Cauchy theory for the gravity water waves system with non localized initial data

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ABSTRACT. In this article, we develop the local Cauchy theory for the gravity water waves system, for rough initial data which do not decay at infinity. We work in the context of L^2-based uniformly local Sobolev spaces introduced by Kato ([27]). In this context we prove a classical well-posedness result (without loss of derivatives). Our result implies also a local well-posedness result in Hölder spaces (with loss of $d/2$ derivatives). As an illustration, we solve a question raised by Boussinesq in 1910 [13] on the water waves problem in a canal. We take benefit of an elementary (though seemingly previously unnoticed) observation to show that the strategy suggested in [13] does indeed apply to this setting.

1. Introduction

We are interested in this paper in the free boundary problem describing the motion of an incompressible, irrotational fluid flow moving under the force of gravitation, without surface tension, in case where the initial data are neither localized nor periodic. There are indeed two cases where the mathematical analysis is rather well understood: firstly for periodic initial data (in the classical Sobolev spaces $H^s(\mathbf{T}^d)$) and secondly when they are decaying to zero at infinity (for instance for data in $H^s(\mathbf{R}^d)$ with s large enough). With regards to the analysis of the Cauchy problem, we refer to the recent papers of Lannes [30], Wu [36, 37] and Germain, Masmoudi and Shatah [23]. We also refer to the introduction of [3] or [10, 28] for more references. However, one can think to the moving surface of a lake or an ocean where the waves are neither periodic nor decaying to zero (see also [20]).

A most natural strategy would be to solve the Cauchy problem in the classical Hölder spaces $W^{k,\infty}(\mathbf{R}^d)$. However even the linearized system at the origin (the fluid at rest) is ill-posed in these spaces (see Remark 2.4 below), and this strategy leads consequently to loss of derivatives. Having this loss of derivatives in mind, the other natural approach is to work in the framework of L^2 based uniformly local Sobolev spaces, denoted by $H_{ul}^s(\mathbf{R}^d)$. These spaces were introduced by Kato (see [27]) in the analysis of hyperbolic systems. Notice however, that compared to general hyperbolic systems, the water waves system appears to be non local, which induces new difficulties. This framework appears to be quite natural

in our context. Indeed, the uniformly local Sobolev spaces $H^s_{ul}(\mathbf{R}^d)$ contain, in particular, the usual Sobolev spaces $H^s(\mathbf{R}^d)$, the periodic Sobolev spaces $H^s(\mathbf{T}^d)$ (spaces of periodic functions on $\mathbf{R}^d$), the sum $H^s(\mathbf{R}^d) + H^s(\mathbf{T}^d)$ and also the Hölder spaces $W^{s,p}(\mathbf{R}^d)$ (and as a by-product of our analysis, we get well-posedness in Hölder spaces, with a loss of derivatives).

The aim of this paper is precisely to prove that the water waves system is locally (in time) well posed in the framework of uniformly local Sobolev spaces. Moreover, following our previous paper [3], the data for which we solve the Cauchy problem are allowed to be quite rough. Indeed we shall assume, for instance, that the initial free surface is the graph of a function which belongs to the space $H^{s+\frac{1}{2}}_{ul}(\mathbf{R}^d)$ for $s > 1 + \frac{d}{2}$. In particular, in term of Sobolev embedding, the initial free surface is merely $W^{\frac{3}{2},\infty}(\mathbf{R}^d)$ thus may have unbounded curvature. On the other hand this threshold should be compared with the scaling of the problem. Indeed it is known that the water wave system has a scaling invariance for which the critical space for the initial free surface is the space $\dot{H}^{1+\frac{d}{2}}(\mathbf{R}^d)$ (or $W^{1,\infty}(\mathbf{R}^d)$). This shows that we solve here the Cauchy problem for data $\frac{1}{2}$ above the scaling. (Notice that in [4] we prove well-posedness, in the classical Sobolev spaces, $\frac{1}{2} - \frac{1}{12}$ above the scaling when $d = 2$). As an illustration of the relevance of this low regularity Cauchy theory in the context of *local* spaces, we solve a question raised by Boussinesq in 1910 [13] on the water waves problem in a canal. In [13], Boussinesq suggested to reduce the water-waves system in a canal to the same system on $\mathbb{R}^3$ with periodic conditions with respect to one variable, by a simple reflection/periodization procedure (with respect to the normal variable to the boundary of the canal). However, for a century, this idea remained unapplicable for the simple reason that the even extension of a smooth function on the half line is in general merely Lipshitz continuous (due to the singularity at the origin). As a consequence, even if one starts with a smooth initial domain, the reflected/periodized domain will only be Lipshitz continuous. Here, we are able to take benefit of an elementary (though seemingly previously unnoticed) observation which shows that actually, as soon as we are looking for *reasonably smooth* solutions, the angle between the free surface and the vertical boundary of the canal is a right angle. Consequently, the reflected/periodized domain enjoy additional smoothness (namely up to C^3), which is enough to apply our *rough data* Cauchy theory and to show that the strategy suggested in [13] does indeed apply. This appears to be the first result on Cauchy theory for the water-wave system in a domain with boundary.

The present paper relies on the strategies developed in our previous paper [3] and we follow the same scheme of proof. In Section 3, we develop the machinery of para-differential calculus in the framework of uniformly local spaces that we need later. We think that this section could be useful for further studies in this framework. In Section 4 we prove that the Dirichlet-Neumann operator is well defined in this framework (notice that this fact is not straightforward, see [22, 19] for related works), and we give a precise description (including sharp elliptic estimates in very rough domains) on these spaces. In Section 5, we symmetrize the system and prove *a priori* estimates. In section 6, we prove contraction estimates and well posedness. In section 7, we give the application to the canal (and swimming pools). Finally, in an appendix, we prove that in the context of Hölder spaces, the water-waves system linearized on the trivial solution (rest) is *ill posed*.

