

# CONCENTRATION OF LAPLACE EIGENFUNCTIONS AND STABILIZATION OF WEAKLY DAMPED WAVE EQUATION

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ABSTRACT. In this article, we prove some universal bounds on the speed of concentration on small (frequency-dependent) neighbourhoods of sub-manifolds of  $L^2$ -norms of quasi modes for Laplace operators on compact manifolds. We deduce new results on the rate of decay of weakly damped wave equations.

## 1. NOTATIONS AND MAIN RESULTS

Let  $(M, g)$  be a smooth compact Riemannian manifold without boundary of dimension  $n$ ,  $\Delta_g$  the Laplace-Beltrami operator on  $M$  and  $d(\cdot, \cdot)$  the geodesic distance on  $M$ .

The purpose of this work is to investigate the concentration properties of eigenfunctions  $\varphi_\lambda$  with eigenvalues  $\lambda^2$  of the Laplace operator  $\Delta_g$  (or more generally quasi-modes), in relation with some control theory results. There are many ways of measuring such possible concentrations. The most classical is by describing semi-classical (Wigner) measures (see the works by Shnirelman [35], Zelditch [50], Colin de Verdière [22], Gérard-Leichtnam [23], Zelditch-Zworski [51], Helffer-Martinez-Robert [24], Anantharaman [1]. Another approach was initiated by Sogge and consists in studying the potential growth of  $\|\varphi_\lambda\|_{L^p(M)}$ , see the works by Sogge [37, 38, 40], Sogge-Zelditch [39], Smith-Sogge [36], Blair-Smith-Sogge [7], Blair [5], Burq-Gérard-Tzvetkov [13, 12, 14], and in the integrable case by Toth [45, 46] and Toth-Zelditch [47, 49, 48]. Finally in [15, 9, 44] the concentration of restrictions on sub-manifolds was considered. Our present approach is connected to works by Blair and Sogge (see also [41, 42]) where the authors studied the concentration (in  $L^2$  norms) on small (frequency dependent) neighbourhoods of geodesics. They recognized that Sogge's  $L^p$  estimates for eigenfunctions implied the impossibility to concentrate on neighbourhoods of width smaller than  $\lambda^{-1/2}$ , and studied the improvements when the manifold has non positive curvature (by showing that it is actually impossible to concentrate on neighbourhoods of size  $\lambda^{-1/2}$ ). Here, on the same question, we follow a different path and are interested on the speed of non concentration and on the extension of such results to higher dimensional sub-manifolds (for which the non concentration property, even with non optimal rates does not follow from Sogge's  $L^p$  estimates). Our first results are the following robust bounds (i.e. independent of the geometry) where, in view of applications to eigenfunctions estimates, we set  $h = \lambda^{-1}$ .

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**Theorem 1.1.** Let  $k \in \{1, \dots, n-1\}$  and  $\Sigma^k$  be a sub-manifold of dimension  $k$  of  $M$ . Let us introduce for  $\beta > 0$ ,

$$(1.1) \quad N_\beta = \{p \in M : d(p, \Sigma^k) < \beta\}.$$

There exists  $C > 0, h_0 > 0$  such that for every  $0 < h \leq h_0$ , every  $\alpha \in (0, 1)$  and every solution  $\psi \in H^2(M)$  of the equation on  $M$

$$(h^2 \Delta_g + 1)\psi = g$$

we have the estimate

$$(1.2) \quad \|\psi\|_{L^2(N_{\alpha h^{1/2}})} \leq C\alpha^{\sigma(k,n)} \left( \|\psi\|_{L^2(M)} + \frac{1}{h} \|g\|_{L^2(M)} \right)$$

where  $\sigma(k, n) = 1$  if  $k \leq n-3$ ,  $\sigma(n-2, n) = 1^-$ ,  $\sigma(n-1, n) = \frac{1}{2}$ .

Here  $1^-$  means that we have a logarithm loss i.e. a bound by  $C\alpha|\log(\alpha)|$ .

**Remark 1.2.**

- (1) As pointed to us by M. Zworski, the result above is *not* invariant by conjugation by Fourier integral operators. Indeed, it is well known that micro-locally,  $-h^2\Delta - 1$  is conjugated by a (micro-locally unitary) FIO to the operator  $hD_{x_1}$ . However the result above is clearly false if one replaces the operator  $-h^2\Delta - 1$  by  $hD_{x_1}$ .
- (2) In the case of curves, Theorem 1.1, with a non optimal exponent  $\sigma = \frac{d-1}{d+1} < \sigma(1, n)$ , follows easily from the  $L^p$  spectral projector estimates by Sogge [37] (see [6] for an improvement on negatively curved manifolds).

Another motivation for our study was the question of stabilization for weakly damped wave equations.

$$(1.3) \quad (\partial_t^2 - \Delta_g + b(x)\partial_t)u = 0, (u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^{s+1}(M) \times H^s(M),$$

where  $0 \leq b \in L^\infty(M)$ . Let

$$E(u)(t) = \int_M \{g_p(\nabla_g u(p), \nabla_g u(p)) + |\partial_t u(p)|^2\} dv_g(p)$$

where  $\nabla_g$  denotes the gradient with respect to the metric  $g$ .

It is known that as soon as the damping  $b \geq 0$  is non trivial, the energy of every solution converge to 0 as  $t$  tends to infinity. On the other hand the rate of decay is *uniform* (and hence exponential) in energy space if and only if the *geometric control condition* [29, 30, 11] is satisfied. Here we want to explore the question when some trajectories are trapped and exhibit decay rates (assuming more regularity on the initial data). This latter question was previously studied in [27, 25], on tori in [16, 28, 2] (see also [17, 18]), on the disc [3], and on hyperbolic manifolds [19, 20, 33, 34, 31, 32, 21, 10], and more recently by Leautaud-Lerner [26] (see also [4] for a singular damping in dimension 1). According to the works by Borichev-Tomilov [8], stabilization results for the wave equation are equivalent to resolvent estimates. On the other hand, Theorem 1.1 implies easily (see Section 2.2) the following resolvent estimate

**Corollary 1.3.** Consider for  $h > 0$  the following operator

$$(1.4) \quad L_h = -h^2 \Delta_g - 1 + ihb, \quad b \in L^\infty(M).$$

Assume that there exists a global compact sub-manifold  $\Sigma^k \subset M$  of dimension  $k$  such that

$$(1.5) \quad b(p) \geq Cd(p, \Sigma^k)^{2\kappa}, \quad p \in M$$

for some  $\kappa > 0$ . Then there exist  $C > 0, h_0 > 0$  such that for all  $0 < h \leq h_0$

$$\|\varphi\|_{L^2(M)} \leq Ch^{-(1+\kappa)} \|L_h \varphi\|_{L^2(M)},$$

for all  $\varphi \in H^2(M)$ .

This result will imply the following one. Notice the loss of one derivative between the energy in the l.h.s. ( $H^1 \times L^2$ -norm) and the r.h.s. ( $H^2 \times H^1$ -norm).

**Theorem 1.4.** Under the geometric assumptions of Corollary 1.3, there exists  $C > 0$  such that for any  $(u_0, u_1) \in H^2(M) \times H^1(M)$ , the solution  $u$  of (1.3) satisfies

$$(1.6) \quad E(u(t))^{\frac{1}{2}} \leq \frac{C}{t^{\frac{1}{\kappa}}} (\|u_0\|_{H^2} + \|u_1\|_{H^1}).$$

**Remark 1.5.** Notice that in Theorem 1.4 the decay rate is worse than the rates exhibited by Leautaud-Lerner [26] in the particular case when the sub-manifold  $\Sigma$  is a torus (and the metric of  $M$  is flat near  $\Sigma$ ). We shall exhibit below examples showing that the rate (1.6) is optimal in general.

A main drawback of the result above (and Leautaud-Lerner's results) is that we were led to *global* assumptions on the geometry of the manifold  $M$  and the trapped region  $\Sigma^k$ . However, the flexibility of Theorem 1.1 is such that we can actually drop all *global* assumptions and keep only a *local* weak controlability assumption.

In the sequel we shall denote by  $\Phi(s)$  the bicharacteristic flow of the Laplacian  $\Delta_g$  defined on the sphere cotangent bundle

$$S^*M = \{(x, \xi) \in T^*M; \|\xi\|_{g(x)} = 1\}.$$

**Theorem 1.6.** Let us assume the following weak geometric control property: for any  $\rho_0 = (p_0, \xi_0) \in S^*M$ , there exists  $s \in \mathbf{R}$  such that the point  $(p_1, \xi_1) = \Phi(s)(\rho_0)$  on the bicharacteristic issued from  $\rho_0$  satisfies

- either  $p_1 \in \omega = \cup\{U \text{ open}; \text{essinf}_U b > 0\}$
- or there exists  $\kappa > 0, C > 0$  and a local sub-manifold  $\Sigma^k$  of dimension  $k \geq 1$  such that  $p_1 \in \Sigma^k$  and near  $p_1$ ,

$$b(p) \geq Cd(p, \Sigma^k)^{2\kappa}.$$

Notice that since  $S^*M$  is compact, we can assume in the assumption above that  $s \in [-T, T]$  is bounded and that a finite number of sub-manifolds (and kappa's) are sufficient. Let  $\kappa_0$  be the largest. Then there exists  $C > 0$  such that for any  $(u_0, u_1) \in H^2(M) \times H^1(M)$ , the solution  $u$  of (1.3) satisfies

$$E(u(t))^{\frac{1}{2}} \leq \frac{C}{t^{\kappa_0}} (\|u_0\|_{H^2} + \|u_1\|_{H^1}).$$

The results in Theorem 1.1 are in general optimal. On spheres  $\mathbf{S}^n = \{\omega \in \mathbf{R}^{n+1} : |\omega| = 1\}$ , an explicit family of eigenfunctions  $e_j(\omega_1, \dots, \omega_{n+1}) = (\omega_1 + i\omega_2)^j$  (eigenvalues  $\lambda_j^2 = j(j+n-1)$ ) is known. We have

$$(1.7) \quad |e_j(\omega)|^2 = (1 - |\omega'|^2)^j = e^{j \log(1 - |\omega'|^2)}, \quad \omega' = (\omega_3, \dots, \omega_{n+1}).$$

$$\text{Set } A_j = \int_{L^2(\mathbf{S}^n)} |e_j(\omega)|^2 d\omega, \quad B_j = \int_{N_{\alpha h_j^{1/2}}} |e_j(\omega)|^2 d\omega, \quad h_j = \lambda_j^{-1} = j^{-1}(1 + o(1)).$$

