Almost sure global existence and scattering for the one-dimensional NLS

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Deterministic results

Consider the one-dimensional defocusing NLS

$$\begin{cases} i\partial_s u + \Delta u = |u|^{r-1}u, & (s,y) \in \mathbb{R}\times\mathbb{R}, \\ u(0,y) = f(y), \end{cases} \qquad (NLS_r)$$

Deterministic theory:
–Global WP: equation is well posed in $L^2(\mathbb{R})$ as soon as $p \leq 5$ and the assumption $p \leq 5$ is known to be optimal in some sense (no continuous flow, Christ-Colliander-Tao and Burq-Gérard-Tzvetkov)
– $p \geq 5$ Scattering in L^2 (Dodson for $p = 5$)
– $3 < p_0 < p$ for initial data in $\mathcal{H}^1 = H^1 \cap L^2/\langle x\rangle$ Scattering in $\mathcal{H}^1$ (Ginibre-Velo Tsutsumi).
– $3 < p$ for initial data in $\mathcal{H}^1 = H^1 \cap L^2/\langle x\rangle$ Scattering in L^2 (Kato?)

Invariant measures

– Measures on the space of initial data (L^2) such that almost surely global existence: flow $\Phi(s)$
– Image of the measure by the flow $\Phi^*(s)\mu$???
– For dispersive equations on $\mathbb{R}^d$ no (non trivial) invariant measures. Indeed, if μ invariant, and supported on $H^s(\mathbb{R}^d)$, so that $\|u\|_{H^s(\mathbb{R}^d)}$ is μ-integrable. Then

$$\mathbb{E}(\|\chi(x)u_0\|_{H^s(\mathbb{R}^d)}) = \mathbb{E}(\|\chi(x)e^{is\Delta_y}u_0\|_{H^s(\mathbb{R}^d)}),$$

But for all $u_0 \in H^s(\mathbb{R}^d)$, because of dispersion

$$\lim_{s\to+\infty} \|\chi(y)e^{is\Delta_y}u_0\|_{H^s(\mathbb{R}^d)} = 0$$

Hence, by dominated convergence

$$\mathbb{E}(\|\chi(y)u_0\|_{H^s(\mathbb{R}^d)}) = \lim_{s\to+\infty} \mathbb{E}(\|\chi(y)e^{is\Delta_y}u_0\|_{H^s(\mathbb{R}^d)}) = 0 \Rightarrow \mu = \delta_{u_0=0}.$$

Objectives

Work on NLS_p

- Exhibit for the one dimensional Schrödinger equation examples of measures μ for which it is possible to describe the evolution by the linear flow $S(t)^*\mu$
- Show that the evolution by the non linear flow of the measure, $\Phi(s)^*\mu$ is absolutely continuous with respect to $S(s)^*\mu$
- Show that almost surely there exists a unique global solution to NLS_p, $\forall p > 1$
- Use this absolute continuity to deduce good estimates for the time evolution $\forall p > 1$ and new almost sure scattering results for NLS_p, $p > 3$

The measures

The one-dimensional harmonic oscillator and Hermite functions

$$H = -\partial_x^2 + x^2, \qquad He_n = \lambda_n^2 e_n, \qquad \lambda_n = \sqrt{2n+1}.$$

form a Hilbert basis of $L^2(\mathbb{R})$. The harmonic Sobolev spaces $\mathcal{H}^s \in \mathbb{R}$, by

$$\mathcal{H}^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}),\ H^{s/2}u \in L^2(\mathbb{R})\},$$

Let $\{g_n\}_{n\geq 0}$ independent complex standard Gaussian variables.
For $\alpha > 0$ the probability measure μ_α on $\mathcal{H}^{-s}, s > 0$ is defined by

$$\Omega \to \mathcal{H}^{-s}(\mathbb{R})$$

$$\omega \mapsto u^\omega = \alpha \sum_{n=0}^{+\infty} \frac{1}{\lambda_n} g_n(\omega) e_n,$$

$$\mu_\alpha(\mathcal{H}^s) = 0, s \geq 0, \quad \mu_\alpha(\mathcal{H}^s) = 1, s < 0$$

$$\mu_\alpha(L^p(\mathbb{R})) = 1, \forall p > 2.$$

μ_α is supported on the harmonic oscillator Besov space $\mathcal{B}^0_{2,\infty}(\mathbb{R})$