2. The problem and the result

In this paper we shall denote by $t \in \mathbf{R}$ the time variable and by $x \in \mathbf{R}^d$ (where $d \geq 1$), $y \in \mathbf{R}$, the horizontal and vertical space variables. We work in a fluid domain with free boundary and fixed bottom on the form

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Omega(t)\} \text{ where}$$
$$\Omega(t) = \{(x, y) : \eta_*(x) < y < \eta(t, x)\}.$$

Here the free surface is described by η, an unknown of the problem, and the bottom by a given function η_*. We shall only assume that η_* is bounded and continuous. We assume that the bottom is the graph of a function for the sake of simplicity: our analysis applies whenever one has the Poincaré inequality given by Lemma 4.1 below.

We shall denote by Σ the free surface and by Γ the bottom,

$$\Sigma = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Sigma(t)\} \quad \text{where}$$
$$\Sigma(t) = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x)\},$$
$$\Gamma = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y = \eta_*(x)\}.$$

We shall use the following notations

$$\nabla_x = (\partial_{x_i})_{1 \leq i \leq d}, \quad \nabla_{x,y} = (\nabla_x, \partial_y), \quad \Delta_x = \sum_{1 \leq i \leq d} \partial_{x_i}^2, \quad \Delta_{x,y} = \Delta_x + \partial_y^2.$$

2.1. The equations. The Eulerian velocity $v : \Omega \to \mathbf{R}^{d+1}$ solves the incompressible and irrotational Euler equation

$$\partial_t v + (v \cdot \nabla_{x,y})v + \nabla_{x,y}P = -g e_y, \quad \operatorname{div}_{x,y} v = 0, \quad \operatorname{curl}_{x,y} v = 0 \quad \text{in } \Omega$$

where $g > 0$ is the acceleration of the gravity, e_y the vector $(x = 0, y = 1)$ and P the pressure. The problem is then given by three boundary conditions:

- a kinematic condition (which states that the free surface moves with the fluid)

$$\partial_t \eta = \sqrt{1 + |\nabla_x \eta|^2}(v \cdot n) \quad \text{on} \quad \Sigma,$$

  where n denotes the unit normal vector to Σ,
- a dynamic condition (that expresses a balance of forces across the free surface)

$$P = 0 \quad \text{on } \Sigma,$$

- the "solid wall" boundary condition on the bottom Γ

$$v \cdot \nu = 0 \quad \text{on } \Gamma,$$

  where ν denotes the normal vector to Γ whenever it exists.

Since the motion is incompressible and irrotational there exists a velocity potential $\phi : \Omega \to \mathbf{R}$ such that $v = \nabla_{x,y}\phi$, thus $\Delta_{x,y}\phi = 0$ in Ω. We shall work with the Zakharov/Craig–Sulem formulation of the water waves equations. We introduce

$$\psi(t, x) = \phi(t, x, \eta(t, x))$$

and the Dirichlet-Neumann operator

$$G(\eta)\psi = \sqrt{1 + |\nabla_x \eta|^2}\Big(\frac{\partial \phi}{\partial n}\Big|_\Sigma\Big)$$
$$= (\partial_y \phi)(t, x, \eta(t, x)) - \nabla_x \eta(t, x) \cdot (\nabla_x \phi)(t, x, \eta(t, x)).$$

Then (see [18]) the water waves system can be written in term of the unknown η, ψ as

(2.1)
$$\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi = -\frac{1}{2}|\nabla_x \psi|^2 + \frac{1}{2}\frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1 + |\nabla_x \eta|^2} - g\eta. \end{cases}$$

It is useful to introduce the vertical and horizontal components of the velocity. We set

(2.2)
$$\begin{cases} B = (v_y)|_\Sigma = \dfrac{\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi}{1 + |\nabla_x \eta|^2}, \\ V = (v_x)|_\Sigma = \nabla_x \psi - B\nabla_x \eta. \end{cases}$$

We recall also that the Taylor coefficient defined by $a = -\frac{\partial P}{\partial y}\big|_\Sigma$ can be defined in terms of η, ψ, B, V only (see §7.2 below and §4.3.1 in [28]).

2.2. The uniformly local Sobolev spaces. We recall here the definition of the uniformly local Sobolev spaces introduced by Kato in [27].

Recall that there exists $\chi \in C^\infty(\mathbf{R}^d)$ with supp $\chi \subset [-1,1]^d, \chi = 1$ near $[-\frac{1}{2}, \frac{1}{2}]^d$ such that

(2.3)
$$\sum_{q \in \mathbf{Z}^d} \chi_q(x) = 1, \quad \forall x \in \mathbf{R}^d$$

where

$$\chi_q(x) = \chi(x - q).$$

Definition 2.1. *For $s \in \mathbf{R}$ the space $H_{ul}^s(\mathbf{R}^d)$ is the space of distributions $u \in H_{loc}^s(\mathbf{R}^d)$ such that*

$$\|u\|_{H_{ul}^s(\mathbf{R}^d)} := \sup_{q \in \mathbf{Z}^d} \|\chi_q u\|_{H^s(\mathbf{R}^d)} < +\infty.$$

The definition of the space $H_{ul}^s(\mathbf{R}^d)$ is independent of the choice of the function χ in $C_0^\infty(\mathbf{R}^d)$ satisfying (2.3) (see Lemma 3.1 below).

Proposition 2.2. *One has the following embeddings:*

(1) If $s > \frac{d}{2}$ and $s - \frac{d}{2} \notin \mathbf{N}$, $H_{ul}^s(\mathbf{R}^d)$ is continuously embedded in $W^{s-\frac{d}{2},\infty}(\mathbf{R}^d)$.
(2) If $m \in \mathbf{N}$, $W^{m,\infty}(\mathbf{R}^d)$ is continuously embedded in $H_{ul}^m(\mathbf{R}^d)$.
(3) If $s \geq 0$, $W^{s+\varepsilon,\infty}(\mathbf{R}^d)$ is continuously embedded in $H_{ul}^s(\mathbf{R}^d)$ for $\varepsilon > 0$.