These eigenfunctions concentrate exponentially on  $j^{-1/2}$  neighbourhoods of the geodesic curve given by  $\{\omega \in \mathbf{S}^n; \omega' = 0\}$  (the equator). As a consequence,  $\lim_{j \rightarrow +\infty} j^{\frac{d-1}{2}} B_j = C > 0$ . Let

$$\Sigma^k = \{\omega = (\omega_1, \omega_2, y, z) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^{n-k} : |\omega| = 1, z = 0\}.$$

Then,  $N_{\alpha h_j^{1/2}} = \{\omega = (\omega_1, \omega_2, y, z) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^{n-k} : |\omega| = 1, |z| \leq \alpha h_j^{\frac{1}{2}}\}$  which we parametrize by:  $|z| \leq \alpha h_j^{\frac{1}{2}}, |y| \leq 1 - |z|^2, \omega_1^2 + \omega_2^2 = 1 - |y|^2 - |z|^2$ . Then

$$B_j = |\mathbf{S}^{n-k-1}| \int_{\rho=0}^{\alpha h_j^{1/2}} \rho^{n-k-1} \int_{|y| \leq \sqrt{1-\rho^2}} \int_{\omega_1^2 + \omega_2^2 = 1 - |y|^2 - |\rho|^2} (1 - |y|^2 - |\rho|^2)^j d\rho dy d\sigma.$$

Setting successively  $y = r\theta, 0 < r < \sqrt{1 - \rho^2}, \theta \in \mathbf{S}^{k-2}, r = \sqrt{1 - \rho^2}s$  then  $\rho = \frac{x}{\sqrt{j}}, s = \frac{t}{\sqrt{j}}$  we see easily that  $j^{\frac{n-1}{2}} A_j \sim C\alpha^{n-k}$ . If  $k = n - 1$  we have  $\frac{n-k}{2} = \frac{1}{2}$  and if  $k = n - 2$  we have  $\frac{n-k}{2} = 1$ . In these two cases the estimate proved in the concentration theorem is optimal (except for the logarithmic loss appearing in the case of co-dimension 2).

On the other hand again on spheres, other particular families of eigenfunctions,  $(f_j, \lambda_j)$  are known (the so called zonal spherical harmonics). These are known to have size of order  $\lambda_j^{(n-1)/2}$  in a neighbourhood of size  $\lambda_j^{-1}$  of two antipodal points (north and south poles). As a consequence, a simple calculation shows that if the sub-manifold contains such a point (which is always achievable by rotation invariance), we have, for  $\alpha = \epsilon h^{1/2}$

$$\|f_j\|_{L^2(N_{\alpha h^{1/2}})}^2 \geq ch \sim \alpha^2,$$

which shows that (1.2) is optimal on spheres (at least in the regime  $\alpha \sim h^{1/2}$ ). To get the full optimality might be possible by studying other families of spherical harmonics. For general manifolds, following the analysis in [38, Section 5] and [15, Section 5]) should give the optimality of our results for *quasi-modes* on any manifold.

The paper is organized as follows. We first show how the non concentration result (Theorem 1.1) imply resolvent estimates for the damped Helmholtz equation, which in turn imply very classically the stabilization results for the damped wave equation. We then focus on the core of the article and prove Theorem 1.1. We first show that the resolvent estimate is implied by a similar estimate for the spectral projector  $T_\lambda = \chi(\sqrt{-\Delta} - \lambda)$ , where  $\chi$  is a non trivial smooth function. Then a classical  $TT^*$  argument reduces the analysis to proving estimates for  $T_\lambda T_\lambda^*$  on  $L^2(N_{\alpha h^{1/2}})$  To prove this latter estimate, we rely on harmonic analysis and the precise description of this spectral projector given in [15], following some earlier works by Sogge [38]. We show that this operator can be decomposed into a sum of operators

$S_\lambda^j, j = 1, \dots, J \sim \log(\lambda)$  the two extremal points  $j = 0, j = J$  corresponding to the two extremal regimes (zonal and highest weight eigenfunctions), and the points in-between to the intermediate regimes. Taking benefit from the precise description of their kernels, each of these pieces can in turn be controlled after suitable change of variables by exploiting on one hand quasi-orthogonality to localize the estimates, and on the other hand oscillations in the phase (despite degeneracies) via yet another  $TT^*$  argument. It remains to sum the contributions of each piece  $S_\lambda^j$  (in the case  $k = n - 2$  each piece gives the same contribution, which leads to the logarithmic loss in (1.2)). We gathered in an appendix several technical results.

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## 2. FROM CONCENTRATION ESTIMATES TO STABILIZATION RESULTS

**2.1. A priori estimates.** Recall that  $(M, g)$  is a compact connected Riemannian manifold. We shall denote by  $\nabla_g$  the gradient operator with respect to the metric  $g$  and by  $dv_g$  the canonical volume form on  $M$ . In all this section we set

$$(2.1) \quad L_h = -h^2 \Delta_g - 1 + ihb$$

We shall first derive some a-priori estimates on  $L_h$ .

**Lemma 2.1.** *Let  $L_h = -h^2 \Delta_g - 1 + ihb$ . Assume  $b \geq 0$  and set  $f = L_h \varphi$ . Then*

$$(2.2) \quad \begin{aligned} (i) \quad & h \int_M b |\varphi(p)|^2 dv_g(p) \leq \|\varphi\|_{L^2(M)} \|f\|_{L^2(M)}, \\ (ii) \quad & h^2 \int_M g_p(\nabla_g \varphi(p), \overline{\nabla_g \varphi(p)}) dv_g(p) \leq \|\varphi\|_{L^2(M)}^2 + \|\varphi\|_{L^2(M)} \|f\|_{L^2(M)}. \end{aligned}$$

*Proof.* We know that  $\Delta_g = \operatorname{div} \nabla_g$  and by the definition of these objects we have

$$A =: \int_M g_p(\nabla_g \varphi(p), \overline{\nabla_g \varphi(p)}) dv_g(p) = - \int_M \Delta_g \varphi(p) \overline{\varphi(p)} dv_g(p).$$

Multiplying both sides by  $h^2$  and since  $-h^2 \Delta_g \varphi = f + \varphi - ihb\varphi$  we obtain

$$h^2 A = \int_M |\varphi(p)|^2 dv_g(p) - ih \int_M b(p) |\varphi(p)|^2 dv_g(p) + \int_M f(p) \overline{\varphi(p)} dv_g(p).$$

Taking the real and the imaginary parts of this equality we obtain the desired estimates.  $\square$

**2.2. Proof of Corollary 1.3 assuming Theorem 1.1.** According to condition (1.5) we have on  $N_{\alpha h^{1/2}}^c$

$$b(p) \geq Cd(p, \Sigma^k)^{2\kappa} \geq C\alpha^{2\kappa} h^\kappa.$$

Writing  $\int_{N_{\alpha h^{1/2}}^c} |\varphi(p)|^2 dv_g(p) = \int_{N_{\alpha h^{1/2}}^c} \frac{1}{b(p)} b(p) |\varphi(p)|^2 dv_g(p)$ , we deduce from Lemma 2.1 that

$$(2.3) \quad \int_{N_{\alpha h^{1/2}}^c} |\varphi(p)|^2 dv_g(p) \leq \frac{1}{C\alpha^{2\kappa}} h^{-(1+\kappa)} \|\varphi\|_{L^2(M)} \|f\|_{L^2(M)}.$$

Therefore we are left with the estimate of the  $L^2(N_{\alpha h^{1/2}})$  norm of  $\varphi$ .

According to (2.1) we see that  $\varphi$  is a solution of

$$(h^2 \Delta_g + 1)\varphi = -f + ihb\varphi =: g_h$$

where  $g_h$  satisfies

$$\|g_h\|_{L^2(M)} \leq \|f\|_{L^2(M)} + Ch\|\varphi\|_{L^2(M)}.$$

It follows from (2.3) and Theorem 1.1 that

$$\|\varphi\|_{L^2(M)} \leq \frac{1}{C^{\frac{1}{2}}\alpha^\kappa} h^{-\frac{1+\kappa}{2}} \|\varphi\|_{L^2(M)}^{\frac{1}{2}} \|f\|_{L^2(M)}^{\frac{1}{2}} + C\alpha^\sigma (\|\varphi\|_{L^2(M)} + \frac{1}{h} \|f\|_{L^2(M)}).$$

Now we fix  $\alpha$  so small that  $C\alpha^\sigma \leq \frac{1}{2}$  and we use the inequality  $a^{\frac{1}{2}}b^{\frac{1}{2}} \leq \varepsilon a + \frac{1}{4\varepsilon}b$  to obtain eventually

$$\|\varphi\|_{L^2(M)} \leq C'h^{-(1+\kappa)}\|f\|_{L^2(M)}$$

which completes the proof of Corollary 1.3.

**2.3. Proof of Theorem 1.4 assuming Corollary 1.3.** The proof is an immediate consequence of a work by Borichev-Tomilov [8] and Corollary 1.3. We quote the following proposition from [26, Proposition 1.5].

**Proposition 2.2.** *Let  $\kappa > 0$ . Then the estimate (1.6) holds if and only if there exist positive constants  $C, \lambda_0$  such that for all  $u \in H^2(M)$ , for all  $\lambda \geq \lambda_0$  we have*

$$C\|(-\Delta_g - \lambda^2 + i\lambda b)u\|_{L^2(M)} \geq \lambda^{1-\kappa}\|u\|_{L^2(M)}.$$

**2.4. Proof of Theorem 1.6 assuming Theorem 1.1.** As before Theorem 1.6 will follow from the resolvent estimate

$$(2.4) \quad \exists C > 0, h_0 > 0 : \forall h \leq h_0 \quad \|\varphi\|_{L^2(M)} \leq Ch^{-(1+\kappa)}\|L_h\varphi\|_{L^2(M)}$$

for every  $\varphi \in C^\infty(M)$ .