The linear evolution

$$S(t) = e^{it\partial_x^2}, (i\partial_t + \partial_x^2)S(t)u_0 = 0, \quad S(0) = \mathrm{Id}$$

The measure $S(t)^*\mu$ is also a Gaussian measure

$$u_t^\omega = \sum_{n=0}^{+\infty} \frac{1}{\lambda_n} g_n(\omega) e_n^t,$$

where e_n^t is a Hilbert basis of L^2 composed of eigenfunctions of a twisted harmonic oscillator

$$H_t e_n^t = \lambda_n^2 e_n^t, \qquad H_t = -(1+4t^2)\partial_x^2 + 2i(tx\partial_x + \partial_x x) + x^2.$$

Proposition

All the measures $\mu_\alpha^t, \alpha \in \mathbb{R}^{*,+}, t \in \mathbb{R}$ are pairwise singular (supported on disjoint sets)

A.s. global existence

Theorem

Let $p > 1$, and $\mu = \mu_1^0 e^{-\|u\|_{L^{p+1}}^{p+1}/(p+1)}$. Then there exists $\sigma > 0$ such that for μ-a.e. initial data u_0, there exists a unique global solution $u = \Psi(t)u_0$,

$$u \in S(t)u_0 + C^0(\mathbb{R}_t; \mathcal{H}^\sigma(\mathbb{R})),$$

For $p \geq 2$, one can take $\sigma = 1/2-$. Furthermore the measure $\Psi(t)^*(\mu)$ is absolutely continuous with respect to $S(t)^*\mu$.
$\forall A \in \mathcal{H}^{-\epsilon}$,

$$S(t)^*\mu(A) \leq \begin{cases} \big(\Psi(t)^*\mu(A)\big)^{t^{\frac{p-5}{2}}} & \text{if } p < 5 \\ \Psi(t)^*\mu(A) & \text{if } p \geq 5 \end{cases} \tag{1}$$

$$\Psi(t)^*\mu(A) \leq \begin{cases} \big(S(t)^*\mu(A)\big)^{t^{\frac{5-p}{2}}} & \text{if } p > 5 \\ S(t)^*\mu(A) & \text{if } p \leq 5 \end{cases} \tag{2}$$

A.s. scattering

Theorem

Let $p > 1$, Then a.s. we have

$$\|\Psi(t)u_0\|_{L^{p+1}} \leq C\frac{\log(t)^{1/(p+1)}}{t^{(\frac{1}{2}-\frac{1}{p+1})}}$$

As a consequence, for $p > 3$ a.s. the solution scatters: there exists $\sigma > 0$ such that for a.e. initial data u_0, there exists $v_0 \in \mathcal{H}^\sigma$ such that

$$\lim_{t\to+\infty} \|\Psi(t)u_0 - S(t)(u_0 + v_0)\|_{\mathcal{H}^\sigma} = 0.$$

(for $p > 2 + \sqrt{5}$ we can take $\sigma = 1/2-$).

Lens transform: compactification of space-time

For $|t| < \frac{\pi}{4}, x \in \mathbb{R}$. define

$$u(t,x) = (\mathcal{L}U)(t,x) = \frac{1}{\cos^{\frac{1}{2}}(2t)} U\Big(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\Big) e^{-\frac{ix^2 \tan(2t)}{2}}, \tag{3}$$

$$\mathcal{L}\big(e^{is\partial_y^2} U\big) = e^{-it(s)H} U, \qquad t(s) = \frac{\arctan(2s)}{2} \tag{4}$$

Moreover, suppose that $U(s,y)$ is a solution NLS_p. Then the function $u(t,x)$ satisfies

$$\begin{cases} i\partial_t u - Hu = \cos^{\frac{p-5}{2}}(2t)|u|^{p-1}u, \quad |t| < \frac{\pi}{4}, x \in \mathbb{R}, \\ u(0,\cdot) = u_0 = U_0. \end{cases} \tag{5}$$