2.3. The main result. The goal of this article is to prove the following result.

Theorem 2.3. *Let $d \geq 1, s > 1 + \frac{d}{2}$. Assume that η_* is a bounded continuous function on $\mathbf{R}^d$. Consider an initial data (η_0, ψ_0) satisfying the following conditions*

(i) $\eta_0 \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d), \psi_0 \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d), V_0 \in H_{ul}^s(\mathbf{R}^d), B_0 \in H_{ul}^s(\mathbf{R}^d)$,
(ii) there exists $h > 0$ such that $\eta_0(x) - \eta_*(x) \geq 2h, \quad \forall x \in \mathbf{R}^d$,

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(iii) there exists $c > 0$ such that $a_0(x) \geq c, \quad \forall x \in \mathbf{R}^d$,

where a_0 denotes the Taylor coefficient at time $t = 0$.

    Then there exists $T > 0$ such that the Cauchy problem for the system (2.1) *with initial data (η_0, ψ_0) at $t = 0$ has a unique solution*

$$(\eta, \psi) \in L^\infty\big([0, T], H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d) \times H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)\big)$$

such that

1. $(V, B) \in L^\infty\big([0, T], H_{ul}^s(\mathbf{R}^d) \times H_{ul}^s(\mathbf{R}^d)\big)$,
2. $\eta(t, x) - \eta_*(x) \geq h, \quad \forall (t, x) \in [0, T] \times \mathbf{R}^d$,
3. $a(t, x) \geq \frac{1}{2}c, \quad \forall (t, x) \in [0, T] \times \mathbf{R}^d$.
4. For any $s' < s$,

$$(\eta, \psi, V, B) \in C^0\big([0, T], H_{ul}^{s'+\frac{1}{2}}(\mathbf{R}^d) \times H_{ul}^{s'+\frac{1}{2}}(\mathbf{R}^d) \times H_{ul}^{s'}(\mathbf{R}^d) \times H_{ul}^{s'}(\mathbf{R}^d)\big).$$

Remark 2.4. • Theorem 2.3 implies local well posedness in Hölder spaces: indeed, assuming that

$$(\eta_0, \psi_0, V_0, B_0) \in W^{\sigma+\frac{1}{2}+\varepsilon,\infty}(\mathbf{R}^d) \times W^{\sigma+\frac{1}{2}+\varepsilon,\infty}(\mathbf{R}^d) \times W^{\sigma+\varepsilon,\infty}(\mathbf{R}^d) \times W^{\sigma+\varepsilon,\infty}(\mathbf{R}^d)$$

$$\subset H_{ul}^{\sigma+\frac{1}{2}+\frac{\varepsilon}{2}}(\mathbf{R}^d) \times H_{ul}^{\sigma+\frac{1}{2}+\frac{\varepsilon}{2}}(\mathbf{R}^d) \times H_{ul}^{\sigma+\frac{\varepsilon}{2}}(\mathbf{R}^d) \times H_{ul}^{\sigma+\frac{\varepsilon}{2}}(\mathbf{R}^d)$$

for some $\sigma > 1 + d/2$ and $\varepsilon > 0$, then we get a solution

$$(\eta, \psi, V, B) \in C^0([0, T], H_{ul}^{\sigma+\frac{1}{2}}(\mathbf{R}^d) \times H_{ul}^{\sigma+\frac{1}{2}}(\mathbf{R}^d) \times H_{ul}^\sigma(\mathbf{R}^d) \times H_{ul}^\sigma(\mathbf{R}^d))$$

$$\subset C^0([0, T], W^{\sigma+\frac{1}{2}-\frac{d}{2},\infty}(\mathbf{R}^d) \times W^{\sigma+\frac{1}{2}-\frac{d}{2},\infty}(\mathbf{R}^d) \times W^{\sigma-\frac{d}{2},\infty}(\mathbf{R}^d) \times W^{\sigma-\frac{d}{2},\infty}(\mathbf{R}^d)).$$

    • It is very likely that this loss of $d/2$ derivatives cannot be completely avoided. Indeed the linearized water waves equation around the zero solution can be written as

$$\partial_t u + i\,|D_x|^{\frac{1}{2}}\,u = 0.$$

The solution of this equation, with initial data u_0, is given by

$$u(t) = S(t)u_0, \qquad S(t) = \exp(-it\,|D_x|^{\frac{1}{2}}).$$

Proposition 8.1 shows that for $t \neq 0$ the operator $S(t)$ is not bounded from the Zygmund space $C_*^\sigma(\mathbf{R}^d)$ to $C_*^s(\mathbf{R}^d)$ if $s > \sigma - \frac{d}{4}$, remembering that $C_*^\sigma(\mathbf{R}^d) = W^{\sigma,\infty}(\mathbf{R}^d)$ if $\sigma \geq 0$, $\sigma \notin \mathbf{N}$. (For positive results see Fefferman and Stein [**21**, page 160]). Thus even in the linear case we have a loss of $\frac{d}{4}$ derivative.

    • The result in the appendix also shows that, in the presence of surface tension, a similar well posedness result in the framework of uniformly local Sobolev space is rather unlikely to hold. Indeed, in the presence of surface tension, the linearized operator around the solution $(\eta, \psi) = (0, 0)$ can be written (see [**1**]) with $u = |D|^{\frac{1}{2}}\eta + i\psi$ as

$$\partial_t u + i|D_x|^{\frac{3}{2}}u = 0, \quad u|_{t=0} = u_0.$$

According to Proposition 8.1 the loss of derivatives in x from u_0 to the solution $u(t, \cdot), t \neq 0$, is at least $\frac{3d}{4}$ whereas an analogue of the above theorem would give a loss of at most $\frac{d}{2}$.

3. Preliminaries

3.1. Invariance. The following result shows that the definition of the uniformly local Sobolev spaces does not depend on the choice of the function χ satisfying (2.3).

Lemma 3.1. *Let E be a normed space of functions from $\mathbf{R}^d$ to $\mathbf{C}$ such that*

$$\forall \theta \in W^{\infty,\infty}(\mathbf{R}^d) \quad \exists C > 0 : \|\theta u\|_E \leq C\|u\|_E \quad \forall u \in E$$

where C depends only on a finite number of semi-norms of θ in $W^{\infty,\infty}(\mathbf{R}^d)$. Then for any $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ there exists $C' > 0$ such that

$$\text{(3.1)} \qquad \sup_{k \in \mathbf{Z}^d} \|\widetilde{\chi}_k u\|_E \leq C' \sup_{q \in \mathbf{Z}^d} \|\chi_q u\|_E$$

where $\widetilde{\chi}_k(x) = \widetilde{\chi}(x - k)$.

