RESONANCE EXPANSIONS IN SEMI-CLASSICAL PROPAGATION

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1. Introduction and statement of results

Resonances, or scattering poles, are complex numbers which mathematically describe meta-stable states: the real part of a resonance gives the rest energy, and its imaginary part, the rate of decay of a meta-stable state. They appear in many branches of physics, chemistry and mathematics, from molecular dynamics to automorphic forms – see [12] for a general introduction (and [5, Sect.1] for a somewhat different point of view).

The purpose of this note is to present an expansion of a semi-classical propagating state in terms of resonances close to the real axis. Since inverse of the distance to the real axis gives the life-span of a resonance, the times at which the expansions are valid have to be large enough to eliminate the contribution of other resonances (see the remark at the end of Sect.3). Our method also gives an expansion of scattered classical waves in terms of scattering poles close to the real axis. This expansion is weaker than the expansion presented in [11] but the advantage is that it does not depend on any hard to verify conditions (which was the case in [11]).

Our semi-classical results are applicable to Schrödinger or wave equations for long range "black box" perturbations [6] – see Sect.2 for a review of definitions. A typical operator to keep in mind is $P(h) = -h^2\Delta + V(x)$, where $|V(x)| \leq C|x|^{-\epsilon}$, $\epsilon > 0$ is analytic in a complex conic neighbourhood of infinity (in this case R_0 in the theorem below can be taken to be 0). We denote by $\mathrm{Res}(P)$ the set of resonances of $P(h)$, that is the set of poles of the meromorphic continuation of $R(z,h) = (P(h) - z)^{-1}$ from $\mathrm{Im}\, z > 0$ to the lower half-plane.

Theorem 1. *Let $P(h)$ be an operator satisfying the assumptions of Sect.2 and let $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$ be equal to one on a neighbourhood of $B(0,R_0)$. Let $\psi \in \mathcal{C}^\infty_{\rm c}((0,\infty))$ and let* chsupp $\psi = [a,b]$. *We put $\mu(z) = z$ or $\sqrt{z}$, with the convention that $\sqrt{z} > 0$ for $z > 0$. There exists $0 < \delta < c(h) < 2\delta$ such that for every $M > M_0$ there exist $L = L(M)$ and $d(h)$, $d(h)h^{-M} = 1 + \mathcal{O}(h)$, and we have*

$$
\begin{aligned}
\chi e^{-it\mu(P)/h}\chi\psi(P) = & \sum_{z\in\Omega(h)\cap \mathrm{Res(P)}} \chi \mathrm{Res}(e^{-it\mu(\bullet)/h}R(\bullet,h),z)\chi\psi(P) \\
& + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty)\,, \quad \text{for} \;\; t > h^{-L}\,, \\
& \Omega(h) = (a - c(h), b + c(h)) - i[0, d(h))\,,
\end{aligned}
\tag{1.1}
$$

and where $\mathrm{Res}(f(\bullet), z)$ denotes the residue of a meromorphic family of operators, f, at z.

The functions $c(h)$ and $d(h)$ depend on the distribution of resonances: roughly speaking we cannot "cut" through a dense cloud of resonances. Even in the very well understood case of the modular surface [1, Theorem 1] there is, currently at least, the need for some non-explicit grouping of terms. This is eliminated by the separation condition [11, (4.4)] which however is hard to verify.

The method of proof is a direct consequence of earlier work of Tang and the second author, which in turn draws heavily from the work of Melrose, Sjöstrand, and Stefanov-Vodev – see [10],[11] and the references given there. The dynamical nature of resonances was emphasized early by Lax and Phillips [3] and their celebrated semi-group still provides the most elegant connection between the stationary and dynamical definitions. In practice, that is, when evolution equations are being considered, the semi-group methods provides expansions in non-trapping situations only – see Sect.3 of [11] for a review of Vainberg's direct proof.

The time dependent theory of resonances has also been investigated recently in [5] and [9]. The difference here lies in considering many resonances at high energies rather than time dependent theory of *a single* resonance obtained by perturbing an embedded eigenvalue. In the energy régimes at which molecular reactions take place there are normally many resonant states and it is their contributions to the propagation of a state that seems to be of interest – see [4] and references given there. The random matrix method is one of the tools used in transition state theory and it still remains inaccessible in rigorous work on quantum mechanics.

