \nabla\varphi + h\Delta_K(\varphi) + h^2(1 - \mu^2)) \ .$$

With the notations of the Appendix A, using $-h^2 \Delta_K = \mathrm{Op}_h(\xi^\intercal K(x) \xi)$, $h\nabla = \mathrm{Op}_h(i\xi)$ and Proposition A.1, we obtain that

$$Q_h = \mathrm{Op}_h \left(\xi^\intercal K(x) \xi - \nabla\varphi^\intercal K(x) \nabla\varphi + 2i \nabla\varphi^\intercal K(x) \xi \right) + \mathcal{O}_{L^2 \to L^2}(h^2) \ .$$

We set $Q_h = Q_h^R + i Q_h^I + \mathcal{O}_{L^2 \to L^2}(h^2)$ with

$$\begin{aligned} Q_h^R &= \mathrm{Op}_h(q_R) = \mathrm{Op}_h(\xi^\intercal K(x) \xi - \nabla\varphi^\intercal K(x) \nabla\varphi) \\ Q_h^I &= \mathrm{Op}_h(q_I) = \mathrm{Op}_h(2 \nabla\varphi^\intercal K(x) \xi) \ . \end{aligned}$$

We use Proposition A.4 in Appendix to check that Q_h^R and Q_h^I are self-adjoint operators and Corollary A.2 shows that

$$\begin{aligned} \|Q_h u\|_{L^2}^2 &= \|Q_h^R u\|_{L^2}^2 + \|Q_h^I u\|_{L^2}^2 + \langle Q_h^R u | i Q_h^I u \rangle_{L^2} + \langle i Q_h^I u | Q_h^R u \rangle_{L^2} + \mathcal{O}(h^2 \|u\|_{L^2}^2) \\ &= \langle ((Q_h^R)^2 + (Q_h^I)^2 + i[Q_h^R, Q_h^I]) u | u \rangle_{L^2} + \mathcal{O}(h^2 \|u\|_{L^2}^2) \\ &\geq h \langle \mathrm{Op}_h(\eta(q_R^2 + q_I^2) + \{q_R, q_I\}) u | u \rangle + \mathcal{O}(h^2 \|u\|_{L^2}^2) \end{aligned} \tag{4.4}$$

where η is any number such that $\eta h \leq 1$. Let

$$B = \left\{ (x, \xi) \in \mathbb{R}^{2d} \ , \ \frac{K_{\inf}}{2K_{\sup}} |\nabla\varphi(x)| \leq |\xi| \leq \frac{2K_{\sup}}{K_{\inf}} |\nabla\varphi(x)| \right\} \ .$$

Notice that q_R is uniformly away from 0 for (x, ξ) outside B. Assume that φ satisfies the sub-ellipticity criterion: There exists $\alpha > 0$ such that

$$\{q_R, q_I\}(x, \xi) \ \geq \ \alpha \ > \ 0 \text{ on } ((\mathbb{R}^d \setminus \omega) \times \mathbb{R}^d) \cap \{(x, \xi); q_R(x, \xi) = q_I(x, \xi) = 0\} \tag{4.5}$$

Then, taking η sufficiently large, $\eta(q_R^2+q_I^2)+\{q_R,q_I\}$ is uniformly positive on $\mathbb{R}^d\setminus\omega$. Moreover, the behaviour for large $|\xi|$ is given by q_R^2, thus we have that $\eta(q_R^2+q_I^2)+\{q_R,q_I\}$ is a symbol of order 4 and there is a positive constant $\alpha>0$ such that

$$\eta(q_R^2+q_I^2)+\{q_R,q_I\}\geq\alpha(1+|\xi|^2)^2 \text{ on } (\mathbb{R}^d\setminus\omega)\times\mathbb{R}^d .$$

Using Gårding inequality stated in Proposition A.6 in Appendix and (4.4), we obtain that, if (4.5) holds, then there is $c>0$ such that

$$\|Q_hu\|_{L^2}^2 \ \geq\ ch\|u\|_{L^2}^2 \text{ for all } u \text{ satisfying } u_{|\omega}\equiv 0 . \tag{4.6}$$

To obtain a contradiction with (4.1) and $\|u_n\|_{L^2}=1$, we proceed as follows. Let $\chi\in\mathcal{C}_b^\infty(\mathbb{R},[0,1])$ be a function such that $\chi(s)=0$ for $s\geq a$ and $\chi\equiv 1$ in a neighbourhood of 0. We have that $\chi\circ\gamma$ vanishes on ω and, since $(1-\chi\circ\gamma)$ vanishes on $\{x,\gamma(x)\leq\nu\}$ for some small $\nu>0$, any derivative of $\chi\circ\gamma$ is controlled by $\kappa\gamma$ for κ large enough. We set $v_n=e^{\varphi/h}(\chi\circ\gamma)u_n$ and compute