PROOF. Let $\underline{\chi} \in C_0^\infty(\mathbf{R}^d)$ be equal to one on the support of χ. We write with $N = d+1$

$$\widetilde{\chi}_k \chi_q u = \langle k - q \rangle^{-N} \left[\frac{\langle k - q \rangle^N}{\langle x - q \rangle^N} \widetilde{\chi}_k \right] \left[\langle x - q \rangle^N \underline{\chi}_q \right] \chi_q u.$$

Since the two functions inside the brackets belong to $W^{\infty,\infty}(\mathbf{R}^d)$ with semi-norms independent of k, q, using the assumption in the lemma we deduce that

$$\|\widetilde{\chi}_k u\|_E \leq \sum_{q \in \mathbf{Z}^d} \|\widetilde{\chi}_k \chi_q u\|_E \leq C \sum_{q \in \mathbf{Z}^d} \langle k - q \rangle^{-N} \sup_{q \in \mathbf{Z}^d} \|\chi_q u\|_E,$$

which completes the proof. $\qquad \square$

Lemma 3.2. *Let $\mu \in \mathbf{R}$ and $N \geq d + 1$. Then there exists $C > 0$ such that*

$$\text{(3.2)} \qquad \sup_{x \in \mathbf{R}^d} \|\langle x - \cdot \rangle^{-N} u\|_{H^\mu(\mathbf{R}^d)} \leq C\|u\|_{H_{ul}^\mu(\mathbf{R}^d)}$$

for all $u \in H_{ul}^\mu(\mathbf{R}^d)$.

PROOF. Indeed we have

$$\|\langle x - \cdot \rangle^{-N} u\|_{H^\mu} \leq \sum_{q \in \mathbf{Z}^d} \|\langle x - \cdot \rangle^{-N} \chi_q u\|_{H^\mu}$$

and we write

$$\langle x - y \rangle^{-N} \chi_q(y) u(y) = \frac{1}{\langle x - q \rangle^N} \frac{\langle x - q \rangle^N}{\langle x - y \rangle^N} \widetilde{\chi}_q(y) \chi_q(y) u(y)$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d), \widetilde{\chi} = 1$ on the support of χ. This implies that

$$\sum_{q \in \mathbf{Z}^d} \|\langle x - \cdot \rangle^{-N} \chi_q u\|_{H^\mu} \leq C_N \sum_{q \in \mathbf{Z}^d} \frac{1}{\langle x - q \rangle^N} \|u\|_{H_{ul}^\mu} \leq C_N' \|u\|_{H_{ul}^\mu},$$

since the function $y \mapsto \frac{\langle x - q \rangle^N}{\langle x - y \rangle^N} \widetilde{\chi}_q(y)$ belongs to $W^{\infty,\infty}(\mathbf{R}^d)$ with semi-norms uniformly bounded (independently of x and q). $\qquad \square$

3.2. Product laws.

Proposition 3.3. (i) *Let* $\sigma_j \in \mathbf{R}, j = 1, 2$ *be such that* $\sigma_1 + \sigma_2 > 0$ *and* $u_j \in H_{ul}^{\sigma_j}(\mathbf{R}^d), j = 1, 2$. *Then* $u_1 u_2 \in H_{ul}^{\sigma_0}(\mathbf{R}^d)$ *for* $\sigma_0 \leq \sigma_j$ *and* $\sigma_0 < \sigma_1 + \sigma_2 - \frac{d}{2}$. *Moreover we have*

$$\|u_1 u_2\|_{H_{ul}^{\sigma_0}(\mathbf{R}^d)} \leq C \|u_1\|_{H_{ul}^{\sigma_1}(\mathbf{R}^d)} \|u_2\|_{H_{ul}^{\sigma_2}(\mathbf{R}^d)}.$$

(ii) *Let* $s \geq 0$ *and* $u_j \in H_{ul}^s(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d), j = 1, 2$. *Then* $u_1 u_2 \in H_{ul}^s(\mathbf{R}^d)$ *and*

$$\|u_1 u_2\|_{H_{ul}^s(\mathbf{R}^d)} \leq C \big(\|u_1\|_{L^\infty(\mathbf{R}^d)} \|u_2\|_{H_{ul}^s(\mathbf{R}^d)} + \|u_2\|_{L^\infty(\mathbf{R}^d)} \|u_1\|_{H_{ul}^s(\mathbf{R}^d)} \big).$$

(iii) *Let* $F \in C^\infty(\mathbf{R}^N, \mathbf{C})$ *be such that* $F(0) = 0$. *Let* $s > \frac{d}{2}$. *If* $U \in \big(H_{ul}^s(\mathbf{R}^d) \big)^N$ *then* $F(U) \in H_{ul}^s(\mathbf{R}^d)$ *and*

$$\|F(U)\|_{H_{ul}^s(\mathbf{R}^d)} \leq G\big(\|U\|_{(L^\infty(\mathbf{R}^d))^N} \big) \|U\|_{(H_{ul}^s(\mathbf{R}^d))^N}$$

for an increasing function $G : \mathbf{R}^+ \to \mathbf{R}^+$.

PROOF. The proofs are straightforward extensions of the proofs in the classical Sobolev spaces case. Indeed let us show (i) for instance. Let χ_q be defined in (2.3) and $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ be equal to one on the support of χ. Then from the classical case we can write

$$\|\chi_q u_1 u_2\|_{H^{s_0}} = \|\chi_q u_1 \widetilde{\chi}_q u_2\|_{H^{s_0}} \leq C \|\chi_q u_1\|_{H^{s_1}} \|\widetilde{\chi}_q u_2\|_{H^{s_2}}$$
$$\leq C \|u_1\|_{H_{ul}^{s_1}} \|u_2\|_{H_{ul}^{s_2}}.$$

The proofs of (ii) and (iii) are similar. $\qquad\qquad\square$

The following spaces will be used in the sequel

Definition 3.4. *Let* $p \in [1, +\infty]$, $J = (z_0, 0), z_0 < 0$ *and* $\sigma \in \mathbf{R}$.