Ideas used in the semi-classical case provide an unconditional result in the case of the classical wave equation. Suppose that the operator P satisfies the assumptions of Sect.2 with $h = 1$. The wave group of P can be defined abstractly by

$$\mathcal{U}(t) = \exp \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix} = \begin{pmatrix} D_t U(t) & U(t) \\ D_t^2 U(t) & D_t U(t) \end{pmatrix} \quad \text{with} \quad U(t) = i \left(\frac{\sin t\sqrt{P}}{\sqrt{P}} \right) \tag{1.2}$$

As usual we define the iterated domain by

$$\mathcal{D}^L = (P + i)^{-L} \mathcal{H}. \tag{1.3}$$

Our result is slightly different depending on finer assumptions on P which we state as three cases:

Case 1	$P\|_{\mathbb{R}^n \setminus B(0,R_0)} = -\Delta\|_{\mathbb{R}^n \setminus B(0,R_0)}$	n odd
Case 2	$P\|_{\mathbb{R}^n \setminus B(0,R_0)} = -\Delta\|_{\mathbb{R}^n \setminus B(0,R_0)}$	n even
Case 3	$P\|_{\mathbb{R}^n \setminus B(0,R_0)} = Q\|_{\mathbb{R}^n \setminus B(0,R_0)}$	any n

where Q is an elliptic operator close to the Laplacian at infinity – see (2.2) with $h = 1$.

Theorem 2. *Let P be an operator satisfying the assumptions of Sect.2 with $h = 1$. Let $\chi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ be equal to one on a neighbourhood of $B(0, R_0)$ and $\Psi \in C^{\infty}(\mathbb{R})$ be an even function such that*

$$\Psi(x) = 1 \quad \begin{cases} \text{for } x \in \mathbb{R} \text{ in cases } 1 \text{ and } 2 \\ \text{for } x \geq 1 \text{ in case } 3 \end{cases} , \qquad \Psi(x) = 0 \quad \text{near } 0 \text{ in case } 3 \tag{1.4}$$

For every $M > M_0$, there exist $\epsilon = \epsilon(M) > 0$, a function $c(t)$ satisfying $|c(t) - t^{\epsilon}| \leq C$, and a curve, $y = \gamma(x) > 0$, such that $\gamma(x)|x|^{-M} = 1 + \mathcal{O}(|x|^{-1})$, and

$$\begin{gathered} \chi U(t) \Psi(\sqrt{P}) \chi = \sum_{\substack{\operatorname{Im} \lambda_j > -\gamma(\operatorname{Re} \lambda_j) \\ 1 < |\operatorname{Re} \lambda_j| < c(t)}} \chi \operatorname{Res}(e^{-it\bullet} R(\bullet), \lambda_j) \chi + E(t) + W(t) \\ \lambda_j^2 \in \operatorname{Res}(P) \,, \ \operatorname{Im} \lambda_j < 0 \,, \\ \|E(t)\|_{\mathcal{D}^L \to \mathcal{H}} \leq \begin{cases} C_M t^{K_0 - \epsilon L} & \text{in cases } 1 \text{ and } 3 \\ C t^{-n+1} & \text{in case } 2 \,, \end{cases} \end{gathered} \tag{1.5}$$

where K_0 is a fixed constant and L is large enough in case 2, and $W(t)$ corresponds to the contribution from the pure point spectrum of P.

We note that when the algebraic multiplicity is equal to the geometric multiplicity we have

$$\operatorname{Res}(e^{-it\bullet} R(\bullet), \lambda_j) = e^{-it\lambda_j} \operatorname{Res}(R(\bullet), \lambda_j) \,,$$

while in general, powers of t will appear in the expansion.

The term $W(t)$ has the usual expression:

$$W(t) = i \sum_{\mu_k \in \sigma_{pp}(P)} \chi \frac{\sin(t\sqrt{\mu}_k)}{\sqrt{\mu}_k} \Psi(\sqrt{\mu}_k) \Pi_{\mu_k} \chi \,,$$

where Π_μ is the orthogonal projection on the eigenspace of μ.

2. Preliminaries

To avoid the analysis of specific aspects of obstacle, potential, or metric scattering we work in the "black box" formalism introduced in [8] and generalized further in [6].

The operator we study acts on $\mathcal{H}$, a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)) \,,$$

where $R_0 > 0$ is fixed and $B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}$.

The corresponding orthogonal projections are denoted by $u|_{B(0,R_0)}$ and $u|_{\mathbb{R}^n \setminus B(0,R_0)}$ or by $\mathbb{1}_{B(0,R_0)} u$ and $\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} u$ respectively, where $u \in \mathcal{H}$.

We work in the semi-classical setting and for each $h \in (0, h_0]$, we have

$$P(h) : \mathcal{H} \longrightarrow \mathcal{H}$$

with the domain $\mathcal{D}$, independent of h, and satisfying

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, R_0))$$

uniformly with respect to h (see [6] or [10] for a precise meaning of this statement).