$$\begin{aligned} Q_hv_n &= h^2e^{\varphi/h}(-P-i\mu\gamma)(\chi\circ\gamma)u_n \\ &= -h^2e^{\varphi/h}(\chi\circ\gamma)Pu_n \ -\ i\mu h^2e^{\varphi/h}\gamma(\chi\circ\gamma)u_n \ -2h^2e^{\varphi/h}\nabla(\chi\circ\gamma)^\intercal K(x)\nabla u_n \\ &\quad -\ h^2e^{\varphi/h}u_n\Delta_K(\chi\circ\gamma) \\ &= o_{L^2}(1) \text{ when } n\to+\infty \text{ and } h>0 \text{ is fixed.} \end{aligned}$$

where we used the fact that γu_n and $\gamma\nabla u_n$ goes to zero in $L^2(\mathbb{R}^d)$. Now, remember that $u_n-(\chi\circ\gamma)u_n$ is supported on $\{x,\gamma(x)\geq\nu\}$ and thus also goes to zero in L^2. For h fixed, $\|v_n\|_{L^2}$ is thus uniformly positive since $e^{\varphi/h}(x)\geq e^{\min\varphi/h}>0$ and $\|(\chi\circ\gamma)u_n\|_{L^2}\to 1$. Since v_n vanishes on ω, this is an obvious contradiction with (4.6) and $Q_hv_n=o_{L^2}(1)$ shown above.

• *Second step: Carleman weight $\varphi=e^{\lambda\psi}$.*
The usual way to obtain a weight φ satisfying Hörmander sub-elliptic assumption (4.5) consists in choosing $\varphi=e^{\lambda\psi}$, with a function ψ, whose critical points belongs to $\omega=\cup_nB(x_n,r)$, and λ large enough. We reproduce here this argument with an obvious care about uniformity.

Assume that $\varphi=e^{\lambda\psi}$ for some constant λ and that $\psi\in\mathcal{C}_b^\infty(\mathbb{R}^d)$ is such that there exists $\alpha>0$ such that $|\nabla\psi(x)|\geq\alpha>0$ for all $x\in\mathbb{R}^d\setminus\omega$. A straightforward computation yields

$$\begin{aligned} \{q_R,q_I\} &= \partial_\xi\left(\xi^\intercal K(x)\xi\right)\partial_x\left(\lambda e^{\lambda\psi}\nabla\psi^\intercal K(x)\xi\right) \\ &\quad -\partial_x\left(\xi^\intercal K(x)\xi-\lambda^2e^{2\lambda\psi}\nabla\psi^\intercal K(x)\nabla\psi\right)\partial_\xi\left(\lambda e^{\lambda\psi}\nabla\psi^\intercal K(x)\xi\right) \\ &\geq \lambda^4e^{3\lambda\psi}(\nabla\psi^\intercal K(x)\nabla\psi)^2+\mathcal{O}(|\xi|^2\lambda e^{\lambda\psi})+\mathcal{O}(\lambda^3e^{3\lambda\psi}) \end{aligned}$$

where the estimations $\mathcal{O}(\cdot)$ hold for $|\xi|$ and λ going to $+\infty$. Notice that, when (x,ξ) belongs to B, $|\xi|$ is of order $\mathcal{O}(\lambda e^{\lambda\psi(x)})$. Since $|\nabla\psi(x)|\geq\alpha>0$ on $\mathbb{R}^d\setminus\omega$, we can fix λ large enough, such that the positive term $\lambda^4e^{3\lambda\psi}(\nabla\psi^\intercal K(x)\nabla\psi)^2$ controls the last two terms (with indefinite sign), when (x,ξ) belongs to B and $x\notin\omega$.

• *Third step: construction of the appropriate Carleman phase ψ.*
To summarise, the above arguments show that, if we are able to construct a suitable phase ψ, then the sub-elliptic criterium (4.5) would hold and also the uniform positivity property (4.6). This will provide a contradiction between (4.1) and $\|u_n\|_{L^2}\equiv 1$, as described below Equation (4.6). This will yield the control of the low frequencies and finished this section.

Thus, we only have to construct $\psi\in\mathcal{C}_b^\infty(\mathbb{R}^d,\mathbb{R})$ such that $|\nabla\psi|$ is uniformly positive outside $\omega=\cup_nB(x_n,r)$. We recall that the points x_n are assumed to form a network in the sense that any point $x \in \mathbb{R}^d$ is at distance at most L of a point x_n. We split $\mathbb{R}^d$ in cubes $(C_k)_{k\in\mathbb{Z}^d}$ of size $4L$ by setting $C_k = 4Lk + [-2L, 2L]^d$. For each $k \in \mathbb{Z}^d$, the center of C_k is $c_k = 4Lk$ and there is at least one of the points x_n which is in $c_k + [-L, L]^d$, let us denote it y_k. For each k, one can find a $\mathcal{C}^\infty-$diffeomorphism with compact support in the interior of C_k such that y_k is mapped onto c_k. We glue all these diffeomorphisms into a diffeomorphism Φ of $\mathbb{R}^d$ mapping all the y_k onto the c_k and we notice that we can make this construction such that Φ and Φ^{-1} belong to $\mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R})$ (an explicit construction is given in [**22**]). Consider

$$\psi(x) = \tilde{\psi} \circ \Phi(x) \ \text{ with } \ \tilde{\psi}(x) = \sum_{i=1}^{d} \cos\left(\frac{\pi x_i}{4L}\right) .$$

Obviously, $|\nabla\tilde{\psi}|$ is uniformly positive outside $\cup_n B(c_n, \rho)$ for any $\rho > 0$. Thus, $|\nabla\psi|$ is uniformly positive outside $\cup_n B(x_n, r)$, which concludes this section.