(1) *The space* $L^p(J, H^\sigma(\mathbf{R}^d))_{ul}$ *is defined as the space of measurable functions* u *from* $\mathbf{R}_x^d \times J_z$ *to* $\mathbf{C}$ *such that*

$$\|u\|_{L^p(J, H^\sigma(\mathbf{R}^d))_{ul}} := \sup_{q \in \mathbf{Z}^d} \|\chi_q u\|_{L^p(J, H^\sigma(\mathbf{R}^d))} < +\infty.$$

(2) *We set*

$$X_{ul}^\sigma(J) = L^\infty(J, H^\sigma(\mathbf{R}^d))_{ul} \cap L^2(J, H^{\sigma + \frac{1}{2}}(\mathbf{R}^d))_{ul}$$
$$Y_{ul}^\sigma(J) = L^1(J, H^\sigma(\mathbf{R}^d))_{ul} + L^2(J, H^{\sigma - \frac{1}{2}}(\mathbf{R}^d))_{ul}$$

endowed with their natural norms.

(3) *We define the spaces* $X^\sigma(J), Y^\sigma(J)$ *by the same formulas without the subscript* ul.

Notice that $L^\infty(J, H^\sigma(\mathbf{R}^d))_{ul} = L^\infty(J, H_{ul}^\sigma(\mathbf{R}^d))$.

Lemma 3.5. *Let* $\sigma_0, \sigma_1, \sigma_2$ *be real numbers such that* $\sigma_1 + \sigma_2 > 0, \sigma_0 \leq \sigma_j, j = 1, 2, \sigma_0 < \sigma_1 + \sigma_2 - \frac{d}{2}$ *and* $2 \leq p \leq +\infty$. *Then*

$$\|uv\|_{L^p(J, H^{\sigma_0}(\mathbf{R}^d))_{ul}} \leq C \|u\|_{L^\infty(J, H^{\sigma_1}(\mathbf{R}^d))_{ul}} \|v\|_{L^p(J, H^{\sigma_2}(\mathbf{R}^d))_{ul}}$$

whenever the right hand side is finite.

The same inequality holds for the spaces without the subscript ul.

PROOF. This follows immediately from Proposition 3.3 (i) and (3.1). $\qquad\square$

Lemma 3.6. *If $\sigma > \frac{d}{2}$ the spaces $X_{ul}^\sigma(J)$ and $X^\sigma(J)$ are algebras.*

Lemma 3.7. *Let $s_0 > 1 + \frac{d}{2}, \mu > 0$ and $J = (-1,0)$. Then we have*

$$(3.3) \qquad \|fg\|_{X_{ul}^\mu} \leq C\big(\|f\|_{L^\infty(J,H^{s_0-1})_{ul}}\|g\|_{X_{ul}^\mu} + \|g\|_{L^\infty(J,H^{s_0-1})_{ul}}\|f\|_{X_{ul}^\mu}\big),$$

$$(3.4) \qquad \|fg\|_{X_{ul}^\mu} \leq C\big(\|f\|_{L^\infty(J,H^{s_0-1})_{ul}}\|g\|_{X_{ul}^\mu} + \|g\|_{L^\infty(J,H^{s_0-\frac{3}{2}})_{ul}}\|f\|_{X_{ul}^{\mu+\frac{1}{2}}}\big).$$

Let $F \in C_b^\infty(\mathbf{C}^N, \mathbf{C})$ be such that $F(0) = 0$. Then there exists a non decreasing function $\mathcal{F}: \mathbf{R}^+ \to \mathbf{R}^+$ such that for $\mu > \frac{d}{2}$ we have

$$(3.5) \qquad \|F(U)\|_{X_{ul}^\mu} \leq \mathcal{F}\big(\|U\|_{L^\infty(J,H^{s_0-1})_{ul}}\big)\|U\|_{X_{ul}^\mu}.$$

PROOF. The first and the third estimates follow easily from the inequalities $(ii),(iii)$ in Proposition 3.3. To prove the second one we start from the inequality (see [**3**], Corollary 2.12)

$$\|\chi_k fg\|_{H^t} \leq C\big(\|\chi_k f\|_{L^\infty}\|\widetilde{\chi}_k g\|_{H^t} + \|\widetilde{\chi}_k g\|_{C_*^{-\frac{1}{2}}}\|\chi_k f\|_{H^{t+\frac{1}{2}}}\big), \quad t > 0$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ is equal to one on the support of χ. Then we use the continuous embeddings: $H^{s_0-1} \subset L^\infty, H^{s_0-\frac{3}{2}} \subset C_*^{-\frac{1}{2}}$ and the above inequality for $t = \mu, t = \mu+\frac{1}{2}$. $\quad\square$

3.3. Continuity of the pseudo-differential operators. We have the following result which reflects the pseudo-local character of the pseudo-differential operators. Recall that $S_{1,0}^m$ is the set of symbols $p \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ such that

$$|D_\xi^\alpha D_x^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|} \quad \forall \alpha, \beta \in \mathbf{N}^d, \forall (x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d.$$

Proposition 3.8. *Let P be a pseudo-differential operator whose symbol belongs to the class $S_{1,0}^m$. Then for every $s \in \mathbf{R}$ there exists a constant $C > 0$ such that*

$$\|Pu\|_{H_{ul}^s(\mathbf{R}^d)} \leq C\|u\|_{H_{ul}^{s+m}(\mathbf{R}^d)},$$

for every $u \in H_{ul}^{s+m}(\mathbf{R}^d)$, where C depends only on semi-norms of the symbol in $S_{1,0}^m$.