We also assume that

$$\mathbb{1}_{B(0,R_0)} (P(h) + i)^{-1} : \mathcal{H} \longrightarrow \mathcal{H}_{R_0} \quad \text{is compact} \,, \tag{2.1}$$

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} P(h) u = Q(h)(u|_{\mathbb{R}^n \setminus B(0,R_0)}), \quad \text{for } u \in \mathcal{D} \,, \tag{2.2}$$

where $Q(h)$ is a formally self-adjoint operator on $L^2(\mathbb{R}^n)$ given by

$$Q(h) v = \sum_{|\alpha| \leq 2} a_\alpha(x; h) (h D_x)^\alpha v \quad \text{for } v \in C_0^\infty(\mathbb{R}^n)$$

such that $a_\alpha(x; h) = a_\alpha(x)$ is independent of h for $|\alpha| = 2$, $a_\alpha(x; h) \in C_b^\infty(\mathbb{R}^n)$ are uniformly bounded with respect to h, here $C_b^\infty(\mathbb{R}^n)$ denotes the space of C^∞ functions on $\mathbb{R}^n$ with bounded derivatives of all orders,

$$\sum_{|\alpha| = 2} a_\alpha(x; h) \xi^\alpha \geq (1/c) |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \,,$$

for some constant $c > 0$, $\sum_{|\alpha| \leq 2} a_\alpha(x; h) \xi^\alpha \longrightarrow \xi^2$ uniformly with respect to h as $|x| \to \infty$.

The meromorphic continuation is guaranteed by the following analyticity assumption: there exist $\theta \in [0, \pi)$, $\epsilon > 0$ and $R \geq R_0$ such that the coefficients $a_\alpha(x; h)$ of $Q(h)$ extend holomorphically in x to

$$\{r\omega : \omega \in \mathbb{C}^n, \operatorname{dist}(\omega, \mathbf{S}^n) < \epsilon, r \in \mathbb{C}, |r| > R, \arg r \in [-\epsilon, \theta_0 + \epsilon)\}$$

with $\sum_{|\alpha| \leq 2} a_\alpha(x; h) \xi^\alpha \longrightarrow \xi^2$ uniformly with respect to h as $|x| \to \infty$ remains valid in this larger set of x's.

We use $P(h)$ to construct a self-adjoint operator $P^\sharp(h)$ on

$$\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0, R_0))$$

as in [8] where $M = (\mathbb{R}/R\mathbb{Z})^n$ for some $R \gg R_0$. Let $N(P^\sharp(h), I)$ denote the number of eigenvalues of $P^\sharp(h)$ in the interval I, we assume

$$N(P^\sharp(h), [-\lambda, \lambda]) = \mathcal{O}((\lambda/h^2)^{n^\sharp/2}), \quad \text{for } \lambda \geq 1\,, \tag{2.3}$$

for some number $n^\sharp \geq n$.

Under the above assumptions on $P(h)$, the resonances close to the real axis can be defined by the method of complex scaling (see [6] and references given there). They coincide with the poles of the meromorphic continuation of the resolvent $(P(h) - z)^{-1}$ from $\operatorname{Im} z > 0$ to a conic neighbourhood of the positive half axis in the lower half plane. The set of resonances of $P(h)$ will be denoted by $\operatorname{Res}P(h)$ and we include them with their multiplicity.

The spectral assumption (2.3) implies (in a non-trivial way – see [6] and references given there) a bound on the number of resonances: let $\Omega \Subset \{z : \operatorname{Im} z \leq 0\,, \operatorname{Re} z > 0\}$, then

$$\#\, \Omega \cap \operatorname{Res}(P(h)) \leq C h^{-n^\sharp}\,, \tag{2.4}$$

where $\#$ denotes the number of elements counted according to their multiplicities.

We will need two lemmas which come from [10]:

Lemma 2.1. *For any simply connected compact set $\tilde{\Omega} \subset S_\theta$ where*

$$S_\theta = \{z \in \mathbb{C} :\ \operatorname{Max}(-\pi, 2\theta - 2\pi) < -\arg z < 2\theta\}$$

(θ as in condition 3 of $P(h)$) and positive function $g(h) \ll 1$ defined on $0 < h < h_0$, there exist constants $A = A(\tilde{\Omega}) > 0$ and h_1 with $0 < h_1 < h_0$ such that

$$\|\chi(P(h) - z)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq A e^{A h^{-n^\sharp} \log \frac{1}{g(h)}} \quad \forall z \in \tilde{\Omega} \setminus \bigcup_{z_j \in \operatorname{Res}P(h)\cap\tilde{\Omega}} D(z_j, g(h))$$

for $h < h_1$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ near $B(0, R_0)$, and

$$D(z_j, g(h)) = \{z \in \mathbb{C},\ |z_j - z| \leq g(h)\}\,.$$

This is a generalization of earlier results on bounds on the resolvent in the non-physical plane (see [10] and references given there, and [11, Proposition 4.1] for a direct proof in a simpler setting), and is based on the work of Sjöstrand [6].