5. Proof of Theorem 1.3

The proof of Theorem 1.3 relies on the same arguments than the ones of Sections 2 and 4. We will only outline them.

Instead of Theorem 2.1, we use the following characterisation of the logarithmic rate of decay given by Theorem 3 of Burq [**7**].

***Theorem 5.1* (Burq, [7, Theorem 3]).** — *Let A be maximal dissipative operator (and hence the generator of e^{At} a $\mathcal{C}^0-$semigroup of contractions) in a Hilbert space X and assume that there exist $C, c > 0$ such that $i\mathbb{R} \subset \rho(A)$ and*

$$\forall \mu \in \mathbb{R}, \|(A - i\mu Id)^{-1}\|_{\mathcal{L}(X)} < Ce^{c|\mu|}. \tag{5.1}$$

Then for any $k > 0$ there exists C_k such that for any $t > 0$,

$$\left\|\frac{e^{tA}}{(1-A)^k}\right\|_{\mathcal{L}(X)} \leq \frac{C_k}{\log(2+t)^k}.$$

First notice that the estimate (5.1) is already proved for low frequencies in Section 4. Thus, it is enough to prove it for large μ (say $\mu = h^{-1}, h \to 0^+$, the case of negative μ being similar). We are consequently in a high frequency regime, but are nevertheless going to use the approach developed in the previous section for low frequencies, based on Carleman weight and Hörmander sub-ellipticity argument. Indeed, a simple adaptation allows to prove similar Carleman estimates in the high-frequency regime (and hence for the semi-classical Helmoltz operator) by tracking the exponential dependence of the constants with respect to the frequency parameter.

As in Section 2, we can assume that γ is smooth by arguing by contradiction: assume that there exist two sequences $(U_n) = (u_n, v_n) \subset D(A) = H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ and $(\mu_n) \longrightarrow +\infty$ such that $\|U_n\|_X^2 = 1$ and $\|(A - i\mu_n)U_n\|_X$ goes to 0 in X faster than any exponential. Once again, we have $\mathrm{Re}(\langle (A - i\mu_n)U_n | U_n\rangle_X) = -\int \gamma(x)|v_n(x)|^2\,dx$, which shows that γv_n is decaying as fast as $(A - i\mu_n)U_n$. As in the second step of Section 2, we can replace γ by a smooth damping $\underline{\gamma} \in \mathcal{C}_b^\infty(\mathbb{R}^d)$ satisfying $\underline{\gamma} \leq \gamma$ without changing the fact that $\|(A - i\mu_n)U_n\|_X$ goes to 0 in X faster than any exponential. Notice that, if γ satisfies Hypothesis (NCC) of Theorem 1.3, then one can easily construct a smooth damping $\underline{\gamma} \leq \gamma$ also satisfying (NCC).

We would like to show (5.1) for large μ. As previously, simple calculations show that it is enough to prove a similar estimate on $((\Delta_K - Id) - i\mu\gamma(x) + \mu^2)^{-1}$:

$$\exists C, c > 0\ ,\ \forall \mu \in \mathbb{R}\ ,\ |||((\Delta_K - Id) - i\mu\gamma(x) + \mu^2)^{-1}|||_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq Ce^{c|\mu|}\ . \tag{5.2}$$

We use the arguments and the notations of Section 4.

Let (u, f) solutions to $((\Delta_K - Id) - i\mu\gamma(x) + \mu^2)u = f$, i.e., setting $h = 1/\mu$,

$$(h^2\Delta_K - ih\gamma(x) + (1 - h^2))u = h^2 f\ .$$

Let

$$\widetilde{Q}_h = e^{\varphi/h}(-h^2\Delta_K + (h^2 - 1)Id)e^{-\varphi/h}\ .$$

We have $\widetilde{Q}_h = \widetilde{Q}_h^R + i\widetilde{Q}_h^I + \mathcal{O}(h^2)$ with

$$\widetilde{Q}_h^R = \mathrm{Op}_h(\widetilde{q}_R) = \mathrm{Op}_h(\xi^\intercal K(x)\xi - \nabla\varphi^\intercal K(x)\nabla\varphi - 1)$$
$$\widetilde{Q}_h^I = \mathrm{Op}_h(\widetilde{q}_I) = \mathrm{Op}_h(2\nabla\varphi^\intercal K(x)\xi)\ .$$

In this setting, we shall assume that the phase function $\varphi \in C_b^\infty(\mathbb{R}^d)$ satisfies Hörmander hypoellipticity condition: there exists $\alpha > 0$ such that

$$\{\widetilde{q}_R, \widetilde{q}_I\}(x, \xi) \geq \alpha > 0 \text{ on } ((\mathbb{R}^d \setminus \omega) \times \mathbb{R}^d) \cap \{(x, \xi); \widetilde{q}_R(x, \xi) = \widetilde{q}_I(x, \xi) = 0\} \tag{5.3}$$

