LONG TIME DYNAMICS FOR DAMPED KLEIN-GORDON EQUATIONS DYNAMIQUE EN TEMPS GRAND DES SOLUTIONS DE L'ÉQUATION DE KLEIN-GORDON AMORTIE

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Abstract. For general nonlinear Klein-Gordon equations with dissipation we show that any finite energy radial solution either blows up in finite time or asymptotically approaches a stationary solution in $H^1 \times L^2$. In particular, any global in positive times solution is bounded in positive times. The result applies to standard energy subcritical focusing nonlinearities $|u|^{p-1}u$, $1 < p < (d+2)/(d-2)$ as well as to any energy subcritical nonlinearity obeying a sign condition of the Ambrosetti-Rabinowitz type. The argument involves both techniques from nonlinear dispersive PDEs and dynamical systems (invariant manifold theory in Banach spaces and convergence theorems).

Résumé. Nous démontrons que toute solution radiale d'énergie finie d'une classe générale d'équations de Klein-Gordon amorties ou bien explose en temps positif fini ou bien converge en temps positif vers une solution stationnaire dans $H^1 \times L^2$. En particulier, toute solution globale en temps positif est bornée en temps positif. Ce résultat s'applique aux non-linéarités focalisantes, sous-critiques pour l'énergie, $|u|^{p-1}u$, $1 < p < (d+2)/(d-2)$, comme à toute non-linéarité, sous-critique pour l'énergie, remplissant une condition de signe de type Ambrosetti-Rabinowitz. La preuve fait appel, à la fois, à des techniques propres aux équations non linéaires dispersives et à des arguments de systèmes dynamiques (variétés invariantes dans des espaces de Banach et théorèmes de convergence).

1. Introduction

Nonlinear dispersive evolution equations such as the wave and Schrödinger equations have been investigated for decades. For defocusing power-type energy subcritical or critical nonlinearities the theory is developed, while the energy supercritical powers are wide open. For semilinear focusing equations the picture is less complete for long-term dynamics. These equations exhibit finite-time blowup, small data global existence and scattering, as well as time-independent solutions (solitons). For the energy critical wave equation

$$\Box u = u^5 , \quad (t,x) \in \mathbb{R}^{1+3} ,$$
$$(u(0), \partial_t u(0)) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) ,$$

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in the radial setting, Duyckaerts, Kenig, and Merle [18] achieved a breakthrough by showing that all global trajectories can be described as a superposition of a finite number of rescalings of the ground state $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$ plus a radiation term which is asymptotic to a free wave. This work introduces the novel *exterior energy* estimates.

The subcritical case appears to require different techniques, however. The focusing subcritical Klein-Gordon equation in $\mathbb{R}^d$, $1 \leqslant d \leqslant 6$ (for the case $d \geqslant 7$, see [7]), takes the form

$$\begin{aligned} &\partial_t^2 u - \Delta u + u - |u|^{\theta-1}u = 0, \\ &(u(0), \partial_t u(0)) = (\varphi_0, \varphi_1) \in \mathcal{H} , \end{aligned} \tag{1.1}$$

where $\mathcal{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $\alpha \geqslant 0$ and

$$1 < \theta < \theta^* \ , \text{ with } \theta^* = \frac{d+2}{d-2} \, . \tag{1.2}$$

We will limit our study to the case of radial functions

$$\mathcal{H}_{rad} = H^1_{rad}(\mathbb{R}^d) \times L^2_{rad}(\mathbb{R}^d) \, .$$

The energy functional E^θ below plays an important role in the analysis of the behaviour of the solutions of (1.1). This energy functional is given by

$$E^\theta(\varphi_0, \varphi_1) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\varphi_0|^2 + \frac{1}{2}\varphi_0^2 + \frac{1}{2}\varphi_1^2 - \frac{1}{\theta+1}|\varphi_0|^{\theta+1} \right) dx \tag{1.3}$$

For the Klein-Gordon equation (1.1), it is known (see [46], [3], [14], [39] and [10] for example) that (1.1) admits a unique positive radial stationary solution $(Q_g, 0)$ (the ground state solution), which minimizes the energy $E^\theta(., 0)$ in the class of all nonzero stationary solutions $(Q, 0)$ in $\mathcal{H}$, that is,

$$0 < E^\theta(Q_g, 0) = \min\{E(Q, 0) \,|\, Q \in H^1(\mathbb{R}^d), Q \neq 0, -\Delta Q + Q - |Q|^{\theta-1}Q = 0\}$$

The behaviour of solutions of (1.1) with initial data $(\varphi_0, \varphi_1) \in \mathcal{H}$ with energy $E^\theta(\varphi_0, \varphi_1) < E^\theta(Q_g, 0)$ is rather well understood since these solutions remain in the so-called Payne-Sattinger sets (see [42]) for all positive times. In these Payne-Sattinger domains, the solutions either blow-up in finite time or globally exist and scatter to 0 (for a description of this phenomenon, we refer for example to the book [40]).

Nakanishi and the third author [40] described the asymptotics of solutions provided the energy $E^\theta(\varphi_0, \varphi_1)$ is only slightly larger than the ground state energy. They showed the following trichotomy in forward time of (i) blowup in finite time (ii) global existence and scattering to zero (iii) global existence and scattering to the ground state. They formulated this trichotomy in terms of the center-stable manifold associated with the ground state $(Q_g, 0)$.

It is also well-known that this equation has an infinite number of radial equilibrium points $(e_\ell, 0)$ with a prescribed number $\ell \geqslant 1$ of zeros (these are called *nodal solutions*, see for example [4]). Unfortunately, one knows almost nothing about the uniqueness and the hyperbolicity of those nodal solutions (In [15] the authors obtain uniqueness results for nodal solutions but for sub-linear nonlinearities). This lack of information prevents the description of the behaviour of the solutions $\vec{u}(t)$ of (1.1) whose initial data (φ_0, φ_1) have an energy $E^\theta(\varphi_0, \varphi_1)$ much larger than the one of the ground state $(Q_g, 0)$.

In 1985 Cazenave [9] established the following dichotomy: solutions of (1.1) either blow up in finite time or are global and bounded in $\mathcal{H}$, provided $1 < \theta < +\infty$, if $d = 1, 2$ with $\theta \leqslant 5$ if $d = 2$ and $1 < \theta \leqslant \frac{d}{d-2}$ if $d \geqslant 3$.

In view of these previous results, a natural conjecture is that any *global, radial, finite energy* solution of (1.1) should scatter toward an equilibrium. However, this result seems to be presently out of reach of the usual approaches. A more accessible model is the focusing subcritical *damped* Klein-Gordon equation

$$\begin{aligned} &\partial_t^2 u - \Delta u + u + 2\alpha \partial_t u - |u|^{\theta-1} u = 0, \\ &(u(0), \partial_t u(0)) = (\varphi_0, \varphi_1) \in \mathcal{H} . \end{aligned} \tag{1.4}$$

In 1998 Feireisl [23], for the dissipative case $\alpha > 0$, gave an independent proof of the boundedness of the global solutions of (1.4), when $d \geqslant 3$ and $1 < \theta < 1 + \min(\frac{2}{d-2}, \frac{4}{d})$ (for the case $d = 1$, see his earlier paper [21]). On the other hand, the results of Cazenave should extend to the damped case. However, the proofs of Cazenave [9] and of Feireisl [23] do not seem to extend to nonlinearities satisfying $\frac{d}{d-2} < \theta < \frac{d+2}{d-2}$, when $d \geqslant 3$, where one needs to use Strichartz estimates in the various a priori estimates rather than Gagliardo-Nirenberg-Sobolev inequalities.

Another motivation for studying the *damped* equation is that, by playing on the damping term and considering the damping $2\alpha(t,x)\partial_t u$ or even the nonlinear damping $2\alpha|\partial_t u|^{\delta-1}\partial_t u$, one should be able to exhibit much richer behaviours (from the dynamics point of view). In this paper, we develop a robust approach to the problem of long-term asymptotics of the general *radial* energy subcritical Klein-Gordon equations with (arbitrarily small) dissipation. Our main result is the following dichotomy.

Theorem 1.1. *Let $\alpha > 0$ and $d \leqslant 6$. Then,*

(1) *either the solutions of* (1.4) *in $\mathcal{H}_{rad}$ blow up in finite positive time,*
(2) *or they are global in positive time and converge to an equilibrium point.*

In particular, all global in positive times solutions are bounded for positive times.

We notice that this theorem is a particular case of Theorem 1.2 below. In [7], we will partly generalise this dichotomy to non-radial solutions.

Actually the above dichotomy holds for some more general nonlinearities and, in this paper, we consider the damped Klein-Gordon equation in $\mathbb{R}^d$, $d \leqslant 6$ (for the case $d \geqslant 7$, see [7]),

$$\begin{aligned} &\partial_t^2 u + 2\alpha \partial_t u - \Delta u + u - f(u) = 0 , \\ &(u(0), \partial_t u(0)) = (\varphi_0, \varphi_1) \in \mathcal{H}_{rad} , \end{aligned} \tag{KG}_\alpha$$

where $f : y \in \mathbb{R} \mapsto f(y) \in \mathbb{R}$ is an odd C^1-function, $f'(0) = 0$, which satisfies the following Ambrosetti-Rabinowitz type condition: there exists a constant $\gamma > 0$ such that

$$(H.1)_f \qquad \int_{\mathbb{R}^d} \big(2(1+\gamma)F(\varphi) - \varphi(x)f(\varphi(x))\big)\, dx \leqslant 0\,, \quad \forall \varphi \in H^1(\mathbb{R}^d)\,,$$

where $F(y) = \int_0^y f(s)ds$. We also need to impose a growth condition on f, when $d \geqslant 2$. We assume that,

$$(H.2)_f \qquad \begin{aligned} &|f'(y)| \leqslant C \max\big(|y|^{\beta}, |y|^{\theta-1}\big)\,, \quad \forall y \in \mathbb{R}\,,\\ &|f'(y_1) - f'(y_2)| \leqslant C|y_1 - y_2|^{\beta}\big(1 + |y_1|^{\theta-1-\beta} + |y_2|^{\theta-1-\beta}\big)\,, \quad \forall y_1, y_2 \in \mathbb{R}\,,\end{aligned}$$

where $1 < \theta < \theta^*$, $0 < \beta < \theta - 1$, $\beta \leqslant 1$, $\theta^* = 2^* - 1$ and where $2^* = \infty$ if $d = 1, 2$ and $2^* = \frac{2d}{d-2}$ if $d \geqslant 3$. We notice that, when $d \geqslant 3$, $\theta^* = \frac{d+2}{d-2}$. In other words, the growth of f is energy subcritical for large $y = 0$, and we also assume that f' is β-Hölder continuous. For sake of simplicity in the proofs below, we may assume, without loss of generality, that $0 < \beta < \min(1, \theta - 1, \frac{2}{d})$.

We remark that our argument does not depend on the existence or uniqueness of a ground state solution. Note that Hypothesis $(H.1)_f$ alone does not imply the existence and uniqueness of a ground state solution. We further note that Hypothesis $(H.1)_f$ may actually be replaced by the following weaker one:

$$(H.1bis)_f \qquad \int_{\mathbb{R}^d} \big(2(1+\gamma)F(\varphi) - \varphi(x)f(\varphi(x))\big)dx \leqslant 0\,, \quad \text{for } \|\varphi\|_{H^1} \text{ large enough.}$$

But, for sake of simplicity, we assume $(H.1)_f$ throughout. A classical example of a function f satisfying hypotheses $(H.1)_f$ and $(H.2)_f$ is as follows:

$$f(u) = \sum_{i=1}^{m_1} a_i |u|^{p_i - 1}u - \sum_{j=1}^{m_2} b_j |u|^{q_j - 1}u\,, \text{ with } 1 < q_j < p_i < \frac{d+2}{d-2}, \forall i, j \quad \text{and } a_i, b_j \geqslant 0, a_{m_1} > 0\,. \tag{1.5}$$

In Section 2, we shall prove that the equation $(KG)_\alpha$ generates a local dynamical system on $\mathcal{H}$ as well as on $\mathcal{H}_{rad}$, for $\alpha \geqslant 0$. We denote $S_\alpha(t)$, $\alpha \geqslant 0$, this local dynamical system. As in the particular case of the Klein-Gordon equation (1.4), we introduce the energy functional (also called Lyapunov functional in the case of positive damping $\alpha > 0$) on $\mathcal{H}$:

$$E(\varphi_0, \varphi_1) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla \varphi_0|^2 + \frac{1}{2}\varphi_0^2 + \frac{1}{2}\varphi_1^2 - F(\varphi_0)\right) dx\,. \tag{1.6}$$

The natural first step in the study of the dynamics of the equation $(KG)_\alpha$ consists in studying the boundedness or unboundedness of its global (in positive times) solutions. As already mentioned above, under restrictions on the growth rate of the nonlinearity, Cazenave [9] and Feireisl [23] established this boundedness. In this paper, taking advantage of the fact that all the functions are radial, we will show the boundedness of the global solutions of $(KG)_\alpha$, for $\alpha > 0$, by using "dynamical systems" arguments. Indeed, we will show that each global solution $\vec{u}(t)$ converges to an equilibrium point as t goes to $+\infty$.

If the equation $(KG)_\alpha$ admits a ground state solution and is Hamiltonian, the functional $K_0 : \varphi \in H^1(\mathbb{R}^d) \mapsto K_0(\varphi) \in \mathbb{R}$ defined as

$$K_0(\varphi) = \int_{\mathbb{R}^d} \left(|\nabla\varphi|^2 + \varphi^2 - \varphi f(\varphi)\right) \, dx \, , \tag{1.7}$$

has played a decisive role in the description of the dynamics of the solutions with initial energy smaller or slightly larger than the one of the ground state (see [42], [40] for example). It will also be important in our situation. First we shall prove in Lemma 2.7, that if

$$\vec{u}(t) = S_\alpha(t)(\varphi_0, \varphi_1)(t) \equiv (u(t), \partial_t u(t))$$

satisfies $K_0(u(t)) \leqslant -\delta$ (where $\delta > 0$), on the maximal interval of existence, the solution blows up in finite time. On the other hand, we will see that, if $K_0(u(t)) \geqslant \eta$ for some finite η on the maximal interval of existence, the solution exists and is bounded for all positive times.

In order to prove that each global solution $\vec{u}(t) = S_\alpha(t)(\varphi_0, \varphi_1)(t)$ converges to an equilibrium point as t goes to $+\infty$, we argue by contradiction. We first show that, for any global solution in forward time, there exists a sequence of times t_n, $t_n \to_{n\to+\infty} +\infty$, such that

$$K_0(u(t_n)) \to_{n\to+\infty} 0$$

Then, using this sequence of times t_n, we show in Theorem 3.3, that the ω-limit set $\omega(\varphi_0, \varphi_1)$ of (φ_0, φ_1) is non-empty and contains at least one equilibrium point $(Q^*, 0)$ of the equation $(KG)_\alpha$. We recall that the ω-limit set $\omega(\varphi_0, \varphi_1)$ of (φ_0, φ_1) is defined as follows:

$$\begin{aligned} \omega(\varphi_0, \varphi_1) = \{\vec{w} \in \mathcal{H}_{rad} \mid \exists \text{ a sequence } \tau_n \geqslant 0, \text{ so that } \tau_n \to_{n\to+\infty} +\infty \, , \\ \text{and } S_\alpha(\tau_n)(\varphi_0, \varphi_1) \to_{n\to+\infty} \vec{w}\} \, . \end{aligned} \tag{1.8}$$

Then, in Section 3.2, taking advantage of the fact that the linearized Klein-Gordon equation around $(Q^*, 0)$ in the space $\mathcal{H}_{rad}$ has a kernel which is at most one-dimensional, we show, by using classical convergence arguments based on invariant manifold theory, that the trajectory converges to this equilibrium point in positive infinite time, and is therefore bounded.

Theorem 1.2. *Let $\alpha > 0$. Assume that $1 \leqslant d \leqslant 6$ and that f satisfies the conditions $(H.1)_f$ and $(H.2)_f$. Let $(\varphi_0, \varphi_1) \in \mathcal{H}_{rad}$, then*

(1) *either $S_\alpha(t)(\varphi_0, \varphi_1)$ blows up in finite time,*
(2) *or $S_\alpha(t)(\varphi_0, \varphi_1)$ exists globally and converges to an equilibrium point $(Q^*, 0)$ of $(KG)_\alpha$, as $t \to +\infty$.*

For the case $d \geqslant 7$, we refer the reader to [7].

To place this result into context, we now briefly recall various related convergence theorems. Since we are considering the equation $(KG)_\alpha$ in the radial setting, the linearized Klein-Gordon operator around the equilibrium $(Q^*, 0)$ has a kernel of dimension less than or equal to 1, that is, either 0 does not belong to the spectrum of the elliptic selfadjoint operator

$$\mathcal{L} \equiv -\Delta + I - f'(Q^*)$$

or 0 is a simple eigenvalue of $\mathcal{L}$ (see Section 2, Lemma 2.10). If 0 is a simple eigenvalue of $\mathcal{L}$, then the dynamical system $S_\alpha(t)$ admits a C^1 local center manifold $W^c((Q^*,0))$ of dimension 1 at $(Q^*,0)$. Since the ω-limit set of any element $(\varphi_0,\varphi_1) \in \mathcal{H}_{rad}$ belongs to the set of equilibria, if the trajectory of $S_\alpha(t)(\varphi_0,\varphi_1) \equiv \vec{u}(t)$ were precompact in $\mathcal{H}_{rad}$, we could directly conclude by using the convergence results contained in [5] or in [26] for example that the whole trajectory $S_\alpha(t)(\varphi_0,\varphi_1)$ converges to $(Q^*,0)$, when t goes to infinity. Unfortunately, we do not know that the trajectory $S_\alpha(t)(\varphi_0,\varphi_1)$ is bounded and thus we do not even know that the ω-limit set of (φ_0,φ_1) is bounded and connected. However, adapting the proof of [5, Lemma 1] and using the asymptotic phase property of the local center unstable and local center manifolds around $(Q^*,0)$ (see Appendix A for these concepts), we easily obtain that the entire trajectory $S_\alpha(t)(\varphi_0,\varphi_1)$ converges to $(Q^*,0)$ as t goes to infinity. An alternative way for showing the convergence of the trajectory $S_\alpha(t)(\varphi_0,\varphi_1)$ towards $(Q^*,0)$ would be to prove a Łojasiewicz-Simon's type inequality (see Sections 3.2 and 3.3 in the monograph of L. Simon [45] and also [28, Theorem 2.1]) and combine it with functional arguments as in Jendoubi and Haraux (see [27] or [28]). The proof of the Łojasiewicz-Simon inequality in [45] uses a Lyapunov-Schmidt decomposition. In the special case where the kernel of $\mathcal{L}$ is one-dimensional, this proof also shows that the set of equilibria of $(KG)_\alpha$ passing through $(Q^*,0)$ is a C^1-curve. Using this Łojasiewicz-Simon's type inequality and introducing an appropriate functional like in [28], we could show that the ω-limit set of every precompact trajectory converges to an equilibrium point. Unfortunately, the trajectory $S_\alpha(t)(\varphi_0,\varphi_1)$ is not a priori bounded and it seems difficult to adapt the functional part of the proof of [28, Theorem 3.1]. Moreover, there is an additional difficulty in the construction of such an appropriate functional coming from the fact that we need to use Strichartz estimates. So we have not been able to follow this route.

The plan of this paper is as follows. Section 2 is devoted to basic properties of the Klein-Gordon equation $(KG)_\alpha$. In particular, we recall the local existence and uniqueness of mild solutions of the equation $(KG)_\alpha$. In Section 2.2, we introduce the functional K_0, which not only plays an important role in the proof of Theorem 1.2 but also defines the well-known Nehari manifold $\mathcal{N}$ as the locus of the radial zeros of the functional K_0. In Lemma 2.7, we give a sufficient condition on K_0 for blow-up in finite time of the solutions of $(KG)_\alpha$. We end this section by describing the spectral properties of the linearized Klein-Gordon equation around a (radial) equilibrium point. Section 3 is the core of this paper. In Section 3.1 (see Theorem 3.3) we show that if a solution $\vec{u}(t)$ does not blow up in finite positive time, then the ω-limit set $\omega(\vec{u}(0))$ contains at least one equilibrium point. In Section 3.2 we show that the whole trajectory $\vec{u}(t)$ converges to this equilibrium point and is therefore bounded. In Section 4, we apply the classical invariant manifold theory, recalled in Appendix A, in order to construct the local unstable, center unstable and center manifolds about equilibrium points of the Klein-Gordon equation $(KG)_\alpha$ and the unstable, center unstable and center manifolds about equilibrium points of the localized Klein-Gordon equation (4.7). In Appendix A, we recall the existence theorems for local center-stable, local center-unstable and local center manifolds together with their foliations and exponential attraction properties with asymptotic phase in the formulation of Chen, Hale and Tan (see [11]). Finally, in Appendix B, we recall the classical convergence theorem (see [1], [25] or [26]) in the generalised form given by Brunovský and Poláčik in [5].

Such a convergence theorem is needed in case the dynamics near the equilibrium exhibits a nontrivial center manifold. As a result of dissipation and the radial condition, this center manifold can be at most one-dimensional. For the nonlinearities (1.5), it is known that the kernel of the linearized operator about the ground state is trivial, see [10]. But, due to the lack of precise description of the bound states, we cannot guarantee that the local center manifold is absent about a bound state. The local strongly unstable manifold is finite-dimensional. The local strongly stable manifold is infinite-dimensional in stark contrast to the Hamiltonian scenario for which the local center manifold is the largest piece. The convergence theorem in [5] then guarantees that, if the ω-limit set is not a single equilibrium point $(Q^*, 0)$, and if $(Q^*, 0)$ is stable for the restriction of $S_\alpha(t)$ to the local center manifold of $(Q^*, 0)$ (for this definition of stability, see (3.40) and Appendix B), then this ω-limit set must contain a point on the unstable manifold of $(Q^*, 0)$, distinct from $(Q^*, 0)$. But this contradicts the fact that, due to the properties of the Lyapunov functional (1.6), the ω-limit set is contained in the set of equilibrium points.

2. Basic properties

2.1. **Local existence results.** Consider the linear equation, with $\alpha \geqslant 0$,

$$\partial_t^2 u + 2\alpha\partial_t u - \Delta u + u = G, \quad (u, \partial_t u)\Big|_{t=0} = (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d). \tag{2.1}$$

Since $v(t) = e^{\alpha t}u(t)$ satisfies

$$v_{tt} - \Delta v + (1-\alpha^2)v = e^{\alpha t}G, \quad (v, v_t)\Big|_{t=0} = (u_0, u_1 + \alpha u_0), \tag{2.2}$$

we deduce that the solution of (2.1) is given by

$$\begin{aligned} u(t) &= e^{-\alpha t}\Big[\cos(t\sqrt{-\Delta+1-\alpha^2}) + \alpha\frac{\sin(t\sqrt{-\Delta+1-\alpha^2})}{\sqrt{-\Delta+1-\alpha^2}}\Big]u_0 \\ &\quad + e^{-\alpha t}\frac{\sin(t\sqrt{-\Delta+1-\alpha^2})}{\sqrt{-\Delta+1-\alpha^2}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta+1-\alpha^2})}{\sqrt{-\Delta+1-\alpha^2}}e^{-(t-s)\alpha}G(s)\,ds \\ &= \mathcal{S}_{1,\alpha}(t)u_0 + \mathcal{S}_{2,\alpha}(t)u_1 + \int_0^t \mathcal{S}_{2,\alpha}(t-s)G(s)\,ds\,. \end{aligned} \tag{2.3}$$

Clearly, the regimes $0 \leqslant \alpha < 1$, $\alpha = 1$, and $\alpha > 1$ exhibit quite different behaviours. The dispersion relation for $\alpha < 1$ is that of Klein-Gordon (the characteristic variety is a hyperboloid), whereas for $\alpha = 1$ it is that of the wave equation (the characteristic variety is a cone).

If X is a Banach space, then we let $L_t^{p,\beta}(X)$ be the space with norm

$$\|f\|_{L_t^{p,\beta}(X)} = \|e^{\beta t}\|f(t)\|_X\|_{L_t^p}, \quad \beta \in \mathbb{R}$$

In this section, the β in these weighted estimates has nothing to do with the regularity in $((H.2)_f)$.

Lemma 2.1. *Let $0 \leqslant \alpha < 1$ and assume $d \geqslant 3$ for simplicity. Set $p = \frac{2d}{d-2}$ and $\sigma = \frac{1}{2} - \frac{1}{d}$, $\sigma' = 1 - \sigma$. The solution u of* (2.1) *satisfies the following Strichartz-type estimates for any $0 \leqslant \beta \leqslant \alpha$,*

$$\|u\|_{L_t^{2,\beta} B^{\sigma}_{p,2} \cap L_t^{\infty,\beta} H^1_x} \leqslant C(\alpha)\Big[\|(u_0, u_1)\|_{H^1 \times L^2} + \|G\|_{L_t^{2,\beta} B^{\sigma'}_{p',2} + L_t^{1,\beta} L^2_x}\Big] \tag{2.4}$$

where $C(\alpha)$ is uniform on compact intervals of $[0, 1)$.