The second lemma is the "semi-classical" maximum principle (see [11, Lemma 4.1]):

Lemma 2.2. *Fix an integer $n^\sharp$ and let $0 < h < 1$. Suppose $F(z, h)$ is a family of holomorphic function (in z) defined in a neighbourhood of*

$$\Omega(h) = [E(h) - 5h^{\tilde{k}}, E(h) + 5h^{\tilde{k}}] + i[-4h^{k+n^\sharp+2}, 4h^{k+1}]$$

for $k > n^\sharp$, $\tilde{k} \leq k - n^\sharp$, and a real number $E(h)$. If $F(z, h)$ satisfies

$$\begin{aligned} |F(z,h)| &\leq A e^{A h^{-n^\sharp} \log \frac{1}{h}} \ \text{ on } \Omega(h) \\ |F(z,h)| &\leq \frac{1}{\operatorname{Im} z} \ \text{ on } \Omega(h) \cap \{\operatorname{Im} z < 0\}, \end{aligned}$$

then, there exist $h(k) > 0$ and $C > 0$ such that

$$|F(z,h)| \leq \frac{C}{h^{k+n^\sharp+2}} \ \text{ for } z \in \tilde{\Omega}(h) = [E(h) - h^{\tilde{k}}, E(h) + h^{\tilde{k}}] + i[-2h^{k+n^\sharp+2}, 2h^{k+n^\sharp+2}] \tag{2.5}$$

A direct consequence of these two lemmas is the following

Lemma 2.3. *Suppose that the operator $P(h)$ has no resonances in*

$$\Omega(h) = [E(h) - 5h^{\tilde{k}}, E(h) + 5h^{\tilde{k}}] + i[-4h^{k+n^\sharp+2}, 4h^{k+1}]$$

for $k > n^\sharp$, $\tilde{k} \leq k - n^\sharp$, and a real number $E(h)$.

then, there exist $h(k) > 0$ and $C > 0$ such that

$$|\chi(P(h) - z)^{-1}\chi| \leq \frac{C}{h^{k+n^\sharp+2}} \tag{2.6}$$
$$\text{for } z \in \tilde{\Omega}(h) = [E(h) - h^{\tilde{k}}, E(h) + h^{\tilde{k}}] + i[-2h^{k+n^\sharp+2}, 2h^{k+n^\sharp+2}]$$

3. The semi-classical expansions

For simplicity of presentation we assume that there is *no* point spectrum. The contribution of eigenvalues of P to (1.1) is immediate and the modification of the argument is clear.

Let us write

$$R_\pm(z, h) = (P(h) - z)^{-1}, \quad \text{analytic for} \quad \pm \operatorname{Im} z > 0 ,$$

using the same notation for the meromorphic continuation. The spectral projection is then given by Stone's formula:

$$dE_\lambda = (2\pi i)^{-1}(R_-(\lambda) - R_+(\lambda)) ,$$

and the left hand side of (1.1) can be rewritten as

$$\chi e^{-it\mu(P)/h}\chi\psi(P) = \frac{1}{2\pi i}\int_0^\infty e^{-it\mu(\lambda)}\chi(R_-(\lambda) - R_+(\lambda))\chi d\lambda\psi(P) . \tag{3.1}$$

We would like to deform the contour in this integral with some modifications necessary, as we would like to commute χ and $\psi(P)$, and as $\psi \in \mathcal{C}^\infty_c((0,\infty))$.

As recalled in Sect.2, the number of resonances in $[a-2\delta, b+2\delta]-i[0,\delta]$ is bounded by $\mathcal{O}(h^{-n^\sharp})$. Hence there exists $c(h)$,

$$0 < \delta < c(h) < 2\delta ,$$

such that

$$D(a - c(h), 2h^{n^\sharp+1}) \cap \operatorname{Res}(P) = D(b + c(h), 2h^{n^\sharp+1}) \cap \operatorname{Res}(P) = \emptyset .$$

We construct an h-dependent function, ψ_h, satisfying,

$$\psi_h \in \mathcal{C}^\infty(\mathbb{R}) , \quad \operatorname{supp}\psi_h \subset [a - c(h) - h^{n^\sharp+1}, b + c(h) + h^{n^\sharp+1}] , \tag{3.2}$$
$$\psi_h \equiv 1 \text{ on } [a - c(h) + h^{n^\sharp+1}, b + c(h) - h^{n^\sharp+1}] .$$