The same proof as in the previous section shows that, under this condition, if v vanished in ω, then for $h > 0$ small enough,

$$\|v\|_{L^2} \leq \frac{C}{h}\|\widetilde{Q}_h v\|_{L^2}.$$

Coming back to u, applying the previous estimate to $v = e^{\varphi(x)/h}\tilde{\chi}u$ with a cutoff $\tilde{\chi} = \chi \circ \gamma$ as in the previous section, we get

$$\|u\|_{L^2} \leq Che^{c/h}\|\tilde{\chi}f - (2(\nabla\tilde{\chi})^\intercal K(x)\nabla + \Delta_K\tilde{\chi})u\|_{L^2} + Ce^{c/h}\|\gamma\tilde{\chi}u\|_{L^2}\ ,$$

where $c = \sup_{x,y\in\mathbb{R}^2} |\varphi(x) - \varphi(y)|$. Remember that all the terms involving u in the right-hand side are controlled by γu, itself being controlled by the usual computation $\int \gamma|u|^2 = -h\mathrm{Re}(\int fu)$. We can now proceed by contradiction and conclude the proof of the estimate (5.2) exactly as in the previous section: it is impossible to have sequences $(\mu_n) = (1/h_n)$, (u_n) and (f_n) such that $\|u_n\| = 1$, $h_n \to 0$ and (f_n) goes to zero faster than any exponential $e^{-\kappa\mu_n}$.

To conclude, it remains to construct the Carleman weight $\varphi = e^{\lambda}\psi$ satisfying Hörmander hypoellipticity condition. This is done exactly as in Section 4. Notice that the only difference here with the low frequency case is that $\widetilde{q}_R = q_R - 1$. However, during the construction, this additional term 1 generates terms which are of order $\lambda^2 e^{\lambda\Psi}$. Thus, this will not perturb the exponential bound of the estimate.

6. Applications to other problems

The exponential decay of the linear semigroup e^{At} is an essential assumption for obtaining several dynamical properties of the damped wave equations. In this section, we emphasise different results, which are corollaries of Theorem 1.2. Each result was already known with stronger assumptions implying the exponential decay of e^{At}. Since we have obtained this decay with weaker conditions, we can improve on these results.

6.1. Stabilisation in uniformly local Sobolev spaces. — Theorem 1.2 concerns the solutions of the damped wave equation with finite energy. In an unbounded domain, a natural question would be to consider solutions with infinite energy. For this reason, we introduce the uniformly local Sobolev spaces as follows. For any $u \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, we set

$$\|u\|_{L^2_{\mathrm{ul}}} = \sup_{\xi\in\mathbb{R}^d}\Big(\int_{B(\xi,1)}|u(x)|^2dx\Big)^{1/2} = \sup_{\xi\in\mathbb{R}^d}\|u\|_{L^2(B(\xi,1))} . \tag{6.1}$$

The uniformly local Lebesgue space is defined as

$$L^2_{\mathrm{ul}}(\mathbb{R}^d) = \Big\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^d)\,\Big|\,\|u\|_{L^2_{\mathrm{ul}}}<\infty\,,\ \lim_{\xi\to 0}\|u(\cdot-\xi)-u\|_{L^2_{\mathrm{ul}}}=0\Big\}\ , \tag{6.2}$$

In a similar way, for any $k\in\mathbb{N}$, we introduce the uniformly local Sobolev space

$$H^k_{\mathrm{ul}}(\mathbb{R}^d) = \Big\{u\in H^k_{\mathrm{loc}}(\mathbb{R}^d)\,\Big|\,\partial^j_{x_i}u\in L^2_{\mathrm{ul}}(\mathbb{R}^d)\text{ for }i=1,\dots,d\text{ and }j=0,1,\dots,k\Big\}\ , \tag{6.3}$$

which is equipped with the natural norm $\|u\|_{H^k_{\mathrm{ul}}} = \Big(\sum_{i=1}^d\sum_{j=0}^k\|\partial^j_{x_i}u\|^2_{L^2_{\mathrm{ul}}}\Big)^{1/2}$. As shown in [**11**], the damped wave equation (1.1) is well defined on $H^1_{\mathrm{ul}}(\mathbb{R}^d)\times L^2_{\mathrm{ul}}(\mathbb{R}^d)$. The assumption $\lim_{\xi\to0}\|u(\cdot-\xi)-u\|_{L^2_{\mathrm{ul}}}=0$ in (6.2) introduces a continuity with respect to translations, which plays the role of the uniform continuity for continuous functions. It could be possible to work without this assumption, however $H^1_{\mathrm{ul}}(\mathbb{R}^d)$ would not be dense in $L^2_{\mathrm{ul}}(\mathbb{R}^d)$ in this case, which is troublesome. That is why the assumption $\lim_{\xi\to0}\|u(\cdot-\xi)-u\|_{L^2_{\mathrm{ul}}}=0$ in (6.2) may be important for the functional analysis.

We have the following result.