Proof. This follows from (2.2) and the Keel-Tao endpoint for the Klein-Gordon equation, see for example Lemma 2.46 in [40]. □

Lemma 2.1 does not hold for $\alpha \geqslant 1$. Indeed, for $\alpha = 1$ we would need to replace the Strichartz estimates for Klein-Gordon in (2.4) with those for the wave equation. We set $\beta(\alpha) = \alpha$ if $0 \leqslant \alpha \leqslant 1$ and

$$\beta(\alpha) = \alpha - \sqrt{\alpha^2 - 1}$$

if $\alpha > 1$. Exploiting the exponential decay in (2.3) we can now state the following space-time averaged estimates.

Lemma 2.2. *Let $\alpha > 0$. In all dimensions $d \geqslant 1$ the solution u of* (2.1) *satisfies the following energy bounds with decay*

$$\sup_{t \geqslant 0} e^{t\beta(\alpha)}\|(u, \partial_t u)(t)\|_{H^1 \times L^2} \leqslant C(\alpha)\Big[\|(u_0, u_1)\|_{H^1 \times L^2} + \int_0^\infty e^{s\beta(\alpha)}\|G(s)\|_2 \, ds\Big] \tag{2.5}$$

as well as the exponentially weighted Strichartz estimates, in dimensions $d \geqslant 2$, and with $0 \leqslant \beta < \beta(\alpha)$,

$$\|u\|_{L_t^{q,\beta} L_x^p} \leqslant C(\alpha, \beta)\big[\|(u_0, u_1)\|_{H^1 \times L^2} + \|G\|_{L_t^{\tilde{q}',\beta} L_x^{\tilde{p}'}}\big] \tag{2.6}$$

where $\frac{1}{q} + \frac{d}{p} = \frac{d}{2} - 1 = \frac{1}{\tilde{q}'} + \frac{d}{\tilde{p}'} - 2$, $2 \leqslant p, \tilde{p} < \infty$, $2 \leqslant q, \tilde{q}$, and $\frac{1}{q} + \frac{d-1}{2p} \leqslant \frac{d-1}{4}$, $\frac{1}{\tilde{q}} + \frac{d-1}{2\tilde{p}} \leqslant \frac{d-1}{4}$. The constant $C(\alpha, \beta)$ is uniform on compact subsets of

$$\{(\alpha, \beta) \mid \alpha \in (0, \infty),\ 0 \leqslant \beta < \beta(\alpha)\}$$

Proof. Taking the Fourier transform of (2.3) yields

$$\hat{u}(t, \xi) = m_\alpha(t, \xi)\widehat{u_0}(\xi) + \tilde{m}_\alpha(t, \xi)\widehat{u_1}(\xi) + \int_0^t \tilde{m}_\alpha(t - s, \xi)e^{-(t-s)\alpha}\widehat{G}(s, \xi) \, ds$$

The multipliers satisfy the estimates

$$|m_\alpha(t, \xi)| + |\tilde{m}_\alpha(t, \xi)| \leqslant C(\alpha)e^{-\beta(\alpha)t}$$

which proves (2.5). For (2.6) we introduce the Littlewood-Paley decomposition

$$1 = P_{\lesssim\alpha} + \sum_j P_j = P_{\lesssim\alpha} + P_{>\alpha}$$

where the P_j are associated to frequencies $2^j > \alpha$ and $P_{\lesssim\alpha} f = f$ for all Schwartz functions with support in $\{|\xi| \leqslant 1 + 2\alpha\}$. Let $K^{\pm}_{\lambda}(t)$ be the propagator defined by, cf. (2.3),

$$[K^{\pm}_{\lambda}(t)f](x) = e^{-\alpha t} \int_{\mathbb{R}^d} e^{\pm it\sqrt{\xi^2+1-\alpha^2}} e^{ix\cdot\xi} \chi(\xi/\lambda) \hat{f}(\xi)\, d\xi$$

where χ is the usual Littlewood-Paley bump function supported on an annulus, and $\lambda > \alpha + 1$ (and ignoring multiplicative constants). Then the root is smooth, and we may apply stationary phase to conclude that

$$\|K^{\pm}_{\lambda}(t)\|_{\infty} \leqslant e^{-\alpha t} \lambda^d \langle t\lambda\rangle^{-\frac{d-1}{2}} \lesssim e^{-\alpha t} t^{-\frac{d-1}{2}} \lambda^{\frac{d+1}{2}}$$

for all $t > 0$. Proceeding as for the wave equation (see Keel-Tao [33]), and ignoring the exponential decay for the frequencies $\gtrsim \alpha$, yields the Strichartz estimates (2.6) for $P_{>\alpha}u$ with $\beta = 0$. On the other hand, by the same logic we can also derive Strichartz estimates for the transformed equation (2.2) which yields (2.6) with $\beta = \alpha$ for the piece $P_{>\alpha}u$. Interpolating between these two cases we obtain Strichartz inequalities for all $0 \leqslant \beta \leqslant \alpha$ for those frequencies. Smaller frequencies require smaller β. Indeed, for the remaining piece $P_{\lesssim\alpha}u$ we use the energy bound (2.5) and Bernstein's inequality. To be precise, the energy estimate

$$\|P_{\lesssim\alpha}u(t)\|_2 \leqslant C(\alpha)\Big[e^{-t\beta(\alpha)}\|(u_0,u_1)\|_{H^1\times L^2} + \int_0^t e^{-(t-s)\beta(\alpha)}\|P_{\lesssim\alpha}G(s)\|_2\, ds\Big]$$

implies via Bernstein's inequality that

$$e^{\beta t}\|P_{\lesssim\alpha}u(t)\|_p \leqslant C(\alpha)\Big[e^{-t(\beta(\alpha)-\beta)}\|(u_0,u_1)\|_{H^1\times L^2} + \int_0^t e^{-(t-s)(\beta(\alpha)-\beta)} e^{\beta s}\|P_{\lesssim\alpha}G(s)\|_{\tilde{p}'}\, ds\Big]$$

Taking L^q_t norms on both sides, and applying Young's inequality to the Duhamel integral yields (2.6) for all frequencies. □

We now turn to the nonlinear equation $(KG)_\alpha$. We write $\vec{u} = (u, \partial_t u)$.

Theorem 2.3. *Let $d \leqslant 6$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 odd function, satisfying the assumption $(H.2)_f$. Then for every data $\vec{u}_0$ in $\mathcal{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ (resp. in $\mathcal{H}_{rad}$) the equation $(KG)_\alpha$ has a unique strong solution*

$$u \in X \equiv X_T := C([0,T], H^1(\mathbb{R}^d)) \cap C^1([0,T], L^2(\mathbb{R}^d))$$

(resp. in $C([0,T], H^1_{rad}(\mathbb{R}^d)) \cap C^1([0,T], L^2_{rad}(\mathbb{R}^d))$), where T only depends on $\|\vec{u}_0\|_{\mathcal{H}}$. Moreover, if $3 \leqslant d \leqslant 6$, the solution belongs to

$$L^{\theta^*}((0,T), L^{2\theta^*}(\mathbb{R}^d))$$

where $\theta^* = \frac{d+2}{d-2}$ and the estimate (2.21) *below holds. Furthermore, the following properties hold.*

(1) *The solution*

$$(t,\vec{u}_0)\in[0,T]\times\mathcal{H}\mapsto\vec{u}(t)\equiv(u(t),\partial_t u(t))\in\mathcal{H}$$

is continuous.

(2) *For any* $0\leqslant\tau\leqslant T$, *the map* $\vec{u}_0\in\mathcal{H}\mapsto S_\alpha(\tau)\vec{u}_0\equiv\vec{u}(\tau)\in\mathcal{H}$ *is Lipschitz continuous on the bounded sets of* $\mathcal{H}$ *(see* (2.23)*).*

(3) *The map* $\vec{u}_0\in\mathcal{H}\mapsto u(t)\in X\cap L^{\theta^*}((0,T),L^{2\theta^*}(\mathbb{R}^d))$ *is a* C^1*-map.*

(4) *Let* T^* *be the maximal time of existence. If* $T^*<\infty$, *then*

$$\limsup_{t\to T^*}\|\vec{u}(t)\|_{\mathcal{H}}=+\infty$$

(5) *If* $\vec{u}_0\in H^2(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$, *then*

$$u\in C([0,T),H^2(\mathbb{R}^d))\cap C^1([0,T),H^1(\mathbb{R}^d))$$

(6) *The energy* (1.6) *decreases: for any* $t_2\geqslant t_1\geqslant 0$, *we have,*

$$E(\vec{u}(t_2))-E(\vec{u}(t_1))=-2\alpha\int_{t_1}^{t_2}\|\partial_t u(s)\|_{L^2}^2\,ds \tag{2.7}$$

and, in particular,

$$E(\vec{u}(t_2))+2\alpha\int_0^{t_2}\|\partial_t u(s)\|_{L^2}^2\,ds\leqslant E(\vec{u}(0)) \tag{2.8}$$

(7) *If* $\|\vec{u}(0)\|_{\mathcal{H}}\ll 1$, *then the solution exists globally, and* $\|\vec{u}(t)\|_{\mathcal{H}}$ *converges exponentially to* 0 *as* $t\to\infty$.

Proof. We first recall the main lines of the proof of the local existence and uniqueness of the solution in the case $d\geqslant 3$. The cases $d=1,2$ are easier and left to the reader. The local existence is proved by using the classical strict contraction fixed point theorem with parameters. In the fixed point argument below, we will use the Strichartz inequality (2.6) given in Lemma 2.2. Let $\theta^*=2^*-1=\frac{d+2}{d-2}$, $(\tilde{p}',\tilde{q}')=(2,1)$ and $(p,q)=(2\theta^*,\theta^*)$. We remark that these pairs satisfy the conditions of Lemma 2.2 and in particular $q\geqslant 2$ if $d\leqslant 6$.

Let $K_0>0$ be a fixed constant. In what follows, we denote $B_{\mathcal{H}}(0,K_0)$ the ball of center 0 and radius K_0 in $\mathcal{H}$. Using the notation of the previous lemma, we set

$$M_0\equiv M_0(\alpha)=4(\mathrm{C}(\alpha)+\mathrm{C}(\alpha,0))K_0\equiv 4C_1(\alpha)K_0 \tag{2.9}$$

and $T>0$ will be a positive constant, to be determined later.
We introduce the following space

$$\begin{aligned}Y\equiv Y_T\equiv\{\vec{u}\in L^\infty((0,T),\mathcal{H})\text{ with }u\in L^{\theta^*}((0,T),L^{2\theta^*}(\mathbb{R}^d))\\ \mid \|u\|_{L^\infty(H^1)\cap W^{1,\infty}(L^2)\cap L^{\theta^*}(L^{2\theta^*})}\leqslant M_0\}\,.\end{aligned} \tag{2.10}$$

We consider the mapping

$$\mathcal{F}:(\vec{u}_0,\vec{u})\in B_{\mathcal{H}}(0,K_0)\times Y\mapsto\mathcal{F}(\vec{u}_0,\vec{u})\equiv(\mathcal{F}_1,\mathcal{F}_2)(\vec{u}_0,\vec{u})\in Y\,,$$

defined by

$$(\mathcal{F}_1(\vec{u}_0,\vec{u}))(t) = \mathcal{S}_{1,\alpha}(t)u_0 + \mathcal{S}_{2,\alpha}(t)u_1 + \int_0^t \mathcal{S}_{2,\alpha}(t-s)f(u(s))\,ds\,, \tag{2.11}$$

and $\mathcal{F}_2(\vec{u}_0,\vec{u}) = \partial_t\mathcal{F}_1(\vec{u}_0,\vec{u})$, where $\vec{u}_0 = (u_0,u_1)$ and $\vec{u} = (u,\partial_t u)$. Fix some $\vec{u}_0 \in \mathcal{H}$ with $\|\vec{u}_0\|_{\mathcal{H}} < K_0$. Consider the map $\mathcal{F}(\vec{u}_0,.) : \vec{u} \in Y \mapsto \mathcal{F}(\vec{u}_0,\vec{u}) \in Y$ and simply write $\mathcal{F}(\vec{u}_0,\vec{u}) = \mathcal{F}\vec{u}$.

An application of Lemma 2.2 implies

$$\|\mathcal{F}(u_0,0)\|_Y \leqslant C_1(\alpha)K_0 \leqslant \frac{M_0}{4}\,. \tag{2.12}$$

Applying again Lemma 2.2 and using the hypothesis $(H.2)_f$, we get

$$\begin{aligned}\|\mathcal{F}\vec{u}-\mathcal{F}\vec{v}\|_Y \leqslant C_1(\alpha)C\big[T\|u-v\|_{L^\infty(L^2)} &+ \int_0^T \||u(s)|^{\theta-1}|u(s)-v(s)|\,\|_{L^2}\,ds \\ &+ \int_0^T \||v(s)|^{\theta-1}|u(s)-v(s)|\,\|_{L^2}\,ds\big]\end{aligned} \tag{2.13}$$

where $C = C(f)$. Applying the Hölder inequality to the term B below, we obtain

$$B := \int_0^T \||u(s)|^{\theta-1}|u(s)-v(s)|\,\|_{L^2}\,ds \leqslant \int_0^T \|u(s)\|^{\theta-1}_{L^{2\theta}}\|u(s)-v(s)\|_{L^{2\theta}}\,ds\,. \tag{2.14}$$

We set

$$\eta = \frac{d+2-\theta(d-2)}{4} \tag{2.15}$$

and write 2θ as $2\theta = 2\eta + 2(1-\eta)\theta^*$. The condition $1 < \theta < \theta^*$ implies $0 < \eta < 1$. Using the above decomposition of θ in (2.14) together with a Hölder inequality, we get

$$B \leqslant \|u(s)\|^{\frac{(\theta-1)\eta}{\theta}}_{L^\infty(L^2)}\|u(s)-v(s)\|^{\frac{\eta}{\theta}}_{L^\infty(L^2)}\int_0^T \|u(s)\|^{\frac{\theta^*(\theta-1)(1-\eta)}{\theta}}_{L^{2\theta^*}}\|u(s)-v(s)\|^{\frac{\theta^*(1-\eta)}{\theta}}_{L^{2\theta^*}}\,ds\,. \tag{2.16}$$

Applying again the Hölder inequality to the integral term, we obtain,

$$\begin{aligned}\int_0^T \|u(s)\|^{\frac{\theta^*(\theta-1)(1-\eta)}{\theta}}_{L^{2\theta^*}}\|u(s)-v(s)\|^{\frac{\theta^*(1-\eta)}{\theta}}_{L^{2\theta^*}}\,ds \leqslant & T^\eta\Big(\int_0^T \|u(s)-v(s)\|^{\theta^*}_{L^{2\theta^*}}\,ds\Big)^{\frac{1-\eta}{\theta}} \\ &\times\Big(\int_0^T \|u(s)\|^{\theta^*}_{L^{2\theta^*}}\,ds\Big)^{\frac{(\theta-1)(1-\eta)}{\theta}}\,.\end{aligned} \tag{2.17}$$

The estimates (2.16) and (2.17) together with the Young inequality give

$$B \leqslant CT^\eta M_0^{\frac{\theta-1}{\theta}(\theta^*(1-\eta)+\eta)}\big[\|u-v\|_{L^\infty(L^2)} + \|u-v\|_{L^{\theta^*}(L^{2\theta^*})}\big]\,. \tag{2.18}$$

We next choose $T_0 > 0$ so that

$$C_1(\alpha)C\big[T_0 + 2T_0^\eta M_0^{\frac{\theta-1}{\theta}(\theta^*(1-\eta)+\eta)}\big] = \frac{1}{4}\,. \tag{2.19}$$

The estimates (2.13) to (2.18) imply that, for $0 < T \leqslant T_0$,

$$\|\mathcal{F}\vec{u} - \mathcal{F}\vec{v}\|_Y \leqslant C_1(\alpha)C\big[T + 2T^{\eta}M_0^{\frac{\theta-1}{\theta}(\theta^*(1-\eta)+\eta)}\big]\|\vec{u} - \vec{v}\|_Y \leqslant \frac{1}{4}\|\vec{u} - \vec{v}\|_Y \,. \tag{2.20}$$

From the estimates (2.12) and (2.20), we deduce that $\mathcal{F}$ is a strict contraction and thus has a unique fixed point $\vec{u} \equiv \vec{u}(\vec{u}_0)$ in Y satisfying

$$\|\vec{u}(\vec{u}_0)\|_Y \leqslant C_1(\alpha)\|\vec{u}_0\|_{\mathcal{H}} \tag{2.21}$$

The fact that $\vec{u}(t) = (u(t), \partial_t u(t))$ also belongs to $C([0,T], \mathcal{H})$ is standard and left to the reader. Likewise, we leave it to the reader to verify that the map $(t, \vec{u}_0) \in [0,T] \times \mathcal{H} \mapsto \vec{u}(t) \in \mathcal{H}$ is jointly continuous.

We now turn to property (2). To show that $\vec{u}_0 \in \mathcal{H} \to \vec{u}(\tau) \equiv S_\alpha(\tau)\vec{u}_0 \in \mathcal{H}$ is Lipschitz continuous on the bounded sets of $\mathcal{H}$, we choose $\vec{u}_0$ and $\vec{v}_0$ in the ball $B_{\mathcal{H}}(0, K_0)$. Let $T_0 > 0$ be given by (2.19) and M_0 be defined in (2.9). Arguing as above (see the inequality (2.20)), we obtain the following inequality for $0 \leqslant T \leqslant T_0$,

$$\|\mathcal{F}(\vec{u}_0, \vec{u}) - \mathcal{F}(\vec{v}_0, \vec{v})\|_{Y_T} \leqslant C_1(\alpha)\|\vec{u}_0 - \vec{v}_0\|_{\mathcal{H}} + \frac{1}{4}\|\vec{u} - \vec{v}\|_{Y_T} \,, \tag{2.22}$$

and thus, the fixed points $\vec{u}(\vec{u}_0)$ and $\vec{v}(\vec{v}_0)$ satisfy:

$$\|\vec{u}(\vec{u}_0) - \vec{v}(\vec{v}_0)\|_{Y_T} \leqslant \frac{4}{3}C_1(\alpha)\|\vec{u}_0 - \vec{v}_0\|_{\mathcal{H}} \,. \tag{2.23}$$

If the solutions $\vec{u}(\vec{u}_0)$ and $\vec{v}(\vec{v}_0)$ exist on a time interval $[0, T^*)$, where $T^* > T_0$, we repeat the above proof by considering now the ball in $\mathcal{H}$ of center $\vec{u}(\vec{u}_0)(T_0)$ and radius $K_1 > 0$ large enough so that $v(\vec{v}_0)(T_0)$ also belongs to this new ball and replacing the non-linearity $f(.)$ by $f(. + u(\vec{u}_0)(T_0)) - f(u(\vec{u}_0)(T_0))$. Repeating this process a finite number of times shows that the map is Lipschitz continuous up to any time $\tilde{T} < T^*$ and therefore on all of $[0, T^*)$. The above inequality also implies the uniqueness of the solution of $(KG)_\alpha$.

We next want to show the property (3), namely that the map

$$\vec{u}_0 \in \mathcal{H} \mapsto u(\vec{u}_0) \in X \cap L^{\theta^*}((0,T), L^{2\theta^*}(\mathbb{R}^d))$$

is a C^1-map. To this end, we will first go back to the mapping

$$\mathcal{F} : (\vec{u}_0, \vec{u}) \in B_{\mathcal{H}}(0, K_0) \times Y \mapsto \mathcal{F}(\vec{u}_0, \vec{u}) \in Y$$

which has been defined by (2.11), and then, for $t \geqslant T_0$, proceed like in the proof of the property (2). Clearly the map $\mathcal{F}(\vec{u}_0, \vec{u})$ is differentiable with respect to the variable $\vec{u}_0$ since it is a linear map in $\vec{u}_0$. The differentiability with respect to the variable $\vec{u} \in Y$ is proved as follows (we only indicate the main arguments and leave the details to the reader). Let $\vec{h} = (h, k) \in Y$ be small. Applying Lemma 2.2, one sees that the proof of the differentiability reduces to proving that

$$\|f(u + h) - f(u) - f'(u)h\|_{L^1((0,T),L^2)} = o(\|\vec{h}\|_Y) \,. \tag{2.24}$$

As above, using the hypothesis $(H.2)_f$, the fact that $0 < \beta \leqslant \frac{2}{d-2}$ and classical Sobolev embeddings, we write

$$\|f(u+h) - f(u) - f'(u)h\|_{L^1((0,T),L^2)} \tag{2.25}$$
$$\leqslant C\int_0^T \||h(s)|^{\beta+1} + |h(s)|^{\theta} + |h(s)|^{\beta+1}|u(s)|^{\theta-(1+\beta)}\|_{L^2}\,ds$$
$$\leqslant C\big[T\|h\|^{1+\beta}_{L^\infty(H^1)} + \int_0^T \||h(s)|^{\theta} + |h(s)|^{\beta+1}|u(s)|^{\theta-(1+\beta)}\|_{L^2}\,ds\big]\,.$$

The last term in the right-hand side of the inequality (2.25) can be estimated by using Strichartz norms and arguing as in the inequalities (2.16) and (2.17). We thus deduce from (2.25) that

$$\|f(u+h) - f(u) - f'(u)h\|_{L^1((0,T),L^2)} = O(\|\vec{h}\|^{1+\delta}_Y)\,, \tag{2.26}$$

where $\delta > 0$. Thus, $\mathcal{F}(\vec{u}_0, \vec{u})$ is differentiable with respect to the variable $\vec{u} \in Y$. The derivative of $\mathcal{F}(\vec{u}_0, \vec{u})$ with respect to $(\vec{u}_0, \vec{u})$ is given by $D\mathcal{F}(\vec{u}_0, \vec{u}) = (D\mathcal{F}_1, D\mathcal{F}_2)(\vec{u}_0, \vec{u})$, where $D\mathcal{F}_2(\vec{u}_0, \vec{u}) = \partial_t D\mathcal{F}_1(\vec{u}_0, \vec{u})$ and

$$(D\mathcal{F}_1(\vec{u}_0, \vec{u})(\vec{v}_0, \vec{v}))(t) = \mathcal{S}_{1,\alpha}(t)v_0 + \mathcal{S}_{2,\alpha}(t)v_1 + \int_0^t \mathcal{S}_{2,\alpha}(t-s)f'(u(s))v(s)\,ds\,. \tag{2.27}$$

We let to the reader to check that this derivative is continuous with respect to $(\vec{u}_0, \vec{u})$. Finally, we remark that, with the choice of the time T_0 made in (2.19), the mapping $\mathcal{F}(\vec{u}_0, .) : \vec{u} \in Y_T \mapsto \mathcal{F}(\vec{u}_0, \vec{u}) \in Y_T$ is a uniform contraction on $B_{\mathcal{H}}(0, K_0)$. We then apply the *uniform contraction principle* as stated for example in [12, Theorem 2.2 on Page 25], which implies that $\vec{u}_0 \in B_{\mathcal{H}}(0, K_0) \mapsto \vec{u}(\vec{u}_0) \in Y_T$ is of class C^1.

We next turn to the $H^2 \times H^1$-regularity question, that is, prove the regularity property (5). Assuming this regularity for now, taking a derivative of $(KG)_\alpha$ yields

$$\partial_t^2 v + 2\alpha\partial_t v - \Delta v + v - f'(u)v = 0 \tag{2.28}$$

where v stands for any of the derivatives $\partial_{x_j} u$, $1 \leqslant j \leqslant d$. The data for (2.28) belong to $\mathcal{H}$ by assumption. We now perform the same estimates as in (2.13)-(2.18) to conclude that

$$\|\vec{v}\|_Y \leqslant C\|(u_0, u_1)\|_{H^2\times H^1} + \frac{1}{2}\|\vec{v}\|_Y,$$

see especially (2.18), (2.20). As above, these estimates require T to be sufficiently small. To be precise, the smallness here is determined by u alone through the constant M_0, see (2.18). It follows that

$$\|\vec{v}\|_Y \leqslant 2C\|(u_0, u_1)\|_{H^2\times H^1}$$

which is the desired regularity estimate. In order to pass from an a priori bound to a regularity statement we follow a standard procedure involving difference quotients: letting $\vec{e}_j$ be the coordinate vectors in $\mathbb{R}^d$ we define with $h > 0$

$$v_j^{(h)}(x) := h^{-1}(u(x + h\vec{e}_j) - u(x))\,.$$

By the argument leading to the a priori estimate we obtain

$$\|\vec{v}_j^{(h)}\|_Y \leqslant 2C\|(u_0,u_1)\|_{H^2\times H^1}$$

uniformly in $h>0$. Passing to suitable weak limits, we obtain the $H^1\times L^2$ regularity of the derivatives of u, as desired.

We now show the energy properties stated in (6). Using the density of $H^2(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$, one shows that

$$E(\vec{u}(t))\in C^1((-\tilde{T},T)),\ \text{and}\ \frac{d}{dt}E(\vec{u}(t)) = -2\alpha\|\partial_t u(t)\|^2_{L^2}. \tag{2.29}$$

Integrating this implies the properties (2.7) and (2.8) for the energy.