We have the following slight modification of the standard *almost analytic continuation* lemma (see for instance [2, Sect.8]):

Lemma 3.1. *The function ψ_h satisfying* (3.2) *can be extended to a function in $\mathcal{C}^\infty(\mathbb{C})$ satisfying*

$$\bar{\partial}_z\tilde{\psi}_h(z) = \begin{cases} \mathcal{O}(|\operatorname{Im} z/h^{n^\sharp+1}|^\infty) \\ 0 \quad \textit{if } |\operatorname{Re} z - a + c(h)| > h^{n^\sharp+1} \textit{ and } |\operatorname{Re} z - b - c(h)| > h^{n^\sharp+1} \end{cases} \tag{3.3}$$

We now introduce fixed cut-off functions $\psi_i \in \mathcal{C}^\infty_c((0,\infty))$:

$$\operatorname{supp}\psi_1 \subset [a - 3\delta, b + 3\delta] , \quad \operatorname{supp}\psi_2 \subset [a - \delta, b + \delta] , \tag{3.4}$$
$$\psi_1 \equiv 1 \quad \text{on} \quad [a - 2\delta, b + 2\delta] , \quad \psi_2 \equiv 1 \quad \text{on} \quad [a - \delta/2, b + \delta/2] , .$$

We then rewrite the right hand side of (3.1) as

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_0^\infty e^{-it\mu(\lambda)}\chi(R_-(\lambda)-R_+(\lambda))\psi_h(\lambda)\chi\psi(P)d\lambda + \\
&\frac{1}{2\pi i}\int_0^\infty e^{-it\mu(\lambda)}\chi(R_-(\lambda)-R_+(\lambda))(\psi_1-\psi_h)(\lambda)\left((1-\psi_2)(P)\chi\psi(P)\right)d\lambda + \\
&\frac{1}{2\pi i}\int_0^\infty e^{-it\mu(\lambda)}\chi(R_-(\lambda)-R_+(\lambda))\left((1-\psi_1)(P)\chi\psi(P)\right)d\lambda\,,
\end{aligned}
\tag{3.5}
$$

and we claim that the two last terms give $\mathcal{O}(h^\infty)$ contributions. For that we need the following lemma which comes essentially from Sect.4 of [6] (see also Sect.5 of [8] for a simpler version, and Sect.3 of [7]):

Lemma 3.2. *Let χ, ψ, and ψ_i have the same properties as in* (3.5)*. Then*

$$
\begin{aligned}
\langle P\rangle^m\psi_i(P)(1-\chi) &= \langle Q\rangle^m\psi_i(Q)(1-\chi)+\mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty)\\
\langle P\rangle^m(1-\chi)\psi(P) &= \langle Q\rangle^m(1-\chi)\psi(Q)+\mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty)\,,
\end{aligned}
$$

where $Q=Q(h)$ is the same as in (2.2)*.*

Hence to show that the last terms in (3.5) are negligible we observe that

$$
\langle Q\rangle^m(1-\psi_j)(Q)\chi\psi(Q)=\mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty)\,,
$$

which, in view of the support properties of ψ and ψ_j, follows from the semi-classical functional calculus (see Sect.8 of [2]). We needed the powers of $\langle P\rangle$ to guarantee the convergence of the spectral measure integral on the real line.

We now deform the contour in the first term of (3.5) (see Fig. 1) using the Green formula and noting that in the support of $\bar\partial_z\tilde\psi_h$ there are no poles of $R_-(z,h)-R_+(z,h)$. Thus

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_0^\infty e^{-it\mu(z)}\chi(R_-(z)-R_+(z))\psi_h(z)\chi\psi(P)dz = \\
&\sum_{z\in\Omega(h)\cap\mathrm{Res(P)}}\chi\mathrm{Res}(e^{-it\mu(\bullet)/h}R(\bullet,h),z)\chi\psi(P)+ \\
&\frac{1}{2\pi i}\int_{\Gamma(h)} e^{-it\mu(z)}\chi(R_-(z)-R_+(z))\tilde\psi_h(z)\chi\psi(P)dz + \\
&\frac{1}{\pi}\int\int e^{-it\mu(z)}\chi(R_-(z)-R_+(z))\bar\partial_z\tilde\psi_h(z)\chi L(dz)\psi(P)\,,
\end{aligned}
\tag{3.6}
$$

where $\Omega(h)$ is as in Theorem 1, $\Gamma(h)$ is the interval $[a-2\delta,b+2\delta]-id(h)$ with the positive orientation, and $L(dz)$ is the Lebesgue $(dxdy)$ measure in $\mathbb{C}$. Here $d(h)=h^M(1+\mathcal{O}(h))$ is chosen so that there are no resonances in $[a-2\delta,b+2\delta]-i[d(h)-h^{M_1},d(h)+h^{M_1}]$, and that $d(\Gamma(h),\mathrm{Res}(P(h)))>h^{M_2}$, for some fixed M_2 (the constants M_1 and M_2 depend on M and $n^\sharp$). The existence of $d(h)$ follows again from the polynomial upper bound on the number of resonances.

