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CONTROL FOR SCHRÖDINGER OPERATORS ON TORI

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Abstract. A well known result of Jaffard states that an arbitrary region on a torus controls, in the L^2 sense, solutions of the free stationary and dynamical Schrödinger equations. In this note we show that the same result is valid in the presence of a potential, that is for Schrödinger operators, $-\Delta + V$, $V \in \mathcal{C}^\infty$.

1. Introduction

We show how simple methods introduced in [8], [6], [18] (see also [12] and [17]) for the study of the equation

$$(-\Delta - \lambda)u(z) = f(z)\,, \quad z \in \mathbb{T}^2 := \mathbb{R}^2/A\mathbb{Z} \times B\mathbb{Z}\,, \quad A, B \in \mathbb{R} \setminus \{0\}\,,$$

where $\lambda \to \infty$, and the control of

$$i\partial_t u(t,z) = -\Delta u(t,z)\,, \quad z \in \mathbb{T}^2\,.$$

can be adapted to obtain similar results for the equations

$$(-\Delta + V(z) - \lambda)u(z) = f(z)\,, \quad z \in \mathbb{T}^2\,, \tag{1.1}$$

and

$$i\partial_t u(t,z) = (-\Delta + V(z))u(t,z)\,, \quad z \in \mathbb{T}^2\,, \tag{1.2}$$

where $V \in \mathcal{C}^\infty(\mathbb{T}^2)$ is a smooth real valued potential.

The first theorem concerns solutions of the stationary Schrödinger equation and is applicable to high energy eigenfunctions:

Theorem 1. *Let $\Omega \subset \mathbb{T}^2$ be a nonempty open set. There exists a constant $K = K(\Omega)$, depending only on Ω, such that for any solution of* (1.1) *we have*

$$\|u\|_{L^2(\mathbb{T}^2)} \leq K\left(\|f\|_{L^2(\mathbb{T}^2)} + \|u\|_{L^2(\Omega)}\right)\,. \tag{1.3}$$

This means that u on $\mathbb{T}^2$ is *controlled* by u in Ω, in the L^2 sense. The next result, which is in fact more general, concerns the dynamical Schrödinger equation:

Theorem 2. *Let $\Omega \subset \mathbb{T}^2$ be any (non empty) open set and let $T > 0$. There exists a constant $K = K(\Omega, T)$, depending only on Ω and T, such that for any solution of* (1.2) *we have*

$$\|u(0,\bullet)\|^2_{L^2(\mathbb{T}^2)} \leq K \int_0^T \|u(t,\bullet)\|^2_{L^2(\Omega)} dt\,. \tag{1.4}$$

An estimate of this type is called *an observability* result. Once we have it, the HUM method (see [16]) automatically provides the following *control* result:

Theorem 3. *Let $\Omega \subset \mathbb{T}^2$ be any (non empty) open set and let $T > 0$. For any $u_0 \in L^2(\mathbb{T}^2)$, there exists $f \in L^2((0,T)\times\Omega)$ such that the solution of the equation*

$$(i\partial_t + \Delta - V(z))u(t,z) = f\mathbb{1}_{(0,T)\times\Omega}(t,z)\,, \qquad u(0,\bullet) = u_0\,,$$

satisfies

$$u(T,\bullet) \equiv 0\,.$$

By applying Theorem 2 to the initial data $u(0,\bullet) = u$, it is easy to see that Theorem 1 follows from Theorem 2 and the Duhamel formula. As a consequence, we will restrict our attention to Theorem 2.

We remark that Theorem 1 implies Theorem 2 for *some* large T – see [7] and [19]. If Theorem 1 held with $\|f\|_{L^2}$ on the right hand side replaced by $\|f\|_{H^{-\epsilon}}$, for any $\epsilon > 0$, then it would imply Theorem 2 – see [7, Theorem 4]. In particular, this holds when the geometric control condition of Bardos-Lebeau-Rauch [2] and Lebeau [15] is satisfied. We stress that the ϵ improvement of regularity in Theorem 1 is not possible.

In the case of $V \equiv 0$ the estimates (1.3) and (1.4) were proved by Jaffard [13] and Haraux [11] using Kahane's work [14] on lacunary Fourier series.

For a presentation of control theory for the Schrödinger equation we refer to [15] – see also [4],[20], and [7, §3].

We conclude this introduction with comments about a natural class of potentials for which the theorems above should hold. When $V \in L^\infty$ and $\|V\|_{L^\infty} \ll 1$ a perturbation argument shows that (1.3) and (1.4) follow from results with $V = 0$.

The methods of this paper can be extended to the case of $V \in C^0(\mathbb{T}^2)$ by first showing that the constant in the high frequency estimate (3.1) is independent of V for V in a bounded subset of L^∞ and then using approximation and a perturbation argument. The restriction that V is real is not essential but makes the writing easier as we can use the calculus of self-adjoint operators.

Conjecture. *Theorems 1,2,3 hold for $V \in L^\infty(\mathbb{T}^2;\mathbb{C})$. Theorems 2 and 3 hold for time dependent potentials $V(t,z) \in L^\infty(\mathbb{R}\times\mathbb{T}^2;\mathbb{C})$.*

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2. Preliminaries

In this section we will recall the basic control result [3],[8] for rectangles, and the normal form theorem based on Moser averaging method [21].

The following result [3] is related to some earlier control results of Haraux [11] and Jaffard [13][†]:

[†]We remark that as noted in [3] the result holds for any product manifold $M = M_x \times M_y$, and the proof is essentially the same.

Proposition 2.1. *Let Δ be the Dirichlet, Neumann, or periodic Laplace operator on the rectangle $R = [0,a]_x \times [0,b]_y$. Then for any open non-empty $\omega \subset R$ of the form $\omega = \omega_x \times [0,a]_y$, there exists C such that for any solutions of*

$$(\Delta - z)u = f \quad \text{on } R,\ u\restriction_{\partial R} = 0 \tag{2.1}$$

we have

$$\|u\|^2_{L^2(R)} \leq C\left(\|f\|^2_{L^2([0,b]_y);H^{-1}([0,a]_x)} + \|u\|^2_{L^2(\omega)}\right) \tag{2.2}$$

Proof. We will consider the Dirichlet case (the proof is the same in the other two cases) and decompose u, f in terms of the basis of $L^2([0,b])$ formed by the Dirichlet eigenfunctions $e_k(y) = \sqrt{2/b}\sin(2k\pi y/b)$,

$$u(x,y) = \sum_k e_k(y)u_k(x), \qquad f(x,y) = \sum_k e_k(y)f_k(x) \tag{2.3}$$

we get for u_k, f_k the equation

$$\left(-\Delta_x - \left(z - (2k\pi/b)^2\right)\right)u_k = f_k, \qquad u_k(0) = u_k(1) = 0 \tag{2.4}$$

We now claim that

$$\|u_k\|^2_{L^2([0,1]_x)} \leq C\left(\|f_k\|^2_{H^{-1}([0,1]_x)} + \|u_k\restriction_{\omega_x}\|^2_{L^2(\omega)}\right) \tag{2.5}$$

from which, by summing the squares in k, we get (2.2).