Finally, we turn to the case of small data. We will only provide a sketch of the main argument. In the hypothesis $(H.2)_f$, we can choose $\beta>0$ arbitrarily small. In particular, we choose $0<\beta<1$. We recall that, for any $y\in\mathbb{R}$,

$$|f(y)|\leqslant C(|y|^\beta+|y|^{\theta-1})|y|\leqslant C(|y|^{1+\beta}+|y|^{\theta^*})\ . \tag{2.30}$$

Proceeding as before, applying Lemma 2.2, using the inequality (2.30), one gets, for $t\geqslant 0$,

$$\begin{aligned}\|u\|_{L^{\theta^*,\beta}((0,t),L^{2\theta^*})}+\|e^{\beta s}\vec{u}\|_{L^\infty((0,t),\mathcal{H})}\leqslant C[&\|(u_0,u_1)\|_{H^1\times L^2}+\|u^{1+\beta}\|_{L^{1,\beta}((0,t),L^2)}\\ &+\||u|^{\theta^*}\|_{L^{1,\beta}((0,t),L^2)}]\ .\end{aligned}$$

Applying the Hölder inequality, one deduces from the above inequality that, for $t\geqslant 0$,

$$\begin{aligned}\|u\|_{L^{\theta^*,\beta}((0,t),L^{2\theta^*})}+\|e^{\beta s}\vec{u}\|_{L^\infty((0,t),\mathcal{H})}\leqslant C[&\|(u_0,u_1)\|_{H^1\times L^2}+\|e^{\beta s}\vec{u}\|^{1+\beta}_{L^\infty((0,t),\mathcal{H})}\\ &+\|u\|^{\theta^*}_{L^{\theta^*,\beta}((0,t),L^{2\theta^*})}]\ ,\end{aligned} \tag{2.31}$$

where we used that $\beta>0$. For small data the method of continuity implies global existence and smallness of the norms on the left-hand side. In particular, we have exponential convergence to zero in the energy (see also [35]). □

In Section 3, we will linearize the equation $(KG)_\alpha$ around an equilibrium point. More generally, we can linearize the Klein-Gordon equation $(KG)_\alpha$ along any solution of the equation $(KG)_\alpha$. This leads us to consider the following affine equation

$$w_{tt}+2\alpha w_t-\Delta w+w-f'(u^*(t,x))w=G\ ,\quad (w,w_t)(0)\equiv\vec{w}(0)=\vec{w}_0\in\mathcal{H}\ , \tag{2.32}$$

where $u^*(t,x)\in X_{\tau_0}\cap L^{\theta^*}((0,\tau_0),L^{2\theta^*}(\mathbb{R}^d))$, $\tau_0>0$, and $G\in L^1((0,\tau_0),L^2(\mathbb{R}^d))$. The existence (and uniqueness) of a solution $\vec{w}\equiv(w,\partial_t w)\in C([0,\tau_0),\mathcal{H})$ is classical if the dimension d is equal to $1,2$. So we will state this existence result and the corresponding Strichartz estimates only in the case where $d\geqslant 3$.

Proposition 2.4. *Let $d\geqslant 3$ and $\alpha\geqslant 0$. Assume that $u^*(t,x)\in X_{\tau_0}\cap L^{\theta^*}((0,\tau_0),L^{2\theta^*}(\mathbb{R}^d))$ and that $G\in L^1((0,\tau_0),L^2(\mathbb{R}^d))$. Then the equation* (2.32) *admits a unique solution $\vec{w}\equiv(w,\partial_t w)\in C([0,\tau_0),\mathcal{H})$. Moreover, the solution $\vec{w}$ of* (2.32) *satisfies the following bound, for $0\leqslant\tau<\tau_0$,*

$$\|\vec{w}\|_{L^\infty((0,\tau),\mathcal{H})}+\|w\|_{L^q((0,\tau),L^p_x)}\leqslant C(\alpha,\tau)\big[\|\vec{w}_0\|_{\mathcal{H}}+\|G\|_{L^1((0,\tau),L^2_x)}\big]\ , \tag{2.33}$$

where

$$\frac{1}{q} + \frac{d}{p} = \frac{d}{2} - 1, \quad 2 \leqslant p < \infty, \quad q \geqslant 2,$$

and $\frac{1}{q} + \frac{d-1}{2p} \leqslant \frac{d-1}{4}$. *The constant* $C(\alpha,\tau) \equiv C(\alpha,\tau,u^*) \geqslant 1$ *depends only on* α, τ *and the norm of* u^* *in the space* $X_\tau \cap L^{\theta^*}((0,\tau), L^{2\theta^*}(\mathbb{R}^d))$. *If* u^*, G *and the initial data are radial functions, then* $\vec{w}$ *is a radial solution.*

Proof. This proposition can be proved in the same way as Theorem 2.3, by considering the term $f'(u^*(t,x))w + G$ as a non-linearity. The changes are minor in the fixed point argument used in the proof of Theorem 2.3. Here Y and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) = (\mathcal{F}_1, \partial_t \mathcal{F}_1)$ simply become:

$$Y \equiv Y_T \equiv \{\vec{w} \in L^\infty((0,\tau_0), \mathcal{H}) \text{ with } w \in L^{\theta^*}((0,\tau_0), L^{2\theta^*}(\mathbb{R}^d))\} .$$

and

$$(\mathcal{F}_1(\vec{w}_0, \vec{w}))(t) = \mathcal{S}_{1,\alpha}(t)w_0 + \mathcal{S}_{2,\alpha}(t)w_1 + \int_0^t \mathcal{S}_{2,\alpha}(t-s)(f'(u^*(s))w(s) + G(s))\, ds .$$

We obtain estimates similar to (2.20), where now M_0 is replaced by the norm of u^* in $X_\tau \cap L^{\theta^*}((0,\tau), L^{2\theta^*}(\mathbb{R}^d))$. If the time T_0 defined in (2.19) is larger than τ_0, then we have proved the existence (and uniqueness) of the solution $\vec{w}(\vec{w}_0) \in Y_T$ and the estimates (2.33) follow from Lemma 2.2. If $T_0 < \tau_0$, we repeat the above proof by taking as initial data $(\vec{w}(\vec{w}_0))(T_0)$ and by replacing

$$f'(u^*(t,x))w(t,x) + G(t,x)$$

by

$$f'(u^*(t+T_0,x))w(t+T_0,x) + G(t+T_0,x)$$

We repeat this argument a finite number of times till we reach the time τ_0. ◻

2.2. **Definition of the functional K_0 and the Nehari manifold.** We introduce the functional $K_0 : \varphi \in H^1(\mathbb{R}^d) \mapsto K_0(\varphi) \in \mathbb{R}$, defined by

$$K_0(\varphi) = \int_{\mathbb{R}^d} (|\nabla\varphi|^2 + \varphi^2 - \varphi f(\varphi))\, dx ,$$

and introduce the Nehari manifold

$$\mathcal{N} = \{\varphi \in H^1_{rad}(\mathbf{R}^d) \mid K_0(\varphi) = 0\} . \tag{2.34}$$

The Nehari manifold arises naturally in the study of elliptic equations. The "Ambrosetti-Rabinowitz" hypothesis $(H.1)_f$ allows to prove the following lemmas, which will be used along this paper. The first one is trivial.

Lemma 2.5. *Assume that Hypothesis* $(H.1)_f$ *holds. Then, for any* $(\varphi,\psi) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, *we have*

$$\gamma(\|\varphi\|^2_{H^1} + \|\psi\|^2_{L^2}) \leqslant 2(1+\gamma)E((\varphi,\psi)) - K_0(\varphi) . \tag{2.35}$$

Proof. We simply write

$$\begin{aligned}\gamma(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2) =&2(1+\gamma)E((\varphi,\psi)) - K_0(\varphi) - \|\psi\|_{L^2}^2 \\ &+ \int_{\mathbb{R}^d} \big(2(1+\gamma)F(\varphi) - \varphi(x)f(\varphi(x))\big)\, dx \\ \leqslant& 2(1+\gamma)E((\varphi,\psi)) - K_0(\varphi)\,,\end{aligned} \tag{2.36}$$

where the integral is nonpositive by $(H.1)_f$. □

Corollary 2.6. *Suppose $\vec{u}(t) = (u(t), \partial_t u(t))$ is a strong solution of $(KG)_\alpha$ defined on the maximal interval $0 \leqslant t < T^*$. Assume*

$$\inf_{0\leqslant t<T^*} K_0(u(t)) > -\infty\,.$$

Then $T^* = \infty$, i.e., the solution is global.

Proof. By Lemma 2.5, we have for some finite M and all $0 \leqslant t < T^*$

$$\begin{aligned}\|\vec{u}(t)\|_{\mathcal{H}} &\leqslant 2(1+\gamma)E(u(t), \partial_t u(t)) + M \\ &\leqslant 2(1+\gamma)E(u(0), \partial_t u(0)) + M\end{aligned}$$

where the second line holds by the decrease of the energy. Since finite time blowup means that $\|\vec{u}(t)\|_{\mathcal{H}}$ goes to infinity in finite time along some subsequence, we obtain the result. □

The proof of the next lemma uses a convexity argument and follows the lines of the proof of [42] and [40, Corollary 2.13]. We denote the nonlinear evolution by $S_\alpha(t)$.

Lemma 2.7. *Assume that the hypotheses $(H.1)_f$ and $(H.2)_f$ hold. Assume that $(u(t), \partial_t u(t))$ is a solution of $(KG)_\alpha$ defined on $[0, T^*)$ where $T^* \in (0, \infty]$ is maximal. If $K_0(u(t)) \leqslant -\delta$ (where $\delta > 0$), for $t_0 \leqslant t < T^*$, then $T^* < \infty$, i.e., the solution blows up in finite time.*

From Lemmas 2.5 and 2.7 we immediately deduce the following result.

Corollary 2.8. *Assume that the initial energy $E(\vec{u}_0)$ is negative. Then the solution blows-up in finite time $T^* < +\infty$.*

Proof of Lemma 2.7. We assume without loss of generality that $t_0 = 0$. We also assume towards a contradiction that $T^* = \infty$. In order to show that $S_\alpha(t)(u_0, u_1)$ blows up in finite time, we use a convexity argument as in [42]. Assume that $S_\alpha(t)(u_0, u_1)$ exists for all $t \geqslant 0$ and let

$$y(t) = \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2\, ds.$$

We have

$$\begin{aligned}\dot{y}(t) &= (u(t), \dot{u}(t)) + \alpha\|u(t)\|_{L^2}^2 \\ &= (u(t), \dot{u}(t)) + \alpha\|u(0)\|_{L^2}^2 + 2\alpha \int_0^t (u(s), \dot{u}(s))\, ds\end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \ddot{y}(t) &= \|\dot{u}(t)\|_{L^2}^2 + (u(t), \ddot{u}(t) + 2\alpha\dot{u}(t)) \\ &= \|\dot{u}(t)\|_{L^2}^2 + (u(t), (\Delta u - u + f(u))(t)) \\ &= \|\dot{u}(t)\|_{L^2}^2 - K_0(u(t)) \,. \end{aligned} \tag{2.38}$$

Thus,

$$\ddot{y}(t) \geqslant \|\dot{u}(t)\|_{L^2}^2 + \delta \geqslant \delta. \tag{2.39}$$

We deduce from (2.39) that $\lim_{t\to+\infty} \dot{y}(t) = +\infty$, and therefore $\lim_{t\to+\infty} y(t) = +\infty$.

Next, we note that

$$\begin{aligned} \ddot{y}(t) &= \|\dot{u}(t)\|_{L^2}^2 - K_0(u(t)) \\ &= (2+\gamma)\|\dot{u}(t)\|_{L^2}^2 + \gamma\|u(t)\|_{H^1}^2 - 2(1+\gamma)E(t) \\ &\quad - \int_{\mathbb{R}^d} \big(2(1+\gamma)F(u(t)) - u(t)f(u(t))\big)\,dx \end{aligned} \tag{2.40}$$

where we have set for simplicity $E(t) = E((u(t), \dot{u}(t)))$. But, we have

$$\dot{E}(t) = -2\alpha\|\dot{u}(t)\|_{L^2}^2$$

and

$$E(t) = E(0) + \int_0^t \dot{E}(s)\,ds = E(0) - 2\alpha\int_0^t \|\dot{u}(s)\|_{L^2}^2\,ds.$$

Using $(H.1)_f$, we can also write, for $t \geqslant 0$,

$$\ddot{y}(t) \geqslant (2+\gamma)\|\dot{u}(t)\|_{L^2}^2 + \gamma\|u(t)\|_{H^1}^2 - 2(1+\gamma)E(0) + 4\alpha(1+\gamma)\int_0^t \|\dot{u}(s)\|_{L^2}^2\,ds. \tag{2.41}$$

For the sake of illustration, assume first that $\alpha = 0$. Since $y(t) \to \infty$, we infer from (2.41) that for large t

$$\ddot{y}(t) \geqslant (2+\gamma)\|\dot{u}(t)\|_{L^2}^2 \tag{2.42}$$

Then $|\dot{y}(t)| \leqslant \|u(t)\|_{L^2}\|\dot{u}(t)\|_{L^2}$ whence

$$\ddot{y}(t) \geqslant \frac{2+\gamma}{2}\frac{\dot{y}^2(t)}{y(t)}$$

This implies that $\frac{d^2}{dt^2}(y^{-\eta}(t)) < 0$ where $\eta = \gamma/2$. Since $y^{-\eta}(t) \to 0$ as $t \to \infty$ we must have $\frac{d}{dt}(y^{-\eta})(t) < 0$ for some $t = t_1 > 0$ whence also $\frac{d}{dt}(y^{-\eta})(t) \leqslant \frac{d}{dt}(y^{-\eta})(t_1) < 0$ for all $t \geqslant t_1$. But then $y^{-\eta}(t_2) = 0$ for some $t_2 > t_1$ which is a contradiction.

For $\alpha > 0$, we claim that there exists $c > 1$ so that for large times

$$\ddot{y}(t)y(t) - c\dot{y}(t)^2 > 0 \tag{2.43}$$

If so, then

$$\frac{d^2}{dt^2}\big(y^{-(c-1)}\big)(t) = -(c-1)y^{-c-1}(t)(\ddot{y}(t)y(t) - c\dot{y}^2(t)) < 0$$

which leads to a contradiction as before.

It remains to verify (2.43). Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} y(t)\ddot{y}(t) - c\dot{y}^2(t) &\geqslant \Big(\frac{1}{2}\|u\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2\, ds\Big) \\ &\cdot \Big((2+\gamma)\|\dot{u}(t)\|_{L^2}^2 + \gamma\|u(t)\|_{H^1}^2 - 2(1+\gamma)E(0) + 4\alpha(1+\gamma)\int_0^t \|\dot{u}(s)\|_{L^2}^2\, ds\Big) \\ &- c\Big[\|u\|_{L^2}\|\dot{u}\|_{L^2} + 2\alpha\Big(\int_0^t \|u(s)\|_{L^2}^2\, ds\Big)^{\frac{1}{2}}\Big(\int_0^t \|\dot{u}(s)\|_{L^2}^2\, ds\Big)^{\frac{1}{2}} + \alpha\|u(0)\|_{L^2}^2\Big]^2. \end{aligned} \tag{2.44}$$

But, for any $\varepsilon > 0$, we estimate the term in brackets as follows:

$$\begin{aligned} &c\Big[\|u\|_{L^2}\|\dot{u}\|_{L^2} + 2\alpha\Big(\int_0^t \|u(s)\|_{L^2}^2\, ds\Big)^{\frac{1}{2}}\Big(\int_0^t \|\dot{u}(s)\|_{L^2}^2\, ds\Big)^{\frac{1}{2}} + \alpha\|u(0)\|_{L^2}^2\Big]^2 \\ &\quad\leqslant c(1+\varepsilon)\Big(\|u\|_{L^2}\|\dot{u}\|_{L^2} + 2\alpha\Big(\int_0^t \|u(s)\|_{L^2}^2\, ds\Big)^{\frac{1}{2}}\Big(\int_0^t \|\dot{u}(s)\|_{L^2}^2\, ds\Big)^{\frac{1}{2}}\Big)^2 \\ &\qquad + c\Big(1+\frac{1}{\varepsilon}\Big)\alpha^2\|u(0)\|_{L^2}^4 \\ &\quad\leqslant c(1+\varepsilon)\Big(\frac{1}{2}\|u\|_{L^2}^2 + \alpha\int_0^t \|u(s)\|_{L^2}^2\, ds\Big)\Big(2\|\dot{u}\|_{L^2}^2 + 4\alpha\int_0^t \|\dot{u}(s)\|_{L^2}^2\, ds\Big) \\ &\qquad + c\Big(1+\frac{1}{\varepsilon}\Big)\alpha^2\|u(0)\|_{L^2}^4. \end{aligned}$$

Setting $b = c(1+\varepsilon)$, $C = c\alpha^2(1+\frac{1}{\varepsilon})\|u(0)\|_{L^2}^4$, we may replace the right-hand side of this inequality by

$$\leqslant y(t)\Big(2b\|\dot{u}\|_{L^2}^2 + 4b\alpha\int_0^t \|\dot{u}(s)\|_{L^2}^2\, ds\Big) + C$$

From the last inequality and from (2.44), we deduce that

$$\begin{aligned} y\ddot{y}(t) - c\dot{y}^2(t) &\geqslant y(t)\Big\{(2+\gamma-2b)\|\dot{u}(t)\|_{L^2}^2 + 4\alpha(1+\gamma-b)\int_0^t \|\dot{u}(s)\|_{L^2}^2\, ds + \gamma\|u(t)\|_{H^1}^2 \\ &\quad - 2(1+\gamma)E(0)\Big\} - C \\ &= y(t)\Psi(t) - C \end{aligned} \tag{2.45}$$

where $\Psi(t)$ is defined by the term in braces.

We now adjust the constants $c > 1$ and $\varepsilon > 0$ so that $2+\gamma-2b > 0$, $1+\gamma-b > 0$. We now pick $\eta > 0$ so small that

$$2+\gamma-2b > \eta, \qquad \gamma - \frac{\eta}{2} - \alpha\eta > 0$$

This allows us to bound $\Psi(t)$ from below:

$$\begin{aligned}\Psi(t) = \Big[&\Big(2+\gamma-2b-\frac{\eta}{2}\Big)\|\dot u(t)\|_{L^2}^2 + 4\alpha(1+\gamma-b)\int_0^t \|\dot u(t)\|_{L^2}^2\,ds + \gamma\|\nabla u(t)\|_{L^2}^2 \\ &+ \Big(\gamma - \frac{\eta}{2} - \alpha\eta\Big)\|u(t)\|_{L^2}^2 + \eta\dot y(t) - 2(1+\gamma)E(0)\Big] \\ \geqslant{}& \eta \dot y(t) - 2(1+\gamma)E(0) + q(t)\end{aligned}$$

where $q(t) \geqslant 0$. From (2.45), we infer that, for $t \geqslant 0$,

$$y(t)\ddot y(t) - c\dot y^2(t) \geqslant y(t)[\eta\dot y(t) - 2(1+\gamma)E(0) + q(t)] - C. \tag{2.46}$$

Since $y(t), \dot y(t) \to \infty$ as $t \to \infty$, we are done. $\square$

2.3. **Spectral properties.** Suppose we have a stationary solution $\varphi_0 \in H^1(\mathbb{R}^d)$ to $(KG)_\alpha$, namely,

$$-\Delta\varphi_0 + \varphi_0 - f(\varphi_0) = 0$$

By elliptic theory, see for example [3, 4], these solutions are exponentially decaying, and lie in $C^{3,\beta}$ for some $\beta > 0$. Solving $(KG)_\alpha$ for $u = \varphi_0 + v$ yields

$$v_{tt} + 2\alpha v_t - \Delta v + v - f'(\varphi_0)v = N(\varphi_0, v) \tag{2.47}$$

where $N(\varphi_0, v) = f(\varphi_0 + v) - f(\varphi_0) - f'(\varphi_0)v$. Set $\mathcal{L} = -\Delta + I - f'(\varphi_0)$. Rewrite (2.47) in the form

$$\partial_t \begin{pmatrix} v \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mathcal{L} & -2\alpha \end{pmatrix} \begin{pmatrix} v \\ v_t \end{pmatrix} + \begin{pmatrix} 0 \\ N(\varphi_0, v) \end{pmatrix} \tag{2.48}$$

Denoting the matrix operator on the right-hand side by A_α, and setting $\vec v := \binom{v}{v_t}$, we may write (2.48) in the form

$$\partial_t \vec v = A_\alpha \vec v + \vec N$$

The spectral properties of $\mathcal{L}$ stated in the following lemma are standard, see for example [32] and the references cited here.

Lemma 2.9. *The operator $\mathcal{L}$ is self-adjoint with domain $H^2(\mathbb{R}^d)$. The spectrum $\sigma(\mathcal{L})$ consists of an essential part $[1, \infty)$, which is absolutely continuous, and finitely many eigenvalues of finite multiplicity all of which fall into $(-\infty, 1]$. The eigenfunctions are $C^{2,\beta}$ with $\beta > 0$ and the ones associated with eigenvalues below 1 are exponentially decaying. Over the radial functions, all eigenvalues are simple.*

Proof. The essential spectrum equals $[1, \infty)$ by the Weyl criterion. The Agmon-Kuroda theory on asymptotic completeness guarantees that there are no imbedded eigenvalues and no singular continuous spectrum. Thus, the spectral measure restricted to $[1, \infty)$ is purely absolutely continuous. The Birman-Schwinger criterion shows (due to the rapid decay of the potential $f'(\varphi_0)$) that there are only finitely many eigenvalues of $\mathcal{L}$ which are $\leqslant 1$, counted with multiplicity. The $C^{2,\beta}$ property of the eigenfunctions is standard elliptic regularity (Schauder estimates) since φ_0 is smooth, and so $f'(\varphi_0)$ is Hölder regular.

For the sake of completeness we remark that the threshold 1 may be an eigenvalue or a resonance. To illustrate what this means, consider $\mathbb{R}^3$. Then this distinction refers to the fact that solutions to $\mathcal{L}\psi = \psi$ either decay like $|x|^{-2}$ (which means $\psi \in L^2$ is an eigenfunction) or like $|x|^{-1}$, the latter implying that $\psi \notin L^2(\mathbb{R}^3)$ (this is the resonant case). We remark that over the radial functions only the resonant case can occur. However, none of this finer analysis at energy 1 is relevant for our purposes.

The exponential decay of the eigenfunctions with eigenvalues below 1 is known as Agmon's estimate. The simplicity of the radial eigenfunctions is immediate from the reduction to an ODE on $(0, \infty)$ with a Dirichlet condition at $r = 0$. Let us elaborate on the kernel of $\mathcal{L}$, since it is important in our construction. We set $\mathcal{L}v = 0$, $v \neq 0$ radial and in H^1. Then

$$-\Delta v + v - f'(\varphi_0)v = 0$$

We already note that $v \in C^{2,\beta}(\mathbb{R}^d)$, and that $v(r)$ decays exponentially. Set $u(r) = r^{\frac{d-1}{2}} v(r)$. Then $u(0) = 0$, $u(r) \to 0$ as $r \to \infty$ (exponentially in fact), and it satisfies the equation

$$-u''(r) + u(r) - (\frac{d-1}{2})(\frac{d-3}{2})\frac{u(r)}{r^2} - f'(\varphi_0)u(r) = 0\,, \quad r > 0 \tag{2.49}$$

This ODE has a fundamental system consisting of a solution growing like e^r and one decaying like e^{-r} as $r \to \infty$. Only the latter can lie in the kernel and it does so if and only if it satisfies the boundary condition $u(0) = 0$. In this case the kernel has dimension 1 otherwise it consists only of $\{0\}$. □

We now analyze the spectral properties of the matrix operator A_α.

Lemma 2.10.