The last two terms of the right hand side of (3.6) are negligible. In fact, Lemma 2.1 and the choice of $\Gamma(h)$ guarantee that

$$
\chi(R_-(z,h)-R_+(z,h))\chi=\mathcal{O}_{\mathcal{H}\to\mathcal{H}}(1)e^{h^{-K_1}}\,,
$$

where K_1 depends on M_2 and $n^\sharp$ only. Hence the integral over $\Gamma(h)$ is bounded by

$$
\exp(-th^M)e^{h^{-K_1}}=\mathcal{O}(h^\infty)\,,
$$

if $t>h^{-L}$, $L>K_1+M$.

The last term on the right hand side of (3.6) is estimated using Lemma 2.3 as we recall that there are no poles in the support of $\bar{\partial}_z\tilde{\psi}_h$. Hence we bound the norm of the integrand by $|\operatorname{Im} z/h^{n^\sharp+1}|^\infty h^{-K_2}$. Since $|\operatorname{Im} z| \leq h^M$, this is $\mathcal{O}(h^\infty)$ if $M > n^\sharp + 1$.

Remark. As was observed by Laurence Nedelec, the method of proof of Theorem 1 easily allows replacing $\Omega(h)$ by $\widetilde{\Omega}(h) = (a - c(h), b + c(h)) - i[0, \tilde{d}(h))$ where $\tilde{d}(h) = \epsilon + \mathcal{O}(h)$, $0 < \epsilon < 1/C$. That of course shows the expected fact that

$$\sum_{z \in \widetilde{\Omega}(h) \cap \mathrm{Res(P)} \setminus \Omega(\mathrm{h})} \chi \mathrm{Res}(e^{-it\mu(\bullet)/h} R(\bullet, h), z) \chi \psi(P) = \mathcal{O}(h^\infty) \quad \text{for } t > h^{-L}.$$

Again, without a separation condition on resonances, the bound is very unstable and the domains $\widetilde{\Omega}(h)$ and $\Omega(h)$ have to be carefully chosen.

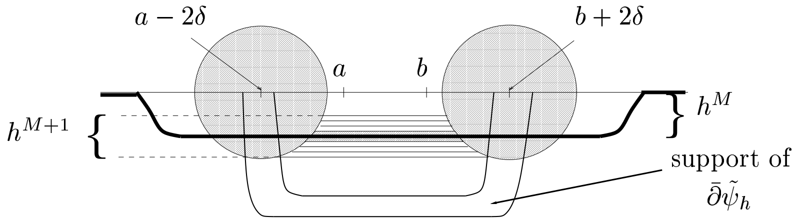

FIGURE 1. The contour deformation in the semi-classical case (there are no resonances in the shaded region)

4. The expansions for the wave equation

In this section we come back to the classical case and assume that the operator P satisfies the asumptions of Sect. 2 for $h = 1$. As in the previous section we assume for simplicity that P has no point spectrum.

For $C > 0$ sufficiently large we write $\tilde{\Omega} = \{|\mathrm{Im} z| < |\mathrm{Re} z|^{-2}/C\,,\ |z| > 1\}$. In the classical case, Lemma 2.1 gives, by a scaling argument, the following estimates on the truncated resolvent:

$$\|\chi(P - z)^{-1}\chi\|_{\mathcal{H} \to \mathcal{H}} \leq A e^{A|z|^{n^\sharp}/2 \log(1/g(1/|z|))} \quad \forall z \in \tilde{\Omega} \setminus \cup_{z_j \in \mathrm{Res}(P) \cap \tilde{\Omega}} D(z_j, g(1/|z|)) \tag{4.1}$$

and for $z_0 \gg 1$ if P has no resonances in the set

$$\Omega(z_0) = [z_0 - 5|z_0|^{-\tilde{k}/2}, z_0 + 5|z_0|^{-\tilde{k}/2}] + i[-4|z_0|^{-(k+n^\sharp+2)/2}, 4|z_0|^{-(k+n^\sharp+2)/2}]$$

then Lemma 2.3 gives

$$\begin{gathered} |\chi(P - z)^{-1}\chi| \leq \frac{C}{|z|^{(k+n^\sharp+2)/2}} \\ \text{for } z \in \tilde{\Omega}(z_0) = [z_0 - |z_0|^{-\tilde{k}/2}, z_0 + |z_0|^{-\tilde{k}/2}] + i[-2|z_0|^{-(k+n^\sharp+2)/2}, 2|z_0|^{-(k+n^\sharp+2})/2]\,. \end{gathered} \tag{4.2}$$

In the case of the wave equation it is convenient to make the change of variables $z = \lambda^2$, with $\mathbb{C} \setminus [0, \infty$ corresponding to $\operatorname{Im} \lambda > 0$. We then write $R(\lambda) = (P - \lambda^2)^{-1}$ and we now have a meromorphic continuation to a conic neighbourhood of $\mathbb{R}$.