A.s. global existence, harmonic version

Theorem

Let $p > 1$, and $\mu = \mu_1^0 e^{-\|u\|_{L^{p+1}}^{p+1}/(p+1)}$. Then there exists $\sigma > 0$ such that for μ-a.e. initial data u_0, there exists a unique solution $u = \Psi(t,0)u_0$ on $(-\pi/4, \pi/4)$,

$$u \in S(t)u_0 + C^0(\mathbb{R}_t; \mathcal{H}^\sigma(\mathbb{R})),$$

For $p \geq 2$, one can take $\sigma = 1/2-$. Furthermore the measure $\Psi(t,0)^*(\mu)$ is absolutely continuous with respect to μ. $\forall A \in \mathcal{H}^{-\epsilon}$,

$$\mu(A) \leq \begin{cases} \left(\Psi(t)^*\mu(A)\right)^{\cos(2t)^{\frac{p-5}{2}}} & \text{if } p < 5 \\ \Psi(t)^*\mu(A) & \text{if } p \geq 5 \end{cases} \tag{6}$$

$$\Psi(t)^*\mu(A) \leq \begin{cases} \left(\mu(A)\right)^{\cos(2t)^{\frac{5-p}{2}}} & \text{if } p > 5 \\ \mu(A) & \text{if } p \leq 5 \end{cases} \tag{7}$$

A.s. scattering, harmonic version

Theorem

Let $p > 1$, Then a.s. we have

$$\|\Psi(t,0)u_0\|_{L^{p+1}} \leq C \log(t)^{1/(p+1)}$$

As a consequence, for $p > 3$ a.s. the solution exists on $[-\frac{\pi}{4}, \frac{\pi}{4}]$: there exists $\sigma > 0$ such that for a.e. initial data u_0, there exists $v_0 \in \mathcal{H}^\sigma$ such that

$$\lim_{t \to \pm\pi/4} \|\Psi(t,0)u_0 - S(t)(u_0 + v_0)\|_{\mathcal{H}^\sigma} = 0.$$

(for $p > 2 + \sqrt{5}$ we can take $\sigma = 1/2-$).

Properties of the measures

Linear properties $q < +\infty$

$$\mathbb{P}(\|S(t)u\|_{L^q_t;W^{1/4-,4}} > \lambda) \leq Ce^{-c\lambda^2},$$

$$\mathbb{P}(\|S(t)u\|_{L^q_t W^{1/6,\infty}} > \lambda) \leq Ce^{-c\lambda^2},$$

$$\mathbb{P}(\|\langle x\rangle^{-1/2}S(t)u\|_{L^2_t;H^{1/2-}} > \lambda) \leq Ce^{-c\lambda^2},$$

Bilinear properties

$$\mathbb{P}(\|(S(t)u)^2\|_{L^q_t;H^{1/2-}} > \lambda) \leq Ce^{-c\lambda^2},$$

$$\mathbb{P}(\||S(t)u|^2\|_{L^q_t;H^{1/2-}} > \lambda) \leq Ce^{-c\lambda^2},$$

a.s. local Cauchy theory

Theorem

Let $p > 1$, as long as we stay away from $\pm\pi/4$, then a.s. there exists T and a unique solution to

$$\begin{cases} i\partial_t u - Hu = \cos^{\frac{p-5}{2}}(2t)|u|^{p-1}u, & |t| < \frac{\pi}{4}, x \in \mathbb{R}, \\ u(0,\cdot) = u_0 = U_0. \end{cases} \tag{8}$$

in the space

$$S(t)u_0 + C_t^0; \mathcal{H}^\sigma.$$

Furthermore, the time existence is bounded from below by a negative power of the norms appearing in previous slide, hence

$$\mathbb{P}(T < \tau) \le Ce^{-c\tau^{-\delta}}$$

and on such time interval, the solution remains essentially bounded by initial data

An argument by Bourgain (case of invariant measures)