To see (2.5) we can use the propagation result below in dimension one, but in this case an elementary calculation is easily available – see [8]. □

The next proposition is the dynamical version of Proposition 2.1 which will be crucial in the proof of Theorem 2. However we change the assumptions on u.

Proposition 2.2. *Let $R = [0,a]_x \times [0,b]_y$, and let $\omega = \omega_x \times [0,b]$, where ω_x is an open subset of $[0,a]$. Suppose that for $W \in \mathcal{C}^\infty(\mathbb{R})$, $W(x+a) = W(x)$,*

$$i\partial_t u(t,x,y) = (-\Delta + W(x))u(t,x,y) \text{ on } \mathbb{R}\times\mathbb{R}^2\,, \tag{2.6}$$

and that, for some $\gamma \in \mathbb{R}$, u satisfies the following periodicity condition:

$$u(t, x+ka, y+\ell b) = u(t,x,y+k\gamma)\,, \quad k,\ell \in \mathbb{Z}\,. \tag{2.7}$$

Then for all $T > 0$ there exists $K > 0$ such that

$$\|u(0,\bullet)\|^2_{L^2(R)} \leq K\int_0^T \|u(t,\bullet)\|^2_{L^2(\omega)}dt\,. \tag{2.8}$$

Remark 2.3. Unitarity of the propagator $\exp(-it(-\Delta+W))$ shows that the $(0,T)$ range integration on the right hand side of (2.6) can be replaced by (T',T) for any $0 \leq T' < T$. Same statement is true in the case of (1.4).

Proof. As in the proof of Proposition 2.1 we reduce the estimate to an estimate in one dimension.

To do that we see that (2.7) implies that u is periodic in y and hence can be expanded into a Fourier series:

$$u(t,x,y)=\sum_{n\in\mathbb{Z}} e^{-it(n\pi/b)^2}u_n(t,x)e^{2\pi iny/b}\,,$$

$$u_n(t,x):=e^{it(n\pi/b)^2}\frac{1}{b}\int_0^b u(t,x,y)e^{-2\pi iny/b}dy\,.$$

The condition (2.7) now means that

$$u_n(t,x+a)=e^{2\pi i\gamma n/b}u_n(t,x)=e^{2\pi i\gamma_n}u_n(t,x)\,,$$
$$\gamma_n=\gamma n/b-[\gamma n/b]\,,\quad 0\le\gamma_n<1\,,$$

that is, the periodicity in x is replaced by a Floquet periodicity condition.

Proposition 2.2 then follows from Lemma 2.4 below. □

Lemma 2.4. *Let $\omega_x\subset[0,a]$ be any open set. Suppose that $v\in L^2_{loc}(\mathbb{R}\times[0,a])$ solves*

$$(i\partial_t-D_x^2-W(x))v=0\,,\quad W(x+a)=W(x)$$

and for some α, $0\le\alpha<1$, v satisfies a Floquet periodicity condition,

$$v(t,x+a)=e^{2\pi i\alpha}v(x)\,.$$

Then for any T there exists C, independent of α, such that

$$\|v(0,\bullet)\|^2_{L^2([0,a])}\le C\int_0^T\|v(t,\bullet)\|^2_{L^2(\omega_x)}dt\,. \tag{2.9}$$

Proof. We use the semi-classical approach developed by Lebeau [15, Theorem 3.1] though the situation is simpler here as we are dealing with internal controls in dimension 1.

Writing $w(x):=e^{-2\pi i\alpha x/a}v(x)$, we obtain a periodic function w satisisfying

$$(i\partial_t-(D_x+\beta)^2-W(x))w=0\,,\quad \beta:=\frac{2\pi\alpha}{a}\,. \tag{2.10}$$

The argument from [15], (used in §§3,4, below – see Remark 3.2 and also [5]) applies and shows uniformity in α. For reader's convenience we provide more details in the appendix. □

Next we present a slight variation of the well known normal form result – see [21] where it was used in the case of Zoll manifolds (of which the circle is a trivial example). Our version can also be seen as a special case of the normal form in [9]

We start by introducing some notation: we have the spaces of standard pseudodifferential operators $\Psi^m(\mathbb{T})$, $\Psi^m(\mathbb{T}^2)$ while

$$\mathcal{C}^\infty\otimes\Psi^m:=\mathcal{C}^\infty(\mathbb{T}^1_x)\otimes\Psi^m(\mathbb{T}_y)\,, \tag{2.11}$$

denotes the space of semiclassical pseudodifferential operators (of order m) in y, depending smoothly on x as a parameter.

To makes things transparent we first present normal form results for tori.

Proposition 2.5. *Let $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^2)$ be equal to 0 in a neighbourhood of $\eta = 0$. Suppose that $V(x,y) \in \mathcal{C}^\infty(\mathbb{T}^1 \times \mathbb{T}^1)$. Then there exist operators*

$$Q(x,y,hD_y) \in \mathcal{C}^\infty \otimes \Psi^0\,, \quad R(x,y,hD_x,hD_y) \in \Psi^0(\mathbb{T}^2)\,,$$

such that

$$\begin{aligned}(I+hQ)\left(D_y^2+V(x,y)\right)\chi(hD_x,hD_y)\\ = (D_y^2+V_0(x))(I+hQ)\chi(hD_x,hD_y)+hR\,,\end{aligned} \tag{2.12}$$

where

$$V_0(x) = \frac{1}{2\pi}\int_{\mathbb{T}^1} V(x,y)dy\,.$$

Proof. Indeed, we have

$$\begin{aligned}(I+hQ)\left(D_y^2+V(x,y)\right)\chi(hD_x,hD_y) - (D_y^2+V_0(x))(\,\mathrm{Id}+hQ)\chi(hD_x,hD_y)\\ = \Big(h[Q,D_y^2]+V(x,y)-V_0(x)+hR_1\Big)\chi(hD_x,hD_y)\end{aligned} \tag{2.13}$$

with $R_1 \in \mathcal{C}^\infty \otimes \Psi^0$. The pseudodifferental calculus shows that to obtain (2.12), it is enough to find $q \in \mathcal{C}^\infty_{\rm c}(\mathbb{T}^2 \times \mathbb{R})$ such that

$$\Big(-\frac{2}{i}\eta\,\partial_y q(x,y,\eta)+V(x,y)-V_0(x)\Big)\chi(\xi,\eta) = 0 \tag{2.14}$$