PROOF. Write

$$(3.6) \qquad \chi_k Pu = \sum_{|k-q|\leq 2} \chi_k P\chi_q u + \sum_{|k-q|\geq 3} \chi_k P\chi_q u =: A + \sum_{|k-q|\geq 3} B_{k,q}.$$

The first sum is finite depending only on the dimension. To bound it in $H^s(\mathbf{R}^d)$ we use the usual continuity of pseudo-differential operators. For the second one let $n_0 \in \mathbf{N}, n_0 \geq s$. We shall prove that

$$(3.7) \qquad \|D_x^\alpha B_{k,q}\|_{L^2(\mathbf{R}^d)} \leq \frac{C_d}{\langle k - q\rangle^{d+1}}\|u\|_{H_{ul}^{s+m}}, \quad |\alpha| \leq n_0$$

which will complete the proof of Proposition 3.8.

Notice that, due to the presence of χ_k, we have $\|D_x^\alpha B_{k,q}\|_{L^2} \leq C\|D_x^\alpha B_{k,q}\|_{L^\infty}$. We have

$$D_x^\alpha B_{k,q}(x) = \big\langle D_x^\alpha K(x,\cdot), \chi_q u\big\rangle$$

with

$$K(x, y) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{i(x-y)\cdot\xi} p(x, \xi) d\xi \, \chi_k(x)\widetilde{\chi}_q(y)$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d), \widetilde{\chi} = 1$ on the support of χ.

Now on the support of $\chi_k(x)\widetilde{\chi}_q(y)$ we have $|x - y| \geq \delta|k - q|, \delta > 0$. Integrating by parts N times (with large N depending on d, n_0) with the vector field $L = \sum_{j=1}^d \frac{x_j - y_j}{|x-y|^2}\partial_{\xi_j}$ we see that for all $\beta \in \mathbf{N}^n$ we have

$$|D_x^\beta K(x, y)| \leq \frac{C_{d,\beta}}{\langle k - q \rangle^{d+1}}|\widetilde{\chi}_q(y)|, \quad \forall(x, y) \in \mathbf{R}^d \times \mathbf{R}^d.$$

It follows that

$$|D_x^\alpha B_{k,q}(x)| \leq \|D_x^\alpha K(x, \cdot)\|_{H^{-(s+m)}}\|\chi_q u\|_{H^{s+m}}$$
$$\leq \frac{C_{d,\beta}}{\langle k - q \rangle^{d+1}}\|\chi_q u\|_{H^{s+m}}$$

which proves (3.7) and hence concludes the proof. □

In a particular case the proof above gives the following more precise result.

Proposition 3.9. *Let* $m \in \mathbf{R}$, $h(\xi) = \widetilde{h}\left(\frac{\xi}{|\xi|}\right)|\xi|^m\psi(\xi)$ *where* $\widetilde{h} \in C^\infty(\mathbf{S}^{d-1})$ *and* $\psi \in C^\infty(\mathbf{R}^d)$ *is such that* $\psi(\xi) = 1$ *if* $|\xi| \geq 1$, $\psi(\xi) = 0$ *if* $|\xi| \leq \frac{1}{2}$. *Then for every* $\mu \in \mathbf{R}$ *there exists a constant* C *such that*

$$\|h(D_x)u\|_{H_{ul}^\mu(\mathbf{R}^d)} \leq C\|\widetilde{h}\|_{H^{d+1}(\mathbf{S}^{d-1})}\|u\|_{H_{ul}^{\mu+m}(\mathbf{R}^d)}$$

for all $u \in H_{ul}^{\mu+m}(\mathbf{R}^d)$.

We shall use the following result when $p(\xi) = \langle\xi\rangle$ and $p(\xi) = |\xi|^2$.

Lemma 3.10. *Let* $d \geq 1, r > 0, m \in \mathbf{R}$. *Let* $p \in S_{1,0}^r(\mathbf{R}^d), a \in S_{1,0}^m(\mathbf{R}^d)$ *two symbols with constant coefficients. We assume that one can find* $c_0 > 0$ *such that for all* $\xi \in \mathbf{R}^d$ *we have* $p(\xi) \geq c_0|\xi|^r$. *Then for all* $\sigma \in \mathbf{R}$ *and every interval* $I = [0, T]$, *one can find a positive constant* C *such that, with* $H^s = H^s(\mathbf{R}^d)$

$$(3.8) \qquad \|e^{-tp(D)}a(D)u\|_{L^\infty(I, H^\sigma)_{ul}} + \|e^{-tp(D)}a(D)u\|_{L^2(I, H^{\sigma+\frac{r}{2}})_{ul}} \leq C\|u\|_{H_{ul}^{\sigma+m}}$$

for all $u \in H_{ul}^{\sigma+m}$.

PROOF. The estimate of the first term in (3.8) follows from Proposition 3.8 since $e^{-tp(D)}a(D)$ is a pseudo-differential operator of order m whose symbol has semi-norms in $S_{1,0}^m$ bounded by constants depending only on T. Let us look at the second term. Set

$$I_q = \|\chi_q e^{-tp(D)}a(D)u\|_{L^2(I, H^{\sigma+\frac{r}{2}})}.$$

One can write

$$(3.9) \quad \begin{cases} I_q = A_q + B_q \\ A_q = \displaystyle\sum_{|k-q|\le 2} \|\chi_q e^{-tp(D)} a(D)\chi_k u\|_{L^2(I,H^{\sigma+\frac{r}{2}})}, \\ B_q = \displaystyle\sum_{|k-q|\ge 3} \|\chi_q e^{-tp(D)} a(D)\chi_k u\|_{L^2(I,H^{\sigma+\frac{r}{2}})}. \end{cases}$$

Since the number of terms in the sum defining A_q is bounded by a fixed constant (depending only on d) using a classical computation we can write

$$(3.10) \quad \begin{aligned} A_q &\le C_1 \sup_{k\in\mathbf{Z}^d} \|e^{-tp(D)} a(D)\chi_k u\|_{L^2(I,H^{\sigma+\frac{r}{2}})} \\ A_q &\le C_2 \sup_{k\in\mathbf{Z}^d} \|a(D)\chi_k u\|_{H^\sigma} \le C_3 \sup_{k\in\mathbf{Z}^d} \|\chi_k u\|_{H^{\sigma+m}} \le C_3 \|u\|_{H^{\sigma+m}_{ul}}. \end{aligned}$$