- A nice local Cauchy theory: initial data of size smaller than R the solution exists (+ nice estimates for $t \in [0, T]$, $T \sim CR^{-\gamma}$.
- A measure, ρ, which is (at least formally) invariant by the flow of the equation, $\Psi(t)$, for any time $t \in \mathbb{R}$.
- Set of initial data larger than λ measure smaller than $e^{-c\lambda^2}$

Target time N, λ a size. E_1, the first set of bad initial data (no solution on $[0, \lambda^{-\kappa}]$,

$$\rho(E_1) \leq Ce^{-c\lambda^2}.$$

E_2 second bad initial data: solutions on $[0, \lambda^{-\kappa}]$ but not on $[\lambda^{-\kappa}, 2\lambda^{-\kappa}]$ Then

$$E_2 = \{u_0; \Psi(\lambda^{-\kappa})u_0 \in E_1\}.$$

Invariance of measure implies $\rho(E_2) = \rho(E_1)$ Iterating gives that the set of initial data for which cannot solve up to time N, $E = \cup_{n=1}^{N\lambda^\kappa} E_n$

$$\rho(E) \leq e^{-c\lambda^2} N\lambda^\kappa$$

The quasi-invariant measures

In the Hermite L^2 basis

$$u = \sum_n u_n e_n(x), \mu = \otimes_n \mathcal{N}(0, 1/\sqrt{2n+1}) = \otimes_n e^{-(2n+1)\frac{|u_n|^2}{2}}(2n+1)du_n$$

where du_n is Lebesgue measure on $\mathbb{C}$. Formally

$$\mu = e^{-\frac{\|u\|^2_{\mathcal{H}^1}}{2}} \otimes_n (2n+1)du_n$$

Let

$$\nu_t = \mu e^{-\frac{\cos(2t)^{(p-5)/2}}{p+1}\|u\|^{p+1}_{L^{p+1}}} = e^{-\mathcal{E}_t(u)} \otimes_n (2n+1)du_n$$

where

$$\mathcal{E}_t(u) = \|u\|^2_{\mathcal{H}^1} + \frac{\cos(2t)^{(p-5)/2}}{p+1}\|u\|^{p+1}_{L^{p+1}}$$

is not invariant along the flow but satisfies

$$\frac{d\mathcal{E}_t(u)}{dt} = -(p-5)\tan(2t)\cos(2t)^{(p-5)/2}\frac{\|u\|^{p+1}_{L^{p+1}}}{p+1}.$$

Quasi-invariance

Compare $\nu_0(A)$ and $\nu_t(\Psi(t,0)A)$.

$$\frac{d}{dt}\nu_t(\Psi(t,0)(A))$$

$$= \frac{d}{dt}\int_{v\in\Psi(t,0)(A)} e^{-\frac{1}{2}\|\sqrt{H}\,v\|^2_{L^2(\mathbb{R})}-\frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|v\|^{p+1}_{L^{p+1}(\mathbb{R})}}dv$$

$$= \frac{d}{dt}\int_{u_0\in A} e^{-\frac{1}{2}\|\sqrt{H}\,u(t)\|^2_{L^2(\mathbb{R})}-\frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}}du_0$$

$$= \int_A \frac{(p-5)\sin(2t)\cos^{\frac{p-7}{2}}(2t)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}e^{-\mathcal{E}_t(u(t))}du_0$$

$$= (p-5)\tan(2t)\int_A \alpha(t,u)e^{-\mathcal{E}_t(u(t))}du_0,$$

where $\alpha(t,u) = \frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}$.

Quasi-invariance continued

Assume that $t \geq 0$. Then by Hölder, for any $k \geq 1$,

$$\frac{d}{dt}\nu_t(\Psi(t,0)(A))$$
$$\leq |p-5|\tan(2t)\Big(\int_A \alpha^k(t,u)e^{-\mathcal{E}_t(u(t))}du_0\Big)^{\frac{1}{k}}\Big(\int_A e^{-\mathcal{E}_t(u(t))}du_0\Big)^{1-\frac{1}{k}}$$
$$= |p-5|\tan(2t)\Big(\int_A \alpha^k(t,u)e^{-\alpha(t,u)-\frac{1}{2}\|\sqrt{H}\,u(t)\|^2_{L^2(\mathbb{R})}}du_0\Big)^{\frac{1}{k}}$$
$$\times\ \Big(\nu_t(\Psi(t,0)(A))\Big)^{1-\frac{1}{k}}.$$