We prove (2.4) by contradiction. If it is false one can find sequences  $(\varphi_j), (h_j), (f_j)$  such that

$$(2.5) \quad (-h_j^2 \Delta_g - 1 + ih_j b)\varphi_j = f_j \quad \text{and} \quad \|\varphi_j\|_{L^2(M)} > \frac{j}{h_j^{1+\kappa}} \|f_j\|_{L^2(M)}.$$

Then  $\|\varphi_j\|_{L^2(M)} > 0$  and we may therefore assume that  $\|\varphi_j\|_{L^2(M)} = 1$ . It follows that

$$(2.6) \quad \|f_j\|_{L^2(M)} = o(h_j^{1+\kappa}), \quad j \rightarrow +\infty.$$

Let  $\mu$  be a semiclassical measure for  $(\varphi_j)$ . By Lemma 2.1 we have

$$\left| \int_M \{ |h_j \nabla_g \varphi_j(p)|^2 - |\varphi_j(p)|^2 \} dv_g(p) \right| \leq \|f_j\|_{L^2(M)}.$$

It follows that  $(\varphi_j)$  is  $h_j$ -oscillating which implies that  $\mu(S^*(M)) = 1$ . We therefore shall reach a contradiction if we can show that  $\text{supp } \mu = \emptyset$  and (2.4) will be proved. First of all by elliptic regularity we have

$$(2.7) \quad \text{supp } \mu \subset \{(p, \xi) \in S^*(M) : g_p(\xi, \xi) = 1\}.$$

On the other hand using Lemma 2.1 we have

$$(2.8) \quad \int b(p) |\varphi_j(p)|^2 dv_g(p) \leq \frac{1}{h_j} \|f_j\|_{L^2(M)}$$

since  $\|\varphi_j\|_{L^2(M)} = 1$ . We deduce from (2.5), (2.8) and (2.6) that

$$(2.9) \quad (h_j^2 \Delta_g + 1)\varphi_j = G_j, \quad \text{where} \quad \|G_j\|_{L^2(M)} = o(h_j^{1+\frac{\kappa}{2}}) \rightarrow 0, j \rightarrow +\infty.$$

This shows that the support of  $\mu$  is invariant by the geodesic flow. Let  $\rho_0 \in S^*(M)$  and  $\rho_1 = (p_1, \xi_1) \in S^*(M)$  belonging to the geodesic issued from  $\rho_0$ . Then

$$\rho_0 \notin \text{supp } \mu \iff \rho_1 \notin \text{supp } \mu.$$

But according to our assumption of weak geometric control, either a neighbourhood of  $p_1$  belongs to the set  $\{b(p) \geq c > 0\}$  or  $p_1 \in \Sigma^k$  and  $b(p) \geq Cd(p, \Sigma^k)^{2\kappa}$  near  $p_1$ . In the first case in a neighbourhood of  $\rho_1$  the essential inf of  $b$  is positive and hence by (2.8)  $\rho_1 \notin \text{supp } \mu$ . In the second case taking a small neighbourhood  $\omega$  of  $p_1$  we write

$$\int_{\omega} |\varphi_j(p)|^2 dv_g(p) = \int_{\omega \cap N_{\alpha h_j^{1/2}}} |\varphi_j(p)|^2 dv_g(p) + \int_{\omega \cap N_{\alpha h_j^{1/2}}^c} |\varphi_j(p)|^2 dv_g(p) = (1) + (2).$$

By Theorem 1.1 and (2.9) we have

$$(1) \leq C\alpha^\kappa \left(1 + \frac{1}{h_j} \|g_j\|_{L^2(M)}\right) \leq C\alpha^\kappa (1 + o(h_j^{\frac{\kappa}{2}}))$$

Using the assumption  $b(p) \geq Cd(p, \Sigma^k)$  and (2.8) we get

$$(2) \leq \frac{C}{\alpha^{2\kappa} h_j^\sigma} \int_M b(p) |\varphi_j(p)|^2 dv_g(p) \leq \frac{C' \|f_j\|_{L^2(M)}}{\alpha^{2\kappa} h_j^{1+\kappa}} = \frac{o(1)}{\alpha^{2\kappa}}.$$

It follows that

$$\int_{\omega} |\varphi_j(p)|^2 dv_g(p) \leq C\alpha^\kappa + \frac{o(1)}{\alpha^{2\kappa}}.$$

Let  $\varepsilon > 0$ . We first fix  $\alpha(\varepsilon) > 0$  such that  $C\alpha(\varepsilon)^\sigma \leq \frac{1}{2}\varepsilon$  then we take  $j_0$  large enough such that for  $j \geq j_0$ ,  $o(1) \leq \alpha(\varepsilon)^{2\kappa} \frac{1}{2}\varepsilon$ . It follows that for  $j \geq j_0$  we have  $\int_{\omega} |\varphi_j|^2 dv_g \leq \varepsilon$ . This shows that  $\lim_{j \rightarrow +\infty} \int_{\omega} |\varphi_j|^2 dv_g = 0$  which implies that  $\rho_1 \notin \text{supp } \mu$  thus  $\rho_0 \notin \text{supp } \mu$ . Since  $\rho_0$  is arbitrary we deduce that  $\text{supp } \mu = \emptyset$  which is the desired contradiction.

The rest of the paper will be devoted to the proof of Theorem 1.1.

### 3. CONCENTRATION ESTIMATES (PROOF OF THEOREM 1.1)

The Laplace-Beltrami operator  $-\Delta_g$  with domain  $D = \{u \in L^2(M) : \Delta_g u \in L^2(M)\}$  has a discrete spectrum which can be written

$$0 = \lambda_0^2 < \lambda_1^2 < \dots < \lambda_j^2 \dots \rightarrow +\infty.$$

Moreover we can write  $L^2(M) = \bigoplus_{j=0}^{+\infty} \mathcal{H}_j$ , where  $\mathcal{H}_j$  is the subspace of eigenvectors associated to the eigenvalue  $\lambda_j^2$  and  $\mathcal{H}_j \perp \mathcal{H}_k$  if  $j \neq k$ , and for  $\varphi \in \mathcal{H}_j$ ,  $-\Delta_g \varphi = \lambda_j^2 \varphi$ .

For  $\lambda \geq 0$  we define the spectral projector  $\Pi_\lambda : L^2(M) \rightarrow L^2(M)$  by

$$(3.1) \quad L^2(M) \ni f = \sum_{j \in \mathbf{N}} \varphi_j, \mapsto \Pi_\lambda f = \sum_{j \in \Lambda_\lambda} \varphi_j, \quad \Lambda_\lambda = \{j \in \mathbf{N} : \lambda_j \in [\lambda, \lambda + 1)\}.$$

Then  $\Pi_\lambda$  is self adjoint and  $\Pi_\lambda^2 = \Pi_\lambda$ .

Theorem 1.1 will be a consequence of the following one. Recall  $N_{\alpha h^{1/2}}$  has been defined in (1.1).

**Proposition 3.1.** *There exist  $C > 0, h_0 > 0$  such that for every  $h \leq h_0$  and every  $\alpha \in (0, 1)$*

$$(3.2) \quad \|\Pi_{\lambda} u\|_{L^2(N_{\alpha h^{1/2}})} \leq C \alpha^{\sigma} \|u\|_{L^2(M)}, \quad \lambda = \frac{1}{h},$$

for every  $u \in L^2(M)$ , Here  $\sigma = 1$  if  $k \leq n - 3$ ,  $\sigma = 1^-$  if  $k = n - 2$ ,  $\sigma = \frac{1}{2}$  if  $k = n - 1$ .

Here, as before,  $1^-$  means that we have an estimate by  $C\alpha|\log(\alpha)|$ .

**3.1. Proof of Theorem 1.1 assuming Proposition 3.1.** If  $\psi = \sum_{j \geq 0} \varphi_j$  we have  $g = (h^2 \Delta_g + 1)\psi = \sum_{j \geq 0} (h^2 \Delta_g + 1)\varphi_j$ . Therefore by orthogonality

$$(3.3) \quad \|g\|_{L^2(M)}^2 = \sum_{j \geq 0} |1 - h^2 \lambda_j^2|^2 \|\varphi_j\|_{L^2(M)}^2.$$

Let  $\varepsilon_0$  be a fixed number in  $]0, 1[$ . With  $N = [\varepsilon_0 \lambda]$  we write

$$\psi = \sum_{k=-N}^N \Pi_{\lambda+k} \psi + R_N.$$

Recall that  $\Pi_{\lambda+k} \psi = \sum_{j \in E_k} \varphi_j$ , where  $E_k = \{j \geq 0 : \lambda_j \in [\lambda + k, \lambda + k + 1]\}$ .