Let us look at the term B_q. Let N_0 be an integer such that $N_0 \ge \sigma + \frac{r}{2}$. Then B_q is bounded by a finite sum of terms of the form

$$\sum_{|k-q|\ge 3} \|(D^\alpha \chi_q)(D^\beta e^{-tp(D)} a(D))\chi_k u\|_{L^2(I,L^2)}$$

with $|\alpha| + |\beta| \le N_0$. Due to the presence of the function $D^\alpha \chi_q$, B_q is therefore bounded by a finite number of terms of the form

$$\sum_{|k-q|\ge 3} \|(D^\alpha \chi_q)(D^\beta e^{-tp(D)} a(D))\chi_k u\|_{L^2(I,L^\infty)}.$$

Now we can write

$$(3.11) \quad F(t,x) := (D^\alpha \chi_q)(D^\beta e^{-tp(D)} a(D))\chi_k u(x) = \langle K(t,x,\cdot), (\chi_k u)(\cdot)\rangle$$

with

$$K(t,x,y) = (2\pi)^{-d}(D^\alpha \chi_q)(x)\widetilde{\chi}_k(y) \int e^{i(x-y)\cdot\xi} q(t,\xi)\, d\xi$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ is equal to one on the support of χ and $q(t,\xi) = \xi^\beta e^{-tp(\xi)} a(\xi)$. It follows that for fixed (t,x) we have

$$(3.12) \quad |F(t,x)| \le \|K(t,x,\cdot)\|_{H^{-(\sigma+m)}} \|\chi_k u\|_{H^{\sigma+m}}.$$

Let $N_1 \in \mathbf{N}$ be fixed such that $N_1 \ge -(\sigma + m)$. We shall show that for every $N \in \mathbf{N}$ one can find $C_N = C_N(T) > 0$ such that for every $(t,x) \in I \times \mathbf{R}^d$ we have

$$(3.13) \quad \|K(t,x,\cdot)\|_{H^{N_1}} \le \frac{C_N}{\langle k-q\rangle^N} |(D^\alpha \chi_q)(x)|.$$

Indeed for $|\mu| \le N_1$, $D_y^\mu K(x,y)$ is a finite linear combination of terms of the form

$$J(t,x,y) := (D_x^\alpha \chi_q)(x)(D_y^\nu \widetilde{\chi}_k)(y) \int e^{i(x-y)\cdot\xi} \xi^\lambda q(t,\xi)\, d\xi$$

where $|\nu| + |\lambda| = |\mu|$.

We notice that for all $\gamma \in \mathbf{N}^d$ we have

$$(3.14) \quad |D_\xi^\gamma(\xi^\lambda q(t,\xi))| \le C_\gamma(T)\langle\xi\rangle^{N_0+N_1+m-|\gamma|}.$$

Now let $N \in \mathbf{N}$ be such that $N \geq \max(d+1, N_0 + N_1 + m + d + 1)$ and $\gamma \in \mathbf{N}^d$ with $|\gamma| = N$. Then

$$(x-y)^\gamma J(t,x,y) = (D^\alpha \chi_q)(x)(D_y^\nu \widetilde{\chi}_k)(y) \int e^{i(x-y)\cdot\xi}(-D_\xi)^\gamma(\xi^\lambda q(t,\xi)) \, d\xi.$$

It follows from (3.14) that

$$|(x-y)^\gamma J(t,x,y)| \leq C_1(T)|(D^\alpha \chi_q)(x)||(D_y^\nu \widetilde{\chi}_k)(y)|.$$

Now since $|k-q| \geq 3$, on the support of $(D^\alpha \chi_q)(x)(D_y^\nu \widetilde{\chi}_k)(y)$ we have $|x-y| \geq \frac{1}{3}|k-q|$. It follows that

$$|J(t,x,y)| \leq \frac{C_2(T)}{\langle k-q\rangle^N}|(D^\alpha \chi_q)(x)||(D_y^\nu \widetilde{\chi}_k)(y)|$$

which proves (3.13). According to (3.12) and (3.13) we obtain

$$\|F\|_{L^2(I,L^\infty)} \leq \frac{C_3(T)}{\langle k-q\rangle^N}\|\chi_k u\|_{H^{\sigma+m}} \leq \frac{C_3(T)}{\langle k-q\rangle^N}\|u\|_{H_{ul}^{\sigma+m}}$$

which implies that $B_q \leq C_4(T)\|u\|_{H_{ul}^{\sigma+m}}$. Combined with (3.9) and (3.10) this proves the estimate of the second term in (3.8). $\qquad\square$

Corollary 3.11. *Let $m \in \mathbf{R}$ and $a \in S_{1,0}^m(\mathbf{R}^d)$. Then for every $\sigma \in \mathbf{R}$ there exists $C > 0$ such that*

$$\|e^{\delta z \langle D_x \rangle}a(D_x)u\|_{X_{ul}^{\sigma-\frac{1}{2}}(-1,0)} \leq C\|u\|_{H_{ul}^{\sigma+m-\frac{1}{2}}(\mathbf{R}^d)}$$

for every $\delta > 0$ and every $u \in H_{ul}^{\sigma+m-\frac{1}{2}}(\mathbf{R}^d)$.

3.4. An interpolation Lemma. We shall use the following interpolation lemma for which we refer to [**32**] Théorème 3.1.

Lemma 3.12. *Let $J = (-1,0)$ and $t \in \mathbf{R}$. Let $f \in L_z^2(J, H^{t+\frac{1}{2}}(\mathbf{R}^d))$ be such that $\partial_z f \in L_z^2(J, H^{t-\frac{1}{2}}(\mathbf{R}^d))$. Then $f \in C_z^0([-1,0], H^t(\mathbf{R}^d))$ and there exists an absolute constant $C > 0$ such that*

$$\sup_{z\in[-1,0]}\|f(z,\cdot)\|_{H^t(\mathbf{R}^d)} \leq C\|f\|_{L_z^2(J,H^{t+\frac{1}{2}}(\mathbf{R}^d))} + C\|\partial_z f\|_{L_z^2(J,H^{t-\frac{1}{2}}(\mathbf{R}^d))}.$$

3.5. Para-differential operators.

3.5.1. *Symbolic calculus.* In this section we quote some results which concern the symbolic calculus for para-differential operators in the framework of the uniformly local Sobolev spaces. Of course, here, the theory for the classical Sobolev spaces will be assumed to be known (see [**33**]).