We are going to use the estimates (4.2) and (4.1) to perform a contour deformation in the integral:

$$\chi U(t)(P + i)^{-L} \Psi(\sqrt{P}) \chi = \frac{i}{2\pi} \int_{-\infty}^{+\infty} e^{-it\lambda} \chi R(\lambda)(\lambda^2 + i)^{-L} \Psi(\lambda) \chi d\lambda\,, \quad t > 0\,, \tag{4.3}$$

were we noted that the contribution of $R(-\lambda)$ in the spectral projection can be eliminated by contour deformation when $t > 0$ – see [11, Sect.4]. We first consider case 1: odd dimensions and $P = -\Delta$ in the exterior of a (large) ball. Let is fix $M > (k + n^\sharp + 2)/2$. In view of the estimate on the number of resonances, we can construct a smooth curve

$$\tilde{\Gamma} = \{ z = x + i\gamma(x) \ : \ x \in \mathbb{R} \} \tag{4.4}$$

such that

$$\forall x \in \mathbb{R}, \ \ \gamma(x) > 0 \qquad \text{and} \qquad \gamma(x)|x|^{-M} = 1 + \mathcal{O}(|x|^{-1}) \tag{4.5}$$

$$\forall z \in \tilde{\Gamma}, \ \operatorname{dist}\,(z, \operatorname{Res}(P)) > |z|^{-M} \tag{4.6}$$

and such that the length of the part of $\tilde{\Gamma}$ lying between $-x$ and x is smaller than $3x$. This can be achieved by considering $h_j = 2^{-j/2}$, $j \in \mathbb{N}$ and scaling: for $|x| \in [2^j, 2^{j+1}]$ the curve is obtained from the curve $\operatorname{Im} z = d(h_j)$ constructed in the proof of Theorem 1. It is then enough to modify the curves slightly near each of the points $x = 2^j$ in order to join them together.

Take $\epsilon > 0$ to be fixed later. Using again the estimate on the number of resonances, there exists a function $c(t) \sim t^\epsilon$ such that the operator P has no resonances in the balls $D(\pm c(t), t^{-M})$. Now, we deform the contour in (4.2) to the following new contour:

$$\Gamma = \{ z = x + i\gamma(x) \ : \ |x| < c(t) \} \cup c(t) \times -i[0, \gamma(c(t))] \cup \{ |x| \geq c(t) \} \tag{4.7}$$

$$= \Gamma_1 + \Gamma_2 + \Gamma_3 \tag{4.8}$$

and we obtain

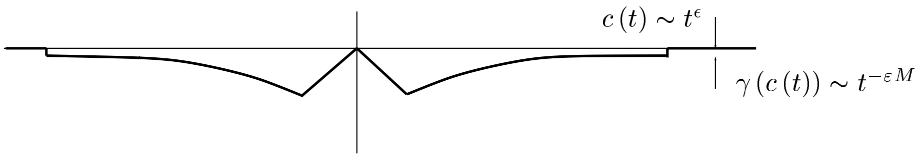

FIGURE 2. The contour deformation in the classical case (for case 2)

$$\begin{aligned} \chi U(t)(P+i)^{-L}\chi = \frac{i}{2\pi} \int_{\Gamma_1+\Gamma_2+\Gamma_3} & e^{-it\lambda}\chi R(\lambda)(\lambda^2+i)^{-L}\chi d\lambda \\ & + \sum_{\substack{\operatorname{Im}\lambda_j > -\gamma(\operatorname{Re}\lambda_j) \\ |\operatorname{Re}\lambda_j| < c(t)}} \chi \operatorname{Res}(e^{-it\bullet}R_-(\bullet), \lambda_j)(P+i)^{-L}\chi \end{aligned} \tag{4.9}$$