Assume  $|k| \geq 2$ . Since  $\lambda + k \leq \lambda_j < \lambda + k + 1$  we have  $|\lambda_j - \lambda| \geq \frac{1}{2}|k|$  which implies that  $|\lambda_j^2 - \lambda^2| \geq \frac{1}{2}|k|\lambda$ . By orthogonality we have

$$\begin{aligned} \|\Pi_{\lambda+k} \psi\|_{L^2(M)}^2 &= \sum_{j \in E_k} \|\varphi_j\|_{L^2(M)}^2 = \sum_{j \in E_k} \frac{1}{|\lambda_j^2 - \lambda^2|^2} |\lambda_j^2 - \lambda^2|^2 \|\varphi_j\|_{L^2(M)}^2 \\ &\leq \frac{4}{|k|^2 \lambda^2} \sum_{j \in E_k} |\lambda_j^2 - \lambda^2|^2 \|\varphi_j\|_{L^2(M)}^2 \leq \frac{4\lambda^2}{|k|^2} \sum_{j \in E_k} |h^2 \lambda_j^2 - 1|^2 \|\varphi_j\|_{L^2(M)}^2. \end{aligned}$$

Since  $\Pi_{\lambda+k}^2 = \Pi_{\lambda+k}$ , using Proposition 3.1 and the above estimate we obtain

$$\begin{aligned} \left\| \sum_{2 \leq |k| \leq N} \Pi_{\lambda+k} \psi \right\|_{L^2(N_{\alpha h^{1/2}})} &\leq \sum_{2 \leq |k| \leq N} \|\Pi_{\lambda+k} \psi\|_{L^2(N_{\alpha h^{1/2}})} \leq C \alpha^{\sigma} \sum_{2 \leq |k| \leq N} \|\Pi_{\lambda+k} \psi\|_{L^2(M)} \\ &\leq 2C \alpha^{\sigma} \lambda \sum_{2 \leq |k| \leq N} \frac{1}{|k|} \left( \sum_{j \in E_k} |h^2 \lambda_j^2 - 1|^2 \|\varphi_j\|_{L^2(M)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using Cauchy-Schwarz inequality, (3.3) and the fact that the  $E_k$  are pairwise disjoint we obtain eventually

$$(3.4) \quad \left\| \sum_{2 \leq |k| \leq N} \Pi_{\lambda+k} \psi \right\|_{L^2(N_{\alpha h^{1/2}})} \leq C \alpha^{\sigma} \frac{1}{h} \|g\|_{L^2(M)}.$$

Now a direct application of Proposition 3.1 shows that

$$(3.5) \quad \left\| \sum_{|k| \leq 1} \Pi_{\lambda+k} \psi \right\|_{L^2(N_{\alpha h^{1/2}})} \leq C \alpha^{\sigma} \|\psi\|_{L^2(M)}.$$

Eventually let us consider the remainder  $R_N$ . We have

$$R_N = \sum_{j \in A} \varphi_j + \sum_{j \in B} \varphi_j, \quad A = \{j : \lambda_j \leq \lambda - N\}, \quad B = \{j : \lambda_j \geq \lambda + N + 1\}.$$

The two sums are estimated by the same way since in both cases we have  $|\lambda_j - \lambda| \geq c\lambda$  thus  $|\lambda_j^2 - \lambda^2| \geq c\lambda^2$ . Then by orthogonality we write

$$\begin{aligned} \left\| \sum_{j \in A} \varphi_j \right\|_{L^2(M)}^2 &= \sum_{j \in A} \|\varphi_j\|_{L^2(M)}^2 = \sum_{j \in A} \frac{1}{|\lambda_j^2 - \lambda^2|^2} |\lambda_j^2 - \lambda^2|^2 \|\varphi_j\|_{L^2(M)}^2 \\ &\leq \frac{C}{\lambda^4} \sum_{j \in A} |\lambda_j^2 - \lambda^2|^2 \|\varphi_j\|_{L^2(M)}^2 \leq \sum_{j \in \mathbf{N}} |h^2 \lambda_j^2 - 1|^2 \|\varphi_j\|_{L^2(M)}^2 = \|g\|_{L^2(M)}^2. \end{aligned}$$

It follows that  $\|R_N\|_{L^2(M)} \leq C\|g\|_{L^2(M)}$ . Now  $(h^2 \Delta_g + 1)R_N = \sum_{j \in A \cup B} (1 - h^2 \lambda_j^2) \varphi_j =: g_N$  and  $\|g_N\|_{L^2(M)} \leq \|g\|_{L^2(M)}$ . So using Lemma A.1 we obtain

$$(3.6) \quad \|R_N\|_{L^2(N_{\alpha h^{1/2}})} \leq C \frac{\alpha^\sigma}{h} \|g\|_{L^2(M)}$$

where  $\sigma = \frac{1}{2}$  if  $k = n - 1$ ,  $\sigma = 1$  if  $1 \leq k \leq n - 2$ .

Then Theorem 1.1 follows from (3.4), (3.5) and (3.6).

**3.2. Proof of Proposition 3.1.** This result will be a consequence of the following one.

**Proposition 3.2.** *There exists  $\chi \in \mathcal{S}(\mathbf{R})$  such that  $\chi(0) = 1$  and there exist  $C > 0, h_0 > 0$  such that for every  $h \leq h_0$ , every  $\alpha \in (0, 1)$ , and every  $u \in L^2(M)$  we have*

$$(3.7) \quad \|\chi(\sqrt{-\Delta_g} - \lambda)u\|_{L^2(N_{\alpha h^{1/2}})} \leq C\alpha^\sigma \|u\|_{L^2(M)}, \quad \lambda = \frac{1}{h}$$

where  $\chi(\sqrt{-\Delta_g} - \lambda)u = \sum_{j \in \mathbf{N}} \chi(\lambda_j - \lambda) \varphi_j$  if  $u = \sum_{j \in \mathbf{N}} \varphi_j$ .

*Proof of Proposition 3.1 assuming Proposition 3.2.* There exists  $\delta = \frac{1}{N} > 0$  and  $c > 0$  such that  $\chi(t) \geq c$  for every  $t \in [-\delta, \delta]$ . Now let  $E = \{j \in \mathbf{N} : \lambda_j \in [\lambda, \lambda + \delta]\}$  and set  $\Pi_\lambda^\delta u = \sum_{j \in E} \varphi_j$ . On  $E$  we have  $\chi(\lambda_j - \lambda) \geq c > 0$  therefore we can write

$$1_E(j) = \chi(\lambda_j - \lambda) \frac{1_E(j)}{\chi(\lambda_j - \lambda)}.$$

It follows that

$$\Pi_\lambda^\delta u = \chi(\sqrt{-\Delta_g} - \lambda) \circ Ru$$

where  $R$  is continuous from  $L^2(M)$  to itself with norm bounded by  $\frac{1}{c}$ . Therefore assuming Proposition 3.2 we can write

$$(3.8) \quad \|\Pi_\lambda^\delta u\|_{L^2(N_{\alpha h^{1/2}})} \leq C\alpha^\sigma \|Ru\|_{L^2(M)} \leq \frac{C}{c} \alpha^\sigma \|u\|_{L^2(M)},$$

where the constants in the right are independent of  $\lambda$ . Now since

$$\{j : \lambda_j \in [\lambda, \lambda + 1)\} = \cup_{k=0}^{N-1} \{j : \lambda_j \in [\lambda + k\delta, \lambda + (k+1)\delta)\}$$

where the union is disjoint, one can write  $\Pi_\lambda u = \sum_{k=0}^{N-1} \Pi_{\lambda+k\delta}^\delta$ . It follows from (3.8) that

$$\|\Pi_\lambda u\|_{L^2(N_{\alpha h^{1/2}})} \leq C' \alpha^\sigma \|u\|_{L^2(M)}$$

which proves Proposition 3.1. □

It remains to prove Proposition 3.2. Until the end of this section  $\sigma$  will be a real number such that

$$\sigma = 1 \text{ if } k \leq n - 3, \quad \sigma = 1^- \text{ if } k = n - 2, \quad \sigma = \frac{1}{2} \text{ if } k = n - 1.$$

As before for every  $p \in \Sigma^k$  one can find an open neighbourhood  $U_p$  of  $p$  in  $M$ , a neighbourhood  $B_0$  of the origin in  $\mathbf{R}^n$  a diffeomorphism  $\theta$  from  $U_p$  to  $B_0$  such that

$$(3.9) \quad \begin{aligned} (i) \quad & \theta(U_p \cap \Sigma^k) = \{x = (x_a, x_b) \in (\mathbf{R}^k \times \mathbf{R}^{n-k}) \cap B_0 : x_b = 0\}, \\ (ii) \quad & \theta(U_p \cap N_{\alpha h^{1/2}}) \subset B_{\alpha, h} =: \{x \in B_0 : |x_b| \leq \alpha h^{\frac{1}{2}}\}. \end{aligned}$$

Now  $\Sigma^k$  and  $N_{\alpha h^{1/2}}$  for  $h$  small, are covered by a finite number of such open neighbourhoods i.e.  $N_{\alpha h^{\frac{1}{2}}} \subset \cup_{j=1}^{n_0} U_{p_j}$ . We take a partition of unity relative to this covering i.e.  $(\zeta_j) \in C^\infty(M)$  with  $\text{supp } \zeta_j \subset U_{p_j}$  and  $\sum_{j=1}^{n_0} \zeta_j = 1$  in a fixed neighbourhood  $\mathcal{O}$  of  $\Sigma^k$  containing  $N_{\alpha h^{1/2}}$ . For  $p \in \mathcal{O}$  we can therefore write

$$(3.10) \quad \chi(\sqrt{-\Delta_g} - \lambda)u(p) = \sum_{j=1}^{n_0} \chi(\sqrt{-\Delta_g} - \lambda)(\zeta_j u)(p).$$

Our aim being to bound each term of the right hand side, we shall skip the index  $j$  in what follows. Moreover we shall set for convenience

$$\chi_\lambda =: \chi(\sqrt{-\Delta_g} - \lambda).$$

We shall use some results by Sogge (see also [15, Theorem 4]).

**Theorem 3.3** ([38, Lemma 5.1.3]). *There exists  $\chi \in \mathcal{S}(\mathbf{R})$  such that  $\chi(0) = 1$  and for any  $p_0 \in \Sigma^k$  there a diffeomorphism  $\theta$  as above, open sets  $W \subset V = \{x \in \mathbf{R}^n : |x| \leq \varepsilon_0\}$ , a smooth function  $a : W_x \times V_y \times \mathbf{R}_\lambda^+ \rightarrow \mathbf{C}$  supported in the set*

$$\{(x, y) \in W \times V : |x| \leq c_0\varepsilon \leq c_1\varepsilon \leq |y| \leq c_2\varepsilon \ll 1\}$$

satisfying

$$\forall \alpha \in \mathbf{N}^{2n}, \exists C_\alpha > 0 : \forall \lambda \geq 0, |\partial_{x,y}^\alpha a(x, y, \lambda)| \leq C_\alpha,$$

an operator  $\mathcal{R}_\lambda : L^2(M) \rightarrow L^\infty(M)$  satisfying

$$\|\mathcal{R}_\lambda u\|_{L^\infty(M)} \leq C \|u\|_{L^2(M)},$$

such that for every  $x \in U =: W \cap \{x : |x| \leq c\varepsilon\}$ , setting  $\tilde{u} = \zeta u \circ \theta^{-1}$  we have

$$(3.11) \quad \chi_\lambda(\zeta u)(\theta^{-1}(x)) = \lambda^{\frac{n-1}{2}} \int_{y \in V} e^{i\lambda\psi(x,y)} a(x, y, \lambda) \tilde{u}(y) dy + (\mathcal{R}_\lambda(\zeta u))(\theta^{-1}(x))$$

where  $\psi(x, y) = -d_g((\theta^{-1}(x)), (\theta^{-1}(y)))$  is the geodesic distance on  $M$  between  $\theta^{-1}(x)$  and  $\theta^{-1}(y)$ . Furthermore the symbol  $a$  is real non negative, does not vanish for  $|x| \leq c\varepsilon$  and  $d_g((\theta^{-1}(x)), (\theta^{-1}(y))) \in [c_3\varepsilon, c_4\varepsilon]$ .