Since χ vanishes near $\eta = 0$, we can find $\zeta \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}\setminus\{0\})$ equal to 1 on the support of χ, and we can solve (2.14) by taking

$$q(x,y,\eta) = \frac{i\zeta(\eta)}{2\eta}\int_0^y (V(x,y')-V_0(x))dy' \tag{2.15}$$

We notice that by construction, $V_0(x)-V(x,y)$ has y-mean equal to 0 and consequently the function q defined in (2.15) is periodic. □

Corollary 2.6. *There exists operators*

$$W = W(x,y,hD_x,hD_y) \in \Psi^0(\mathbb{T}^2)\,, \quad R = R(x,y,hD_x,hD_y) \in \Psi^0(\mathbb{T}^2)\,,$$

such that

$$\begin{aligned}(I+hQ)\left(D_x^2+D_y^2+V(x,y)\right)\chi(hD_x,hD_y)\\ = \Big(\big(D_x^2+D_y^2+V_0(x)\big)(I+hQ)+W\Big)\chi(hD_x,hD_y)+hR\,,\\ W(x,y,0,\eta)\equiv 0\,.\end{aligned} \tag{2.16}$$

Proof. Indeed, the same calculation as above shows that by symbolic calculus, we can take

$$W(x,y,\xi,\eta) = \frac{2}{i}\xi\partial_x q(x,y,\eta)\widetilde{\chi}(\xi,\eta)\,,$$

where $\widetilde{\chi} \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^2)$ is equal to one on the support of χ. □

In the case of irrational tori $\mathbb{T}^2 \simeq [0,A]\times[0,B]$, $A/B \notin \mathbb{Q}$, we need slightly more complicated versions of Proposition 2.5 and Corollary 2.6. They involve covering $\mathbb{T}^2$ by a strip.

Let us consider a constant rational vector field on the torus given by a direction

$$\Xi_0 = c(nA,mB)\,, \quad n,m \in \mathbb{Z}\,, \quad c \in \mathbb{R}\setminus\{0\}\,.$$

In our argument below the directions corresponding to phase space concentration of the solution to (1.2) will be necessarily rational and Ξ_0 will be chosen to be an isolated direction of that type.

As shown in Fig. 1 we can find a strip bounded in the direction of Ξ_0 and covering $\mathbb{T}^2$. If the torus is itself rational (that is $A/B \in \mathbb{Q}$ in (3.4)) we can find a rectangle R with sides parallel to Ξ_0 and $\Xi_0^\perp$ which covers $\mathbb{T}^2$.

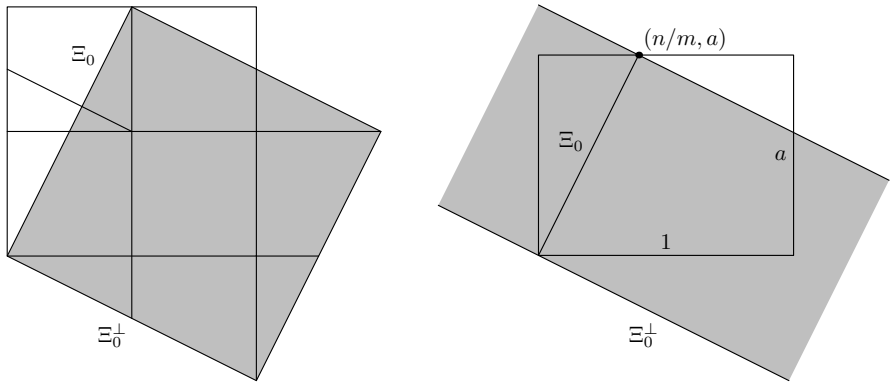

FIGURE 1. On the left, a rectangle, R, covering a rational torus $\mathbb{T}^2$. In that case we obtain a periodic solution on R. On the right, the irrational case: the strip with sides $m\Xi_0 \times \mathbb{R}\Xi_0^\perp$, $\Xi_0 = (n/m, a)$ (not normalized to have norm one), also covers the torus $[0,1] \times [0,a]$. Periodic functions are pulled back to functions satisfying (2.19).

Let us normalize Ξ_0 to have norm one,

$$\Xi_0 = \frac{1}{\sqrt{n^2A^2 + m^2B^2}}(nA, mB)\,, \quad \Xi_0^\perp = \frac{1}{\sqrt{n^2A^2 + m^2B^2}}(-mB, nA)\,. \tag{2.17}$$

The change of coordinates in $\mathbb{R}^2$,

$$F \,:\, (x,y) \longmapsto z = F(x,y) = x\Xi_0^\perp + y\Xi_0\,, \tag{2.18}$$

is orthogonal and hence $-\Delta_z = D_x^2 + D_y^2$.

We have the following simple lemma:

Lemma 2.7. *Suppose that Ξ_0 and F are given by* (2.17) *and* (2.18)*. If $u = u(z)$ is perodic with respect to $A\mathbb{Z} \times B\mathbb{Z}$ then*

$$F^*u(x + ka, y + \ell b) = F^*u(x, y - k\gamma)\,, \quad k, \ell \in \mathbb{Z}\,, \quad (x,y) \in \mathbb{R}^2\,, \tag{2.19}$$

where, for any fixed $p, q \in \mathbb{Z}$,

$$a = \frac{(qn - pm)AB}{\sqrt{n^2A^2 + m^2B^2}}\,, \quad b = \sqrt{n^2A^2 + m^2B^2}\,, \quad \gamma = -\frac{pnA^2 + qmB^2}{\sqrt{n^2A^2 + m^2B^2}}\,.$$

When $B/A = r/s \in \mathbb{Q}$, $r, s \in \mathbb{Z} \setminus \{0\}$, then

$$F^*(x + k\widetilde{a}, y + \ell b) = F^*u(x,y)\,, \qquad k, \ell \in \mathbb{Z}\,, \quad (x,y) \in \mathbb{R}^2\,,$$

for $\widetilde{a} = (n^2s^2 + m^2r^2)a$.