Theorem 6.1. — *Assume that the assumptions of Theorem 1.2 hold. Then the semigroup generated by the damped wave equation* (1.1) *on* $H^1_{ul}(\mathbb{R}^d)\times L^2_{ul}(\mathbb{R}^d)$ *is exponentially decreasing. There exist* M *and* $\lambda>0$ *such that, for any solution* $U(t)=(u(t),\partial_tu(t))$ *of* (1.1) *with* $U(0)\in H^1_{ul}(\mathbb{R}^d)\times L^2_{ul}(\mathbb{R}^d)$, *we have*

$$\|U(t)\|_{H^1_{ul}\times L^2_{ul}}\le Me^{-\lambda t}\|U(0)\|_{H^1_{ul}\times L^2_{ul}}$$

Proof: It is sufficient to show that there exists a time $T>0$ and $C\in(0,1)$ such that

$$\|U(T)_{|B(\xi,1)}\|_{H^1\times L^2}\le C\|U(0)\|_{H^1_{\mathrm{ul}}\times L^2_{\mathrm{ul}}} \tag{6.4}$$

for all solutions of (1.1) and all $\xi\in\mathbb{R}^d$. Due to the finite speed of propagation of informations and due to the bounds on the metric $K(x)$, $U(T)_{|B(\xi,1)}$ only depends on the values of $U(0)_{|B(\xi,1+\kappa T)}$ for some $\kappa>0$. Applying Theorem 1.2 to the solution corresponding to a compactly supported truncation of $U(0)$, we have that

$$\|U(T)_{|B(\xi,1)}\|_{H^1\times L^2}\le Me^{-\lambda T}\|U(0)_{|B(\xi,2+\kappa T)}\|_{H^1\times L^2}\le Ne^{-\lambda T}T^d\|U(0)\|_{H^1_{\mathrm{ul}}\times L^2_{\mathrm{ul}}}$$

since the ball $B(\xi,2+\kappa T)$ can be covered by a number $\mathcal{O}(T^d)$ of balls of radius 1. For T large enough, we obtain (6.4), which shows the result. □

6.2. Linear control. — By HUM Method of Lions (see [**23**]), the exponential decay of the linear semigroup e^{At} is equivalent to the controllability of the linear wave equation (in large time). We denote by $\mathbb{1}_\omega$ the function $\mathbb{1}_\omega \equiv 1$ on ω and 0 elsewhere.

Corollary 6.2. — *Let ω be an open subset of $\mathbb{R}^d$ and assume that the hypotheses of Theorem 1.2 hold with $\gamma = \mathbb{1}_\omega$. Then, there exists $T > 0$ such that, for any $(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and any $(\tilde{u}_0, \tilde{u}_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, there exists a control $v \in L^1((0,T), L^2(\omega))$ such that the solution u of*

$$\left\{ \begin{array}{ll} \partial_{tt}^2 u - \mathrm{div}(K(x)\nabla u) + u = \mathbb{1}_\omega v(x,t) & (t,x) \in (0,T) \times \mathbb{R}^d \ , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) & \end{array} \right.$$

satisfies $(u, \partial u)(\cdot, T) = (\tilde{u}_0, \tilde{u}_1)$.

6.3. Global attractor and stabilisation for the non-linear equation. — Another related problem is the asymptotic behaviour of the non-linear equation as studied in [**33**], [**10**], [**9**] or [**16**]. One considers the non-linear equation

$$\left\{ \begin{array}{ll} \partial_{tt}^2 u + \gamma(x)\partial_t u = \mathrm{div}(K(x)\nabla u) - u - f(x,u) & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \ , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) & \end{array} \right. \tag{6.5}$$

with $f(x,s) \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ compactly supported in x, satisfying

$$|f(x,s)| \leq C(1+|s|)^p \quad \text{and} \quad |f'(x,s)| \leq C(1+|s|)^{p-1} \tag{6.6}$$

with $1 \leq p < (d+2)/(d-2)$ (or any $p \geq 1$ if $d < 3$) and

$$\liminf_{|s| \to +\infty} \ \max_{x \in \mathrm{supp}(f)} \ f(x,s)s \ \geq \ 0 \ . \tag{6.7}$$

To each solution of (6.5), one can associate the energy

$$E(u) := E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^d} (|\partial_t u|^2 + |\nabla u^\intercal . K(x) . \nabla u| + |u|^2) + \int_{\mathbb{R}^d} V(x,u) \ ,$$

where $V(x,u) = \int_0^u f(x,s)ds$. This energy is non-increasing since

$$\partial_t E(u(t)) = -\int_{\mathbb{R}^d} \gamma(x) |\partial_t u(x,t)|^2 \, dx \ .$$

Notice that (6.6) implies that the energy is well defined and that (6.7) shows that the energy is bounded from below and that the bounded sets of $X = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ are equivalent to the sets of bounded energy. Thus any trajectory of (6.5) has a non-increasing energy and stays bounded in X. Assume now that the semigroup e^{At} is exponentially stable, then any trajectory $U = (u, \partial_t u)$ satisfies

$$U(t) = e^{At} U(0) + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(x, u(x,s)) \end{pmatrix} ds \ . \tag{6.8}$$