Let us set

$$(3.12) \quad \mathcal{T}_\lambda \tilde{u}(x) = \int_{y \in V} e^{i\lambda\psi(x,y)} a(x, y, \lambda) \tilde{u}(y) dy.$$

It follow from (3.11) that

$$(3.13) \quad \|\chi_\lambda(\zeta u)\|_{L^2(N_{\alpha,h})} \leq \lambda^{\frac{n-1}{2}} \|\mathcal{T}_\lambda \tilde{u}\|_{L^2(B_{\alpha,h})} + \|\mathcal{R}_\lambda(\zeta u)\|_{L^2(N_{\alpha,h})}$$

Let us look to the contribution of  $\mathcal{R}_\lambda$ . Since (see (3.9)) the volume of  $N_{\alpha h^{1/2}}$  is bounded by  $C(\alpha h^{\frac{1}{2}})^{n-k}$  we can write

$$\|\mathcal{R}_\lambda(\zeta u)\|_{L^2(N_{\alpha h^{1/2}})} \leq C(\alpha h^{\frac{1}{2}})^{\frac{n-k}{2}} \|\mathcal{R}_\lambda(\zeta u)\|_{L^\infty(M)} \leq C(\alpha h^{\frac{1}{2}})^{\frac{n-k}{2}} \|u\|_{L^2(M)}.$$

If  $k = n - 1$  we have  $\alpha^{\frac{n-k}{2}} = \alpha^{\frac{1}{2}}$  and if  $1 \leq k \leq n - 2$  we have  $\alpha^{\frac{n-k}{2}} \leq \alpha$ . Therefore we get

$$(3.14) \quad \|\mathcal{R}_\lambda(\zeta u)\|_{L^2(N_{\alpha h^{1/2}})} \leq C\alpha^\sigma \|u\|_{L^2(M)}.$$

According to (3.13) Proposition 3.2 will be a consequence of the following result.

**Proposition 3.4.** *There exists positive constants  $C, \lambda_0$  such that*

$$(3.15) \quad \lambda^{\frac{n-1}{2}} \|\mathcal{T}_\lambda \tilde{u}\|_{L^2(B_{\alpha,h})} \leq C\alpha^\sigma \|u\|_{L^2(M)}$$

for every  $\lambda \geq \lambda_0$  and every  $u \in L^2(M)$ .

*Proof of Proposition 3.4.* Set  $\mathcal{S}_\lambda = \mathcal{T}_\lambda \mathcal{T}_\lambda^*$  and denote by  $1_B$  the indicator function of the set  $B_{\alpha,h}$ . By the usual trick (3.15) will be a consequence of the following estimate.

$$(3.16) \quad \|1_B \mathcal{S}_\lambda 1_B v\|_{L^2(\mathbf{R}^n)} \leq Ch^{n-1} \alpha^{2\sigma} \|v\|_{L^2(\mathbf{R}^n)}, \quad h = \frac{1}{\lambda}.$$

Let  $\mathcal{K}_\lambda(x, x')$  be the kernel of  $\mathcal{S}_\lambda$ . By (3.12) it is given by

$$(3.17) \quad \mathcal{K}_\lambda(x, x') = \int e^{i\lambda[\psi(x,y) - \psi(x',y)]} a(x, y, \lambda) \bar{a}(x', y, \lambda) dy.$$

We shall decompose

$$(3.18) \quad \begin{cases} \mathcal{K}_\lambda = \mathcal{K}_\lambda^1 + \mathcal{K}_\lambda^2, \\ \mathcal{K}_\lambda^1 = 1_{\{|x-x'| \leq \frac{1}{\lambda}\}} \mathcal{K}_\lambda, & \mathcal{K}_\lambda^2 = 1_{\{\frac{1}{\lambda} < |x-x'| \leq \varepsilon\}} \mathcal{K}_\lambda, \\ \mathcal{S}_\lambda = \sum_{j=1}^2 \mathcal{S}_\lambda^j, & \mathcal{S}_\lambda^j \tilde{u}(x) = \int \mathcal{K}_\lambda^j(x, x') \tilde{u}(x') dx' \end{cases}$$

and treat separately each piece.

**3.3. Estimate of  $\mathcal{S}_\lambda^1$ .** When  $|x - x'| \leq \frac{1}{\lambda}$  the kernel  $\mathcal{K}_\lambda$  is uniformly bounded. Therefore  $|\mathcal{K}_\lambda^1| \leq C 1_{\{|x-x'| \leq \frac{1}{\lambda}\}}$ , so by Schur lemma we have

$$\|\mathcal{S}_\lambda^1 v\|_{L^2(\mathbf{R}^n)} \leq Ch^n \|v\|_{L^2(\mathbf{R}^n)}.$$

Therefore

$$(3.19) \quad \|1_B \mathcal{S}_\lambda^1 1_B v\|_{L^2(\mathbf{R}^n)} \leq Chh^{n-1} \|v\|_{L^2(\mathbf{R}^n)}.$$

Now writing  $x = (x_a, x_b) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$ ,  $x' = (x'_a, x'_b) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$ , we have

$$\|\mathcal{S}_\lambda^1 v(\cdot, x_b)\|_{L^2(\mathbf{R}^k)} \leq C \int_{\mathbf{R}^{n-k}} 1_{\{|x_b - x'_b| \leq h\}} \left\| \int_{\mathbf{R}^k} 1_{\{|x_a - x'_a| \leq h\}} v(x'_a, x'_b) dx'_a \right\|_{L^2(\mathbf{R}^k)} dx'_b.$$

Again by Schur lemma we get

$$\|\mathcal{S}_\lambda^1 v\|_{L^\infty(\mathbf{R}^{n-k}, L^2(\mathbf{R}^k))} \leq Ch^k \|v\|_{L^1(\mathbf{R}^{n-k}, L^2(\mathbf{R}^k))}.$$

We deduce that

$$\|1_B \mathcal{S}_\lambda^1 1_B v\|_{L^2(\mathbf{R}^n)} \leq C(\alpha h^{\frac{1}{2}})^{n-k} h^k \|v\|_{L^2(\mathbf{R}^n)}.$$

This estimate can be rewritten as

$$(3.20) \quad \|1_B \mathcal{S}_\lambda^1 1_B v\|_{L^2(\mathbf{R}^n)} \leq C\alpha^{2\sigma} \alpha^{n-k-2\sigma} h^{\frac{n-k}{2}+k} \|v\|_{L^2(\mathbf{R}^n)}.$$

Now if  $h^{\frac{1}{2}} \leq \alpha$  we use (3.19) and we obtain

$$\|1_B \mathcal{S}_\lambda^1 1_B v\|_{L^2(\mathbf{R}^n)} \leq C\alpha^2 h^{n-1} \|v\|_{L^2(\mathbf{R}^n)}.$$

If  $\alpha \leq h^{\frac{1}{2}}$  we use instead (3.20). Since  $n-k-2\sigma \geq 0$  we can write

$$\begin{aligned} \|1_B \mathcal{S}_\lambda^1 1_B v\|_{L^2(\mathbf{R}^n)} &\leq C\alpha^{2\sigma} h^{\frac{1}{2}(n-k-2\sigma) + \frac{1}{2}(n-k)+k} \|v\|_{L^2(\mathbf{R}^n)} = C\alpha^{2\sigma} h^{n-\sigma} \|v\|_{L^2(\mathbf{R}^n)} \\ &\leq C\alpha^{2\sigma} h^{n-1} \|v\|_{L^2(\mathbf{R}^n)}. \end{aligned}$$

Therefore in all cases we have

$$(3.21) \quad \|1_B \mathcal{S}_\lambda^1 1_B v\|_{L^2(\mathbf{R}^n)} \leq C\alpha^{2\sigma} h^{n-1} \|v\|_{L^2(\mathbf{R}^n)}.$$

**3.4. Estimate of  $\mathcal{S}_\lambda^2$ .** To complete the proof of Proposition 3.1, the strategy is the following: we first decompose the kernel  $\mathcal{K}_\lambda^2$  (and hence the operator  $\mathcal{T}_\lambda \mathcal{T}_\lambda^*$ ) into pieces corresponding to  $|x-x'| \sim 2^j$ ,  $j=0, \dots, J = \log(\lambda)/\log(2)$ , and bound separately the contribution of each piece. We then use quasi-orthogonality and rescaling arguments to reduce the analysis to estimating operators with kernels of the type

$$e^{i\mu_j \tilde{\psi}_j(X, X')} \chi_0(X-X') a(2^{-j}X, 2^{-j}X', \lambda),$$

where  $\mu_j = \lambda 2^{-j}$ ,  $\chi_0 \in C_0^\infty(\mathbb{R}^n)$  is supported in a shell  $\{\frac{1}{2} \leq |X| \leq 2\}$ , and  $\tilde{\psi}_j$  are phases depending nicely on  $j$  as a parameter (they are asymptotically close as  $j \rightarrow +\infty$  to the euclidean distance  $|X-X'|$ ). Finally, we can apply to these operators a version of a classical lemma (see Theorem A.2) allowing to bound the operator norm of such operators with oscillatory kernels, assuming a lower bound on the rank of the differential of the phase (see Theorem A.4).

Our starting point is the description of the kernel  $\mathcal{K}$  given in [15].