Proof. The proof is a calculation: we need to find a, b, and γ so that for any $k, \ell \in \mathbb{Z}$ there exist $P, Q \in \mathbb{Z}$ so that

$$ka\Xi_0^\perp + (\ell b + k\gamma)\Xi_0 = PA(1,0) + QB(0,1)\,.$$

Taking $b = \sqrt{n^2A^2 + m^2B^2}$ we only need to check that this relation holds with $k = 1$ and $\ell = 0$:

$$a\Xi_0^\perp + \gamma\Xi_0 = pA(1,0) + qB(0,1)\,,$$

which can be solved for a and γ for any p and q. By taking inner products with Ξ_0, $\Xi_0^\perp$ we obtain formulæ for a and γ.

When $B/A = r/s$, $r, s \in \mathbb{Z} \setminus \{0\}$ we need to find $M \in \mathbb{Z} \setminus \{0\}$ so that $M\gamma = Kb$ for some $K \in \mathbb{Z}$. We check that $M = n^2s^2 + r^2m^2$ works and hence we obtain periodicity. □

Remark 2.8. Condition (2.19) for $w = F^*u$ is in fact equivalent to periodicity of u with respect to

$$\mathbb{Z}\,\vec{v}_1 \,\oplus\, \mathbb{Z}\,\vec{v}_2\,, \quad \vec{v}_1 = (nA, mB)\,,\ \vec{v}_2 := (pA, qB)\,.$$

That periodicity is of course implied by periodicity with respect to $A\mathbb{Z} \times B\mathbb{Z}$.

Remark 2.9. A natural choice of p and q which excludes the degenerate cases $p = q = 0$ and $p = n$, $q = m$, can be obtained by assuming (without loss of generality) that n and m are relatively prime and then taking p and q satisfying

$$nq - mp = 1\,,$$

which is possible by Bezout's theorem. This will be the choice we make in what follows.

We can now give a generalized version of Proposition 2.5:

Proposition 2.10. *Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is given by (2.18) and that $V \in \mathcal{C}^\infty(\mathbb{R}^2)$ is periodic with respect to $A\mathbb{Z} \times B\mathbb{Z}$. Let a, b and γ be as in (2.19).*

Let $\chi \in \mathcal{C}_{\rm c}^\infty(\mathbb{R}^2)$ is equal to 0 near the set $\eta = 0$. There exist operators

$$Q(x, y, hD_y) \in \mathcal{C}^\infty(\mathbb{R}) \otimes \Psi^0(\mathbb{R})\,, \quad R(x, y, hD_y, hD_x) \in \Psi^0(\mathbb{R}^2)\,,$$

such that $(F^{-1})^*QF^*$ and $(F^{-1})^*RF^*$ preserve $A\mathbb{Z} \times B\mathbb{Z}$ periodicity, and

$$\begin{aligned}(I + hQ)\left(D_y^2 + F^*V(x,y)\right)\chi(hD_x, hD_y)\\ = (D_y^2 + V_0(x))(I + hQ)\chi(hD_x, hD_y) + hR\,,\end{aligned} \tag{2.20}$$

where

$$V_0(x) := \frac{1}{b}\int_0^b F^*V(x,y)dy\,,$$

satisfies $V_0(x + ka) = V_0(x)$, $k \in \mathbb{Z}$.

Proof. We proceed as in the proof of Proposition 2.5. We need to solve equation (2.14) but now q has to satisfy the twisted periodicity condition (2.19), which will follow if $q_F(z,\eta) := (F^{-1})^* q(\bullet,\eta)(z)$ is $A\mathbb{Z}\times B\mathbb{Z}$ periodic. The equation (2.19) is then equivalent to

$$n\eta\langle \Xi_0, \partial_x\rangle q_F(z,\eta) = V(z) - (F^{-1})^* V_0(z)\,, \tag{2.21}$$

on the support of $\chi(\xi,\eta)$. We note that $(F^{-1})^*V_0$ is the average of V over the (closed) orbit of $\langle \Xi_0, \partial_z\rangle$. In particular, the average of the right hand side is 0.

An equation of this form can be solved on any compact Riemannian manifold: if X is a length one vectorfield with closed integral curves, and f is function integrating (with respect to the length parameter) to 0 along those curves, then there exists u, smooth on M, satisfying $Xu = f$. To see this we solve the equation on each curve, demanding that u integrates to zero on that curve. That determines u uniquely and hence provides a global smooth solution. Note that this is not the solution we took in (2.15). In the notation of (2.15) the current solution corresponds to

$$\begin{aligned} q(x,y,\eta) &= \frac{i\zeta(\eta)}{2\eta}\left(\int_0^y (V(x,y)-V_0(x))dy - q_0(x)\right)\,,\\ q_0(x) &:= \frac{1}{B}\int_0^B\int_0^y (V(x,y')-V_0(x))dy'dy\,. \end{aligned}$$

□

Finally, we have the corresponding analogue of Corollarry 2.6.

Corollary 2.11. *In the notation of Proposition 2.10, there exists operators*

$$W = W(x,y,hD_x,hD_y)\in\Psi^0(\mathbb{R}^2)\,,\quad R = R(x,y,hD_x,hD_y)\in\Psi^0(\mathbb{R}^2)\,,$$

such that $(F^{-1})^*WF^*$ and $(F^{-1})^*RF^*$ preserve $A\mathbb{Z}\times B\mathbb{Z}$ periodicity, and

$$\begin{aligned} &(I+hQ)\left(D_x^2+D_y^2+V(x,y)\right)\chi(hD_x,hD_y)\\ &\quad = \Big(\big(D_x^2+D_y^2+V_0(x)\big)(I+hQ)+W\Big)\chi(hD_x,hD_y)+hR\,,\\ &\qquad W(x,y,0,\eta)\equiv 0\,. \end{aligned} \tag{2.22}$$

3. A semiclassical estimate

The purpose of this section is to prove the main step towards Theorem 2, its semiclassically localized version:

Proposition 3.1. *Let $\chi\in\mathcal{C}^\infty_{\rm c}(-1,1)$ be equal to 1 near 0, and define*

$$\Pi_{h,\rho}(u_0) := \chi\left(\frac{h^2(-\Delta+V)-1}{\rho}\right)u_0\,,\quad \rho>0\,.$$

Then for any $T>0$ there exists $\rho, C, h_0>0$ such that for any $0<h<h_0$, u_0, we have

$$\|\Pi_{h,\rho}u_0\|^2_{L^2}\le C\int_0^T \|e^{-it(-\Delta+V)}\Pi_{h,\rho}u_0\|^2_{L^2(\Omega)}dt\,. \tag{3.1}$$