- *The operator A_α has discrete spectrum if and only if $\mathcal{L}$ does. The essential spectrum of A_α lies strictly to the left of the imaginary axis, i.e., in $\Re(z) < -\delta(\alpha)$ for some $\delta(\alpha) > 0$. The spectrum of A_α on the imaginary axis is either empty or $\{0\}$. In the latter case, 0 is an eigenvalue of A_α and this occurs if and only if 0 is an eigenvalue of $\mathcal{L}$. Then $\dim(\mathrm{Ker}(\mathcal{L})) = 1$, in which case 0 is a simple eigenvalue. The eigenvalues of A_α are precisely*

$$-\alpha \pm \sqrt{\alpha^2 - \mu}$$

  where $\mu \in \sigma(\mathcal{L})$ is an eigenvalue.
  - *If $\alpha \geqslant 1$, then the discrete spectrum of A_α lies only on the real axis.*
  - *If $0 < \alpha < 1$, in addition to real eigenvalues, there may also be eigenvalues on the line $\Re(z) = -\alpha$ resulting from eigenvalues of $\mathcal{L}$ in the gap $(0, 1]$.*
- *The essential spectrum of $\mathcal{L}$ gives rise to essential spectrum $\sigma_{\mathrm{ess}}(A_\alpha)$ of A_α as follows:*
  - *If $0 < \alpha \leqslant 1$, $\sigma_{\mathrm{ess}}(A_\alpha)$ is contained in the line $\Re(z) = -\alpha$ and consists of $-\alpha \pm ib$, $b \geqslant \sqrt{1-\alpha^2}$.*
  - *If $\alpha > 1$, $\sigma_{\mathrm{ess}}(A_\alpha)$ consists of the entire line $\Re(z) = -\alpha$ and of the interval*

$$[-\alpha - \sqrt{\alpha^2 - 1}, -\alpha + \sqrt{\alpha^2 - 1}]$$

Proof. We need to address the solvability of the system

$$A_\alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = z \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

over the domain $H^2_{rad}(\mathbb{R}^d) \times H^1_{rad}(\mathbb{R}^d)$ of A_α. This means that

$$\begin{aligned} u_2 &= z u_1 \\ -\mathcal{L} u_1 - 2\alpha u_2 &= z u_2 \end{aligned}$$

which is the same as

$$\begin{aligned} u_2 &= z u_1 \\ (\mathcal{L} + 2\alpha z + z^2) u_1 &= 0 \end{aligned}$$

There exists a solution in the domain of A_α if and only if

$$2\alpha z + z^2 \in \sigma(-\mathcal{L})$$

Taking $\lambda \in \sigma(\mathcal{L})$, this means that

$$z = -\alpha \pm \sqrt{\alpha^2 - \lambda}, \quad \lambda \in \sigma(\mathcal{L})\,. \tag{2.50}$$

This relation establishes all the claims concerning the point spectrum of A_α. Let now τ belong to the resolvent set $\rho(A_\alpha)$ of A_α. Then, for any $(0, v_2) \in \mathcal{H}_{rad}$, the system

$$(A_\alpha - \tau Id) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \tag{2.51}$$

has a unique solution (u_1, u_2) in $\mathcal{H}_{rad}$, which implies that

$$-\mathcal{L} u_1 - (\tau^2 + 2\alpha\tau) u_1 = v_2$$

has a unique solution u_1 and thus $\tau^2 + 2\alpha\tau \equiv -\lambda$ does not belong to the spectrum of $-\mathcal{L}$, that is,

$$\tau \neq -\alpha \pm \sqrt{\alpha^2 - \lambda}, \quad \lambda \in \sigma(\mathcal{L})$$

and we are done. □

The discrete spectrum of A_α (and therefore of $\mathcal{L}$) is important to our analysis. In fact, the strongly unstable manifold of the linear evolution e^{tA_α} as $t \to \infty$ corresponds exactly to spectrum of A_α in the right-half plane which occurs if and only if $\mathcal{L}$ exhibits negative eigenvalues. In the generality we assume here we cannot determine whether this is the case or not, and so our arguments need to be flexible enough to account for both possibilities.

However, consider the following additional condition, where γ is as in $(H.1)_f$: for any $\phi \in H^1$,

$$\int_{\mathbb{R}^d} \left[\phi^2(x) f'(\phi(x)) - (1 + 2\gamma)\phi(x) f(\phi(x))\right] dx \geqslant 0 \tag{2.52}$$

Let $\varphi_0 \neq 0$ be a stationary solution as before. Then it follows from (2.52) that

$$
\begin{aligned}
\langle \mathcal{L}\varphi_0, \varphi_0 \rangle &= \int_{\mathbb{R}^d} (|\nabla\varphi_0|^2 + \varphi_0^2 - f'(\varphi_0)\varphi_0^2)\, dx \\
&= -2\gamma \int_{\mathbb{R}^d} f(\varphi_0)\varphi_0\, dx + \int_{\mathbb{R}^d} [(1+2\gamma) f(\varphi_0)\varphi_0 - f'(\varphi_0)\varphi_0^2]\, dx \\
&\leqslant -2\gamma \|\varphi_0\|_{H^1}^2 < 0
\end{aligned}
\tag{2.53}
$$

where we used that $K_0(\varphi_0) = 0$. Therefore, $\mathcal{L}$ has negative eigenvalues. We leave it to the reader to check that the class of nonlinearities f given by a sum and difference of pure powers as in (1.5) satisfy (2.52). Hence, for such nonlinearities all nonzero stationary solutions are linearly unstable. In other words, under the additional condition (2.52) all nonzero equilibria give rise to a strongly unstable manifold of e^{tA_α}.

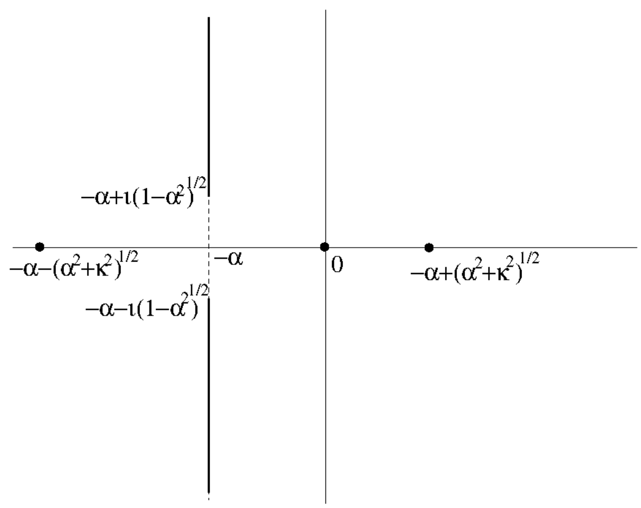

Figure 1. The spectrum of A_α for $0 < \alpha < 1$

3. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. To this end, given $(\varphi_0, \varphi_1) \in \mathcal{H}_{rad}$, we will first show that, if $S_\alpha(t)(\varphi_0, \varphi_1)$ does not blow up in finite time, then there exists a sequence of times t_n going to $+\infty$ such that $S_\alpha(t_n)(\varphi_0, \varphi_1)$ converges to an equilibrium point $(u^*, 0)$.

3.1. **Convergence to an equilibrium** $(u^*, 0)$ **along a subsequence.** Denote the evolution operator of $(KG)_\alpha$ by $S_\alpha(t)$ and for $(\varphi_0, \varphi_1) \in \mathcal{H}_{rad}$, let $\vec{u}(t) := S_\alpha(t)(\varphi_0, \varphi_1)$. We have the following trichotomy for the forward evolution of $(KG)_\alpha$:

(FTB) $\vec{u}(t)$ blows up in finite positive time.

(GEB) $\vec{u}(t)$ exists globally and the trajectory $\{\vec{u}(t), t \geqslant 0\}$ is bounded in $\mathcal{H}_{rad}$,
(GEU) $\vec{u}(t)$ exists globally and the trajectory $\{\vec{u}(t), t \geqslant 0\}$ is unbounded in $\mathcal{H}_{rad}$.

Later in Section 3.2, we shall show that (GEU) cannot occur.

Remark 3.1. Several remarks have to be made at this stage.

(i): From Corollary 2.8, we know that if $E(\varphi_0, \varphi_1) < 0$, then $S_\alpha(t)(\varphi_0, \varphi_1)$ blows up in finite time. Thus, in the study of the cases (GEB) and (GEU), we only need to consider solutions $\vec{u}(t) \equiv S_\alpha(t)(\varphi_0, \varphi_1)$ such that, for any $t \geqslant 0$,

$$E(u(t), \partial_t u(t)) \geqslant 0 . \tag{3.1}$$

(ii): Assume now that a solution $\vec{u}(t) \equiv S_\alpha(t)(\varphi_0, \varphi_1)$ of $(KG)_\alpha$ satisfies the properties $(H.1)_f$, $(H.2)_f$ and (3.1). Assume moreover, that the exponent θ in $(H.2)_f$ satisfies the bound

$$\theta < 1 + \frac{4}{d} . \tag{3.2}$$

Then, arguing exactly as in [23, Lemma 4.2], one can prove that every global solution $S_\alpha(t)(\varphi_0, \varphi_1)$ is bounded in $\mathcal{H}$. In this proof, the upper bound (3.2) of θ plays a crucial role.

(iii): Now, let us turn to the case where $1 + \frac{4}{d} \leqslant \theta \leqslant \frac{d}{d-2}$. We consider a global solution $(u(t), \partial_t u(t)) = S_\alpha(t)(\varphi_0, \varphi_1)$. In this case, arguing as in [23, Page 59] by introducing the auxiliary equation satisfied by $\partial_t \vec{u}(t) := (\partial_t u(t), \partial_t^2 u(t))$, one shows that $\partial_t \vec{u}(t)$ converges to $(0,0)$ in $L^2(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$. From this convergence property, we deduce that $K_0(u(t))$ converges to 0 as t goes to infinity.

Proposition 3.2. *Assume that the hypotheses $(H.1)_f$ and $(H.2)_f$ hold. In the cases (GEU) and (GEB), there exist a sequence of times t_n and a sequence of numbers δ_n such that $t_n \to +\infty$ as $n \to +\infty$ and that*

$$K_0(u(t_n)) = \delta_n \,, \; \textit{with} \lim_{n\to+\infty} \delta_n = 0 . \tag{3.3}$$

We remark, that, in the (GEU) case, the sequence t_n can be chosen so that $\delta_n \leqslant 0$ for every n.

Proof. Let $\vec{u}(t) := (u(t), \partial_t u(t)) = S_\alpha(t)(\varphi_0, \varphi_1)$. By Remark 3.1, we may assume without loss of generality that, for any $t \geqslant 0$,

$$E(u(t), \partial_t u(t)) \geqslant 0 .$$

To show that there exist two sequences t_n and δ_n satisfying the properties of the proposition, we will argue by contradiction. If such sequences do not exist, there exist a time T_0 and a constant $\kappa_0 > 0$ such that,

(1) either $K_0(u(t)) \leqslant -\kappa_0$ for any $t \geqslant T_0$,
(2) or $K_0(u(t)) \geqslant \kappa_0$ for any $t \geqslant T_0$.

In the case (1), Lemma 2.7 implies finite time blow–up, which contradicts the hypotheses (GEU) or (GEB). Thus, the case (1) cannot occur. In the case (2), by Lemma 2.5 the solution

$\vec{u}(t)$ is bounded in $\mathcal{H}_{rad}$. In particular, the function $|\dot{y}(t)|$ defined in (2.37) is bounded. By (2.38), we have for any $t \geqslant T_0$,

$$\ddot{y}(t) = \|\dot{u}(t)\|_{L^2}^2 - K_0(u(t)) \leqslant \|\dot{u}(t)\|_{L^2}^2 - \kappa_0 \,, \tag{3.4}$$

which in turn implies that, for any $T > T_0$,

$$\begin{aligned} \dot{y}(T) - \dot{y}(T_0) = \int_{T_0}^{T} \ddot{y}(t)dt &\leqslant \int_{T_0}^{T} \|\dot{u}(t)\|_{L^2}^2 dt - \kappa_0(T - T_0) \\ &\leqslant E(\vec{u}(T_0)) - E(\vec{u}(T)) - \kappa_0(T - T_0) \\ &\leqslant E(\vec{u}(T_0)) - \kappa_0(T - T_0) \,, \end{aligned} \tag{3.5}$$

which contradicts the boundedness of $\dot{y}(T)$ as $T \to +\infty$. This proves Proposition 3.2. The above proof also shows that, in the (GEU) case, the sequence t_n can be chosen so that $\delta_n \leqslant 0$ for every n. □

Next, by means of these vanishing results for K_0, we deduce the convergence to an equilibrium along a subsequence.

Theorem 3.3. *Let $\alpha > 0$ and $\vec{u}_0 := (\varphi_0, \varphi_1) \in \mathcal{H}_{rad}$ so that the solution $\vec{u}(t)$ exists for all times $t > 0$. Let t_n be a sequence of times such that $K_0(u(t_n)) = \delta_n$ converges to 0, then there exists an equilibrium point $\vec{u}^* = (u^*, 0) \in \mathcal{H}_{rad}$ such that (after possibly extracting a subsequence), $\vec{u}(t_n)$ converges to $(u^*, 0)$ in $\mathcal{H}$.*

Proof. From Lemma 2.5 we conclude that

$$\sup_{n \geqslant 0} \|(u(t_n), \partial_t u(t_n))\|_{\mathcal{H}} < \infty$$

We recall that without loss of generality, we may assume that

$$E(u(t), \partial_t u(t)) \geqslant 0, \quad \forall t \geqslant 0.$$

Since the left-hand side is non-increasing, there exists $\ell \geqslant 0$ such that

$$\lim_{t \to +\infty} E(u(t), \partial_t u(t)) = \ell \geqslant 0. \tag{3.6}$$

In fact, from the equality valid for any $t_1 \leqslant t_2$,

$$E(u(t_1), \partial_t u(t_1)) - E(u(t_2), \partial_t u(t_2)) = 2\alpha \int_{t_1}^{t_2} \|\partial_t u(s)\|_{L^2}^2 \, ds \,,$$

we deduce that $\int_{t_1}^{t_2} \|\partial_t u(s)\|_{L^2}^2 \, ds$ tends to 0, as $t_1, t_2 \to \infty$.

We consider the equations

$$(KG)_\alpha^n \qquad \begin{cases} \partial_{tt} u_n + 2\alpha \partial_t u_n - \Delta u_n + u_n - f(u_n) = 0 \\ (u_n(0), \partial_t u_n(0)) = (u(t_n), \partial_t u(t_n)) \end{cases}$$

By Theorem 2.3, there exists $T > 0$ and $C > 0$ such that, for any n, the solution $(u_n(t), \partial_t u_n(t))$ is in $C^0([-T, T], \mathcal{H})$ and, for $-T \leqslant t \leqslant T$,

$$\|(u_n(t), \partial_t u_n(t))\|_{\mathcal{H}} \leqslant C \,. \tag{3.7}$$

In the case $d = 1$ or $d = 2$, the inequality (3.7) implies that $\|u_n\|_{L^\infty((-T,T),L^p)} \leqslant C$, for any $2 \leqslant p < +\infty$. In the case $3 \leqslant d \leqslant 6$, the estimate (2.21) in Theorem 2.3 also implies that

$$\|u_n\|_{L^{\theta*}((0,T),L^{2\theta*})} \leqslant C\,. \tag{3.8}$$

where $\theta^* = \frac{d+2}{d-2}$. By uniqueness, $u_n(t) = u(t_n + t)$. For any $s, t \in [-T, T]$,

$$\begin{aligned}\int_{\mathbb{R}^d} |u_n(t) - u_n(s)|^2\,dx &= \int_{\mathbb{R}^d} \left|\int_s^t \partial_t u_n(\sigma)\,d\sigma\right|^2 dx \\ &\leqslant |t-s| \int_{\mathbb{R}^d}\int_s^t |\partial_t u_n(\sigma)|^2\,d\sigma dx \\ &\leqslant |t-s| \int_{s+t_n}^{t+t_n} \|\partial_t u(\sigma)\|_{L^2}^2\,d\sigma\end{aligned}$$

whence

$$\begin{aligned}\|u_n(t) - u_n(s)\|_{L^2}^2 &\leqslant |t-s| \int_{s+t_n}^{t+t_n} \|\partial_t u(\sigma)\|_{L^2}^2\,d\sigma \\ &\leqslant 2T \int_{t_n-T}^{t_n+T} \|\partial_t u(\sigma)\|_{L^2}^2\,d\sigma \longrightarrow 0 \quad \text{as } n \to +\infty.\end{aligned} \tag{3.9}$$

For $s, t \in [-T, T]$, and fixed $p \in (2, 2^*)$, interpolation gives the existence of $a \in (0, 1)$ such that

$$\begin{aligned}\|u_n(t) - u_n(s)\|_{L^p} &\leqslant \|u_n(t) - u_n(s)\|_{L^{2*}}^a \|u_n(t) - u_n(s)\|_{L^2}^{1-a} \\ &\lesssim |t-s|^{\frac{1-a}{2}} \left(\int_{t_n-T}^{t_n+T} \|\partial_t u(\sigma)\|_{L^2}^2\,d\sigma\right)^{\frac{1-a}{2}}\end{aligned} \tag{3.10}$$

with a uniform constant in n. Fix $2 < p_0 < p_1 < 2^*$ and set $X := L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d)$. The choice of p_0, p_1 depends on the nonlinearity $f(u)$ through the parameters β, θ in $(H.2)_f$. With the hypotheses made on β (see Hypothesis $(H.2)_f$), we can choose $2 < p_0 < p_1 < 2^*$ so that, in addition, $p_2 \equiv 2\beta p_0/(p_0 - 2)$ satisfies the inequality

$$2 \leqslant p_2 \leqslant p_1\,. \tag{3.11}$$

This property will be used later in the inequality (3.21).

We consider the family of functions $(u_n(t))_n$ in $C^0([-T, T]; X)$. By the property (3.7),

$$\bigcup_{\substack{n\in\mathbb{N},\\ t\in[-T,T]}} u_n(t) \subset \text{ bounded set of } H^1_{rad}(\mathbb{R}^d).$$

Due to the compact embedding of $H^1_{rad}(\mathbb{R}^d)$ into X, we deduce that

$$\bigcup_{\substack{n\in\mathbb{N},\\ t\in[-T,T]}} u_n(t) \subset \text{ compact set of } X$$

Moreover, by (3.10), the family $(u_n(t))_n$ is equicontinuous in $C^0([-T,T];X)$. Thus, by the theorem of Ascoli, (after possibly extracting a subsequence) the sequence $u_n(t)$ converges in $C^0([-T,T];X)$ to a function $u^*(t) \in C^0([-T,T];X)$.

Moreover, by (3.9) and (3.10), $u^*(t)$ is constant on the time interval $[-T,T]$. We shall simply write $u^*(t) \equiv u^*$. Remark that we deduce from $K_0(u_n(0)) \to 0$ and the convergence of $u_n(t)$ towards u^* in $C^0([-T,T];X)$ that

$$\lim_{n\to+\infty} \|u_n(0)\|_{H^1}^2 = \int_{\mathbb{R}^d} f(u^*)u^*\, dx. \tag{3.12}$$

For this implication we need to choose p_0, p_1 close to $2, 2^*$, respectively, depending on $(H.2)_f$.

To summarize, we know that

- $u_n(t) \to u^*$ as $n \to +\infty$ in $C^0([-T,T];X)$ and $u^* := u^*(t)$
- $\partial_t u_n(t) \to 0$ as $n \to +\infty$ in $L^2((-T,T);L^2(\mathbb{R}^d))$
- $(u_n(t), \partial_t u_n(t))_n$ is uniformly bounded in n in $L^\infty((-T,T);\mathcal{H})$ and, in particular in $L^2((-T,T);\mathcal{H})$.

Taking these properties into account, one shows that $(u_n, \partial_t u_n)$ converges in the sense of distributions (i.e., $\mathcal{D}'((-T,T)\times\mathbb{R}^d)$) towards $(u^*,0)$ as $n \to +\infty$ and that $(u^*,0)$ is an equilibrium point of $(KG)_\alpha$. Since $(u_n(0), \partial_t u_n(0))$ is uniformly bounded in $\mathcal{H}$, with respect to n, there exists a subsequence (that we still label by n) such that $u_n(0) \rightharpoonup u^*$ as $n \to +\infty$ weakly in $H^1(\mathbb{R}^d)$.

Since u^* is an equilibrium point of $(KG)_\alpha$, the following equality holds:

$$\int_{\mathbb{R}^d} f(u^*)u^*\, dx = \int_{\mathbb{R}^d} (|\nabla u^*|^2 + (u^*)^2)\, dx. \tag{3.13}$$

The equalities (3.12) and (3.13) imply that

$$\lim_{n\to+\infty} \|u_n(0)\|_{H^1}^2 = \|u^*\|_{H^1}^2 \tag{3.14}$$

and thus, since $u_n(0) \rightharpoonup u^*$ as $n \to +\infty$ weakly in $H^1(\mathbb{R}^d)$, the convergence of $u_n(0)$ towards u^* takes place in the strong sense in $H^1(\mathbb{R}^d)$. Moreover, the strong convergence of $u_n(0)$ towards u^* in $L^2(\mathbb{R}^d)$ and the property (3.9) imply the strong convergence of $u_n(s)$ towards u^* in $L^2(\mathbb{R}^d)$, uniformly in $s \in [-T,T]$. In summary,

$$u_n(.) \to u^* \text{ in } C^0((-T,T), L^2(\mathbb{R}^d)).$$

To finish the proof of Theorem 3.3 it remains to prove

$$\partial_t u_n(0) \to 0 \text{ in } L^2(\mathbb{R}^d). \tag{3.15}$$

As a first step towards the proof of property (3.15), we consider the equation satisfied by $\tilde{u}_n := u_n - u^*$, namely

$$\begin{cases} \partial_{tt}\tilde{u}_n - \Delta\tilde{u}_n + \tilde{u}_n = f(u_n) - f(u^*) - 2\alpha\partial_t\tilde{u}_n \\ \tilde{u}_n(0) = u_n(0) - u^* \to 0 \quad \text{as } n \to +\infty \quad \text{in } H^1(\mathbb{R}^d) \\ \partial_t\tilde{u}_n(0) = \partial_t u_n(0) \end{cases} \tag{3.16}$$

We write $u_n - u^* = w_n + v_n$ where w_n and v_n are solutions of the following equations:

$$\begin{cases} \partial_{tt}w_n - \Delta w_n + w_n = f(u_n) - f(u^*) - 2\alpha\partial_t u_n \\ w_n(0) = u_n(0) - u^* \\ \partial_t w_n(0) = 0 \end{cases} \tag{3.17}$$

and

$$\begin{cases} \partial_{tt}v_n - \Delta v_n + v_n = 0 \\ v_n(0) = 0 \\ \partial_t v_n(0) = \partial_t u_n(0). \end{cases} \tag{3.18}$$

The classical energy estimates for the Klein-Gordon equation imply that, for $-T \leqslant t \leqslant T$,

$$\begin{aligned} \|(w_n, \partial_t w_n)(t)\|_{\mathcal{H}} \leqslant C\Big[\|u_n(0) - u^*\|_{H^1} + 2\alpha\sqrt{2T}\left(\int_{-T}^{T} \|\partial_t u_n(s)\|_{L^2}^2\, ds\right)^{\frac{1}{2}} \\ + \int_{-T}^{T} \|f(u_n)(s) - f(u^*)\|_{L^2}\, ds\Big] \end{aligned} \tag{3.19}$$

Taking into account Hypothesis $(H.2)_f$, one has

$$\begin{aligned} &\int_{-T}^{T} \|f(u_n)(s) - f(u^*)\|_{L^2}\, ds \\ &\leqslant C\int_{-T}^{T} \|(u_n(s) - u^*)(|u_n|^\beta + |u^*|^\beta + |u_n|^{\theta-1} + |u^*|^{\theta-1})\|_{L^2}\, ds \end{aligned} \tag{3.20}$$

We recall that we have chosen $2 < p_0 < p_1 < 2^*$ so that $p_2 \equiv 2\beta p_0/(p_0 - 2)$ satisfies $2 \leqslant p_2 \leqslant p_1$. Applying the Hölder inequality, we obtain,

$$\begin{aligned} &\int_{-T}^{T} \|(u_n(s) - u^*)(|u_n|^\beta + |u^*|^\beta)\|_{L^2}\, ds \\ &\leqslant CT\|u_n - u^*\|_{L^\infty(I,L^{p_0})}(\|u_n\|^\beta_{L^\infty(I,L^{p_2})} + \|u^*\|^\beta_{L^\infty(I,L^{p_2})}) \\ &\leqslant CT\|u_n - u^*\|_{L^\infty(I,L^{p_0})}(\|u_n\|^\beta_{L^\infty(I,H^1)} + \|u^*\|^\beta_{L^\infty(I,H^1)})\,. \end{aligned} \tag{3.21}$$

Since $u_n \to u^*$ in $C(I, X)$, we conclude that the right-hand side of (3.21) vanishes in the limit $n \to \infty$. We next estimate the term

$$\int_{-T}^{T} \|(u_n - u^*)|u^*|^{\theta-1})\|_{L^2}\, ds \leqslant 2T\|u_n - u^*\|_{L^\infty(L^2)}\|u^*\|^{\theta-1}_{L^\infty(L^\infty)} \leqslant C\|u_n - u^*\|_{L^\infty(L^2)}\,, \tag{3.22}$$

which tends to 0 as $n \to \infty$. To bound the remaining term in (3.20), we argue as in the proof of Theorem 2.3. Indeed, from the estimates (2.13) to (2.17), we deduce that

$$\begin{aligned}\int_{-T}^{T} \|(u_n(s) - u^*(s))|u_n(s)|^{\theta-1}\|_{L^2}\, ds \leqslant& (2T)^{\eta} C\|u_n\|^{\frac{(\theta-1)\eta}{\theta}}_{L^\infty(I,L^2)}\|u_n - u^*\|^{\frac{\eta}{\theta}}_{L^\infty(I,L^2)} \\ &\times \left[\|u_n\|^{(1-\eta)\theta^*}_{L^{\theta^*}(I,L^{2\theta^*})} + (2T)^{1-\eta}\|u^*\|^{(1-\eta)\theta^*}_{L^\infty(I,L^{2\theta^*})}\right] \\ \leqslant& C(1+T)\|u_n - u^*\|^{\frac{\eta}{\theta}}_{L^\infty(I,L^2)}\,,\end{aligned} \tag{3.23}$$

where, by (2.15), $\eta = \frac{d+2-\theta(d-2)}{4}$. The right-hand side of the inequality (3.23) tends to 0 as n goes to infinity.

Finally, in view of (3.19), (3.20), (3.21), (3.22) and (3.23), we conclude that

$$\|(w_n(t), \partial_t w_n(t)\|_{\mathcal{H}} \to 0 \ \text{ as } \ n \to +\infty, \tag{3.24}$$

uniformly in $-T \leqslant t \leqslant T$.