**Lemma 3.5** ([15] Lemma 6.1). *There exists  $\varepsilon \ll 1$ ,  $(a_p^\pm, b_p)_{p \in \mathbf{N}} \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R})$  such that for  $|x-x'| \gtrsim \lambda^{-1}$  and any  $N \in \mathbf{N}^*$  we have*

$$(3.22) \quad \mathcal{K}_\lambda(x, x') = \sum_{\pm} \sum_{p=0}^{N-1} \frac{e^{\pm i\lambda \tilde{\psi}(x, x')}}{(\lambda|x-x'|)^{\frac{n-1}{2}+p}} a_p^\pm(x, x', \lambda) + b_N(x, x', \lambda),$$

where  $\tilde{\psi}(x, x')$  is the geodesic distance between the points  $\theta^{-1}(x)$  and  $\theta^{-1}(x')$ . Moreover  $a_p^\pm$  are real, have supports of size  $\mathcal{O}(\varepsilon)$  with respect to the two first variables and are uniformly bounded with respect to  $\lambda$ . Finally

$$|b_N(x, x', \lambda)| \leq C_N (\lambda|x-x'|)^{-\left(\frac{d-1}{2}+N\right)}.$$

Recall that  $\mathcal{K}_\lambda$  is the kernel of  $\mathcal{T}_\lambda \mathcal{T}_\lambda^*$ , and notice that the description of the kernel of  $\mathcal{T}_\lambda$  given in [38, Lemma 5.1.3] which was of course an important point for the analysis in [15] would not be sufficient for our purpose, as we shall perform later yet another  $TT^*$  argument (one never has enough of a good thing), using the partial non-degeneracy of the phase below (see Theorem A.2). In particular, in the stationary phase argument leading to (3.22), the identification of the critical points (and hence the two critical values of the phase,  $\pm\tilde{\psi}(x, x')$ ) is of crucial importance for our analysis below.

We cut the set  $\frac{1}{\lambda} \leq |x - x'| \leq \varepsilon$  into pieces

$$|x - x'| \sim 2^{-j}, \quad \frac{1}{\lambda} \leq 2^{-j} \leq \varepsilon$$

and we estimate the contribution of each term. According to Lemma 3.5 we are lead to work with the operator

$$A_j v(x) = \int k_j(x, x', \lambda) v(x') dx'$$

where

$$(3.23) \quad k_j(x, x', \lambda) = (\lambda 2^{-j})^{-\frac{n-1}{2}} \chi_0(2^j(x - x')) e^{i\lambda \tilde{\psi}(x, x')} \sum_{p=0}^{N-1} \lambda^{-p} a_p(x, x', \lambda).$$

Now there exists  $\chi \in C^\infty(\mathbf{R}^n)$  such that  $\text{supp } \chi \subset \{x : |x| \leq 1\}$ ,  $\chi(x) = 1$  if  $|x| \leq \frac{1}{2}$  and

$$\sum_{p \in \mathbf{Z}^n} \chi(x - p) = 1, \forall x \in \mathbf{R}^n.$$

We now write

$$(3.24) \quad \begin{aligned} k_j(x, x', \lambda) &= \sum_{p, q \in \mathbf{Z}^n} k_{j p q}(x, x', \lambda) \\ k_{j p q}(x, x', \lambda) &= \chi(2^j x - p) k_j(x, x', \lambda) \chi(2^j x' - q) \end{aligned}$$

and we denote by  $A_{j p q}$  the operator with kernel  $k_{j p q}$ .

Notice that the sum appearing in (3.24) is to be taken only for  $|p - q| \leq 2$ .

We claim that by quasi orthogonality in  $L^2$  we have

$$(3.25) \quad \|1_B A_j 1_B\|_{L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n)} \leq C \sup_{|p-q| \leq 2} \|1_B A_{j p q} 1_B\|_{L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n)}.$$

Indeed let us forget  $1_B$  which plays any role. We have

$$\|A_j v\|_{L^2(\mathbf{R}^n)} = \sum_{|p-q| \leq 2} \sum_{|p'-q'| \leq 2} \int A_{j p q} [\tilde{\chi}(2^j \cdot - q)v](x) A_{j p' q'} [\tilde{\chi}(2^j \cdot - q')v](x) dx$$

where  $\tilde{\chi} \in C_0^\infty(\mathbf{R}^n)$ ,  $\tilde{\chi} = 1$  on the support of  $\chi$  and  $\sum_{p \in \mathbf{Z}^n} [\tilde{\chi}(x - p)]^2 \leq M, \forall x \in \mathbf{R}^n$ . Due to the presence of  $\chi(2^j x - p)$ ,  $\chi(2^j x - p')$  and  $\chi_0(2^j(x - x'))$  inside the above integral one must also have  $|p - p'| \leq 2$  in the sum. Therefore we are summing on the set  $E = \{(p, q, p', q') : |p - q| \leq 2, |p - p'| \leq 2, |p' - q'| \leq 2\}$ . We have

$$\begin{aligned} E &\subset E_1 = \{(p, q, p', q') : |p - q| \leq 2, |p' - q| \leq 4, |q' - q| \leq 6\}, \\ E &\subset E_2 = \{(p, q, p', q') : |p' - q'| \leq 2, |p - q'| \leq 4, |q - q'| \leq 6\}. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality that  $\|A_j v\|_{L^2(\mathbf{R}^n)}$  can be bounded by

$$\left( \sum_{E_1} \|A_{j p q}\|_{L^2 \rightarrow L^2}^2 \|\tilde{\chi}(2^j \cdot - q) v\|_{L^2(\mathbf{R}^n)}^2 \right)^{\frac{1}{2}} \left( \sum_{E_2} \|A_{j p' q'}\|_{L^2 \rightarrow L^2}^2 \|\tilde{\chi}(2^j \cdot - q') v\|_{L^2(\mathbf{R}^n)}^2 \right)^{\frac{1}{2}}$$

and therefore by the choice of  $\tilde{\chi}$  by  $C \sup_{|p-q| \leq 2} \|A_{j p q}\|_{L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n)} \|v\|_{L^2(\mathbf{R}^n)}$  which proves our claim.

Now let us consider the operator  $Q_{j p q}$  defined by

$$(3.26) \quad \begin{aligned} Q_{j p q} v(X) &= \int_{\mathbf{R}^n} \sigma_{j p q}(X, X', \lambda) v(X') dX' \\ \sigma_{j p q}(X, X', \lambda) &= \chi(X - p) k_j(2^{-j} X, 2^{-j} X', \lambda) \chi(X' - q). \end{aligned}$$

Then by the change of variables ( $x = 2^{-j} X, x' = 2^{-j} X'$ ) we can see easily that

$$(3.27) \quad \|1_{2^j B} Q_{j p q} 1_{2^j B} v\|_{L^2(\mathbf{R}^n)} \leq K_j \|v\|_{L^2(\mathbf{R}^n)} \quad \text{implies}$$

$$(3.28) \quad \|1_B A_{j p q} 1_B v\|_{L^2(\mathbf{R}^n)} \leq 2^{-j n} K_j \|v\|_{L^2(\mathbf{R}^n)}.$$

Setting

$$(3.29) \quad \mu_j = \lambda 2^{-j}, \quad \tilde{\psi}_j(X, X') = 2^j \tilde{\psi}(2^{-j} X, 2^{-j} X'),$$

we deduce from (3.23) and (3.26) we have

$$(3.30) \quad \begin{aligned} \sigma_{j p q}(X, X', \lambda) &= \mu_j^{-\frac{n-1}{2}} e^{i \mu_j \tilde{\psi}_j(X, X')} \chi(X - p) \chi(X - q) \chi_0(X - X') \\ &\quad \cdot \sum_{p=0}^{N-1} \lambda^{-p} a_p(2^{-j} X, 2^{-j} X', \lambda). \end{aligned}$$

We shall derive two estimates of the left hand side of (3.27). On one hand using Theorem A.4 with  $p = k - 1$  we can write,

$$\begin{aligned} \|1_{2^j B} Q_{j p q} 1_{2^j B} v\|_{L^2(\mathbf{R}^n)} &\leq C (\alpha h^{\frac{1}{2}} 2^j)^{\frac{n-k}{2}} \|Q_{j p q} 1_{2^j B} v\|_{L^\infty(\mathbf{R}_{x_b}^{n-k} \times \mathbf{R}_{x_{a1}} \times L^2(\mathbf{R}_{x'_a}^{k-1}))}, \\ &\leq C \mu_j^{-\frac{n-1}{2}} (\alpha h^{\frac{1}{2}} 2^j)^{\frac{n-k}{2}} \mu_j^{-\frac{k-1}{2}} \|1_{2^j B} v\|_{L^1(\mathbf{R}_{x_b}^{n-k} \times \mathbf{R}_{x_{a1}} \times L^2(\mathbf{R}_{x'_a}^{k-1}))}, \\ &\leq C \mu_j^{-\frac{n-1}{2}} (\alpha h^{\frac{1}{2}} 2^j)^{n-k} \mu_j^{-\frac{k-1}{2}} \|v\|_{L^2(\mathbf{R}^n)}. \end{aligned}$$

We deduce from (3.28) and (3.25) that

$$(3.31) \quad \|1_B A_j 1_B v\|_{L^2(\mathbf{R}^n)} \leq C h^{n-1} \alpha^{n-k} 2^{j(\frac{n-k}{2}-1)} \|v\|_{L^2(\mathbf{R}^n)}.$$

On the other hand using Theorem A.2 with  $p = n - 1$  we can write

$$\|1_{2^j B} Q_{j p q} 1_{2^j B} v\|_{L^2(\mathbf{R}^n)} \leq \|Q_{j p q} 1_{2^j B} v\|_{L^2(\mathbf{R}^n)} \leq C \mu_j^{-\frac{n-1}{2}} \mu_j^{-\frac{n-1}{2}} \|v\|_{L^2(\mathbf{R}^n)},$$

from which we deduce using (3.28) and (3.25) that

$$(3.32) \quad \|1_B A_j 1_B v\|_{L^2(\mathbf{R}^n)} \leq C 2^{-j n} (2^j h)^{n-1} \leq C h^{n-1} 2^{-j}.$$

Recall that we have  $S_\lambda^2 = \sum_{j \in E} A_j$  where  $E = \{j : \frac{1}{\varepsilon} \leq 2^j \leq \lambda\}$ . Then we write

$$(3.33) \quad \begin{aligned} 1_B S_\lambda^2 1_B v &= \sum_{j \in E_1} 1_B A_j 1_B v + \sum_{j \in E_2} 1_B A_j 1_B v = (1) + (2), \quad \text{where} \\ E_1 &= \{j : \frac{1}{\varepsilon} \leq 2^j \leq \alpha^{-2}\}, \quad E_2 = \{j : \alpha^{-2} \leq 2^j \leq \lambda\}. \end{aligned}$$

To estimate the term (1) we use (3.31). We obtain

$$\|(1)\|_{L^2(\mathbf{R}^n)} \leq Ch^{n-1} \alpha^{n-k} \sum_{j \in E_1} 2^{j(\frac{n-k}{2}-1)} \|v\|_{L^2(\mathbf{R}^n)}.$$

Then we have three cases.