The first term of (6.8) is decaying exponentially fast and the integral term is compact since f is compactly supported in x and due to either the compact Sobolev embedding $H^1 \hookrightarrow L^{2p}$ for $p < d/(d-2)$ or to more technical arguments based on the Strichartz estimates for $p \in [d/(d-2), (d+2)/(d-2))$ (see [**9**] and see [**16**]). Thus, following the ideas of [**9**] and [**16**], we obtain that any solution is asymptotically compact and converges to a trajectory with constant energy. Now, we would like to show that the energy E associated to (6.5) is a Lyapounov function, that is that it is non-increasing and cannot be constant along a solution $u(t)$, except of course if $u(t)$ is an equilibrium point. If (6.5) admits a Lyapounov function, one says that the corresponding dynamical system is gradient. In particular, it cannot admit periodic orbits, homoclinic orbits… The gradient structure of (6.5), together with its asymptotic compactness, will also ensure the existence of a compact global attractor, that is a compact invariant set of X which attracts all the trajectories of (6.5). This set is a central object of the theory of dynamical systems. It contains all the solutions $u(t)$, which exist for all $t \in \mathbb{R}$ and which are uniformly bounded in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$ (as equilibrium points, heteroclinic orbits etc.). See for example [**13**] and [**31**] for a review on the concepts of compact global attractors, of asymptotic compactness or of gradient structure.

To show that the energy E is a Lyapounov function, that is that it cannot be constant along a trajectory, except if this trajectory is an equilibrium point, one has to use a suitable unique continuation property. In [**33**] and [**9**] (see also [**10**]), the authors use a unique continuation property, which needs geometric assumptions stronger than the one required for the exponential decay of e^{At}. However, we have shown in [**16**] that the geometric assumptions required for the exponential decay of e^{At} are sufficient if we assume that f is smooth and partially analytic. Thus, we can improve the result of [**16**] by using a weaker assumption than the one that $\gamma \geq \alpha > 0$ outside a compact set.

Corollary 6.3. — *Assume that the hypotheses of Theorem 1.2 hold. Also assume that $f \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies* (6.6) *and* (6.7) *and that f is compactly supported in x and analytic with respect to u. Then the dynamical system generated by* (6.5) *in $H^1_0(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ is gradient and admits a compact global attractor $\mathcal{A}$.*

Moreover, if $f(x,u)u \geq 0$ for any $(x,u) \in \mathbb{R}^{d+1}$, then the semilinear damped wave equation (6.5) *is stabilised in the sense that for any $E_0 \geq 0$, there exist $K > 0$ and $\lambda > 0$ such that, for all solutions u of* (6.5) *with $E(u(0)) \leq E_0$, $E(u(t)) \leq Me^{-\lambda t}E(u(0))$ for all $t \geq 0$.*

Of course, in the one-dimensional case $d = 1$, the unique continuation property holds without any additional assumption. We obtain the following result, where one can omit the assumption $\gamma \geq \alpha > 0$ close to $\pm\infty$ used in [**33**].

Corollary 6.4. — *Assume that the hypotheses of Theorem 1.2 hold in dimension $d = 1$ and that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies* (6.6) *and* (6.7) *and that f is compactly supported in x. Then the dynamical system generated by* (6.5) *in $H^1_0(\mathbb{R}) \times L^2(\mathbb{R})$ is gradient and admits a compact global attractor $\mathcal{A}$.*

Moreover, if $f(x,u)u \geq 0$ for any $(x,u) \in \mathbb{R}^2$, then the semilinear damped wave equation (6.5) *is stabilised.*

Notice that Corollaries 6.3 and 6.4 are not exactly generalisations of the results of [**16**] and [**33**]. Indeed, f is assumed to be compactly supported in x. In [**16**] and [**33**], because of the assumption $\gamma \geq \alpha > 0$ outside a compact set, one is able to deal with nonlinearities f being not compactly supported.

This type of non-linear stabilisation results is also closely related to the problem of global control of the non-linear wave equation, see [**9**], [**16**] and [**17**]. For example, one gets the following result in dimension $d = 1$.

Corollary 6.5. — *Let ω be an open subset of $\mathbb{R}$. Assume that there exist $L > 0$ and $\varepsilon > 0$ such that ω contains an interval of length ε in any interval $[x, x+L]$, $x \in \mathbb{R}$. Let also $f \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ compactly supported in x and satisfying* (6.7).

Then, for all $E_0 \geq 0$, there exists $T > 0$ such that, for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1) \in H^1(\mathbb{R})\times L^2(\mathbb{R})$ with energy E less than E_0, there exists a control $v \in L^1((0,T), L^2(\omega))$ such that the solution u

of

$$\begin{cases} \partial_{tt}^2 u - \operatorname{div}(K(x)\nabla u) + u + f(x,u) = \mathbb{1}_\omega v(x,t) & (t,x) \in (0,T) \times \mathbb{R} \ , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \end{cases}$$

satisfies $(u, \partial u)(\cdot, T) = (\tilde{u}_0, \tilde{u}_1)$.

A. Appendix: pseudo-differential semiclassical calculus

In section, we recall the main results and notations of pseudo-differential calculus, which are used in this paper. The details and the proofs could be found in many textbooks, as [**34**], [**24**], [**1**] or [**19**].