Proof. We first observe that if the estimate (3.1) is true for some $\rho>0$, then is is true for all $0<\rho'<\rho$. As a consequence, if (3.1) were false, there would exist $T>0$ and sequences

$$h_n\longrightarrow 0,\quad \rho_n\longrightarrow 0,\quad u_{0,n} = \Pi_{h_n,\rho_n}(v_{0,n})\in L^2,$$

$$i\partial_t u_n(t,z) = (-\Delta + V(z))u_n(t,z)\,, \quad u_n(0,z) = u_{0,n}(z)\,,$$

such that

$$1 = \|u_{0,n}\|^2_{L^2}, \qquad \int_0^T \|u_n(t,\bullet)\|^2_{L^2(\Omega)}dt \longrightarrow 0\,.$$

The sequence (u_n) is bounded in $L^2_{\rm loc}(\mathbb{R}\times\mathbb{T}^2)$ and consequently, after possibly extracting a subsequence, there exists a semi-classical defect measure μ on $\mathbb{R}_t\times T^*(\mathbb{T}^2_z)$ such that for any function $\varphi\in\mathcal{C}^0_0(\mathbb{R}_t)$ and any $a\in\mathcal{C}^\infty_{\rm c}(T^*\mathbb{T}^2_z)$, we have

$$\langle\mu,\varphi(t)a(z,\zeta)\rangle = \lim_{n\to\infty}\int_{\mathbb{R}_t\times\mathbb{T}^2}\varphi(t)(a(z,h_nD_z)u_n)(t,z)\overline{u}_n(t,z)dtdz\,.$$

Furthermore, standard arguments[‡] show that the measure μ satisfies

- $$\mu((t_0,t_1)\times T^*\mathbb{T}^2_z) = t_1 - t_0\,. \tag{3.2}$$
- The measure μ on $\mathbb{R}_t\times T^*(\mathbb{T}^2)$ is supported in the set
$$\{(t,z,\zeta)\in\mathbb{R}_t\times\mathbb{T}^2_z\times\mathbb{R}^2_\zeta; |\zeta| = 1\}$$
and is invariant under the action of the geodesic flow:
$$2\langle\zeta,\partial_z\rangle\,\mu_\varphi = 0, \quad \mu_\varphi(E) := \int_{\mathbb{R}\times E}\varphi(t)d\mu, \quad E\subset T^*\mathbb{T}^2.$$
We shall only use that the support of the measure μ is invariant:
$$(t_0,z_0,\zeta_0)\in{\rm supp}(\mu) \implies (t_0,z+s\zeta_0,\zeta_0)\in{\rm supp}(\mu)\,, \quad \forall s\in\mathbb{R} \tag{3.3}$$
- The measure μ vanishes on $(0,T)\times T^*\Omega$.

We are going to show that the measure μ is identically equal to 0 on $(0,T)\times T^*\mathbb{T}^2$. This will provide a contradiction with (3.2).

Remark 3.2. In the case of geometric control, as in the work by Lebeau, the vanishing of $\mu\mid_{(0,T)}$ is a direct consequence of the invariance property. Actually, in Lebeau's work, which concerns boundary value problems, the difficult part is to precisely to prove (analogues of) this invariance property. See the appendix for more details.

The z projection of a trajectory associated to a irrational direction ζ is dense. Consequently, the support of $\mu\mid_{t\in(0,T)}$ contains only points (t,z,Ξ_0) with rational Ξ_0:

$$\mathbb{T}^2\simeq[0,A]_x\times[0,B]_y\,, \quad \Xi_0 = \alpha(A/B,n/m)\,, \quad n,m\in\mathbb{Z}\,, \quad \alpha\in\mathbb{R}\setminus\{0\}\,. \tag{3.4}$$

In fact, that is the condition implying that the trajectory $s\longmapsto z_0+s\Xi_0$ is closed when projected to $\mathbb{T}^2$, for any $z_0\in\mathbb{R}^2$. Any other trajectory is dense.

Define

$$M_\mu := \pi_1({\rm supp}\,\mu\cap\{(t,z,\zeta); t\in(0,T)\})\,, \quad \pi_1:(t,z,\zeta)\longmapsto\zeta\,. \tag{3.5}$$

The discussion above shows that M contains only rational directions and hence it is countable and closed. This in turn implies that it contains an isolated point, Ξ_0 (perfect sets cannot be countable).

[‡]see [1] for a review of recent results about measures used for the Schrödinger equation.

We now consider the Schrödinger equation on on the strip (or rectangle) $R = \mathbb{R}_x \times [0,b]_y$ ($R = [0,a]_x \times [0,b]_y$, respectively) using the function F given in (2.18). In this coordinate system, $\Xi_0 = (0,1)$ – see Fig. 1.

Let $\chi(hD_z)$ be a Fourier multiplier with a symbol supported in a neighborhood of Ξ_0 containing no other points in the intersection with $(0,T)\times T^*\mathbb{T}^2$ of the support of μ, and define

$$\widetilde{u}_n = \chi(h_n D_z) u_n \,,$$

We denote by $\widetilde{\mu}$, the semiclassical measure of the sequence $\widetilde{u}_n$. We clearly have

$$\widetilde{\mu} = (\chi(\zeta))^2 \mu \,,$$

and consequently, we know that the ζ-projection, π_1, of the intersection with $(0,T)\times T^*\mathbb{T}^2$ of the support of the measure $\widetilde{\mu}$ is equal to $\{\Xi_0\}$:

$$M_{\widetilde{\mu}} = \{\Xi_0\} = \{(0,1)\} \,, \tag{3.6}$$

where we used the coordinates (x,y) in the last identification.

Using Proposition 2.10 (or, in the easier case of rational tori, Proposition 2.5) we define

$$v_n = \Big(1 + hQ\Big)\widetilde{u}_n \,.$$

Since the operator Q is bounded on L^2, the semiclassical defect measures associated to v_n and $\widetilde{u}_n$ are equal We now consider the time dependent Schrödinger equation satisfied by v_n. With

$$Q_n := Q(x,y,h_nD_y) \,, \quad R_n := R(x,y,h_nD_x,h_nD_y) \,, \quad W_n := W(x,y,h_nD_x,h_nD_y) \,,$$

given in (2.22) and $\chi_n := \chi(h_nD_z)$, we have

$$\begin{aligned}(i\partial_t + \Delta - V_0(x))v_n &= (I + h_nQ_n)(i\partial_t + \Delta - V(x,y))\chi u_n - W_n\chi u_n - h_nR_nu_n \\ &= -W_n\chi_n u_n + [V,\chi]u_n + o_{L^2}(1) \\ &= -W_n\chi_n u_n + o_{L^2_{x,y}}(1)\end{aligned} \tag{3.7}$$