We use that $\alpha^k(t,u)e^{-\alpha(t,u)} \leq k^k e^{-k}$, then

$$\frac{d}{dt}\nu_t(\Psi(t,0)(A)) \quad \leq \quad |p-5|\tan(2t)\frac{k}{e}\Big(\nu_t(\Psi(t,0)(A))\Big)^{1-\frac{1}{k}}$$

Optimize with $k = -\ln(\nu_t(\Psi(t,0)A))$, gives

$$\frac{d}{dt}\nu_t(\Psi(t,0)(A)) \leq -|p-5|\tan(2t)\ln(\nu_t(\Psi(t,0)A))\nu_t(\Psi(t,0)(A)).$$

Quasi-invariance continued

Theorem

$$\nu_t(\Psi(t,0)(A)) \le \big(\nu_0(A)\big)^{(\cos(2t))^{\frac{|p-5|}{2}}} \le \big(\mu_0(A)\big)^{(\cos(2t))^{\frac{|p-5|}{2}}}.$$

Reverse inequality obtained by backward integration of the estimate and reads similarly

$$\nu_0(A) \le \big(\nu_t(\Psi(t,0)A)\big)^{(\cos(2t))^{\frac{|p-5|}{2}}} \le \big(\mu_0(\Psi(t,0)A)\big)^{(\cos(2t))^{\frac{|p-5|}{2}}}.$$

Back to global existence

As long as we stay away from $\pm\pi/4$ in

$$\nu_0(A) \le \left(\mu_0(\Psi(t,0)A)\right)^{(\cos(2t))^{\frac{|p-5|}{2}}}.$$

$(\cos(2t))^{\frac{|p-5|}{2}}$ remains bounded from below by α. Hence Bourgain's globalization argument applies, with subgaussian estimates $\nu_0(E_k) \le e^{-c\lambda^{2\alpha}}$ Hence global existence on $(-\pi/4, \pi/4)$ with bounds on norms growing like

$$(\cos(2t))^{-\frac{|p-5|}{4}}$$

hence back to the original problem on $\mathbb{R}_s$ bounds

$$s^{\frac{|p-5|}{4}}.$$

Back to L^{p+1} estimates for NLS_p, $3 < p < 5$

Fix t and Let

$$A = \{u_0; \|\Psi(t,0)A)\|_{L^{p+1}} > \lambda\}.$$

Then

$$\nu_0(A) \leq \big(\nu_t(\Psi(t,0)A)\big)^{(\cos(2t))^{\frac{5-p}{2}}}$$
$$\leq \Big(e^{-\cos(2t)^{\frac{p-5}{2}}\frac{\lambda^{p+1}}{p+1}}\mu_0(\Psi(t,0)A)\Big)^{(\cos(2t))^{\frac{5-p}{2}}} \leq e^{-\frac{\lambda^{p+1}}{p+1}} \quad (9)$$

The log loss comes from passing from estimates for a fixed time to estimates for all times.

Back to scattering for NLS_p, $3 < p < 5$

$$i\partial_t u - Hu = \cos^{\frac{p-5}{2}}(2t)|u|^{p-1}u,$$

from L^{p+1} estimates, good estimates on $|u|^{p-1}u$ in $L^{\frac{p+1}{p}} \to \mathcal{H}^{-\sigma}$, and ($p > 3$)

$$\cos^{\frac{p-5}{2}}(2t) \in L^1(-\frac{\pi}{4}, \frac{\pi}{4}),$$

hence convergence in $\mathcal{H}^{-\sigma}$. To improve to positive regularity (mandatory to be able to come back via the lens transform), write

$$u = e^{itH}u_0 + v$$

and estimate (using L^{p+1} estimate)

$$\frac{d}{dt}\|v\|^2_{\mathcal{H}^s},$$

and show that for $s > 0$ sufficiently small this norm remains bounded (use again integrability of $\cos^{\frac{p-5}{2}}(2t)$).