Let $h > 0$ be a small parameter, say that $h \in (0, 1]$. We say that $a(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is a *symbol* of order m if, for any multi-indices α and β, there exists $C_{\alpha,\beta}$ such that

$$\sup_{(x,\xi)\in\mathbb{R}^{2d}} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|}$$

and m is the smallest number such that this bounds holds. To each symbol a, we associate the *pseudo-differential semiclassical operator* denoted by $\operatorname{Op}_h(a)$ and defined by Weyl quantization

$$\operatorname{Op}_h(a)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2d}} e^{i(x-y).\xi}\, a\left(\frac{x+y}{2}, h\xi\right) u(y)\, dy\, d\xi \ . \tag{A.1}$$

If a is of order m, then, for any $s > 0$, $\operatorname{Op}_h(a)$ is a bounded operator from $H^s(\mathbb{R}^d)$ into $H^{s-m}(\mathbb{R}^d)$, uniformly with respect to $h \in (0, 1]$.

Let $f \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ be a smooth bounded function such that all its derivative are also bounded functions of $L^\infty(\mathbb{R}^d)$. It is not so trivial but well known that the simple operator $u \mapsto f(x)u$ has for symbol $f(x)$, which is of order 0 (see for example Chapter 4 of [**34**]). More classically, we have that the operator $h\nabla$ has for symbol $i\xi$, which is of order 1. Using Proposition A.1 below, one can check that the operator $h^2\Delta_K = h^2\operatorname{div}(K(x)\nabla\cdot)$ has for principal symbol $-\xi^\intercal.K(x).\xi$ in the sense that

$$h^2\Delta_K = \operatorname{Op}_h(-\xi^\intercal.K(x).\xi) + \mathcal{O}_{L^2\to L^2}(h^2) \ .$$

Composing two pseudo-differential operators, one obtain a pseudo-differential operator, which symbol can be express by an asymptotic development. In this paper, we will simply use the following cases, see [**34**], [**24**], [**1**] or any other textbooks on pseudo-differential calculus for more precise developments and for proofs.

Proposition A.1 **(Composition)**. — *Let a and b be two symbols of order m and n respectively. Assume that $m + n \leq 2$, then $\operatorname{Op}_h(a) \circ \operatorname{Op}_h(b)$ is a pseudo-differential operator of order $m + n$ and its symbol $(a\#b)$ satisfies*

$$a\#b = ab - \frac{ih}{2}\{a, b\} + \mathcal{O}_{L^2\to L^2}(h^2)$$

where $\{a, b\} = \partial_\xi a \partial_x b - \partial_\xi b \partial_x a$ is the Poisson bracket of a and b and is of order $m + n - 1$. In particular, if $m + n \leq 1$, then

$$\operatorname{Op}_h(a) \circ \operatorname{Op}_h(b) = \operatorname{Op}_h(ab) + \mathcal{O}_{L^2\to L^2}(h) \ .$$

Corollary A.2 **(Commutators)**. —

i) *If a is of order 1 or less and if b is of order 0, then the commutator $[\operatorname{Op}_h(a), \operatorname{Op}_h(b)] = \operatorname{Op}_h(a) \circ \operatorname{Op}_h(b) - \operatorname{Op}_h(b) \circ \operatorname{Op}_h(a)$ is of order 0 or less and of estimate $\mathcal{O}_{L^2\to L^2}(h)$.*

ii) *If a is of order 2 and if b is of order 0, then their commutator is of order 1 and*

$$[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = -ih\mathrm{Op}_h(\{a,b\}) + \mathcal{O}_{L^2\to L^2}(h^2)\ .$$

***Corollary A.3* (Inverse).** — *Assume that a is a symbol of order $m \geq 0$ such that there exists $\alpha > 0$ such that $|a(x,\xi)| \geq \alpha$ for all $(x,\xi) \in \mathbb{R}^{2d}$. Then $b = 1/a$ is a symbol of order $-m$ and $\mathrm{Op}_h(b)$ is a first order inverse of a in the sense that*

$$\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b) = Id + \mathcal{O}_{L^2\to L^2}(h) \quad \text{and} \quad \mathrm{Op}_h(b) \circ \mathrm{Op}_h(a) = Id + \mathcal{O}_{L^2\to L^2}(h)$$

In particular, $\mathrm{Op}_h(a)$ is invertible for h sufficiently small.

Simply using the definition (A.1) and a straightforward computation, we obtain the expression of the adjoint operator.

***Proposition A.4* (Adjoint operator).** — *Let a be a symbol of order $m \geq 0$ and $u \in H^m(\mathbb{R}^d)$. Then,*

$$(\mathrm{Op}_h(a))^* = \mathrm{Op}_h(\overline{a}) \quad \text{and} \quad \mathrm{Re}(\langle\, \mathrm{Op}_h(a)u \,|\, u\,\rangle_{L^2}) = \langle\, \mathrm{Op}_h(\mathrm{Re}(a))u \,|\, u\,\rangle_{L^2}\ .$$

We will also use a version of Gårding inequality. We give a complete proof since in many cases, the result is stated for functions defined in a compact domain rather than in $\mathbb{R}^d$, which avoids the question of uniformity of the constants.