The finitely many additional resonances appearing in (4.9) (compared to (1.5) – all of them near 0) give an exponentially small contribution. To obtain Theorem 2 it is now sufficient to control the error term given by the integral on the deformed contour. Let us consider first the contribution of Γ_1. We use (4.1) and get, for any $f \in \mathcal{H}$,

$$\begin{aligned} \| \int_{\Gamma_1} e^{-it\lambda}\chi R(\lambda)(\lambda^2+i)^{-L}\chi d\lambda f \| & \leq \int_{\Gamma_1} e^{-\alpha t |z|^M} A e^{A|z|^{n^\sharp}} dz \|f\| \\ & \leq \int_{-c(t)}^{c(t)} e^{-\alpha t t^{-\epsilon M}} A e^{A t^{\epsilon n^\sharp}} dz \|f\| \,, \quad \alpha > 0 \,, \end{aligned} \tag{4.10}$$

and for $\epsilon(M+n^\sharp) < 1$, this term is decaying faster than any polynomial. We now fix $\epsilon < 1/(M+n^\sharp)$ and consider the contribution of Γ_2, where we use (4.2):

(4.11)
$$\begin{aligned}\| \int_{\Gamma_2} e^{-it\lambda}\chi R(\lambda)(\lambda^2+i)^{-L}\chi d\lambda f\| &\leq \int_0^{\gamma(c(t))} e^{-s\alpha t t^{\epsilon M}} \frac{|z|^{(k+n^\sharp+2)/2}}{(1+|z|)^L} ds\|f\| \\ &\leq \int_0^{2t^{-\epsilon M}} t^{\epsilon(k+n^\sharp+2)/2-\epsilon L} ds\|f\| \leq Ct^{\epsilon[(k+n^\sharp+2)/2-L]}\|f\|\,.\end{aligned}$$

This is exactly the decay stated in Theorem 2. For the contribution coming from Γ_3, we use the spectral calculus, to obtain

$$\| \int_{\Gamma_3} e^{-it\lambda}\chi(R(\lambda)-R(-\lambda))(\lambda^2+i)^{-L}\chi d\lambda f\| \leq \sup_{\Gamma_3} |\lambda^2+i|^{-L}\|f\| \leq \frac{1}{t^{\epsilon L}}\|f\|\,, \tag{4.12}$$

where (see [11, Sect.4]) we can insert the term $R(-\lambda)$ and use the bounds on the spectral projection.

To deal with case 2, that is with n even, we have to modify our argument because the resolvent has a branching point at $\lambda = 0$. Hence we have to deform the contour which near 0 is equal to

$$\{z = x - i\beta x, x \geq 0\} \cup \{z = x + i\beta x, x \leq 0\},\ \text{for } \beta \text{ small} \tag{4.13}$$

and to use the usual estimates for the resolvent near 0:

$$\|\chi R(z)\chi\| \leq C_{a,b}|z|^{n-2}|\log(z)| \tag{4.14}$$

in any sector $\arg z \in [-a,b]$.

To deal with the general case, we consider an almost analytic extension of the function Ψ, $\widetilde{\Psi}$ such that $\bar\partial\widetilde{\Psi}$ is supported in a set where P has no resonances. We deform (4.3) to a contour which for $|z|>1$ is the same as before, and for $|z|<1$ it is as in Fig. 3. By Stokes's formula we

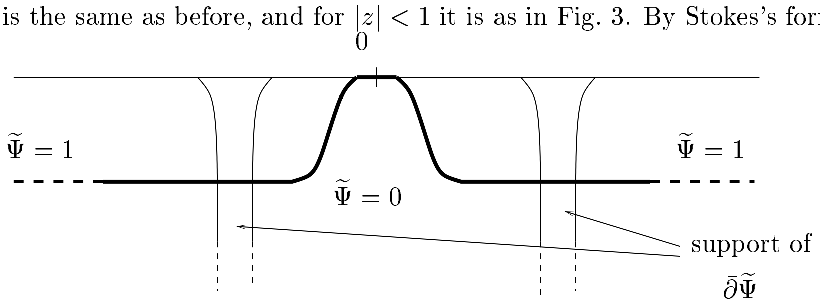

FIGURE 3. The contour deformation in the "black box" case

get exactly the same contributions as in case 1 (since near 0, $\widetilde{\Psi} = 0$) with an additional term

$$\frac{i}{2\pi}\int_\Omega \bar\partial\widetilde{\Psi}(z)\, e^{-it\lambda}\chi(R(\lambda)(\lambda^2+i)^{-L}\chi dL(z) \tag{4.15}$$

where Ω is the domain between the real axis and the new contour (in shadedin Fig. 3). Using that $\bar\partial\widetilde{\Psi}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$, we obtain that this last term is $\mathcal{O}(t^{-\infty})$ (in the energy norm).

Finally, we point out that the remark made at the end of Sect.3 applies also in the classical case: we can sum over resonances in a larger region but the additional contribution can be absorbed into the error term $E(t)$.

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