By construction, $v_n = (u_n - u^*) - w_n$ and, in particular, $\partial_t v_n = \partial_t u_n - \partial_t w_n$. From (3.24) and the properties of $\|\partial_t u_n\|_{L^2(I;L^2(\mathbb{R}^d))}$, we infer that

$$\|\partial_t v_n\|_{L^2((-T,T);L^2(\mathbb{R}^d))} \leqslant \|\partial_t u_n\|_{L^2((-T,T);L^2(\mathbb{R}^d))} + \sqrt{2T}\|\partial_t w_n\|_{C^0([-T,T];L^2(\mathbb{R}^d))} \to 0 \tag{3.25}$$

as $n \to \infty$.

In the final step of the proof we shall turn this L^2_t averaged vanishing of $\|\partial_t v_n(t)\|_{L^2_x}$ as $n \to \infty$ into vanishing in the uniform sense in t. The main tool for this is the following "observation inequality" for equation (3.18).

Lemma 3.4. *For any $T_0 > 0$, there exists a positive constant $c(T_0) > 0$, independent of n, such that*

$$\|\partial_t v_n(0)\|^2_{L^2(\mathbb{R}^d)} \leqslant c(T_0) \int_{-T_0}^{T_0} \int_{\mathbb{R}^d} |\partial_t v_n|^2\, dxds. \tag{3.26}$$

Proof. For sake of simplicity, we set:

$$\partial_t v_n(0) \equiv \partial_t u_n(0) = v_{n1}.$$

If $\hat{v}_n$ denotes the Fourier transform of v_n, we have

$$\hat{v}_n(t,\xi) = \frac{\sin\left(t\sqrt{|\xi|^2+1}\right)}{\sqrt{|\xi|^2+1}}\hat{v}_{n1}(\xi)$$

and therefore

$$\|\partial_t \hat{v}_n(t,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}^d} \left|\sin\left(t\sqrt{|\xi|^2+1}\right)\right|^2 |\hat{v}_{n1}(\xi)|^2\, d\xi$$

as well as

$$\begin{aligned}\int_{-T_0}^{T_0} \|\partial_t \hat{v}_n(t,\cdot)\|_{L^2}^2\, dt &= \int_{-T_0}^{T_0}\int_{\mathbb{R}^d} \left|\sin\left(t\sqrt{|\xi|^2+1}\right)\right|^2 |\hat{v}_{n1}(\xi)|^2\, d\xi dt \\ &= \int_{\mathbb{R}^d}\left(\int_{-T_0}^{T_0} \left|\sin\left(t\sqrt{|\xi|^2+1}\right)\right|^2 dt\right) |\hat{v}_{n1}(\xi)|^2\, d\xi \\ &\geqslant \tilde{c}(T_0)|\int_{\mathbb{R}^d} \hat{v}_{n1}(\xi)|^2\, d\xi\,,\end{aligned} \tag{3.27}$$

where $\tilde{c}(T_0) > 0$, since $T_0 > 0$. Indeed

$$\begin{aligned}\int_{-T_0}^{T_0} \left|\sin\left(t\sqrt{|\xi|^2+1}\right)\right|^2 dt &= \int_{-T_0}^{T_0}\left(\frac{1-\cos\left(2t\sqrt{|\xi|^2+1}\right)}{2}\right) dt \\ &= T_0 - \frac{\sin\left(2T_0\sqrt{|\xi|^2+1}\right)}{2\sqrt{|\xi|^2+1}}\,.\end{aligned}$$

One easily sees that, for any $T_0 > 0$, there exists $\tilde{c}(T_0) > 0$ such that, for any $|\xi|$,

$$T_0 - \frac{\sin\left(2T_0\sqrt{|\xi|^2+1}\right)}{2\sqrt{|\xi|^2+1}} \geqslant \tilde{c}(T_0)\,. \tag{3.28}$$

The estimate (3.26) is then a direct consequence of (3.27), (3.28) and Plancherel's theorem. □

From the property (3.26) and the estimate (3.25), one deduces that

$$\|\partial_t u_n(0)\|_{L^2} \leqslant c(T)\left[\|\partial_t u_n\|_{L^2((-T,T);L^2)} + \sqrt{2T}\|\partial_t w_n\|_{C^0([-T,T];L^2(\mathbb{R}^d))}\right] \to 0 \quad \text{as } n \to +\infty \tag{3.29}$$

and the theorem is proved. □

3.2. **Convergence property.** Let $\vec{u}_0 = (\varphi_0, \varphi_1) \in \mathcal{H}_{rad}$ be so that the solution $\vec{u}(t) = S_\alpha(t)\vec{u}_0 \equiv (u(t), \partial_t u(t))$ exists globally and may be unbounded. Theorem 3.3 asserts that there exists a sequence of times $t_n \to +\infty$ such that $\vec{u}(t_n) \to (Q^*, 0)$ strongly in $\mathcal{H}_{rad}$, where Q^* is an equilibrium of $(KG)_\alpha$. We shall now show by contradiction that then necessarily $\vec{u}(t) \to (Q^*, 0)$ strongly in $\mathcal{H}_{rad}$ as $t \to \infty$ and hence the trajectory is bounded. In other words, Theorem 3.3 implies that the ω-limit set $\omega(\vec{u}_0)$ is not empty and contains an equilibrium point $(Q^*, 0) \in \mathcal{H}_{rad}$. We recall that the ω-limit set of $\vec{u}_0$ is defined as

$$\omega(\vec{u}_0) = \{\vec{w} \in \mathcal{H}_{rad} \mid \exists \text{ a sequence } s_n \geqslant 0, \text{ so that } s_n \to_{n\to+\infty} +\infty , \\ \text{and } S_\alpha(s_n)\vec{u}_0 \to_{n\to+\infty} \vec{w}\} .$$

Below we will show that the ω-limit set $\omega(\vec{u}_0)$ reduces to the singleton $(Q^*, 0)$, and that the entire trajectory converges to this point in the strong sense. And this concludes the proof of Theorem 1.2.

Before proving that the entire trajectory $\vec{u}(t) = S_\alpha(t)\vec{u}_0$ converges to $(Q^*, 0)$, we will emphasize that the ω-limit set $\omega(\vec{u}_0)$ is contained in the set $\mathcal{E}_{rad}$ of radial equilibrium points of $(KG)_\alpha$.

Lemma 3.5. *The ω-limit set $\omega(\vec{u}_0)$ satisfies the property*

$$\omega(\vec{u}_0) \subset \mathcal{E}_{rad} . \tag{3.30}$$

Proof. Let $\vec{v}_0 = (v_0, v_1) \in \omega(\vec{u}_0)$. Then, there exists a sequence $s_n \to_{n\to+\infty} +\infty$ such that $S_\alpha(s_n)\vec{u}_0 \equiv \vec{u}(s_n) \to_{n\to+\infty} \vec{v}_0$. On the one hand, we know by (3.6) that the energy satisfies

$$E(\vec{u}(s_n)) \to \ell = E((Q^*, 0))$$

as $n \to +\infty$, and

$$E(\vec{u}(s_n)) \to E(\vec{v}_0).$$

If $\vec{v}_0$ is not an equilibrium point, then for some time $\sigma > 0$,

$$E(S_\alpha(\sigma)\vec{v}_0) \leqslant E(\vec{v}_0) - \delta = \ell - \delta \tag{3.31}$$

where $\delta > 0$. Since

$$E(\vec{u}(s_n + \sigma)) \to \ell$$

and

$$E(\vec{u}(s_n + \sigma)) \to E(S_\alpha(\sigma)\vec{v}_0) ,$$

we arrive at a contradiction and (3.30) holds. □

Remark 3.6. *Let us fix a positive time $\tau > 0$ and introduce the ω-limit set $\omega_\tau(\vec{u}_0)$ of the discrete dynamical system defined by the iterates $S_\alpha(\tau)^m$, $m \in \mathbb{N}$, that is,*

$$\omega_\tau(\vec{u}_0) = \{\vec{w} \in \mathcal{H}_{rad} \mid \exists \textit{ a sequence } k_n \geqslant 0, \textit{ so that } k_n \to_{n\to+\infty} +\infty , \\ \textit{and } S_\alpha(\tau)^{k_n}\vec{u}_0 \to_{n\to+\infty} \vec{w}\} .$$

Obviously, $\omega_\tau(\vec{u}_0) \subset \omega(\vec{u}_0)$. Using the fact that $\omega(\vec{u}_0)$ is contained in $\mathcal{E}_{rad}$ and that the Lipschitz property of $S_\alpha(t) : \vec{v}_0 \in \mathcal{H} \to S_\alpha(t)\vec{v}_0 \in \mathcal{H}$, which is uniform with respect to $t \in [0, \tau]$ (see the arguments in Step 1 of Section 4 and especially the estimates (4.11), (4.12)*, and* (4.13)*), one can show that*

$$\omega_\tau(\vec{u}_0) = \omega(\vec{u}_0) . \tag{3.32}$$

To prove that the ω-limit $\omega(\vec{u}_0)$ is a singleton and that the entire trajectory converges to this point, we will apply a generalization of the classical convergence theorem of Aulbach [1], Hale-Massat [25] and Hale-Raugel [26], due to Brunovský and P. Poláčik [5], which uses local invariant manifold theory. For more details on these convergence theorems, we refer the reader to Appendix B and especially to Lemma B.3 that we shall apply below. The behaviour of $S_\alpha(t)\vec{u}_0 = \vec{u}(t)$ heavily depends on the spectral properties of the linearized operator $\mathcal{L}$ about Q^* and the linearized operator $\tilde{\Sigma}_\alpha(t) = e^{A_\alpha t}$ about $(Q^*, 0)$ (see the definitions (2.47), (2.48) or (4.3) with $\varphi = Q^*$). Lemma 2.10 describes the spectrum of the operator A_α. Before proving this convergence result, we need to recall some notation given in Section 4. There we introduce the modified (localized) Klein-Gordon equation (4.7) and show that this localized equation defines a globally defined flow $\bar{S}_\alpha(t)$ on $\mathcal{H}_{rad}$, such that,

$$\vec{u}(t) = S_\alpha(t)((Q^*, 0) + \vec{v}_0) = (Q^*, 0) + \bar{S}_\alpha(t)\vec{v}_0 \ , \text{ as long as } \vec{u}(t) \in B_{r_1} \ , \tag{3.33}$$

where $B_{r_1} \equiv B((Q^*, 0), r_1)$ is the open ball of center $(Q^*, 0)$ and radius $r_1 > 0$, with $r_1 \leqslant (8C(\alpha, \tau_0))^{-1} r_0$ (see Remark 4.2). In other terms, if we set

$$S^*_\alpha(t)\vec{u}_0 = (Q^*, 0) + \bar{S}_\alpha(t)(\vec{u}_0 - (Q^*, 0)) \ ,$$

then $S_\alpha(t)\vec{u}_0$ and $S^*_\alpha(t)\vec{u}_0$ coincide as long as $S_\alpha(t)\vec{u}_0 \in B_{r_1}$. In Section 4, we define the (global) stable, unstable, center stable, center unstable, and center manifolds $W^{i*}((Q^*, 0))$ of $S^*_\alpha(t)$ about $(Q^*, 0)$, where $i = s, u, cs, cu, c$ respectively. Since $S_\alpha(t)\vec{u}_0$ and $S^*_\alpha(t)\vec{u}_0$ coincide as long as $S_\alpha(t)\vec{u}_0 \in B_{r_1}$, we may define the local stable, unstable, center stable, center unstable, and center manifolds $W^i_{loc}((Q^*, 0))$ of $S_\alpha(t)$ about $((Q^*, 0))$ as follows:

$$W^i_{loc}((Q^*, 0)) = W^{i*}((Q^*, 0)) \cap B_{r_1} \ , \quad i = s, u, cs, cu, c \ . \tag{3.34}$$

We begin our proof with the particular case where $(Q^*, 0)$ is the (hyperbolic) trivial equilibrium $(0, 0)$ of $(KG)_\alpha$. We remark that in that case $\mathcal{L} = -\Delta + I$ and the entire spectrum of A_α lies in a half-plane of the form $\Re z < -\delta < 0$. In the terminology of Section 4 and of Appendix A, this means that the local stable manifold $W^u_{loc}((0, 0))$ is a whole neighborhood of $(0, 0)$ and that then necessarily $(0, 0)$ is an isolated equilibrium, and the perturbative equation (2.47) around $(0, 0)$ exhibits exponential decay of solutions in $\mathcal{H}_{rad}$ for small data. Actually, this exponential decay to zero had already been proved in Theorem 2.3. In particular, $\vec{u}(t) \to (0, 0)$ in that case as $t \to \infty$.

Let us come back to the general case. If $Q^* \neq 0$, then Lemma 2.10 states that A_α has either a trivial kernel, or a one-dimensional kernel. The former case means that the dynamics near $(Q^*, 0)$ is *hyperbolic*, whereas in the latter case it is not. In the hyperbolic scenario, we have no central part, which means that the invariant manifolds constructed in Section 4 and in Appendix A only involve stable and unstable manifolds $W^s_{loc}((Q^*, 0))$ and $W^u_{loc}((Q^*, 0))$. In both cases, the (local) unstable manifold $W^u_{loc}(Q^*, 0)$ is finite-dimensional since $\mathcal{L}$ has only finitely many eigenvalues (and thus only finitely many eigenvalues with positive real part).

In the non-hyperbolic case, the kernel of A_α is one-dimensional, the local center manifold $W^c_{loc}((Q^*,0))$ is a C^1-curve containing $(Q^*,0)$. We notice that we can also choose $r_1 > 0$ small enough so that $W^c_{loc}((Q^*,0)) = W^{c*}((Q^*,0)) \cap \overline{B}_{r_1}$ is a connected curve. Moreover, as remarked above, the (local) unstable manifold $W^u_{loc}(Q^*,0)$ is finite-dimensional. In order to prove the convergence to $(Q^*,0)$, we would like to directly apply the classical convergence theorem of [5] or [26], which is the case (1) of Theorem B.4. However, we do not know that the trajectory $\vec{u}(t)$ is bounded and thus we also cannot ascertain that the ω-limit set $\omega(\vec{u}_0)$ is connected. So we will apply the more general convergence Theorem B.2 of Brunovský and Poláčik, and more precisely their local Lemma B.3, which are recalled in Appendix B. To this end, we need to show that $(Q^*,0)$ is stable for $S_\alpha(t)$ restricted to the local center manifold (see the definition (3.40) below). In order to prove this stability, we shall use the same arguments as Brunovský and Poláčik in the proof of Lemma B.3. Like them, we will make use of the attraction of the center unstable manifold with asymptotic phase of Section 4 (see also Appendix A). Notice that the hyperbolic case can be considered as a special case, where the local center unstable (respectively, center) manifold reduces to the local unstable manifold (respectively, to $(Q^*,0)$). In the non-hyperbolic case, the center manifold is present and the dynamics is more delicate to analyze.

We proceed by contradiction and assume that $\vec{u}(t) \nrightarrow (Q^*,0)$. Since $\vec{u}(t)$ does not converge to $(Q^*,0)$, there exists $\beta_0 > 0$, $\beta_0 < \frac{r_1}{2}$ with the following property: for any $0 < \beta \leqslant \beta_0$, if $\vec{u}(t_0) \in B_{\mathcal{H}}((Q^*,0),\beta)$, there exists a first time $\tau_0 > 0$ such that $\vec{u}(t_0+\tau) \in B_\beta$, for $0 \leqslant \tau < \tau_0$, and $\vec{u}(t_0+\tau_0) \notin B_\beta$. In other words, $\vec{u}(t_0+\tau_0)$ belongs to the sphere $S((Q^*,0),\beta)$.

We first fix $\beta > 0$, $\beta \leqslant \beta_0$. By Theorem 3.3, there exists $n(\beta)$ such that, for $n \geqslant n(\beta)$, $\vec{u}(t_n) \in B_\beta$. Moreover, there exists a first time $\tau_n(\beta) > 0$ such that

$$\begin{aligned} &\vec{u}(t_n+\tau) \in B_\beta \quad \text{for } 0 \leqslant \tau < \tau_n(\beta) \\ &\vec{u}(t_n+\tau) \notin B_\beta \quad \text{for } \tau = \tau_n(\beta)\,. \end{aligned} \tag{3.35}$$

Since $\vec{u}(t_n) \to (Q^*,0)$ as $n \to +\infty$, we remark that $\tau_n(\beta) \to +\infty$ as $n \to +\infty$. We now invoke the attraction with asymptotic phase property of the center-unstable manifold, see (A.9) (or also (4.29) in Theorem 4.1). Thus, there exists $\xi_n := \xi(\vec{u}(t_n)) \in W^{cu}_{loc}((Q^*,0))$ such that, for $t \geqslant 0$,

$$\|S^*_\alpha(t)\vec{u}(t_n) - S^*_\alpha(t)\xi_n\|_{\mathcal{H}} \leqslant c_0\rho_0^t\|\vec{u}(t_n) - \xi_n\|_{\mathcal{H}}, \tag{3.36}$$

where $0 < \rho_0 < 1$. And, by continuity of the map $\xi(\cdot)$,

$$\xi_n \to (Q^*,0) \quad \text{as } n \to +\infty.$$

In particular, (3.36) implies that

$$\|S_\alpha(\tau_n(\beta))\vec{u}(t_n) - S^*_\alpha(\tau_n(\beta))\xi_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to +\infty. \tag{3.37}$$

Since $W^{cu*}((Q^*,0))$ is finite-dimensional and by (3.37), $S^*_\alpha(\tau_n(\beta))\xi_n$ is bounded, the sequence $S^*_\alpha(\tau_n(\beta))\xi_n$, $n \in \mathbb{N}$, contains a convergent subsequence. We conclude that up to passing to a subsequence one has

$$\vec{u}(t_n + \tau_n(\beta)) = S_\alpha(\tau_n(\beta))\vec{u}(t_n) \to (\tilde{u}_0, \tilde{u}_1) \in \bar{B}_\beta \text{ as } n \to +\infty.$$

By the invariance property of $W^{cu*}((Q^*, 0))$ and by (3.37),

$$(\tilde{u}_0, \tilde{u}_1) \in W^{cu}_{loc}((Q^*, 0)) \;. \tag{3.38}$$

We remark that, by (3.30) and (3.35),

(3.39)

$$(\tilde{u}_0, \tilde{u}_1) \text{ is an equilibrium point } (\tilde{Q}, 0) \equiv (\tilde{Q}(\beta), 0) \text{ and } \|(\tilde{Q}(\beta), 0) - (Q^*, 0)\|_{\mathcal{H}} = \beta.$$

If $(Q^*, 0)$ is an isolated equilibrium point, then (3.39) with $\beta \leqslant \frac{r_1}{2}$ leads to a contradiction. We remark that, in the hyperbolic case, $(Q^*, 0)$ is necessarily an isolated equilibrium which ends the proof in this case.

Let us now focus on the case where $(Q^*, 0)$ is not isolated. Before completing the proof in this case, we recall a definition of Brunovský and Poláčik, see Appendix B. We say that $(Q^*, 0)$ is **stable for** $S_\alpha(t)|_{W^c_{loc}((Q^*,0))}$ if, $\forall \epsilon > 0$, $\exists \theta > 0$ such that, for any $\vec{v}_0 \in W^c_{loc}((Q^*, 0))$,

$$\|\vec{v}_0 - (Q^*, 0)\|_{\mathcal{H}} \leqslant \theta$$

implies that, for $t \geqslant 0$,

$$\|S_\alpha(t)\vec{v}_0 - (Q^*, 0)\|_{\mathcal{H}} \leqslant \epsilon. \tag{3.40}$$

We now complete our proof. By construction and (3.38), the element $(\tilde{Q}(\beta), 0)$ belongs to $W^{cu}_{loc}((Q^*, 0))$. Since $(\tilde{Q}(\beta), 0)$ is an equilibrium point, it necessarily belongs to the local center manifold $W^c_{loc}((Q^*, 0))$ (see Section 4 and Appendix A for more explanations), which, as we saw earlier, is a C^1 one-dimensional embedded manifold passing through $(Q^*, 0)$.

Since (3.39) holds for any small $\beta > 0$, we see that this curve segment contains equilibria in the *omega*-limit set $\omega(\vec{u}_0)$ which are arbitrarily close to, but distinct from, $(Q^*, 0)$. In fact, we can say even more than that. First, we place an order on the curve $\tilde{W}^c_{r_1}((Q^*, 0))$ if $r_1 > 0$ is small enough. We adopt the notation $v^- < (Q^*, 0) < v^+$ if v^- (respectively v^+) is to the "left" (resp. "right") of $(Q^*, 0)$ on the curve segment $\tilde{W}^c_{r_1}((Q^*, 0))$. By intersecting the tangent line to this curve at $(Q^*, 0)$ with the spheres of radius β for all small β, we see that there are two possibilities:

(1) Either there exist two families of equilibria $(Q^-_m, 0)$ and $(Q^+_m, 0)$ with $(Q^-_m, 0) < (Q^*, 0) < (Q^+_m, 0)$ such that

$$(Q^{\pm}_m, 0) \to (Q^*, 0) \quad \text{ as } m \to +\infty. \tag{3.41}$$

A simple dynamical argument based on (3.41) implies that $S_\alpha(t)|_{W^c_{loc}((Q^*,0))}$ is in fact stable. We can now directly apply Lemma B.3 of Brunovský and Poláčik to the time 1 map $S_\alpha(1)$, which implies that the ω-limit set $\omega_1(\vec{u}_0)$ and thus the ω-limit set $\omega(\vec{u}_0)$ contain an element of $W^u_{loc}((Q^*, 0))\backslash(Q^*, 0)$. This contradicts the fact that $\omega(\vec{u}_0) \in \mathcal{E}_\ell$. Instead of directly applying Lemma B.3 to the map $S_\alpha(1)$, we can also argue for the flow $S_\alpha(t)$ as at the end of the proof of [5, Lemma 1] of Brunovský and Poláčik and directly show that $(\tilde{Q}(\beta), 0) \in W^u_{loc}((Q^*, 0)) \backslash (Q^*, 0)$, where $\tilde{Q}(\beta)$ is as in (3.39). But this contradicts the fact that $(\tilde{Q}(\beta), 0)$ is an equilibrium and so we again obtain the desired convergence.

(2) Or there exists $\beta_2 > 0$ such that there is no equilibrium point from the family $(\tilde{Q}(\beta), 0)$ on the "left" (say) of $(Q^*, 0)$ in $W^c_{loc}((Q^*, 0)) \cap B_{2\beta_2}$. But then, the above arguments (and in particular the properties (3.39)) imply that, for every $0 \leqslant \beta \leqslant \beta_2$, there exists an equilibrium $(\tilde{Q}^+(\beta), 0)$ in $\omega(\vec{u}_0)$ satisfying the properties (3.39). This implies that on the right of $(Q^*, 0)$, $W^c_{loc}((Q^*, 0))$ consists only of equilibria and that the ω-limit set $\omega(\vec{u}_0)$ contains a curve C of equilibria with end point $(Q^*, 0)$ (as for an interval). We then choose an equilibrium $(\tilde{Q}^+(\beta), 0)$ in the interior of C and close to $(Q^*, 0)$. We repeat the above proof with $(Q^*, 0)$ replaced by $(\tilde{Q}^+(\beta), 0)$. And we again obtain the same contradiction as in Case (1).

Remark 3.7. In the particular case of a wave type or reaction-diffusion equation, the proof of the Łojasiewicz-Simon inequality (see Sections 3.2 and 3.3 in the monograph of L. Simon [45] and also [28, Theorem 2.1]) shows that, when the kernel of $\mathcal{L}$ is one-dimensional, the set of equilibria of $(KG)_\alpha$ passing through $(Q^*, 0)$ is a C^1-curve. Adapting this approach, we could have avoided the last arguments and applied Theorem B.2. However, in view of possible further extensions, we chose not to follow this path.

4. Invariant manifold theory for the Klein-Gordon equation

In Section 3.2, in order to prove the convergence of any global solution (in positive time) towards an equilibrium point $(\varphi_0, 0)$ of $(KG)_\alpha$, we used the properties of the local unstable, local center unstable and local center manifolds $W^i_{loc}((\varphi_0, 0))$, $i = u, cu, c$ about $(\varphi_0, 0)$ for the flow $S_\alpha(t)$. There, we defined these local manifolds as the intersections of the global manifolds $W^{i*}((\varphi_0, 0))$, $i = u, cu, c$ about $(\varphi_0, 0)$ for the global flow $S^*_\alpha(t)$, with the ball of center $(\varphi_0, 0)$ and radius $r_1 > 0$, where $r_1 > 0$ is small enough. We recall that the global flow $S^*_\alpha(t)$ was defined by

$$S^*_\alpha(t)\vec{u}_0 = (\varphi_0, 0) + \bar{S}_\alpha(t)(\vec{u}_0 - (\varphi_0, 0)) ,$$

where $\bar{S}_\alpha(t)$ is the global flow defined by the localized Klein-Gordon equation (4.7) below. In this section, we construct the global invariant manifolds $W^i((0,0))$, $i = u, cu, c$, for the global flow $\bar{S}_\alpha(t)$ and obtain the attraction property of $W^{cu}((0,0))$ by applying the general invariant manifold theory recalled in Appendix A.