***Proposition A.5* (Gårding inequality).** — *Let a be a symbol of order $m \geq 0$ such that there exists a positive constant α and $0 \leq k \leq m$ such that, for all $(x,\xi) \in \mathbb{R}^{2d}$, $\mathrm{Re}(a(x,\xi)) \geq \alpha(1+|\xi|)^k > 0$. Then, there exists $c > 0$ such that, for any $u \in H^m(\mathbb{R}^d)$ and any $h > 0$ sufficiently small,*

$$\mathrm{Re}(\langle\, \mathrm{Op}_h(a)u \,|\, u\,\rangle_{L^2}) \ \geq\ c\left(\|u\|_{L^2}^2 + h^{k/2}\|u\|_{H^{k/2}}^2\right)\ .$$

Proof: Assume first that $k = 0$. We define $b(x,\xi) = \sqrt{\mathrm{Re}(a(x,\xi))}$. We notice that b is a well defined symbol of order $m/2$ and $\mathrm{Op}_h(b)$ is invertible in the sense of Corollary A.3. In particular, there exists $\kappa > 0$ such that $\|\mathrm{Op}_h(b)u\| \geq \kappa\|u\|^2$ for any h small enough. Using Proposition A.4, we get that

$$\begin{aligned}\mathrm{Re}(\langle\, \mathrm{Op}_h(a)u \,|\, u\,\rangle_{L^2}) &= \langle\, \mathrm{Op}_h(b^2)u \,|\, u\,\rangle_{L^2} = \langle\, \mathrm{Op}_h(b)^2u \,|\, u\,\rangle_{L^2} + \mathcal{O}(h)\|u\|_{L^2}^2\\ &= \langle\, \mathrm{Op}_h(b)u \,|\, \mathrm{Op}_h(b)u\,\rangle_{L^2} + \mathcal{O}(h)\|u\|_{L^2}^2\\ &\geq (\kappa^2 + \mathcal{O}(h))\|u\|_{L^2}^2\ ,\end{aligned}$$

which concludes for h small enough.

Now, if $k > 0$, we consider $\tilde{a} = a - \beta(1+|\xi|^2)^{k/2}$, which satisfies the proposition for $k = 0$ if β is small enough. We get that

$$\mathrm{Re}(\langle\, \mathrm{Op}_h(a)u \,|\, u\,\rangle_{L^2}) \ -\ \beta\langle\, \mathrm{Op}_h((1+|\xi|^2)^{k/2})u \,|\, u\,\rangle_{L^2} \ \geq\ c\|u\|_{L^2}^2\ .$$

To conclude, we only have to remark that $\langle\, \mathrm{Op}_h((1+|\xi|^2)^{k/2})u \,|\, u\,\rangle_{L^2}$ is equivalent to $\|u\|_{L^2}^2 + h^{k/2}\|u\|_{H^{k/2}}^2$. □

It is also common to use Gårding inequality for functions vanishing on some part of $\mathbb{R}^d$. In this case, we have to be more careful about the uniformity of the positive constants, which leads use to work with $k = m$.

Proposition A.6 **(Gårding inequality with truncation).** — *Let a be a symbol of order $m \geq 0$ and ω be a subset of $\mathbb{R}^d$. Assume that there exists a positive constant α such that, for all $x \in \mathbb{R}^d \setminus \omega$ and all $\xi \in \mathbb{R}^d$, $\mathrm{Re}(a(x,\xi)) \geq \alpha(1+|\xi|)^m > 0$. Then, there exists $c > 0$ such that, for any $u \in H^m(\mathbb{R}^d)$ such that $u_{|\omega} \equiv 0$ and any $h > 0$ sufficiently small,*

$$\mathrm{Re}(\langle\, \mathrm{Op}_h(a)u \,|\, u\,\rangle_{L^2}) \;\geq\; c\,\left(\|u\|^2_{L^2} + h^{m/2}\|u\|^2_{H^{m/2}}\right) \;.$$

Proof: We set $\Omega_\varepsilon = \{x \in \mathbb{R}^d,\ d(x, \mathbb{R}^d \setminus \omega) < \varepsilon\}$. Notice that, by definition of a symbol of order $m \geq 0$, we have $\partial_x a(x,\xi) \leq C(1+|\xi|^2)^{m/2}$, which implies that we still have $\mathrm{Re}(a(x,\xi)) \geq \tilde{\alpha}(1+|\xi|^2)^{m/2} > 0$ for $\tilde{\alpha} \in (0,\alpha)$ in Ω_ε for some small $\varepsilon > 0$. Due to the distance $\varepsilon > 0$ between $\mathbb{R}^d \setminus \Omega_\varepsilon$ and $\mathbb{R}^d \setminus \omega$, we can construct a function $\chi \in \mathcal{C}^\infty_b(\mathbb{R}^d, [0,1])$ with support in ω and which is equal to 1 outside Ω_ε. It is then sufficient to apply Proposition A.5 to $\tilde{a}(x,\xi) = a(x,\xi) + (1+|\xi|^2)^{m/2}\chi(x)$. □

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