We also recall that according to Corollary 2.6, the symbol of the operator W vanishes in the set

$$\{x,y,\xi,\eta) \ : \ \xi = 0\} \,.$$

Consequently it vanishes on the intersection with $(0,T)\times T^*\mathbb{T}^2$ of the support of the defect measure of $\chi_n u_n = \widetilde{u}_n$ which, by construction, is included in the set

$$\pi_1^{-1}(M_{\widetilde{\mu}}) = \{(t,x,y,\xi,\eta) \ : \ \xi = 0 \,, \ \eta = 1\}.$$

As a consequence, the semiclassical measure of $W_n\chi_n u_n$ is equal to 0. This implies that

$$(i\partial_t + \Delta - V_0(x))v_n = o_{L^2_{\rm loc}((0,T)\times R)}(1) \,. \tag{3.8}$$

In view of Lemma 2.7, see (2.19), we are now in the setting of Proposition 2.2. To apply it let us choose a band domain $\omega = \omega_x \times [0,b]_y$ where ω_x is a an interval such that any line $\{x\}\times[0,b]_y$, encounters the interior of $\pi_2^{-1}(\Omega)$, where $\pi_2 : R \to \mathbb{T}^2$ – see Fig. 2.

We know that there exists $(t_0,z_0,\Xi_0) \in \mathrm{supp}(\mu)$ for some $t_0 \in (0,T)$.

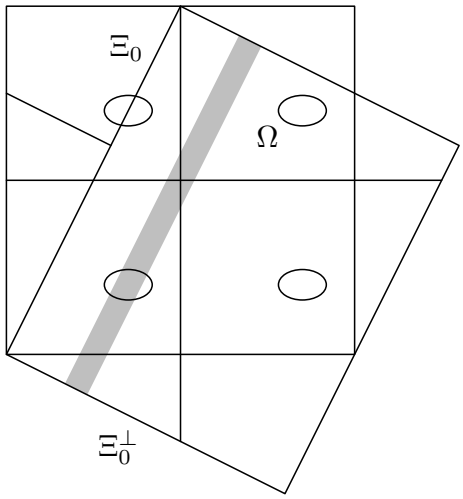

FIGURE 2. The rectangle R, covering a rational torus $\mathbb{T}^2$ and the choice of $\omega = \omega_x \times [0,b]_y$ shown as a shaded region.

Since according to (3.8) on $(0,T)$, the family (v_n) is a family of solutions of the free Schrödinger equations up to $o_{L^2_{\rm loc}((0,T)\times R)}(1)$, we can apply Proposition 2.2 to obtain

$$\begin{aligned}\|v_n\|^2_{L^2((t_0-\epsilon,t_0+\epsilon)\times R)} &\leq \|v_n\|^2_{L^\infty((t_0-\epsilon,t_0+\epsilon);R(\mathbb{T}^2))}\\ &\leq C\int_{t_0-2\epsilon}^{t_0+2\epsilon}\int_\omega |v_n(t,x,y)|^2 dxdy + o(1)\end{aligned}$$

where $\epsilon > 0$ is chosen small enough so that $(t_0-2\epsilon, t_0+2\epsilon) \Subset (0,T)$. This implies that there exists $t_0' \in (t_0-\epsilon, t_0+\epsilon)$, $z_0' \in \omega$, Ξ_0' such that

$$(t_0', z_0', \Xi_0') \in \mathrm{supp}(\widetilde{\mu})\,.$$

From (3.6) we necessarily have $\Xi_0' = \Xi_0$. The invariance of the support of $\widetilde{\mu}$ shows that the whole line

$$(t_0', z_0' + s\Xi_0, \Xi_0) \in \mathrm{supp}(\widetilde{\mu})\,.$$

consequently the support of the measure $\widetilde{\mu}$ does encounter the set $(0,T)\times T^*\Omega$, which gives the contradiction and concludes the proof of Proposition 3.1. □

4. Proof of Theorem 2

To prove Theorem 2 we need to pass from the semiclassical estimate of §3 to an estimate for all frequencies. We start with a result involving an error term:

Proposition 4.1. *For any $T > 0$ and any non empty open set $\Omega \subset \mathbb{T}^2$, there exists $C > 0$ such that for any $u_0 \in L^2(\mathbb{T}^2)$,*

$$\|u_0\|^2_{L^2} \leq C\Big(\int_0^T \int_\Omega |e^{-it(-\Delta+V)}u_0|^2 dzdt + \|u_0\|^2_{H^{-2}}\Big)\,. \tag{4.1}$$

Remark 4.2. The H^{-2} norm on the right hand side of (4.1) can be replaced by the H^{-k} norm for any k. We only need it for some $k \geq 2$ in order to apply the Bardos-Lebeau-Rauch uniqueness-compactness argument at the end of this section.

Proof. Consider a partition of unity

$$1 = \varphi_0(r)^2 + \sum_{j=1}^{\infty} \varphi_j(r)^2\,, \quad \varphi_j(r) := \varphi(R^{-j}|r|)\,, \quad R > 1\,,$$
$$\varphi \in \mathcal{C}^\infty_{\rm c}((R^{-1}, R); [0,1])\,, \quad (R^{-1}, R) \subset \{r \,:\, \chi(r/\rho) \geq 1/2\}\,,$$

where χ and ρ come from Proposition 3.1. Then, we decompose u_0 dyadically:

$$\|u_0\|^2_{L^2} = \sum_{j=0}^{\infty} \|\varphi_j(P_V)u_0\|^2_{L^2}\,. \quad P_V := -\Delta + V\,.$$

Let $\psi \in \mathcal{C}^\infty_{\rm c}((0,T);[0,1])$ satisfy $\psi(t) > 1/2$, on $T/3 < t < 2T/3$. We first observe that in Proposition 3.1 we have actually proved (see the remark after Proposition 2.2) that

$$\|\Pi_h u_0\|^2_{L^2} \leq C \int_{\mathbb{R}} \psi(t)^2 \|e^{-it(-\Delta+V)}\Pi_h u_0\|^2_{L^2(\Omega)} dt\,, \quad 0 < h < h_0\,, \tag{4.2}$$

which is the version we will use.