Let $(\varphi_0, 0) \in \mathcal{H}_{rad}$ be an equilibrium point of $(KG)_\alpha$, that is, φ_0 is a radial solution of the elliptic equation

$$-\Delta\varphi_0 + \varphi_0 - f(\varphi_0) = 0 . \tag{4.1}$$

Solving the equation $(KG)_\alpha$ in the neighborhood of $(\varphi_0, 0)$ leads one to solve the equation

$$v_{tt} + 2\alpha v_t + \mathcal{L}v - g_0(v) = 0 , \quad (v, v_t)(0) \equiv \vec{v}(0) \in \mathcal{H}_{rad} . \tag{4.2}$$

where

$$
\begin{aligned}
&\mathcal{L} = -\Delta + I - f'(\varphi_0)\,, \\
&g_0(v) = f(\varphi_0 + v) - f(\varphi_0) - f'(\varphi_0)v\,.
\end{aligned}
\tag{4.3}
$$

The equation (4.2) can be written in matrix form as follows

$$
\partial_t \begin{pmatrix} v \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mathcal{L} & -2\alpha \end{pmatrix} \begin{pmatrix} v \\ v_t \end{pmatrix} + \begin{pmatrix} 0 \\ g_0(v) \end{pmatrix} \equiv A_\alpha \vec{v} + \begin{pmatrix} 0 \\ g_0(v) \end{pmatrix}
\tag{4.4}
$$

We denote by $\tilde{\Sigma}_\alpha(t) = e^{A_\alpha t}$ the linear group generated by A_α and $\tilde{S}_\alpha(t)$ the local flow defined by the equation (4.2). We notice that

$$
S_\alpha(t)\vec{u}_0 = S_\alpha(t)((\varphi_0, 0) + \vec{v}_0) = (\varphi_0, 0) + \tilde{S}_\alpha(t)\vec{v}_0\,, \quad \text{where } \vec{v}_0 = \vec{u}_0 - (\varphi_0, 0)\,.
\tag{4.5}
$$

When $\alpha > 0$, according to Lemma 2.10, the radius $\rho(\sigma_{ess}(\tilde{\Sigma}_\alpha(\tau)))$ of the essential spectrum of $\tilde{\Sigma}_\alpha(\tau)$ satisfies:

$$
\rho(\sigma_{ess}(\tilde{\Sigma}_\alpha(\tau))) \leqslant \delta(\alpha, \tau) < 1
$$

The operator A_α can have a finite number of negative eigenvalues $\mu_j^-(\alpha) < 0$ (resp. a finite number of positive eigenvalues $\mu_j^+(\alpha) > 0$), in which case, $\lambda_j^-(\tau, \alpha) \equiv \exp(\mu_j^-(\alpha)\tau) < 1$ (resp. $\lambda_j^+(\tau, \alpha) \equiv \exp(\mu_j^+(\alpha)\tau) > 1$).

In addition, if 1 is an eigenvalue of $\tilde{\Sigma}_\alpha(\tau_0)$,$\tau_0 > 0$, it is a simple eigenvalue (and is a simple eigenvalue of $\tilde{\Sigma}_\alpha(\tau)$ for any $\tau > 0$). Since this case plays an important role in the proof of Section 3.2, we assume that 1 is an eigenvalue of $\tilde{\Sigma}_\alpha(\tau_0)$,$\tau_0 > 0$. In this case, we will construct a local center unstable manifold $W^{cu}_{loc}((0,0))$ of the equilibrium $(0,0)$ of $\tilde{S}_\alpha(t)$, a foliation of a neighborhood of $(0,0)$ in $\mathcal{H}_{rad}$ over $W^{cu}_{loc}((0,0))$ as well as a local center manifold $W^{c}_{loc}((0,0))$ by applying Theorems A.2 and A.5 to $\tilde{S}_\alpha(t)$. We choose $\tau_0 > 0$ small enough (τ_0 will be made more precise later). And we set

$$
L = \tilde{\Sigma}_\alpha(\tau_0)\,.
$$

The spectrum $\sigma(L)$ can be decomposed as in Hypothesis (HA.5.1) and one can define constants $C_1 \geqslant 1$, $C_2 \geqslant 1$, $\eta > 0$ and $\varepsilon > 0$ satisfying the estimates (A.20), (A.21), (A.22). Unfortunately, $\tilde{S}_\alpha(t)$ is only a local flow and thus $\tilde{S}_\alpha(\tau_1)$ will not satisfy the hypothesis (HA.3). Moreover, we need to show that the Lipschitz-constant $\mathrm{Lip}(R)$ can be chosen as small as needed, which is not true for $\tilde{S}_\alpha(t)$. Therefore, we need to make a localization in the following way, for instance. Let $r_0 > 0$ be a small constant, which will be made more precise later. We introduce a smooth cut-off function $\chi : \mathbb{R} \to [0,1]$ such that

$$
\chi(s) = \begin{cases} 1 & \text{if } |s| \leqslant 1, \\ 0 & \text{if } |s| \geqslant 2\,. \end{cases}
\tag{4.6}
$$

And, we consider the modified Klein-Gordon equation,

$$
v_{tt} + 2\alpha v_t + \mathcal{L}v - g_0(v)\chi\Big(\frac{\|\vec{v}\|^2_{\mathcal{H}}}{r^2}\Big) = 0\,, \quad \vec{v}(0) = \vec{v}_0 \in \mathcal{H}_{rad}\,,
\tag{4.7}
$$

where $0 < r \leqslant r_0$ is fixed. To simplify the notation, we set

$$h(\vec{v}) = g_0(v)\chi\big(\frac{\|\vec{v}\|^2_{\mathcal{H}}}{r^2}\big)\ .$$

We first show that, for any $\vec{v}_0 \in \mathcal{H}$, the equation (4.7) admits a unique solution $\vec{v}(t) \equiv \bar{S}_\alpha(t)\vec{v}_0 \in C^0([0,+\infty),\mathcal{H})$ (we leave to the reader to show that $\bar{S}_\alpha(t)\vec{v}_0$ also belongs to $C^0((-\infty,0],\mathcal{H})$). To this end, it is sufficient to show that, for any $\vec{v}_0 \in \mathcal{H}$, the solution $\vec{v}(t) \equiv \bar{S}_\alpha(t)\vec{v}_0$ of (4.7) exists on the time interval $[0,\tau_0]$ and remains bounded there. We will do that in two steps. We will give the proof only in the case where $d \geqslant 3$, the case $d \leqslant 2$ being easier. We first recall that the solution $\vec{v}(t)$ of (4.7) is given by the Duhamel formula,

$$\vec{v}(t) = \tilde{\Sigma}_\alpha(t)\vec{v}_0 + \int_0^t \tilde{\Sigma}_\alpha(t-s)(0, g_0(v(s))\chi\big(\frac{\|\vec{v}(s)\|^2_{\mathcal{H}}}{r^2}\big))^t\, ds\ , \tag{4.8}$$

and also remark that, as long as $\vec{v}(s) \notin B_{\mathcal{H}}(0, \sqrt{2}r)$, the term $h(\vec{v}(s))$ vanishes.

Step 1: Let $\vec{v}_0 \in \mathcal{H}$ so that $\|\vec{v}_0\|_{\mathcal{H}} \leqslant mr$ with $(8C(\alpha,\tau_0))^{-1} \leqslant m \leqslant 2$ for example. We set: $M_0 \equiv M_0(mr) = 4C(\alpha,\tau_0)mr$, where $C(\alpha,\tau_0) \geqslant 1$ is the constant given in Proposition 2.4. To show the local existence of the solution $\vec{v}(t)$ on the time interval $[0,\tau_0]$, we argue as in the proof of Theorem 2.3 and introduce the space

$$Y \equiv \{\vec{v} \in L^\infty((0,\tau_0),\mathcal{H}) \text{ with } v \in L^{\theta^*}((0,\tau_0), L^{2\theta^*}(\mathbb{R}^d)) \\ \quad | \ \|v\|_{L^\infty(H^1)\cap W^{1,\infty}(L^2)\cap L^{\theta^*}(L^{2\theta^*})} \leqslant M_0(mr)\}\ .$$

Like there we introduce the mapping $\mathcal{F} : Y \to Y$ defined by

$$(\mathcal{F}\vec{v})(t) = \tilde{\Sigma}_\alpha(t)\vec{v}_0 + \int_0^t \tilde{\Sigma}_\alpha(t-s)(0, h(\vec{v}(s)))^t ds\ .$$

The application of Proposition 2.4 implies

$$\|\mathcal{F}(0)\|_Y \leqslant C(\alpha,\tau_0)mr \leqslant \frac{M_0(mr)}{4}\ . \tag{4.9}$$

We next show that $\mathcal{F}$ is a strict contraction from Y into Y. Using the hypothesis $(H.2)_f$ and the fact that φ_0 belongs to $L^\infty(\mathbb{R}^d)$, we may write, for v_1, v_2 in $H^1(\mathbb{R}^d)$,

(4.10)
$$\begin{aligned}
|(g_0(v_1) - g_0(v_2))(x)| &= |f(\varphi_0(x) + v_1(x)) - f(\varphi_0(x) + v_2(x)) - f'(\varphi_0(x))(v_1(x) - v_2(x))| \\
&= \Big|\int_0^1 (f'(\varphi_0 + v_2 + \sigma(v_1 - v_2)) - f'(\varphi_0))(v_1 - v_2)d\sigma\Big| \\
&\leqslant C|(|v_1|^\beta + |v_2|^\beta + |v_1|^{\theta-1} + |v_2|^{\theta-1})(v_1 - v_2)|\ ,
\end{aligned}$$

where $0 < \beta < \min(\theta - 1, \frac{2}{d-2})$ and $C \equiv C(f, \varphi_0)$ is a constant depending only on f and on φ_0. For $\vec{v}_i \in Y$, $i = 1, 2$, Proposition 2.4 and the inequality (4.10) imply,

$$\begin{aligned}
\|\mathcal{F}\vec{v}_1 - \mathcal{F}\vec{v}_2\|_Y &\leqslant C(\alpha,\tau_0)\int_0^{\tau_0} \|h(\vec{v}_1(s)) - h(\vec{v}_2(s))\|_{L^2} ds \\
&\leqslant C(\alpha,\tau_0)\int_0^{\tau_0} \|(g_0(v_1) - g_0(v_2))\chi\big(\frac{\|\vec{v}_2\|^2_{\mathcal{H}}}{r^2}\big) + g_0(v_1)\big(\chi\big(\frac{\|\vec{v}_1\|^2_{\mathcal{H}}}{r^2}\big) - \chi\big(\frac{\|\vec{v}_2\|^2_{\mathcal{H}}}{r^2}\big)\big)\|_{L^2} ds \\
&\leqslant C(\alpha,\tau_0)C\Big[\int_0^{\tau_0} \|(|v_1(s)|^{\beta} + |v_2(s)|^{\beta})|v_1(s) - v_2(s)|\,\|_{L^2} ds \\
&\quad + \int_0^{\tau_0} \|(|v_1(s)|^{\theta-1} + |v_2(s)|^{\theta-1})|v_1(s) - v_2(s)|\,\|_{L^2} ds \\
&\quad + \int_0^{\tau_0} \|(|v_1(s)|^{\beta+1} + |v_1(s)|^{\theta})\|_{L^2}|\big(\chi\big(\frac{\|\vec{v}_1\|^2_{\mathcal{H}}}{r^2}\big) - \chi\big(\frac{\|\vec{v}_2\|^2_{\mathcal{H}}}{r^2}\big)\big)|ds\Big] \\
&\equiv B_1 + B_2 + B_3 .
\end{aligned} \tag{4.11}$$

Arguing as in the proof of Theorem 2.3, by using the Sobolev embeddings, the Hölder inequality and the fact that $0 < \beta < \frac{2}{d-2}$, we obtain the following inequality for B_1:

(4.12)

$$B_1 \leqslant C(\alpha,\tau_0)C\int_0^{\tau_0} (\|v_1\|^{\beta}_{H^1} + \|v_2\|^{\beta}_{H^1})\|v_1 - v_2\|_{H^1} ds \leqslant 2C(\alpha,\tau_0)\tau_0 C M_0(rm)^{\beta}\|v_1 - v_2\|_{L^\infty(H^1)}$$

The bound of the term B_2 is obtained as in the proof of Theorem 2.3 (see (2.18)):

$$B_2 \leqslant 2C(\alpha,\tau_0)C^2\tau_0^{\eta} M_0(rm)^{\frac{\theta-1}{\theta}(\theta^*(1-\eta)+\eta)}\big[\|v_1 - v_2\|_{L^\infty(L^2)} + \|v_1 - v_2\|_{L^{\theta^*}(L^{2\theta^*})}\big] . \tag{4.13}$$

where $\eta > 0$ is given in the formula (2.15). It remains to bound the term B_3. We first remark that, since $\chi'\big(\frac{\|\vec{w}\|^2_{\mathcal{H}}}{r^2}\big)$ vanishes if $\|\vec{w}\|_{\mathcal{H}} \geqslant \sqrt{2}r$, we may write

(4.14)

$$\begin{aligned}
|\big(\chi\big(\frac{\|\vec{v}_1\|^2_{\mathcal{H}}}{r^2}\big) - \chi\big(\frac{\|\vec{v}_2\|^2_{\mathcal{H}}}{r^2}\big)\big)| &\leqslant \int_0^1 |\chi'\big(\frac{\|\vec{v}_2 + \sigma(\vec{v}_1 - \vec{v}_2)\|^2_{\mathcal{H}}}{r^2}\big)\big(\frac{\vec{v}_2 + \sigma(\vec{v}_1 - \vec{v}_2)}{r^2}, (\vec{v}_1 - \vec{v}_2))_{\mathcal{H}}|d\sigma \\
&\leqslant \frac{\sqrt{2}}{r}\|\vec{v}_1 - \vec{v}_2\|_{\mathcal{H}} .
\end{aligned}$$

The estimate (4.14), together with the estimates (4.12) and (4.13) with $v_2 = 0$, imply that

(4.15)

$$B_3 \leqslant 8\sqrt{2}mC^2C(\alpha,\tau_0)^2[\tau_0 M_0(rm)^{\beta} + 2C(\alpha,\tau_0)\tau_0^{\eta}M_0(rm)^{\frac{\theta-1}{\theta}(\theta^*(1-\eta)+\eta)}]\|\vec{v}_1 - \vec{v}_2\|_{L^\infty(\mathcal{H})} .$$

Choosing $r_0 > 0$ small enough so that

$$\begin{aligned}
K(r_0,\tau_0) \equiv& 2C(\alpha,\tau_0)\tau_0 C M_0(2r_0)^{\beta} + 4C(\alpha,\tau_0)C^2\tau_0^{\eta}M_0(2r_0)^{\frac{\theta-1}{\theta}(\theta^*(1-\eta)+\eta)} \\
&+ 8\sqrt{2}C^2C(\alpha,\tau_0)^2[\tau_0 M_0(2r_0)^{\beta} + 2C(\alpha,\tau_0)\tau_0^{\eta}M_0(2r_0)^{\frac{\theta-1}{\theta}(\theta^*(1-\eta)+\eta)}] \leqslant \frac{1}{4} ,
\end{aligned} \tag{4.16}$$

we deduce from the inequalities (4.11) to (4.16) that

$$\|\mathcal{F}\vec{v}_1 - \mathcal{F}\vec{v}_2\|_Y \leqslant \frac{1}{4}\|\vec{v}_1 - \vec{v}_2\|_Y \,, \tag{4.17}$$

which implies with (4.9), that, for any $\vec{v}_1 \in Y$,

$$\|\mathcal{F}\vec{v}_1\|_Y \leqslant \frac{M_0(mr)}{2} \,. \tag{4.18}$$

Therefore, $\mathcal{F}$ is a strict contraction and admits a unique fixed point $\vec{v}(\vec{v}_0)$ in Y. The uniqueness of the solution $\vec{v}$ of the equation (4.7) on the time interval $[0, \tau_0]$ is proved as in the proof of Theorem 2.3. Let next $\vec{v}_{0,i}$, $i = 1, 2$, be so that $\|\vec{v}_{0,i}\|_{\mathcal{H}} \leqslant mr$, and let $\vec{v}_i$, $i = 1, 2$, be the corresponding solutions of the equation (4.7) on the time interval $[0, \tau_0]$; by the above proof, they belong to Y. Applying Proposition 2.4 and repeating the above proof, we show that

$$\|\vec{v}_1 - \vec{v}_2\|_Y \leqslant \frac{4}{3}C(\alpha, \tau_0)\|\vec{v}_{0,1} - \vec{v}_{0,2}\|_Y \,. \tag{4.19}$$

As in the proof of Theorem 2.3, one also shows that $\vec{v}_0 \in B_{\mathcal{H}}(0, mr) \mapsto \vec{v}(\vec{v}_0) \in Y$ is a C^1-function. In the remaining part of the proof, we set $m = 2$.

Step 2 : We begin by showing that for every $\vec{v}_0 \in \mathcal{H}$, $\vec{v}(t) = \bar{S}_\alpha(t)\vec{v}_0$ exists on $[0, +\infty)$. Let first $\vec{v}_0 \in \mathcal{H}$ satisfying $\|\vec{v}_0\|_{\mathcal{H}} \leqslant 2r$, then, by Step 1, $\vec{v}(t)$ stays in the ball $B_{\mathcal{H}}(0, M_0(2r))$ for $0 \leqslant t \leqslant \tau_0$. Let next $\vec{v}_0 \in \mathcal{H}$ be such that $\|\vec{v}_0\|_{\mathcal{H}} \geqslant 2r$ and let $\vec{v}(t) = \bar{S}_\alpha(t)\vec{v}_0$ be the mild local solution of (4.7). By continuity of this solution, there exists a time $t_1 > 0$ so that $\vec{v}(t) \notin B_{\mathcal{H}}(0, \sqrt{2}r)$, for $0 \leqslant t \leqslant t_1$. We have, for $0 \leqslant t \leqslant t_1$,

$$\vec{v}(t) = \tilde{\Sigma}_\alpha(t)\vec{v}_0 \,, \tag{4.20}$$

and,

$$\|\vec{v}(t)\|_{\mathcal{H}} + \|v\|_{L^{\theta*}((0,t), L^{2\theta*})} \leqslant C(\alpha, \tau_0)\|\vec{v}_0\|_{\mathcal{H}} \,. \tag{4.21}$$

If $t_1 \geqslant \tau_0$, then, in particular, $\vec{v}(t)$ exists on the time interval $[0, \tau_0]$. If $t_1 < \tau_0$, there exists a first time t_2, $0 \leqslant t_2 < t_1$, such that $\vec{v}(t_2)$ enters into the ball $B_{\mathcal{H}}(0, 2r)$ and then, according to Step 1, for $t_2 \leqslant t \leqslant t_2 + \tau_0$, $\vec{v}(t)$ exists, stays in the ball $B_{\mathcal{H}}(0, M_0(2r))$ and satisfies the estimates given in Step 1. We thus have proved that, for every $\vec{v}_0 \in \mathcal{H}$, $\vec{v}(t)$ exists on the time interval $[0, \tau_0]$. Consequently, for every $\vec{v}_0 \in \mathcal{H}$, $\bar{S}_\alpha(t)\vec{v}_0$ exists on $[0, +\infty)$. Likewise, one shows that $\bar{S}_\alpha(t)\vec{v}_0$ exists on $(-\infty, 0])$. Arguing as in the proof of Theorem 2.3, one shows the continuity properties of $\bar{S}_\alpha(t)\vec{v}_0$ with respect to $(t, \vec{v}_0)$ and the fact that, for any $t \in \mathbb{R}$, $\vec{v}_0 \in \mathcal{H} \mapsto \bar{S}_\alpha(t)\vec{v}_0 \in \mathcal{H}$ is a C^1-map.

We are now able to prove that $\bar{S}_\alpha(t)$ satisfies the assumptions $(HA.3)$, $(HA.5.2)$, and $(HA.5.3)$. We first prove the last part of assumption $(HA.3)$, namely that $\bar{S}_\alpha(t)$ is Lipschitz continuous, with a Lipschitz constant which is uniform in $0 \leqslant t \leqslant \tau_0$. The idea is that it is true if $\vec{v}_{0,1}$ and $\vec{v}_{0,2}$ belong to $B_{\mathcal{H}}(0, 2r)$ by (4.19). If $\vec{v}_{0,2} \in B_{\mathcal{H}}(0, 2r)$ and $\vec{v}_{0,1} \notin B_{\mathcal{H}}(0, 2r)$, we estimate the difference up to the first time $t_1 \leqslant \tau_0$ when $\vec{v}_1(t)$ enters the ball $B_{\mathcal{H}}(0, 2r)$, and afterwards, we apply the estimate proved in the first case up to time τ_0. Finally, if both initial data are outside $B_{\mathcal{H}}(0, 2r)$, we apply the linear estimates up to the first time when one solution enters $B_{\mathcal{H}}(0,2r)$ and afterwards, we apply the estimate of the second case. As a consequence, to conclude, it remains to show that, if $\|\vec{v}_{0,1}\|_{\mathcal{H}} \leqslant 2r$ and $\|\vec{v}_{0,2}\|_{\mathcal{H}} \geqslant 2r$ so that $\|\vec{v}_2(t)\|_{\mathcal{H}} \geqslant 2r$ for any $t \geqslant 0$, then $\vec{v}_1 - \vec{v}_2$ satisfies the estimate (4.19). Using Proposition 2.4, the inequalities (4.10), (4.11), and (4.15), we obtain, for $0 \leqslant t \leqslant \tau_0$,

$$\begin{aligned} &\|\vec{v}_1 - \vec{v}_2\|_Y \\ &\leqslant C(\alpha,\tau_0)\big[\|\vec{v}_{0,1} - \vec{v}_{0,2}\|_{\mathcal{H}} + \int_0^{\tau_0} \|h(\vec{v}_1(s))ds\big] \\ &\leqslant C(\alpha,\tau_0)\big[\|\vec{v}_{0,1} - \vec{v}_{0,2}\|_{\mathcal{H}} + \int_0^{\tau_0} \|g_0(v_1)\big(\chi\big(\frac{\|\vec{v}_1\|^2_{\mathcal{H}}}{r^2}\big) - \chi\big(\frac{\|\vec{v}_2\|^2_{\mathcal{H}}}{r^2}\big)\big)\|_{L^2} ds\big] \\ &\leqslant C(\alpha,\tau_0)\|\vec{v}_{0,1} - \vec{v}_{0,2}\|_{\mathcal{H}} + B_3 , \end{aligned} \tag{4.22}$$

where B_3 had already been defined and used in (4.11). As before, the inequality (4.14) holds. Therefore, we deduce from the estimates (4.22), (4.15) and the condition (4.16) that, for $0 \leqslant t \leqslant \tau_0$,

$$\|\vec{v}_1 - \vec{v}_2\|_Y \leqslant C(\alpha,\tau_0)\|\vec{v}_{0,1} - \vec{v}_{0,2}\|_{\mathcal{H}} + \frac{1}{4}\|\vec{v}_1 - \vec{v}_2\|_Y . \tag{4.23}$$

And thus the inequality (4.19) holds. From all the above results, one infers that $\bar{S}_\alpha(t)$ is Lipschitz continuous and that

$$\sup_{0\leqslant t\leqslant\tau_0} \text{Lip } (\bar{S}_\alpha(t)) = D \leqslant \frac{16}{9} C^3(\alpha,\tau_0) . \tag{4.24}$$

Likewise, one shows that this estimate also holds for $-\tau_0, \leqslant t \leqslant 0$. Thus, Hypothesis $(HA.3)$ is satisfied.

We next show that the hypotheses $(HA.5.2)$ and $(HA.5.3)$ hold. To this end, we set

$$\begin{aligned} \bar{S}_\alpha(\tau_0) &= \tilde{\Sigma}_\alpha(\tau_0) + R(\tau_0) \equiv L(\tau_0) + R(\tau_0) \\ \bar{S}_\alpha(-\tau_0) &= \tilde{\Sigma}_\alpha(-\tau_0) + \tilde{R}(\tau_0) \equiv L(\tau_0)^{-1} + \tilde{R}(\tau_0) . \end{aligned} \tag{4.25}$$

Let $\vec{v}_0 \in \mathcal{H}$ and $\vec{v}(t) = \bar{S}_\alpha(t)\vec{v}_0$; then, $R(\tau_0)$ writes

$$R(\tau_0) = \int_0^{\tau_0} \tilde{\Sigma}_\alpha(t-s)(0, h(v(s)))^t ds . \tag{4.26}$$

To prove that the conditions (A.23), (A.24), and (A.29) hold, we will show that $\text{Lip}(R(\tau_0))$ and $\text{Lip}(\tilde{R}(\tau_0))$ go to zero as r_0 goes to zero (we will only show it for $R(\tau_0)$, since the proof is similar for $\tilde{R}(\tau_0)$). To show this property, we are going back to the three cases considered above. If $\vec{v}_{0,1}$ and $\vec{v}_{0,2}$ belong to $B_{\mathcal{H}}(0,2r)$, then the estimates (4.11) to (4.19) imply that

$$\|R(\tau_0)\vec{v}_{0,1} - R(\tau_0)\vec{v}_{0,2}\|_Y \leqslant \frac{4}{3} K(r_0,\tau_0) C(\alpha,\tau_0)\|\vec{v}_{0,1} - \vec{v}_{0,2}\|_{\mathcal{H}} . \tag{4.27}$$

The estimate (4.22) shows that the same property (4.27) holds if $\vec{v}_{0,1}$ belongs to $B_{\mathcal{H}}(0,2r)$ and $\vec{v}_{0,2}$ is so that $\|\vec{v}_2(t)\|_{\mathcal{H}} \geqslant 2r$ for any $0 \leqslant t \leqslant \tau_0$. Finally, we remark that if $\vec{v}_i(t) \notin B_{\mathcal{H}}(0,2r)$,

$i = 1, 2$, for $0 \leqslant t \leqslant \tau_0$, then $R(\tau_0)\vec{v}_{0,1} - R(\tau_0)\vec{v}_{0,2} = 0$. Combining all the above cases and using the estimate (4.24), we finally obtain that, in every case,

$$\|R(\tau_0)\vec{v}_{0,1} - R(\tau_0)\vec{v}_{0,2}\|_Y \leqslant \frac{16}{9}K(r_0, \tau_0)C^3(\alpha, \tau_0)\|\vec{v}_{0,1} - \vec{v}_{0,2}\|_{\mathcal{H}} . \tag{4.28}$$

Since $K(r_0, \tau_0)$ goes to zero as r_0 goes to zero, $\mathrm{Lip}(R(\tau_0))$ goes to zero as r_0 goes to zero and the condition (A.23) is satisfied provided r_0 is chosen small enough. Likewise the conditions (A.24) and (A.29) hold, provided r_0 is chosen small enough. From now on, we fix $r_0 > 0$ small enough so that these conditions are satisfied and we choose $r = r_0$ in (4.7).