If  $\frac{n-k}{2} - 1 > 0$  that is if  $k \leq n - 3$  then

$$\|(1)\|_{L^2(\mathbf{R}^n)} \leq Ch^{n-1} \alpha^{n-k} \left(\frac{1}{\alpha^2}\right)^{\frac{n-k}{2}-1} \|v\|_{L^2(\mathbf{R}^n)} \leq Ch^{n-1} \alpha^2 \|v\|_{L^2(\mathbf{R}^n)}.$$

If  $\frac{n-k}{2} - 1 = 0$  that is if  $k = n - 2$  then

$$\|(1)\|_{L^2(\mathbf{R}^n)} \leq Ch^{n-1} \alpha^2 \log(\alpha^{-1}) \|v\|_{L^2(\mathbf{R}^n)}.$$

If  $k = n - 1$  then

$$\|(1)\|_{L^2(\mathbf{R}^n)} \leq Ch^{n-1} \alpha \sum_{j=0}^{\infty} 2^{-j} \|v\|_{L^2(\mathbf{R}^n)} \leq Ch^{n-1} \alpha \|v\|_{L^2(\mathbf{R}^n)}.$$

To estimate the term (2) we use (3.32). We obtain

$$\|(2)\|_{L^2(\mathbf{R}^n)} \leq Ch^{n-1} \alpha^2 \|v\|_{L^2(\mathbf{R}^n)}.$$

Using these estimates and (3.33) we deduce

$$(3.34) \quad \|1_B S_\lambda^2 1_B v\|_{L^2(\mathbf{R}^n)} \leq C \alpha^{2\sigma} h^{n-1} \|v\|_{L^2(\mathbf{R}^n)}$$

where  $\sigma = 1$  if  $k \leq n - 3$ ,  $\sigma = 1^-$  if  $k = n - 2$ ,  $\sigma = \frac{1}{2}$  if  $k = n - 1$ . □

Gathering the estimates proved in (3.21) and (3.34) we obtain (3.16) which proves Proposition 3.4 and therefore Proposition 3.1. The proof of Theorem 1.1 is complete.

## APPENDIX A. SOME TECHNICAL RESULTS

### A.1. A lemma.

**Lemma A.1.** *Let  $w \in C^\infty(M)$  be a solution of the equation  $(h^2 \Delta_g + 1)w = F$  Then*

$$\|w\|_{L^2(N_{\alpha h^{1/2}})} \leq C \frac{\alpha^\gamma}{h} (\|F\|_{L^2(M)} + \|w\|_{L^2(M)})$$

where  $\gamma = \frac{1}{2}$  if  $k = n - 1$ ,  $\gamma = 1$  if  $1 \leq k \leq n - 2$ .

*Proof.* Setting  $\|\nabla_g w\|_{L^2(M)} = \left( \int_M g_p(\nabla_g w(p), \overline{\nabla_g w(p)}) dv_g(p) \right)^{\frac{1}{2}}$  we deduce from Lemma 2.1 and from the equation that

$$(A.1) \quad \begin{aligned} h\|\nabla_g w\|_{L^2(M)} &\leq C(\|F\|_{L^2(M)} + \|w\|_{L^2(M)}), \\ h^2\|\Delta_g w\|_{L^2(M)} &\leq \|F\|_{L^2(M)} + \|w\|_{L^2(M)}. \end{aligned}$$

Now setting  $\tilde{w}_j = (\zeta_j w) \circ \theta^{-1}$  (see (3.10)), we have

$$(A.2) \quad \|w\|_{L^2(N_{\alpha h^{1/2}})} \leq \sum_{j=1}^{n_0} \|\zeta_j w\|_{L^2(N_{\alpha h^{1/2}})} \leq C \sum_{j=1}^{n_0} \|\tilde{w}_j\|_{L^2(B_{\alpha, h})}.$$

For fixed  $j \in \{1, \dots, n_0\}$  we deduce from (A.1) that

$$(A.3) \quad h\|\tilde{w}_j\|_{H^1(B_{\alpha, h})} + h^2\|\tilde{w}_j\|_{H^2(B_{\alpha, h})} \leq C(\|F\|_{L^2(M)} + \|w\|_{L^2(M)}),$$

from which we deduce that for  $\varepsilon > 0$  small

$$(A.4) \quad h^{1+\varepsilon}\|\tilde{w}_j\|_{H^{1+\varepsilon}(B_{\alpha, h})} \leq C(\|F\|_{L^2(M)} + \|w\|_{L^2(M)}).$$

Using the Sobolev embeddings  $H^1(\mathbf{R}) \subset L^\infty(\mathbf{R})$  and  $H^{1+\varepsilon}(\mathbf{R}^2) \subset L^\infty(\mathbf{R}^2)$ , the fact that  $B_{\alpha, h} \subset \{x = (x_a, x_b) \in \mathbf{R}^k \times \mathbf{R}^{n-k} : |x_b| \leq \alpha h^{\frac{1}{2}}\}$  and (A.3), (A.4) we obtain

$$\begin{aligned} \|\tilde{w}_j\|_{L^2(B_{\alpha, h})} &\leq (\alpha h^{\frac{1}{2}})^{\frac{1}{2}} \|\tilde{w}_j\|_{H^1(B_{\alpha, h})} \leq C \frac{\alpha^{\frac{1}{2}}}{h} (\|F\|_{L^2(M)} + \|w\|_{L^2(M)}), \quad \text{if } k = n-1, \\ \|\tilde{w}_j\|_{L^2(B_{\alpha, h})} &\leq \alpha h^{\frac{1}{2}} \|\tilde{w}_j\|_{H^{1+\varepsilon}(B_{\alpha, h})} \leq C \frac{\alpha}{h} (\|F\|_{L^2(M)} + \|w\|_{L^2(M)}), \quad \text{if } k \leq n-2. \end{aligned}$$

Lemma A.1 follows then from (A.2).  $\square$

**A.2. Stein's lemma.** In this section we prove a version of Stein Lemma [43, Chap 9, Proposition 1.1]. For  $\lambda > 0$  we consider the operator

$$(A.5) \quad T^\lambda u(\Xi) = \int_{\mathbf{R}^n} e^{i\lambda\phi(X, \Xi)} a(X, \Xi, \lambda) u(X) dX$$

where  $\phi : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$  is a smooth real valued phase and  $a$  a smooth symbol.

We shall make the following assumptions.

(H1) there exists a compact  $\mathcal{K} \subset \mathbf{R}^n \times \mathbf{R}^n$  such that  $\text{supp}_{X, \Xi} a \subset \mathcal{K}$ ,  $\forall \lambda > 0$ ,

(H2)  $\text{rank} \left( \frac{\partial^2 \phi}{\partial X_i \partial \Xi_j}(X, \Xi) \right)_{1 \leq i, j \leq n} \geq p \in \{1, \dots, n\}, \forall (X, \Xi) \in \mathcal{K}$ .

Our purpose is to prove the following result.

**Theorem A.2.** *Under the hypotheses (H1) and (H2) there exists  $C > 0$  such that*

$$\|T^\lambda u\|_{L^2(\mathbf{R}^n)} \leq C \lambda^{-\frac{p}{2}} \|u\|_{L^2(\mathbf{R}^n)}$$

for every  $\lambda > 0$  and all  $u \in L^2(\mathbf{R}^n)$ .

**Remark A.3.** We shall actually apply Theorem A.2 for a family of phases  $\phi_j$  and symbols  $a_j$  converging in  $C^\infty$  topology to a fixed phase  $\phi$  and symbol  $a$  and use that in such case the estimates are uniform with respect to the parameter  $j$ , which will be a consequence of the proof given below.

Below we shall prove a slightly stronger result.

First of all by the hypothesis (H1), using partitions of unity, we may assume without loss of generality that with a small  $\varepsilon > 0$

$$\text{supp}_{X,\Xi} a \subset V_{\rho_0} = \{(X, \Xi) \in \mathbf{R}^n \times \mathbf{R}^n : |X - X_0| + |\Xi - \Xi_0| \leq \varepsilon\}, \quad \rho_0 = (X_0, \Xi_0).$$

Moreover changing if necessary the orders of the variables we may assume that near  $\rho_0$

$$X = (x, y) \in \mathbf{R}^p \times \mathbf{R}^{n-p}, \quad \Xi = (\xi, \eta) \in \mathbf{R}^p \times \mathbf{R}^{n-p}$$

and for all  $(X, \Xi) \in V_{\rho_0}$  the  $p \times p$ -matrix

$$(A.6) \quad M_p(X, \Xi) = \left( \frac{\partial^2 \phi}{\partial x_i \partial \xi_j}(X, \Xi) \right)_{1 \leq i, j \leq p}$$

is invertible with  $\|M_p(X, \Xi)^{-1}\| \leq c_0$ .

Then we have

**Theorem A.4.** *There exists a positive constant  $C$  such that for every  $\lambda > 0$  we have*

$$\|T^\lambda u\|_{L^\infty(\mathbf{R}_\eta^{n-p}, L^2(\mathbf{R}_\xi^p))} \leq C \lambda^{-\frac{p}{2}} \|u\|_{L^1(\mathbf{R}_y^{n-p}, L^2(\mathbf{R}_x^p))}$$

for all  $u \in L^1(\mathbf{R}_y^{n-p}, L^2(\mathbf{R}_x^p))$ .