Taking K large enough so that $R^{-K} \leq h_0$, where h_0 is as in Proposition 3.1, we apply (4.2) to the dyadic pieces:

$$\begin{aligned}
\|u_0\|^2_{L^2} &= \sum_j \|\varphi_j(P_V)u_0\|^2_{L^2} \\
&\leq \sum_{j=0}^{K} \|\varphi_j(P_V)u_0\|^2_{L^2} + C \sum_{j=K+1}^{\infty} \int_0^T \psi(t)^2 \|\varphi_j(P_V)e^{-itP_V}u_0\|^2_{L^2(\Omega)} dt \\
&= \sum_{j=0}^{K} \|\varphi_j(P_V)u_0\|^2_{L^2} + C \sum_{j=K+1}^{\infty} \int_{\mathbb{R}} \|\psi(t)\varphi_j(P_V)e^{-itP_V}u_0\|^2_{L^2(\Omega)} dt\,.
\end{aligned}$$

Using the equation we can replace $\varphi(P_V)$ by $\varphi(D_t)$ which meant that we did not change the domain of z integration. We need to consider the commutator of $\psi \in \mathcal{C}^\infty_{\rm c}((0,T))$ and $\varphi_j(D_t) = \varphi(R^{-j}D_t)$. If $\widetilde{\psi} \in \mathcal{C}^\infty_{\rm c}((0,T))$ is equal to 1 on $\operatorname{supp}\psi$ then the semiclassical pseudodifferential calculus with $h = R^{-j}$ (see for instance [10, Chapter 4]) gives

$$\psi(t)\varphi_j(D_t) = \psi(t)\varphi_j(D_t)\widetilde{\psi}(t) + E_j(t, D_t)\,, \quad \partial^\alpha E_j = \mathcal{O}(\langle t\rangle^{-N}\langle\tau\rangle^{-N}R^{-Nj})\,, \tag{4.3}$$

for all N and uniformly in j.

The errors obtained from E_j can be absorbed into the $\|u_0\|_{H^{-2}(\mathbb{T}^2)}$ term on the right hand side. Hence we obtain

$$\begin{aligned}\|u_0\|_{L^2}^2 &\le C\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + C\sum_{j=0}^{\infty}\int_0^T \|\psi(t)\varphi_j(D_t)e^{-itP_V}u_0\|_{L^2(\Omega)}^2 dt \\ &\le \widetilde{C}\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + C\sum_{j=0}^{\infty}\langle \varphi_j(D_t)^2\widetilde{\psi}(t)e^{-itP_V}u_0, \widetilde{\psi}(t)e^{-itP_V}u_0,\rangle_{L^2(\mathbb{R}_t\times\Omega)} \\ &= \widetilde{C}\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + C\int_{\mathbb{R}} \|\widetilde{\psi}(t)e^{-itP_V}u_0\|_{L^2(\Omega)}^2 dt \\ &\le \widetilde{C}\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + C\int_0^T \|e^{-itP_V}u_0\|_{L^2(\Omega)}^2 dt\end{aligned}$$

where the last inequality is the statement of the proposition. □

To eliminate the H^{-2} error term in (4.1) we use the now classical uniqueness-compactness argument of Bardos, Lebeau and Rauch [2]. For reader's convenience we recall the argument.

Let us fix $\delta \ge 0$ and define

$$N_\delta := \{u_0 \in L^2(\mathbb{T}^2) \,:\, e^{-it(-\Delta+V)}u_0 \equiv 0 \text{ on } (0,T-\delta)\times\Omega\}\,.$$

Let $u_0 \in N_0$. We now define

$$v_{\epsilon,0} = \frac{1}{\epsilon}\left(e^{-i\epsilon(-\Delta+V)} - I\right)u_0\,.$$

If $\epsilon \le \delta$, then $e^{-it(-\Delta+V)}v_{\epsilon,0} \equiv 0$ on $(0,T-\delta)\times\Omega$

We write u_0 in terms of orthonormal eigenvectors of $-\Delta+V$:

$$u_0 = \sum_{\lambda\in\sigma(-\Delta+V)} u_{0,\lambda}e_\lambda.$$

Proposition 4.1 applied with T replaced by $T/2$ gives that for any $0<\alpha,\beta<T/2$, we have

$$\begin{aligned}\|v_{\alpha,0}-v_{\beta,0}\|_{L^2}^2 &\le C\|v_{\alpha,0}-v_{\beta,0}\|_{H^{-2}}^2 \\ &\le C\sum_{\lambda\in\sigma(-\Delta+V)}\left|\frac{e^{-i\alpha\lambda}-1}{\alpha} - \frac{e^{-i\beta\lambda}-1}{\beta}\right|^2(1+\lambda)^{-2}|u_{0,\lambda}|^2 \\ &\le C'\sum_{\lambda\in\sigma(-\Delta+V)}\lambda^2|\alpha-\beta|^2(1+\lambda)^{-2}|u_{0,\lambda}|^2 \le C'|\alpha-\beta|^2\,.\end{aligned}$$

Hence $\lim_{\alpha,\beta\to0}\|v_{\alpha,0}-v_{\beta,0}\|_{L^2}=0$, and there exists $v_0\in L^2$ such that

$$L^2\text{-}\lim_{\alpha\to0} v_{\alpha,0} = v_0\,.$$

This limit is necessarily in N_δ for all $\delta>0$, hence in N_0. On the other hand, we have in the sense of distributions,

$$e^{-it(-\Delta+V)}v_0 = \partial_t e^{-it(-\Delta+V)}u_0\,,$$

which implies that

$$v_0 = -i(-\Delta+V)u_0\,.$$

Hence N_0 is an invariant subspace of $-i(-\Delta+V)$. According to Proposition 4.1, $\|u_0\|_{H^{-2}}$ is a norm on a subspace of L^2, N_0. Hence the unit ball of N_0 is compact, and consequently, N_0 is finite dimensional. This means that there exists an eigenvector w,

$$(-\Delta+V)w=\mu w\,,\quad w|_\Omega=0\,.$$

We can now use the the standard unique continuation results for elliptic second order operators to conclude that $w\equiv 0$ which then implies that $N_0=\{0\}$.

Finally, to conclude the proof of Theorem 2, we argue by contradiction: if (1.4) were not true, we could construct a sequence $(u_{n,0})\in L^2(\mathbb{T}^2)$ such that

$$1=\|u_{n,0}\|_{L^2}\,,\qquad \int_0^T\int_\Omega \left|e^{-it(-\Delta+V)}u_{n,0}\right|^2 dxdt\longrightarrow 0\,,\quad n\longrightarrow\infty\,.$$

We could then extract a subsequence $u_{n_k,0}$ converging weakly in L^2 (and hence strongly in H^{-2}) to a limit $u_0\in N$ which would satisfy, according to Proposition 4.1,

$$1=\lim_{k\to\infty}\|u_{n_k,0}\|_{L^2}\le C\int_0^T\int_\Omega \left|e^{-it(-\Delta+V)}u_{n_k,0}\right|^2 dxdt+C\|u_{n_k,0}\|^2_{H^{-2}}\,.$$

That would imply that

$$1\le C\lim_{k\to\infty}\|u_{n_k,0}\|^2_{H^{-2}}=C\|u_0\|^2_{H^{-2}}\,,$$

showing that there exists $u_0\in N, u_0\not\equiv 0$ contradicting our earlier conclusion. This ends the proof of Theorem 2