We have seen that, for $r_0 > 0$ small enough, $\bar{S}_\alpha(t)$ satisfies the hypotheses of Theorems A.2 and A.5. We can thus state the following result concerning the invariant manifolds of $\bar{S}_\alpha(t)$. For the notations and definitions of the different invariant manifolds, we refer the reader to Appendix A below. As in the assumption (HA.5.1), we denote by P_i the spectral (continuous) projection associated to the spectral set σ^i and let $\mathcal{H}_{rad,i}$ be the image $\mathcal{H}_{rad,i} = P_i\mathcal{H}_{rad}$, where $i = cu, cs, u, s, c$.

Theorem 4.1. *Let $\alpha > 0$ be fixed. 1) There exists a C^1 globally Lipschitz continuous map $g_{cu} : \mathcal{H}_{rad,cu} \to \mathcal{H}_{rad,s}$ so that the C^1 center unstable manifold $W^{cu}((0,0))$ of $\bar{S}_\alpha(t)$ at $(0,0)$*

$$W^{cu}((0,0)) = \{\vec{v}_{cu} + g_{cu}(\vec{v}_{cu}) \,|\, \vec{v}_{cu} \in \mathcal{H}_{rad,cu}\}$$

satisfies all the properties given in Theorem A.1.

2) There exists a C^1 globally Lipschitz continuous map $g_u : \mathcal{H}_{rad,u} \to \mathcal{H}_{rad,cs}$ so that the C^1 (strongly) unstable manifold $W^u((0,0))$ of $\bar{S}_\alpha(t)$ at $(0,0)$

$$W^u((0,0)) = \{\vec{v}_u + g_u(\vec{v}_u) \,|\, \vec{v}_u \in \mathcal{H}_{rad,u}\}$$

satisfies all the properties described in the statement (2) of Theorem A.5.

3) Moreover, there exists a continuous mapping $\ell : \mathcal{H}_{rad} \times \mathcal{H}_{rad,s} \to \mathcal{H}_{rad,cu}$, such that, for any $\vec{v} \in \mathcal{H}_{rad}$, the manifold $\mathcal{M}_{\vec{v}} = \{\vec{v} + \ell(\vec{v}, \vec{v}_s) \,|\, \vec{v}_s \in \mathcal{H}_{rad,s}\}$ satisfies all the properties in Theorem A.2. In particular, $\{\mathcal{M}_{\vec{\xi}} \,|\, \vec{\xi} \in W^{cu}((0,0))\}$ is a foliation of $\mathcal{H}_{rad}$ over $W^{cu}((0,0))$.

4) In particular, there exist $\tilde{c} > 1$, $0 < \rho_0 < 1$, and, for any $\vec{v}_0 \in \mathcal{H}_{rad}$, a unique element $\vec{\xi}(\vec{v}_0) \in W^{cu}((0,0))$ such that, for $t \geqslant 0$,

$$\|\bar{S}_\alpha(t)\vec{v}_0 - \bar{S}_\alpha(t)\vec{\xi}(\vec{v}_0)\|_{\mathcal{H}} \leqslant \tilde{c}\rho_0^t\|\vec{v}_0 - \vec{\xi}(\vec{v}_0)\|_{\mathcal{H}} . \tag{4.29}$$

Moreover, the map $\vec{v}_0 \in \mathcal{H}_{rad} \mapsto \vec{\xi}(\vec{v}_0) \in W^{cu}((0,0))$ is continuous.

5) There exists a C^1 globally Lipschitz continuous map $g_c : \mathcal{H}_{rad,c} \to \mathcal{H}_{rad,s} \oplus \mathcal{H}_{rad,u}$ with $g_c(0) = 0$, so that the center manifold $W^c(0)$ of $\bar{S}_\alpha(t)$ at $(0,0)$

$$W^c((0,0)) = \{x_c + g_c(x_c) \,|\, x_c \in \mathcal{H}_{rad,c}\} = W^{cu}((0,0)) \cap W^{cs}((0,0))$$

satisfies all the properties given in statement (4) of Theorem A.5.

Let us go back to the "actual" variable $\vec{u} = \vec{v} + (\varphi_0, 0)^t$. We set

$$S^*_\alpha(t)\vec{u}_0 = (\varphi_0, 0)^t + \bar{S}_\alpha(t)(\vec{u}_0 - (\varphi_0, 0)) .$$

Then the invariant manifolds of $S^*_\alpha(t)$ are defined by

$$W^{i*}((\varphi_0, 0)) = (\varphi_0, 0)^t + W^i((0,0)) , i = cu, c, u, s . \tag{4.30}$$

Remark 4.2. *We emphasize that the proof given in Step 1 above shows that if, for example,* $r = r_0$, $m = (8C(\alpha, \tau_0))^{-1}$, *and* $\|\vec{u}_0\|_{\mathcal{H}} \leqslant m r_0$, *then, for* $0 \leqslant t \leqslant \tau_0$,

$$\|\bar{S}_\alpha(t)\vec{u}_0\|_Y \leqslant r_0/2 ,$$

which implies that, for $0 \leqslant t \leqslant \tau_0$, $\bar{S}_\alpha(t)\vec{u}_0 = S_\alpha(t)\vec{u}_0$. *In other terms, if* $\vec{u}_0$ *belongs to the ball* $B_{\mathcal{H}_{rad}}((\varphi_0, 0), r_1)$ *of center* $(\varphi_0, 0)$ *and radius* $r_1 \leqslant (8C(\alpha, \tau_0))^{-1} r_0$, *then* $S^*_\alpha(t)\vec{u}_0 = S_\alpha(t)\vec{u}_0$. *This allows one to define the local invariant manifolds* $W^i_{loc}((\varphi_0, 0))$ *of* $S_\alpha(t)$ *about* $(\varphi_0, 0)$ *as*

$$W^i_{loc}((\varphi_0, 0)) = W^{i*}((\varphi_0, 0)) \cap B_{\mathcal{H}_{rad}}((\varphi_0, 0), r_1) , i = cu, c, u, s . \tag{4.31}$$

Remark 4.3. *1) In the above theorem,* $\mathcal{M}_0$ *coincides with the (strongly) stable manifold* $\tilde{W}^s((0,0))$. *2) If* $\mathrm{Ker}(\mathcal{L}) = \{0\}$, *then the center unstable manifold* $W^{cu}((0,0))$ *coincides with the unstable manifold* $W^u((0,0))$ *of* $(0,0)$, *while* $\mathcal{M}_0$ *coincides with the stable manifold* $W^s((0,0))$.

Remark 4.4. *In the case where* $\alpha = 0$, *we can also apply Theorems A.1 and A.2 below in order to prove the existence of the strong unstable manifold and the existence of a center stable manifold around any equilibrium point of* $(KG)_\alpha$ *as well as the existence of a foliation of* $\mathcal{H}_{rad}$ *over the unstable manifold. This gives an alternative proof to the construction of a center stable manifold, by the Hadamard method in* [40] *(for more details, see* [7]*).*

Appendix A. Global invariant manifolds and foliations by the Lyapunov-Perron method

In this appendix, we recall the basic properties of invariant manifold theory that we applied to the equation $(KG)_\alpha$ in Section 4. We reproduce the theorems of Chen, Hale and Tan about global invariant manifolds and foliations as given in [11]. For classical results on invariant manifolds, we also refer the reader to the books [8], [29], [30], and [41] for example as well as to [2] and to [13].

Let X be a Banach space with norm $\|\cdot\|_X$ and $S(t) : X \to X$ be a non-linear semigroup, satisfying the following hypotheses:

(HA.1) : $S(.). : (t, x) \in [0, +\infty) \times X \mapsto S(t)x \in X$ is continuous and there exists a constant $\tau_0 > 0$ such that,

$$\sup_{0 \leqslant t \leqslant \tau_0} \mathrm{Lip}(S(t)) = D < +\infty .$$

(HA.2): There exists τ, $0 < \tau \leqslant \tau_0$ such that $S(\tau)$ can be decomposed as

$$S(\tau) = L + R ,$$

where $L : X \to X$ is a bounded linear operator and $R : X \to X$ is a global Lipschitz continuous map, satisfying the following properties.

(HA.2.1): There are subspaces X_i, $i = 1, 2$, of X and continuous projections $P_i : X \to X_i$ such that $P_1 + P_2 = I$, $X = X_1 \oplus X_2$, L leaves X_i, $i = 1, 2$, invariant and L commutes with P_i, $i = 1, 2$. The restrictions L_i of L to X_i satisfy the following properties. The map L_1 has a bounded inverse and there exist constants $0 \leqslant \beta_2 < \beta_1$, $C_i \geqslant 1$, $i = 1, 2$, such that, for $k \geqslant 0$,

$$\begin{aligned} \|L_1^{-k} P_1\|_{L(X,X)} &\leqslant C_1 \beta_1^{-k} , \\ \|L_2^{k} P_2\|_{L(X,X)} &\leqslant C_2 \beta_2^{k} . \end{aligned} \tag{A.1}$$

(HA.2.2): The maps L and R satisfy the condition

$$\frac{(\sqrt{C_1} + \sqrt{C_2})^2}{\beta_1 - \beta_2} \mathrm{Lip}(R) < 1 . \tag{A.2}$$

Chen, Hale and Tan considered the following quantity, for $\gamma \in (\beta_2, \beta_1)$,

$$\lambda(\gamma) = \frac{C_1}{\beta_1 - \gamma} + \frac{C_2}{\gamma - \beta_2} . \tag{A.3}$$

A short computation shows that, under the condition (A.2), there exist γ_i, $i = 1, 2$, with $\beta_2 < \gamma_2 < \gamma_1 < \beta_1$ such that,

$$\lambda(\gamma_1)\mathrm{Lip}(R) = \lambda(\gamma_2)\mathrm{Lip}(R) = 1 , \text{ and } \lambda(\gamma)\mathrm{Lip}(R) < 1 , \quad \forall \gamma \in (\gamma_2, \gamma_1) . \tag{A.4}$$

In the trivial case, where $\mathrm{Lip}(R) = 0$, one sets $\gamma_1 = \beta_1$ and $\gamma_2 = \beta_2$.

We are now able to state the first theorem, concerning the existence of an invariant manifold, which is a graph over X_1.

Theorem A.1. *Assume that the hypotheses (HA.1), (HA.2) hold and that $R(0) = 0$. Then there exists a globally Lipschitz map $g : X_1 \to X_2$ with $g(0) = 0$, and*

$$\mathrm{Lip}(g) \leqslant \min_{\gamma_2 \leqslant \gamma \leqslant \gamma_1} \frac{C_1 C_2 \mathrm{Lip}(R) \gamma}{\beta_1 (\gamma - \beta_2)(1 - \lambda(\gamma)\mathrm{Lip}(R))} , \tag{A.5}$$

so that the Lipschitz submanifold

$$G = \{x_1 + g(x_1) \,|\, x_1 \in X_1\}$$

satisfies the following properties:

(i): *(Invariance) The restriction to G of the semi-flow $S(t)$, $t \geqslant 0$, can be extended to a Lipschitz continuous flow on G. In particular, $S(t)G = G$, for any $t \geqslant 0$, and for any $\xi \in G$, there exists a unique negative semi-orbit $u(t) \in G$ of $S(.)$, $t \leqslant 0$, so that $u(0) = \xi$.*

(ii): *(Lyapunov exponent) If a negative semi-orbit $u(t)$, $t \leqslant 0$, of $S(.)$ is contained in G, then,*

$$\limsup_{t \to -\infty} \frac{1}{|t|} \ln |u(t)| \leqslant -\frac{1}{\tau} \ln \gamma_1 . \tag{A.6}$$

Conversely, if a negative semi-orbit $u(t)$, $t \leqslant 0$, of $S(.)$ is contained in X satisfies

$$\limsup_{t \to -\infty} \frac{1}{|t|} \ln |u(t)| < -\frac{1}{\tau} \ln \gamma_2 . \tag{A.7}$$

then, it is contained in G.

(iii): *(Smoothness) If the map $S(\tau) : X \to X$ is of class C^1, then $g : X_1 \to X_2$ is of class C^1, that is, G is a C^1-submanifold of X.*

The second theorem states the existence of a foliation of X over the invariant manifold G.

Theorem A.2. *Assume that the hypotheses (HA.1), (HA.2) hold and that $R(0) = 0$. Then, there exists an invariant foliation of X over G as follows.*

(i): *(Invariance) There exists a continuous mapping $\ell : X \times X_2 \to X_1$ such that, for any $\xi \in G$, $\ell(\xi, P_2\xi) = P_1\xi$ and the manifold $\mathcal{M}_\xi = \{x_2 + \ell(\xi, x_2) \,|\, x_2 \in X_2\}$ passing through ξ satisfies:*

$$S(t)\mathcal{M}_\xi \subset \mathcal{M}_{S(t)\xi}\ , \quad t \geqslant 0\ , \tag{A.8}$$

and

$$\mathcal{M}_\xi = \{y \in X \,|\, \limsup_{t\to\infty} \frac{1}{t} \ln |S(t)y - S(t)\xi| \leqslant \frac{1}{\tau} \ln \gamma_2\}\ . \tag{A.9}$$

Moreover, the map $\ell : X \times X_2 \to X_1$ is uniformly Lipschitz continuous in the X_2 direction.

(ii): *(Completeness) Suppose in addition that*

$$\Big[\min_{\gamma_2\leqslant\gamma\leqslant\gamma_1} \frac{C_1C_2\mathrm{Lip}(R)}{(\beta_1-\gamma)(1-\lambda(\gamma)\mathrm{Lip}(R))}\Big] \cdot \Big[\min_{\gamma_2\leqslant\gamma\leqslant\gamma_1} \frac{C_1C_2\mathrm{Lip}(R)\gamma}{\beta_1(\gamma-\beta_2)(1-\lambda(\gamma)\mathrm{Lip}(R))}\Big] < 1. \tag{A.10}$$

Then, for any $x \in X$, $\mathcal{M}_x \cap G$ consists of a single point. In particular,

$$\mathcal{M}_\xi \cap \mathcal{M}_\eta = \varnothing\ , \quad \forall \xi, \eta \in G\ , \qquad X = \bigcup_{\xi\in G} \mathcal{M}_\xi\ . \tag{A.11}$$

In other terms, $\{\mathcal{M}_\xi \,|\, \xi \in G\}$ is a foliation of X over G. Moreover, the mapping $x \in X \mapsto \xi(x) = \mathcal{M}_x \cap G$ is a continuous map from X into $G \subset X$.

(iii): *(Smoothness) If the map $S(\tau) : X \to X$ is of class C^1, then $\ell : X \times X_2 \to X_1$ is of class C^1 in the X_2 direction. Hence, $\mathcal{M}_\xi$ is a C^1-submanifold of X, for any $\xi \in G$.*

Comments on the proof of Theorems A.1 and A.2:

Theorems A.1 and A.2 are proved in [11] by first showing the corresponding results for the map $S(\tau)$ and at the end coming back to the continuous dynamical system. This means that Theorems A.1 and A.2 still hold for iterates of maps $S(\tau)$. It suffices to replace $t \in \mathbb{R}$ by $n\tau$, $n \in \mathbb{N}$. Theorems A.1 and A.2 are proved in [11] by using the Lyapunov-Perron method.

The property that the mapping $x \in X \mapsto \xi(x) = \mathcal{M}_x \cap G$ is a continuous map from X into $G \subset X$ is not stated in the main Theorem 1.1 of [11]. It is merely a consequence of the proof of [11, Lemma 3.4]. Indeed, given $x \in X$, the intersection points $\xi(x)$ of $\mathcal{M}_x$ with G are the solutions of

$$\xi(x) \equiv y_2 + \ell(x, y_2) = \ell(x, y_2) + g(\ell(x, y_2))\ , \tag{A.12}$$

where $y_2 \in X_2$. This leads to study the fixed points of the map $F_x(y_2) \equiv F(x, y_2) = g(\ell(x, y_2))$, depending on the parameter $x \in X$. One can check that the condition (A.10) implies that $F_x : X_2 \to X_2$ is a strict contraction and therefore has a unique fixed point $y_2(x)$. The continuity property of $y_2(x)$ with respect to $x \in X$ is a direct consequence of the continuity of F with respect to the variable $x \in X$ and of the *uniform contraction principle* (see [12, Theorem 2.2 on Page 25]). It follows that $\xi(x) = y_2(x) + \ell(x, y_2(x)) \in G$ is also continuous with respect to $x \in X$.

Remark A.3. *If the equilibrium point* 0 *of* $S(.)$ *is hyperbolic, then we may choose* $\beta_2 < 1 < \beta_1$. *In this case,* G *is the classical unstable manifold* $W^u(0)$ *and* M_ξ, $\xi \in G$, *defines an invariant foliation of* X *over* $W^u(0)$, *with* $\mathcal{M}_0$ *being the classical stable manifold* $W^s(0)$. *And the solutions on* $\mathcal{M}_0$ *decay exponentially to* 0, *as* t *goes to* $+\infty$. *If* 0 *is a non-hyperbolic equilibrium point and* $\beta_2 < \beta_1 < 1$ *with* β_1 *close to* 1, *then Theorems A.1 and A.2 allow for the construction of the center-unstable manifold* $G = W^{cu}(0)$ *of* 0 *and a foliation over it. If* 0 *is a non-hyperbolic equilibrium point and* $1 < \beta_2 < \beta_1$ *with* β_2 *close to* 1, *then Theorems A.1 and A.2 give the strongly unstable manifold* $G = W^{su}(0)$ *of* 0 *and a foliation over it. If* $\gamma_2 < 1$, *the existence of the foliation implies that each positive semi-orbit of* $S(t)$ *converges exponentially to an orbit of* G *and is synchronized with this orbit in time. This property is often called "attraction" of* G *with asymptotic phase". We emphasize that the construction in Theorems A.1 and A.2 is also interesting in the case where* $S_\alpha(.)$ *depends on a parameter* α *and* $\beta_2(\alpha) < 1 < \beta_1(\alpha)$ *with* $\beta_2(\alpha)$ *arbitrarily close to* 1 *as* α *converges say to* $\alpha_0 = 0$.

Mutatis mutandis, repeating the arguments of the proofs of Theorems A.1 and A.2, one can also show the existence of a Lipschitz manifold $\tilde{G} = \{x_2 + \tilde{g}(x_2) \,|\, x_2 \in X_2\}$ where $\tilde{g} : X_2 \to X_1$ is a globally Lipschitz map with $\tilde{g}(0) = 0$, such that $\tilde{G}$ is invariant and such that, if a semi-orbit $u(t)$, $t \geqslant 0$, of $S(.)$ is contained in $\tilde{G}$, then,

$$\limsup_{t\to\infty} \frac{1}{t} \ln |u(t)| \leqslant \frac{1}{\tau} \ln \tilde{\gamma}_2^{-1} , \tag{A.13}$$

where $\beta_2 < \tilde{\gamma}_2^{-1} < \tilde{\gamma}_1^{-1} < \beta_1$ is made more precise below, and also the existence of a foliation $\tilde{\mathcal{M}}_\xi$ (in reverse time) of X over $\tilde{G}$.

If $S(t)$ is a non-linear group, these properties can be proved by reversing the time in Theorems A.1 and A.2. In Section 3, the existence of a center manifold played an important role. We can derive this existence by defining the center manifold as the intersection of the center stable and center unstable manifolds. The center stable manifold is constructed like the Lipschitz manifold $\tilde{G} = \{x_2 + \tilde{g}(x_2) \,|\, x_2 \in X_2\}$ described above. Since throughout the paper we are only dealing with groups, we will quickly show the existence of $\tilde{G}$ by reversing the time in Theorem A.1. The constants appearing in the proof below are maybe not optimal, but we are not looking here for optimality.

In addition to the hypothesis (HA.2), we assume now that

(HA.3) : $S(.).:(t,x) \in (-\infty,+\infty) \times X \mapsto S(t)x \in X$ is continuous and there exists a constant $\tau_0 > 0$ such that,

$$\sup_{-\tau_0 \leqslant t \leqslant \tau_0} \mathrm{Lip}(S(t)) = D < +\infty .$$

(HA.4): $S(-\tau)$ can be decomposed as

$$S(-\tau) = L^{-1} + \tilde{R} ,$$

where τ and $L : X \to X$ have been introduced in the hypothesis (HA.2) and where $\tilde{R} : X \to X$ is a global Lipschitz continuous map, satisfying the following property:

$$\frac{(\sqrt{C_1} + \sqrt{C_2})^2}{\beta_1 - \beta_2} \beta_1 \beta_2 \mathrm{Lip}(\tilde{R}) < 1 . \tag{A.14}$$

We remark that the linear map L^{-1} satisfies the hypothesis (HA.2.1) with P_1 (resp. P_2) replaced by P_2 (resp. P_1), C_1 (resp. C_2) replaced by C_2 (resp. C_1), and β_1 (resp. β_2) replaced by β_2^{-1} (resp. β_1^{-1}). Indeed, we have

$$\begin{aligned} \|(L^{-1})^{-k} P_2\|_{L(X,X)} &\leqslant C_2 (\beta_2^{-1})^{-k} , \\ \|(L^{-1})^{k} P_1\|_{L(X,X)} &\leqslant C_1 (\beta_1^{-1})^{k} . \end{aligned} \tag{A.15}$$

We next set

$$\tilde{\lambda}(\tilde{\gamma}) = \frac{C_2}{\beta_2^{-1} - \tilde{\gamma}} + \frac{C_1}{\tilde{\gamma} - \beta_1^{-1}} . \tag{A.16}$$

As above, a short computation shows that, under the condition (A.14), there exist $\tilde{\gamma}_i$, $i = 1, 2$, with $\beta_1^{-1} < \tilde{\gamma}_1 < \tilde{\gamma}_2 < \beta_2^{-1}$ such that,

$$\tilde{\lambda}(\tilde{\gamma}_1)\mathrm{Lip}(\tilde{R}) = \tilde{\lambda}(\tilde{\gamma}_2)\mathrm{Lip}(\tilde{R}) = 1 , \text{ and } \tilde{\lambda}(\tilde{\gamma}_1)\mathrm{Lip}(\tilde{R}) < 1 , \quad \forall \tilde{\gamma} \in (\tilde{\gamma}_1, \tilde{\gamma}_2) . \tag{A.17}$$

We may now apply Theorem A.1 to the nonlinear semigroup $\tilde{S}(t) = S(-t)$ and we obtain the following result.

Theorem A.4. *Assume that the hypotheses (HA.2), (HA.3), and (HA.4) hold and that $R(0) = \tilde{R}(0) = 0$. Then there exists a globally Lipschitz map $\tilde{g} : X_2 \to X_1$ with $\tilde{g}(0) = 0$ and*

$$\mathrm{Lip}(\tilde{g}) \leqslant \min_{\tilde{\gamma}_1 \leqslant \tilde{\gamma} \leqslant \tilde{\gamma}_2} \frac{C_1 C_2 \mathrm{Lip}(\tilde{R}) \beta_1 \beta_2}{(\beta_1 - 1/\tilde{\gamma})(1 - \tilde{\lambda}(\tilde{\gamma})\mathrm{Lip}(\tilde{R}))} , \tag{A.18}$$

so that the Lipschitz submanifold

$$\tilde{G} = \{x_2 + \tilde{g}(x_2) \mid x_2 \in X_2\}$$

satisfies the following properties:

(i): *(Invariance) $\tilde{G}$ is invariant under $S(t)$, i.e., $S(t)\tilde{G} = \tilde{G}$, for any $t \geqslant 0$.*

(ii): *(Lyapunov exponent) If a positive semi-orbit $u(t)$, $t \geqslant 0$, of $S(.)$ is contained in $\tilde{G}$, then,*

$$\limsup_{t\to\infty} \frac{1}{t} \ln |u(t)| \leqslant \frac{1}{\tau} \ln \frac{1}{\tilde{\gamma}_2} .$$

Conversely, if a positive semi-orbit $u(t)$, $t \geqslant 0$, of $S(.)$ in X, satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \ln |u(t)| < \frac{1}{\tau} \ln \frac{1}{\tilde{\gamma}_1} . \tag{A.19}$$

then, it is contained in $\tilde{G}$.