Theorem A.2 follows from Theorem A.4 using (H1) and Hölder inequality.

*Proof of Theorem A.4.* Let us set for  $(y, \eta) \in \mathbf{R}^{n-p} \times \mathbf{R}^{n-p}$

$$(A.7) \quad \phi_{(y,\eta)}(x, \xi) = \phi(x, y, \xi, \eta), \quad a_{(y,\eta)}(x, \xi) = a(x, y, \xi, \eta), \quad u_y(x) = u(x, y),$$

$$(A.8) \quad T_{(y,\eta)}^\lambda u_y(\xi) = \int_{\mathbf{R}^p} e^{i\lambda \phi_{(y,\eta)}(x, \xi)} a_{(y,\eta)}(x, \xi) u_y(x) dx.$$

Then we have

$$(A.9) \quad T^\lambda u(\Xi) = \int_{\mathbf{R}^{n-p}} T_{(y,\eta)}^\lambda u_y(\xi) dy.$$

We claim that there exists  $C > 0$  such that for every  $(y, \eta) \in V_{(y_0, \eta_0)}$  we have

$$(A.10) \quad \|T_{(y,\eta)}^\lambda u_y\|_{L^2(\mathbf{R}_\xi^p)} \leq C \lambda^{-\frac{p}{2}} \|u_y\|_{L^2(\mathbf{R}_x^p)} \quad \forall \lambda > 0.$$

Assuming for a moment that (A.10) is proved we obtain

$$\|T^\lambda u(\cdot, \eta)\|_{L^2(\mathbf{R}_\xi^p)} \leq \int_{\mathbf{R}^{n-p}} \|T_{(y,\eta)}^\lambda u_y\|_{L^2(\mathbf{R}_\xi^p)} dy \leq C \lambda^{-\frac{p}{2}} \int_{\mathbf{R}^{n-p}} \|u(\cdot, y)\|_{L^2(\mathbf{R}_x^p)} dy$$

which implies immediately the conclusion of Theorem A.2.

The claim (A.10) follows immediately from the proof of the Proposition in [43, Chapter IX, Section 1.1, p. 377–379]. However, for the convenience of the reader, we shall give it here.

For simplicity we shall skip the subscript  $(y, \eta)$ , keeping in mind the uniformity, with respect to  $(y, \eta) \in V_{(y_0, \eta_0)}$ , of the constants in the estimates. Therefore we set

$$S_\lambda = T_{(y,\eta)}^\lambda, \quad \phi_{(y,\eta)} = \psi, \quad b = a_{(y,\eta)}.$$

It follows from (A.6) that the matrix

$$N(x, \xi) = \left( \frac{\partial^2 \psi}{\partial x_i \partial \xi_j}(x, \xi) \right)_{1 \leq i, j \leq p}$$

is invertible and  $\|N(x, \xi)^{-1}\| \leq c_0$  where  $c_0$  is independent of  $(y, \eta)$ . Now by the usual trick the estimate (A.10) is satisfied if and only if we have

$$(A.11) \quad \|S_\lambda S_\lambda^* f\|_{L^2(\mathbf{R}^p)} \leq C \lambda^{-p} \|f\|_{L^2(\mathbf{R}^p)}$$

with  $C$  independent of  $(y, \eta)$ . It is easy to see that

$$(A.12) \quad S_\lambda S_\lambda^* f(\xi) = \int_{\mathbf{R}^p} K(\xi, \xi') f(\xi') d\xi'$$

with

$$K(\xi, \xi') = \int_{\mathbf{R}^k} e^{i\lambda(\psi(x, \xi) - \psi(x, \xi'))} b(x, \xi) \bar{b}(x, \xi') dx.$$

Let us set

$$c(x, \xi, \xi') = N(x, \xi)^{-1} \frac{\xi - \xi'}{|\xi - \xi'|}.$$

Then we can write

$$(A.13) \quad c(x, \xi, \xi') \cdot \nabla_x e^{i\lambda(\psi(x, \xi) - \psi(x, \xi'))} = e^{i\lambda(\psi(x, \xi) - \psi(x, \xi'))} i\lambda \Delta(x, \xi, \xi')$$

where

$$\begin{aligned} \Delta(x, \xi, \xi') &= \sum_{j=1}^k c_j(x, \xi, \xi') \left( \frac{\partial \psi}{\partial x_j}(x, \xi) - \frac{\partial \psi}{\partial x_j}(x, \xi') \right), \\ &= \sum_{j,l=1}^k c_j(x, \xi, \xi') \left( \frac{\partial^2 \psi}{\partial x_j \partial \xi_l}(x, \xi) (\xi_l - \xi'_l) + \mathcal{O}(|\xi - \xi'|^2) \right), \\ &= \langle N(x, \xi) c(x, \xi, \xi'), \xi - \xi' \rangle + \mathcal{O}(|\xi - \xi'|^2) = |\xi - \xi'| + \mathcal{O}(|\xi - \xi'|^2), \end{aligned}$$

where  $\mathcal{O}(|\xi - \xi'|^2)$  is independent of  $(y, \eta)$ . Since  $b$  has small support in  $\xi$  we deduce that

$$(A.14) \quad \Delta(x, \xi, \xi') \geq C |\xi - \xi'|.$$

Moreover since the derivatives with respect to  $x$  of  $N(x, \xi)^{-1}$  are products of  $N(x, \xi)^{-1}$  and derivatives of  $N(x, \xi)$ , we see that all the derivatives with respect to  $x$  of  $\Delta(x, \xi, \xi')$  are uniformly bounded in  $(y, \eta)$  near  $(y_0, \eta_0)$ . Let us set

$$L = \frac{1}{i\lambda \Delta(x, \xi, \xi')} c(x, \xi, \xi') \cdot \nabla_x.$$

It follows from (1.4) and the fact that  $b$  has compact support in  $x$  that for every  $N \in \mathbf{N}$  we can write

$$K(\xi, \xi') = \int_{\mathbf{R}^p} e^{i\lambda(\psi(x, \xi) - \psi(x, \xi'))} ({}^t L)^N [b(x, \xi) \bar{b}(x, \xi')] dx.$$

We deduce from (A.14) that for every  $N \in \mathbf{N}$  there exists  $C_N > 0$  independent of  $(y, \eta)$  such that

$$|K(\xi, \xi')| \leq \frac{C_N}{(1 + \lambda |\xi - \xi'|)^N}.$$

Taking  $N > p$  we deduce from (A.12) and Schur lemma that (A.11) holds with a constant  $C$  independent of  $(y, \eta)$ . This completes the proof.  $\square$

**Lemma A.5.** Let  $d \geq 1$ ,  $\delta \in \mathbf{R}$  and  $\varphi_0(x, x') = (\sum_{j=1}^d (x_j - x'_j)^2 + \delta^2)^{\frac{1}{2}}$ . Let  $M = \left( \frac{\partial^2 \varphi_0}{\partial x_j \partial x'_k}(x, x') \right)_{1 \leq j, k \leq d}$ . Then

- (i) if  $\delta \neq 0$   $M$  has rank  $d$  for all  $x, x' \in \mathbf{R}^d$ ,
- (ii) if  $\delta = 0$   $M$  has rank  $d - 1$  for  $x \neq x'$ .

*Proof.* (i) A simple computation shows that

$$M = \varphi_0(x, x')^{-1} (-\delta_{jk} + \omega_j \omega_k), \quad \omega_j = \frac{x_j - x'_j}{\varphi_0(x, x')}$$

where  $\delta_{jk}$  is the Kronecker symbol. For  $\lambda \in \mathbf{R}$  consider the polynomial in  $\lambda$

$$F(\lambda) = \det \left( -\delta_{jk} + \lambda \omega_j \omega_k \right)_{1 \leq j, k \leq d}$$

We have obviously  $F(0) = (-1)^d$ . Now denote by  $C_j(\lambda)$  the  $j^{\text{th}}$  column of this determinant. Then

$$F'(\lambda) = \sum_{k=1}^d \det (C_1(\lambda), \dots, C'_k(\lambda), \dots, C_d(\lambda)).$$

Since  $\det (C_1(0), \dots, C'_k(0), \dots, C_d(0)) = (-1)^{d-1} \omega_j^2$  we obtain  $F'(0) = (-1)^{d-1} \sum_{j=1}^d \omega_j^2$ . Now  $C_j(\lambda)$  being linear with respect to  $\lambda$  we have  $C''_j(\lambda) = 0$ . Therefore

$$F''(\lambda) = \sum_{j=1}^d \sum_{k=1, k \neq j}^d \det (C_1(\lambda), \dots, C'_j(\lambda), \dots, C'_k(\lambda), \dots, C_d(\lambda)).$$

Since  $C'_j(\lambda) = \omega_j(\omega_1, \dots, \omega_d)$  and  $C'_k(\lambda) = \omega_k(\omega_1, \dots, \omega_d)$  we have  $F''(\lambda) = 0$  for all  $\lambda \in \mathbf{R}$ . It follows that  $F(\lambda) = (-1)^d (1 - \lambda \sum_{j=1}^d \omega_j^2)$ . Therefore

$$\det M = (-1)^d (1 - \sum_{j=1}^d \omega_j^2) = (-1)^d \frac{\delta^2}{\varphi_0(x, x')^2} \neq 0.$$

(ii) Since  $x - x' \neq 0$  we may assume without loss of generality that  $\omega_d \neq 0$ . Set

$$A = \left( -\delta_{jk} + \omega_j \omega_k \right)_{1 \leq j, k \leq d-1}.$$

Introducing  $G(\lambda) = \det \left( -\delta_{jk} + \lambda \omega_j \omega_k \right)_{1 \leq j, k \leq d-1}$  the same computation as above shows that

$$\det A = (-1)^{d-1} (1 - \sum_{j=1}^{d-1} \omega_j^2) = (-1)^{d-1} \omega_d^2 \neq 0.$$

□

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