Appendix: Proof of Lemma 2.4

To prove (2.9), we rewrite it as an inequality for periodic functions, that is as an inequality on the circle:

$$\|v_0\|^2_{L^2(\mathbb{T}^1)}\le C\int_0^T\|e^{-it(D+\beta)^2+W)}v_0\|^2_L(\omega_x)^2dt\,. \tag{A.1}$$

As presented in detail in the second part of §4, this follows from the reduction performed in (2.10) and the analogue of estimate (4.1): there exist $C>0$ such that for any $\beta\in[0,2\pi/a]$, and any $v_0\in L^2(0,a)$,

$$\|v_0\|^2_{L^2(\mathbb{T}^1)}\le C\Big(\int_0^T\int_{\omega_x}|e^{-it((D+\beta)^2+W)}v_0|^2dzdt+\|v_0\|^2_{H^{-2}(\mathbb{T}^1)}\Big)\,. \tag{A.2}$$

We remark that the proof in §4 applies to this setting where we consider a family of operators, $(D_x+\beta)^2+V$, $\beta\in[0,2\pi/a]$, it could actually handle the more general case of a family of potentials V, relatively compact in L^∞.

As shown in Proposition 4.1 this in turn follows from the analogue Proposition 3.1: for any $T>0$ there exists $C,h_0>0$ such that for any $\beta\in[0,2\pi/a]$, $0<h<h_0$, and $v_0\in L^2(\mathbb{T}^1)$, we have

$$\|\Pi_{h,\beta}v_0\|^2_{L^2(\mathbb{T}^1)}\le C\int_0^T\|e^{-it((D+\beta)^2+W)}\Pi_{h,\beta}v_0\|^2_{L^2(\omega_x)}dt\,, \tag{A.3}$$

where now, in the notation of Proposition 3.1,

$$\Pi_{h,\beta}v_0:=\chi\left(h^2((D+\beta)^2+W)-1\right)v_0\,,\quad \beta\in[0,2\pi/a]\,.$$

If this were false there would exist $T > 0$ and sequences

$$h_n \longrightarrow 0\,, \quad \beta_n \longrightarrow \beta \in [0, 2\pi/a]\,, \quad v_{0,n} = \Pi_{h_n,\beta_n}(v_{0,n}) \in L^2,$$
$$i\partial_t v_n(t,x) = ((D+\beta_n)^2 + W(x))v_n(t,x)\,, \quad v_n(0,x) = v_{0,n}(x)\,,$$

such that

$$1 = \|v_{0,n}\|^2_{L^2(\mathbb{T}^1)}, \qquad \int_0^T \|u_n(t,\bullet)\|^2_{L^2(\omega_x)}dt \longrightarrow 0\,. \tag{A.4}$$

We associate to the sequence v_n a semiclassical defect measure, ν, on $\mathbb{R}\times T^*\mathbb{T}^1$. As recalled in §3 (see [1] and [17]) the measure satisfies $\nu((t_0,t_1)\times T^*\mathbb{T}^1) = t_1 - t_0$, and its support is invariant under the flow of principal symbol of $(D+\beta)^2 + W(x)$ (since $\beta_n = \beta + o(1)$):

$$(t_0,x_0,\xi_0) \in \operatorname{supp}(\nu) \implies (t_0, x_0 + s\xi_0, \xi_0) \in \operatorname{supp}(\nu)\,, \quad \forall s \in \mathbb{R}\,.$$

In view of the second part of (A.4) the measure ν vanishes on $(0,T)\times T^*\omega_x$ which contradicts the invariance of the support.

References

[1] N. Anantharaman and F. Macia, *The dynamics of the Schrödinger flow from the point of view of semicalssical measures* (2011). Preprint, arxiv: 1102.0970.

[2] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065.

[3] N. Burq, *Control for Schrödinger equations on product manifolds* (1992). Unpublished.

[4] ———, *Contrôle de l'équation des plaques en présence d'obstacles strictement convexes*, Mém. Soc. Math. France (N.S.) (1993), no. 55, 126.

[5] ———, *Semi-classical estimates for the resolvent in nontrapping geometries*, Int. Math. Res. Not. (2002), no. 5, 221–241.

[6] N. Burq and M. Zworski, *Eigenfunctions for partially rectangular billiards* (2003). Unpublished, arxiv:math/0312098.

[7] ———, *Geometric control in the presence of a black box*, J. Amer. Math. Soc. **17** (2004), no. 2, 443–471 (electronic).

[8] ———, *Bouncing ball modes and quantum chaos*, SIAM Rev. **47** (2005), no. 1, 43–49 (electronic).

[9] J.-M. Delort, *Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds*, Int. Math. Res. Not. IMRN (2010), no. 12, 2305–2328.

[10] L. C. Evans and M. Zworski, *Lectures on semiclassical analysis* (2011). Preprint, www.math.berkeley.edu/~ zworski/semiclassical.pdf.

[11] A. Haraux, *Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire*, J. Math. Pures Appl. (9) **68** (1989), no. 4, 457–465 (1990).

[12] A. Hassell, L. Hillairet, and J. Marzuola, *Eigenfunction concentration for polygonal billiards*, Comm. Partial Differential Equations **34** (2009), no. 4-6, 475–485.

[13] S. Jaffard, *Contrôle interne exact des vibrations d'une plaque rectangulaire*, Portugal. Math. **47** (1990), no. 4, 423–429.

[14] J.-P. Kahane, *Pseudo-périodicité et séries de Fourier lacunaires*, Ann. Sci. École Norm. Sup. (3) **79** (1962) 93–150.

[15] G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. (9) **71** (1992), no. 3, 267–291.

[16] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, Vol. 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*, Masson, Paris (1988), ISBN 2-225-81477-5. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[17] F. Macià, *The Schrödinger flow in a compact manifold: High-frequency dynamics and dispersion,*, Modern Aspects of the Theory of Partial Differential Equations, Operator Theory: Advances and Applications **216** (2011) 275289.
[18] J. Marzuola, *Eigenfunctions for partially rectangular billiards*, Comm. Partial Differential Equations **31** (2006), no. 4-6, 775–790.
[19] L. Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal. **218** (2005), no. 2, 425–444.
[20] M. Tucsnak and G. Weiss, Observation and control for operator semigroups, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel (2009), ISBN 978-3-7643-8993-2.
[21] A. Weinstein, *Asymptotics of eigenvalue clusters for the Laplacian plus a potential*, Duke Math. J. **44** (1977), no. 4, 883–892.

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