(iii): *(Smoothness) If the map $S(\tau) : X \to X$ is of class C^1, then $\tilde{g} : X_2 \to X_1$ is of class C^1, that is, $\tilde{G}$ is a C^1-submanifold of X.*

We next consider the classical case where $S(.)$ is a non-linear group satisfying the assumption (HA.3) as well as

(HA.5): The point 0 is an equilibrium point of $S(.)$. And there exists τ, $0 < \tau \leqslant \tau_0$ such that $S(\tau)$ and $S(-\tau)$ can be decomposed as follows

$$S(\tau) = L + R , \quad S(-\tau) = L^{-1} + \tilde{R} ,$$

where $L : X \to X$ is a bounded linear operator, $R : X \to X$ and $\tilde{R} : X \to X$ are global Lipschitz continuous maps, satisfying the following properties.

(HA.5.1): The spectrum $\sigma(L)$ of L can be written as

$$\sigma(L) = \sigma^s \cup \sigma^c \cup \sigma^u ,$$

where σ^s, σ^c and σ^u are closed subsets of $\{\lambda \in \mathbb{C} \,|\, |\lambda| < 1\}$, $\{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$, and $\{\lambda \in \mathbb{C} \,|\, |\lambda| > 1\}$.

There exists $\eta > 0$ such that

$$\sigma^s \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| < 1 - \eta\} , \quad \sigma^u \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| > 1 + \eta\} \tag{A.20}$$

We set: $\sigma^{cu} = \sigma^c \cup \sigma^u$ and $\sigma^{cs} = \sigma^c \cup \sigma^s$. Let P_i be the spectral (continuous) projector associated to the spectral set σ^i and let X_i be the image $X_i = P_i X$, where $i = cu, cs, u, s, c$. We have that $P_{cu} + P_s = I = P_{cs} + P_u$. The linear map L leaves X_i invariant and commutes with P_i, $i = cu, cs, u, s, c$. Now we choose $0 < \varepsilon < \eta/2$. The restrictions L_i of L to X_i satisfy the following properties. There exist constants $C_1 \geqslant 1$ and $C_2 \geqslant 1$ such that, for $k \geqslant 0$,

$$\begin{aligned} \|L_{cu}^{-k} P_{cu}\|_{L(X,X)} &\leqslant C_1 (1-\varepsilon)^{-k} , \\ \|L_s^k P_s\|_{L(X,X)} &\leqslant C_2 (1-\eta)^k , \end{aligned} \tag{A.21}$$

and

$$\begin{aligned} \|(L_{cs}^{-1})^{-k} P_{cs}\|_{L(X,X)} &\leqslant C_2 ((1+\varepsilon)^{-1})^{-k} , \\ \|(L_u^{-1})^k P_u\|_{L(X,X)} &\leqslant C_1 ((1+\eta)^{-1})^k . \end{aligned} \tag{A.22}$$

We further assume that the maps R and $\tilde{R}$ satisfy the conditions.

(HA.5.2): The following inequalities hold

$$\frac{(\sqrt{C_1}+\sqrt{C_2})^2}{\eta-\varepsilon}\mathrm{Lip}(R) < 1\,, \tag{A.23}$$

and

$$\frac{(\sqrt{C_1}+\sqrt{C_2})^2}{\eta-\varepsilon}(1+\varepsilon)(1+\eta)\mathrm{Lip}(\tilde{R}) < 1\,. \tag{A.24}$$

(HA.5.3): We define the function $\lambda(\gamma)$ as in (A.3), that is,

$$\lambda(\gamma) = \frac{C_1}{1-\varepsilon-\gamma} + \frac{C_2}{\gamma-1+\eta}\,, \tag{A.25}$$

and the quantities γ_i, $i=1,2$, with $1-\eta<\gamma_2<\gamma_1<1-\varepsilon$, satisfying (A.4). Likewise, we define the function $\tilde{\lambda}(\tilde{\gamma})$ as in (A.16), that is,

$$\tilde{\lambda}(\tilde{\gamma}) = \frac{C_2}{(1+\varepsilon)^{-1}-\tilde{\gamma}} + \frac{C_1}{\tilde{\gamma}-(1+\eta)^{-1}}\,. \tag{A.26}$$

and the quantities $\tilde{\gamma}_i$, $i=1,2$, with $(1+\eta)^{-1}<\tilde{\gamma}_1<\tilde{\gamma}_2<(1+\varepsilon)^{-1}$, satisfying (A.17). We next introduce the function $\lambda^*(\gamma^*)$:

$$\lambda^*(\gamma^*) = \frac{C_1}{1+\eta-\gamma^*} + \frac{C_2}{\gamma^*-1-\varepsilon}\,, \tag{A.27}$$

and the quantities γ_i^*, $i=1,2$, with $1+\varepsilon<\gamma_2^*<\gamma_1^*<1+\eta$, satisfying

$$\lambda^*(\gamma_1^*)\mathrm{Lip}(R) = \lambda(\gamma_2^*)\mathrm{Lip}(R) = 1\,,\ \text{and}\ \lambda^*(\gamma)\mathrm{Lip}(R) < 1\,,\quad \forall\gamma^*\in(\gamma_2^*,\gamma_1^*)\,. \tag{A.28}$$

We finally require that the following inequality holds:

$$\min_{\gamma_2\leqslant\gamma\leqslant\gamma_1}\frac{C_1C_2\mathrm{Lip}(R)\gamma}{(1-\varepsilon)(\gamma-1+\eta)(1-\lambda(\gamma)\mathrm{Lip}(R))}\times\min_{\tilde{\gamma}_1\leqslant\tilde{\gamma}\leqslant\tilde{\gamma}_2}\frac{C_1C_2\mathrm{Lip}(\tilde{R})(1+\varepsilon)(1+\eta)}{(1+\eta-1/\tilde{\gamma})(1-\tilde{\lambda}(\tilde{\gamma})\mathrm{Lip}(\tilde{R}))} < 1. \tag{A.29}$$

Applying Theorems A.1 and A.4 to the above flow $S(.)$, we obtain the following properties, which are used in Sections 3 and 4.

Theorem A.5. *Assume that the hypotheses (HA.3) and (HA.5) are satisfied. Then, the following properties hold.*

(1) *There exists a globally Lipschitz map $g_{cu}: X_{cu}\to X_s$ with $g_{cu}(0)=0$, so that the Lipschitz center unstable manifold $W^{cu}(0)$*

$$W^{cu}(0) = \{x_c+x_u+g_{cu}(x_c+x_u)\,|\,x_c\in X_c, x_u\in X_u\}$$

satisfies all the properties described in Theorem A.1. In particular, if $S(\tau)$ is of class C^1, then $g_{cu}: X_{cu}\to X_s$ is of class C^1.

(2) *There exists a globally Lipschitz map $g_u : X_u \to X_{cs}$ with $g_u(0) = 0$, so that the Lipschitz unstable (also called strongly unstable) manifold $W^u(0)$*

$$W^u(0) = \{x_u + g_u(x_u) \mid x_u \in X_u\}$$

satisfies all the properties described in Theorem A.1 with γ replaced by γ^* and γ_i replaced by γ_i^*, $i = 1, 2$. In particular, if $S(\tau)$ is of class C^1, then $g_u : X_u \to X_{cs}$ is of class C^1. And, if a negative semi-orbit $u(t)$, $t \leqslant 0$, of $S(.)$ is contained in $W^u(0)$, then,

$$\limsup_{t\to-\infty} \frac{1}{|t|} \ln |u(t)| \leqslant -\frac{1}{\tau} \ln \gamma_1^* \,. \tag{A.30}$$

(3) *There exists a globally Lipschitz map $g_{cs} : X_{cs} \to X_u$ with $g_{cs}(0) = 0$ so that the Lipschitz center stable manifold $W^{cs}(0)$*

$$W^{cs}(0) = \{x_c + x_s + g_{cs}(x_c + x_s) \mid x_c \in X_c, x_s \in X_s\}$$

satisfies all the properties described in Theorem A.4. In particular, if $S(\tau)$ is of class C^1, then $g_{cs} : X_{cs} \to X_u$ is of class C^1.

(4) *There exists a globally Lipschitz map $g_c : X_c \to X_s \oplus X_u$ with $g_c(0) = 0$, so that the Lipschitz center manifold $W^c(0)$*

$$W^c(0) = \{x_c + g_c(x_c) \mid x_c \in X_c\} = W^{cu}(0) \cap W^{cs}(0)$$

satisfies the following properties:
(i) $W^c(0)$ is invariant under $S(t)$, i.e., $S(t)W^c(0) = W^c(0)$, for any $t \geqslant 0$. (ii) The properties (ii) of Theorem A.1 and the properties (ii) of Theorem A.4 hold. In particular, if a trajectory $u(t)$, $t \in (-\infty, \infty)$ of $S(.)$ is contained in $W^c(0)$, then

$$\limsup_{t\to-\infty} \frac{1}{|t|} \ln |u(t)| \leqslant -\frac{1}{\tau} \ln \gamma_1 \,, \quad \limsup_{t\to\infty} \frac{1}{t} \ln |u(t)| \leqslant \frac{1}{\tau} \ln \frac{1}{\tilde{\gamma}_2} \,. \tag{A.31}$$

Moreover, $W^c(0)$ contains all the equilibria of $S(t)$. (iii) If the map $S(\tau) : X \to X$ is of class C^1, then $g_c : X_c \to X_s \oplus X_u$ is of class C^1, that is, $W^c(0)$ is a C^1-submanifold of X.

(5) *If moreover the condition* (A.10) *holds with $\beta_1 = 1 - \varepsilon$ and $\beta_2 = 1 - \eta$, then one has a foliation of X over $W^{cu}(0)$ as defined in Theorem A.2.*

Proof. (1) Statements (1) and (5) are direct consequences of Theorem A.1 and Theorem A.2 respectively, applied to the case where $\beta_1 = 1 - \varepsilon$ and $\beta_2 = 1 - \eta$. (2) Statement (2) is a direct consequence of Theorem A.1, applied to the case where $\beta_1 = 1 + \eta$ and $\beta_2 = 1 + \varepsilon$. (3) Statement (3) is a direct consequence of Theorem A.4, applied to the case where $\beta_2^{-1} = (1 + \varepsilon)^{-1}$ and $\beta_1^{-1} = (1 + \eta)^{-1}$. Let us next prove the statement (4). We are looking for the trajectories $u(t)$, which satisfy both properties of (A.31). These two properties together are satisfied only by the elements in $W^{cu}(0) \cap W^{cs}(0)$. Thus, we are looking for the elements $x = x_c + x_s + x_u$ so that

$$x_c + x_u + g_{cu}(x_c + x_u) = x_c + x_s + g_{cs}(x_c + x_s) = x_c + g_{cu}(x_c + x_u) + g_{cs}(x_c + g_{cu}(x_0 + x_u)) \,, \tag{A.32}$$

or also for the elements $x_u \in X_u$ satisfying

$$x_u = g_{cs}(x_c + g_{cu}(x_c + x_u)) \,. \tag{A.33}$$

In other terms, given $x_c \in X_c$, we are looking for the fixed point of the map $x_u \in X_u \mapsto F(x_c, x_u) = g_{cs}(x_c + g_{cu}(x_c + x_u)) \in X_u$. We notice that the Lipschitz constant of $F(x_c, .)$ satisfies

$$\mathrm{Lip}(F(x_c, .)) \leqslant \mathrm{Lip}(g_{cs}) \times \mathrm{Lip}(g_{cu}) \,.$$

By Theorems A.1 and A.4 and the assumption (A.29), we have, for any $x_0 \in X_0$

(A.34)

$$\mathrm{Lip}(F(x_c, .) \leqslant \min_{\gamma_2 \leqslant \gamma \leqslant \gamma_1} \frac{C_1 C_2 \mathrm{Lip}(R)\gamma}{(1-\varepsilon)(\gamma - 1 + \eta)(1 - \lambda(\gamma)\mathrm{Lip}(R))} \times \min_{\tilde{\gamma}_1 \leqslant \tilde{\gamma} \leqslant \tilde{\gamma}_2} \frac{C_1 C_2 \mathrm{Lip}(\tilde{R})(1+\varepsilon)(1+\eta)}{(1+\eta - 1/\tilde{\gamma})(1 - \tilde{\lambda}(\tilde{\gamma})\mathrm{Lip}(\tilde{R}))} < 1.$$

Therefore, $x_u \in X_u \to F(x_c, x_u) \in X_u$ is a strict contraction, uniformly in x_c. Thus, for any $x_c \in X_c$, there exists a unique fixed point $h(x_c) \in X_u$ of $F(x_c, .)$. And $g_c(x_c)$ is given by

$$g_c(x_c) = x_c + h(x_c) + g_{cu}(x_c + h(x_c)) \,.$$

The regularity of the map g_c is proved by using the regularity of the mappings g_{cu} and g_{cs} and by applying the uniform contraction principle of [12, Theorem 2.2 on Page 25]. □

Remark A.6. *1. If the equilibrium point is hyperbolic (that is, $\sigma^c = \emptyset$), then one can choose $\varepsilon = \eta$ in the hypotheses (HA.5.1) and (HA.5.2). The center unstable manifold $W^{cu}(0)$ and the (strongly) unstable manifold $W^u(0)$ coincide (that is, $g_{cu} = g_u$). And the center manifold $W^c(0)$ reduces to* 0.
2. In the above theorem, we have only stated those properties which are used in this paper. We leave it to the reader to state the existence of the (strongly) stable manifold.

Appendix B. Classical convergence results

In the study of asymptotic behaviour of dynamical systems, one often encounters the following question: knowing that the ω-limit set of a relatively compact trajectory contains an equilibrium point x_0, does this ω-limit set reduce to the point x_0, i.e., does the trajectory converge to x_0? This question is especially interesting in the case of gradient systems (that is, systems which admit a strict Lyapunov functional). In fact, consider a gradient system with a hyperbolic equilibrium x_0. Then x_0 is isolated and the whole trajectory converges to this point x_0. If the equilibrium x_0 is not hyperbolic and the spectrum of the linearized dynamical system around x_0 intersects the unit circle, then x_0 could lie in a continuum of equilibria, which could be contained in the ω-limit set. If x_0 belongs to a normally hyperbolic manifold of equilibria, we can still have convergence to x_0, under additional hypotheses.

In the proof of Theorem 1.2, we use the convergence property to an equilibrium point in order to prove the boundedness of the orbits, which are global in forward time. We recall here the general convergence property in the form proved by Brunovský and Poláčik in [5], who extended earlier convergence results, proved for example by Aulbach [1] in the finite-dimensional frame, or by Hale and Raugel [26], who generalised the convergence property of Aulbach to the infinite-dimensional setting (see also the paper [25] of 1982, and [43] for applications). In the case of the one-dimensional parabolic equation with separate boundary conditions, convergence proofs had been given before in [38] and [47].

Let X be a Banach space and $\Phi : X \to X$ be a continuous map admitting a fixed point y_0. Without loss of generality, we may choose $y_0 = 0$. Brunovský and Poláčik assumed the following hypotheses:

- **(HB.1)** There exists a neighborhood U of 0 in X so that the restriction $\Phi\big|_U : U \to X$ is of class C^1.
- **(HB.2)** The spectrum $\sigma(DF(0))$ can be written as $\sigma(DF(0)) = \sigma^s \cup \sigma^c \cup \sigma^u$, where σ^s, σ^c and σ^u are closed subsets of $\{\lambda \in \mathbb{C} \,|\, |\lambda| < 1\}$, $\{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$, and $\{\lambda \in \mathbb{C} \,|\, |\lambda| > 1\}$.

As in Appendix A, we introduce the spectral projectors P_i of $B = DF(0)$ associated with the spectral sets σ^i, $i = s, c, u$ and the images $X_i = P_i X$. We recall that these spaces are all B-invariant and $X = X_s \oplus X_c \oplus X_u$. We also denote $X_{cu} = X_c \oplus X_u$.

As we have seen in Appendix A, the hypotheses (HB.1) and (HB.2) allow one to construct Lipschitz continuous local center unstable and local center manifolds $W^{cu}_{loc}(0)$, $W^{c}_{loc}(0)$ of Φ at 0 as graphs over X_{cu} and X_c, respectively, and also the local unstable manifold $W^{u}_{loc}(0)$ as a graph over X_u, by extending the map Φ into a global Lipschitz continuous and C^1 mapping $\tilde{\Phi}$, which coincides with Φ on the ball $B_X(0, \delta)$ of center 0 and radius $\delta > 0$ (δ being small enough), and by applying Theorems A.1 and A.5. These local invariant manifolds are defined in the following way

$$W^{i}_{loc}(0) = \tilde{W}^{i}_{\delta}(0)\,, \quad i = cu, c, u\,, \tag{B.1}$$

where $\tilde{W}^{cu}_{\delta}(0)$, $\tilde{W}^{c}_{\delta}(0)$ and $\tilde{W}^{u}_{\delta}(0)$ are the global center stable, center and unstable manifolds of $\tilde{\Phi}$ around 0.

On the other hand, Theorem A.2 in Appendix A on the invariant foliations implies that $W^{cu}_{loc}(0)$ is exponentially attractive in X with asymptotic phase (see Appendix A for more details). Likewise, one can show that $W^{c}_{loc}(0)$ is exponentially attractive in backward time in $W^{cu}_{loc}(0)$ with asymptotic phase. These asymptotic phase properties are among the key arguments in the proof of the main convergence theorem B.2 below.

Remark B.1. *Actually, the hypothesis (HB.1) can be replaced by the weaker hypothesis:*
(HB.1bis) *There exists a neighborhood U of 0 in X so that the restriction $\Phi\big|_U : U \to X$ is Lipschitz continuous and differentiable at every fixed point contained in U.*

Before stating the main convergence result of [5], we introduce the concept of stability restricted to $W^{c}_{loc}(0)$. We say that 0 is stable for the map $\Phi\big|_{W^{c}_{loc}(0)}$, if, for any $\varepsilon > 0$, there exists $\eta > 0$ such that, for any $y \in W^{c}_{loc}(0)$ with $\|y\|_X \leqslant \eta$, we have

$$\|\Phi^n(y)\|_X \leqslant \varepsilon\,, \quad \forall\, n = 0, 1, 2, \dots\,. \tag{B.2}$$

As pointed out in [5], this stability is independent of the choice of the local center manifold $W^{c}_{loc}(0)$. The independence of this stability on the choice of the local center manifold can be proved by using foliations as in the paper of [6], who actually showed that the flows on different local center manifolds are conjugated (under some more restrictive hypotheses, which can be easily removed). As also remarked in [5], the fact that the stability is independent of the choice of the local centre manifold, is not needed in the proof of Theorem B.2 below.

Theorem B.2. *Assume that the hypotheses (HB.1) (or (HB.1bis)) and (HB.2) hold. Let* $x_0 \in X$ *be such that the fixed point* 0 *belongs to the* ω*-limit set* $\omega(x_0)$ *of* x_0*. Assume that either* X_{cu} *is finite-dimensional or that the trajectory* $\Phi^n(x_0)$*,* $n = 1, 2, \cdots$*, of* x_0 *is relatively compact. Assume, moreover, that* 0 *is stable for the map* $\Phi|_{W^c_{loc}(0)}$*, where* $W^c_{loc}(0)$ *is a local center manifold of* 0*. Then either* $\Phi^n(x_0)$ *converges to* 0 *as* $n \to \infty$*, or* $\omega(x_0)$ *contains a point of the local unstable manifold* $W^u_{loc}(0)$ *of* 0*, distinct from* 0*.*

Theorem B.2 generalises the above mentioned convergence result of [26] in two ways. Firstly, the hypotheses do not require that $\omega(x_0)$ consists only of fixed points. Secondly, it does not require that the trajectory $\Phi^n(x_0)$, $n = 1, 2, \cdots$, of x_0 be relatively compact. But, of course, it requires the additional stability property defined above.

In [5], Brunovský and Poláčik have proved the following lemma (see [5, Lemma 1]) and have obtained Theorem B.2 as a direct consequence of it. We emphasize that Lemma B.3 is really a local result anf that Lemma B.3 will hold for any mapping $\Phi^* : y \in \mathcal{U} \mapsto \Phi^* y \in X$ coinciding with Φ in $\mathcal{U}$. In particular, Φ^* need not be well defined outside $\mathcal{U}$, which is the case in our application in Section 3.

Lemma B.3. *Assume that the hypotheses (HB.1) (or (HB.1bis)) and (HB.2) hold, that* $\delta > 0$ *is small enough so that* $B_X(0, \delta) \subset \mathcal{U}$ *and that* 0 *is stable for the map* $\Phi|_{W^c_{loc}(0)}$*. Let* $x_k \in X$ *and* $p_k \in \mathbb{N}$ *be sequences satisfying the following properties:*

(1) $x_k \to 0$ *as* $k \to +\infty$.
(2) $\Phi^j(x_k) \in B_X(0, \beta)$ *for* $j = 0, 1, 2, \ldots, p_k$ *and* $\Phi^{p_k+1}(x_k) \notin B_X(0, \beta)$*, where* $0 < \beta < \delta$.
(3) *In the case, where* $\dim X_{cu} = \infty$*, the set* $\{\Phi^j_*(x_k) \mid k \in \mathbb{N}, j = 0, \ldots, p_k\}$ *is relatively compact.*

Then $\Phi^{p_k}(x_k)$ *contains a subsequence converging to an element of* $W^u_{loc}(0) \backslash \{0\}$*.*

As an easy consequence of Theorem B.2, Brunovský and Poláčik have obtained the following more classical theorem.

Theorem B.4. *Assume that the hypotheses (HB.1) (or (HB.1bis)) and (HB.2) hold. Let* x_0 *be a point in* X *such that the fixed point* 0 *belongs to the* ω*-limit set* $\omega(x_0)$ *of* x_0 *and such that* $\omega(x_0)$ *is contained in the set* $\mathrm{Fix}(\Phi)$ *of fixed points of* Φ*. Assume that either* X_{cu} *is finite-dimensional or that the trajectory* $\Phi^n(x_0)$*,* $n = 1, 2, \cdots$*, of* x_0 *is relatively compact. Assume moreover that one of the following two properties holds:*

(1) $\dim X^c = 1$ *and the trajectory* $\Phi^n(x_0)$*,* $n = 1, 2, \cdots$*, of* x_0 *is relatively compact.*
(2) $\dim X^c = m < \infty$ *and there is a submanifold* $M \subset X$ *with* $\dim M = m$ *such that* $0 \in M \subset \mathrm{Fix}(\Phi)$.

Then $\omega(x_0) = \{0\}$*.*

Proof. We give the proof, because it is short.

First assume that (2) holds. Then, if $\delta > 0$ is small enough, the sets M and $W^u_{loc}(0)$ coincide since $M \subset W^u_{loc}(0)$, and they both have the same dimension m. The assumption $M \subset \mathrm{Fix}(\Phi)$ thus implies that 0 is stable for the map $\Phi|_{W^c_{loc}(0)}$. Since $W^u_{loc}(0)\backslash\{0\}$ contains no fixed point if $\delta > 0$ is small enough and since $\omega(x_0) \in \mathrm{Fix}(\Phi)$, Theorem B.2 implies that $\omega(x_0) = \{0\}$.

In the case (1), we first remark that, since the trajectory $\Phi^n(x_0)$, $n = 1, 2, \cdots$, of x_0 is relatively compact and since $\omega(x_0)$ consists only of fixed points, the omega-limit set $\omega(x_0)$ is connected (see for example [26, Lemma 2.7]). If $\omega(x_0)$ contains more than one fixed point, then all fixed points near 0 are contained in $W^c_{loc}(0)$ and thus 0 belongs to a curve of fixed points. If 0 belongs to the relative interior of this curve, one applies the case (2), which leads to a contradiction. If 0 does not belong to the relative interior of this curve, we consider a fixed point y^* near 0, contained in the relative interior of this curve of fixed points and in $\omega(x_0)$. Replacing Φ by $\Phi(y^* + x)$, we are now back to the case (2). Applying the case (2), we obtain that $\omega(x_0) = y^*$, which also leads to a contradiction. $\square$

In Section 3.2 we encountered the case of an element $u_0 \in \mathcal{H}_{rad}$ for which we did not know that the forward trajectory $\{S_\alpha(t)\vec{u}_0 \mid t \geqslant 0\}$ is bounded. We used there the property that $W^{cu}_{loc}(0)$ is exponentially attractive in X with asymptotic phase together with the fact that $\dim X^c = 1$, to obtain that $S_\alpha(t)$ has the stability property (3.40) (or (B.2)). Then, we applied Theorem B.2 to the time τ-map $\Phi = S_\alpha(\tau)$, where $\tau > 0$ is small enough, in order to obtain the convergence result. Since these arguments did not use the particular properties of $S_\alpha(t)$, this allows us to state the following general result.

Corollary B.5. *Assume that the map $\Phi = S(\tau)$ where $S(t) : \mathbb{R}\times X \to X$ is a continuous dynamical system and that $\tau > 0$ is a small enough positive time, so that $\Phi = S(\tau)$ satisfies the hypotheses (HB.1) (or (HB.1bis)) and (HB.2). Let x_0 be a point in X such that the equilibrium point* 0 *belongs to the ω-limit set $\omega(x_0)$ of x_0 and such that $\omega(x_0)$ is contained in the set of equilibrium points of $S(t)$. Assume that either X_{cu} is finite-dimensional or that the trajectory $\Phi^n(x_0)$, $n = 1, 2, \cdots$, of x_0 is relatively compact. Assume moreover that* $\dim X^c = 1$. *Then $\omega(x_0) = \{0\}$.*

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