The following technical lemma will be used in the sequel.

Lemma 3.13. *Let $\chi \in C_0^\infty(\mathbf{R}^d)$ and $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ be equal to one on the support of χ. Let $\psi, \theta \in \mathcal{S}(\mathbf{R}^d)$. For every $m, \sigma \in \mathbf{R}$ one can find a constant $C > 0$ such that*

$$(3.15) \qquad \sum_{j\geq-1}\|\chi_k\psi(2^{-j}D)((1-\widetilde{\chi}_k)u)\theta(2^{-j}D)v\|_{H^m(\mathbf{R}^d)} \leq C\|u\|_{H_{ul}^\sigma(\mathbf{R}^d)}\|v\|_{L^\infty(\mathbf{R}^d)}.$$

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For every $m, \sigma, t \in \mathbf{R}$ one can find a constant $C > 0$ such that

$$(3.16) \qquad \sum_{j \geq -1} \|\chi_k \psi(2^{-j} D)((1 - \widetilde{\chi}_k)u)\theta(2^{-j}D)v\|_{H^m(\mathbf{R}^d)} \leq C\|u\|_{H^\sigma_{ul}(\mathbf{R}^d)}\|v\|_{H^t_{ul}(\mathbf{R}^d)}.$$

PROOF. We may assume $m \in \mathbf{N}$. Let us call $A_{k,j}$ the term inside the sum in the left hand side of (3.15). Due to χ_k, the term $A_{k,j}$ is a bounded by finite sum of terms of the form

$$A_{k,j,\alpha} := 2^{jm}\|(D^{\alpha_1}\chi_k)\psi_{\alpha_2}(2^{-j}D)((1 - \widetilde{\chi}_k)u)\widetilde{\chi}_k\theta_{\alpha_3}(2^{-j}D)v\|_{L^\infty}$$

where $|\alpha_1| + |\alpha_2| + |\alpha_3| \leq m$ and $\psi_{\alpha_2} = x^{\alpha_2}\psi, \theta_{\alpha_3} = x^{\alpha_3}\theta$. We are going to show that for large $N \in \mathbf{N}$ we have

$$(i) \quad \|(D^{\alpha_1}\chi_k)\psi_{\alpha_2}(2^{-j}D)((1 - \widetilde{\chi}_k)u)\|_{L^\infty} \leq C_N 2^{jM_1(d,\sigma)}2^{-jN}\|u\|_{H^\sigma_{ul}}$$

$$(ii) \quad \|\widetilde{\chi}_k\theta_{\alpha_3}(2^{-j}D)v\|_{L^\infty} \leq C\|v\|_{L^\infty}$$

$$(iii) \quad \|\widetilde{\chi}_k\theta_{\alpha_3}(2^{-j}D)v\|_{L^\infty} \leq C2^{jM_2(d,t)}\|v\|_{H^t_{ul}}$$

where, as indicated, M_j are fixed constants depending only on d, σ, t. Then the lemma will follow from these estimates.

To prove (i) we write

$$(D^{\alpha_1}\chi_k(x))\psi_{\alpha_2}(2^{-j}D)((1 - \widetilde{\chi}_k)u)(x) = 2^{jd}2^{-jN}$$

$$\times \left\langle (2^j|x - \cdot|)^N \widehat{\psi_{\alpha_2}}(2^j(x - \cdot)), \frac{\langle x - \cdot\rangle^N}{|x - \cdot|^N}(D^{\alpha_1}\chi_k(x))((1 - \widetilde{\chi}_k(\cdot))\langle x - \cdot\rangle^{-N}u\right\rangle.$$

The function $y \mapsto \frac{\langle x-y\rangle^N}{|x-y|^N}(D^{\alpha_1}\chi_k(x))((1 - \widetilde{\chi}_k(y))$ belongs to $W^{\infty,\infty}$ with semi-norms uniformly bounded in x. Using the duality $H^{-\sigma} - H^\sigma$ we deduce that

$$\|(D^{\alpha_1}\chi_k)\psi_{\alpha_2}(2^{-j}D)((1 - \widetilde{\chi}_k)u)\|_{L^\infty} \leq C_N 2^{jM_1(d,\sigma)}2^{-jN}\|\langle x - \cdot\rangle^{-N}u\|_{H^\sigma},$$

and we conclude using Lemma 3.2.

The estimate (ii) is easy. To prove (iii) we take $\check{\chi} \in C_0^\infty(\mathbf{R}^d)$ equal to one on the support of $\widetilde{\chi}$ and we write

$$\|\widetilde{\chi}_k\theta_{\alpha_3}(2^{-j}D)v\|_{L^\infty} \leq \|\widetilde{\chi}_k\theta_{\alpha_3}(2^{-j}D)\check{\chi}_k v\|_{L^\infty} + \|\widetilde{\chi}_k\theta_{\alpha_3}(2^{-j}D)(1 - \check{\chi})v\|_{L^\infty}.$$

The second term is bounded exactly by the same method as (i). For the first one we write

$$\widetilde{\chi}_k\theta_{\alpha_3}(2^{-j}D)\check{\chi}_k v(x) = 2^{jd}\widetilde{\chi}_k(x)\langle\widehat{\theta_{\alpha_3}}(2^j(x - \cdot)), \check{\chi}_k(\cdot)v(\cdot)\rangle$$

and we use the $H^{-t} - H^t$ duality. $\qquad\qquad\square$

Remark 3.14. (i) Notice that the sames estimates in (3.15), (3.16) hold if in the left hand side one 2^{-j} is replaced by 2^{-j-j_0} where $j_0 \in \mathbf{Z}$ is fixed.

(ii) Notice also that in the above proof we have proved that for all real numbers m, σ, all $N \in \mathbf{N}$ and all $\psi \in C_0^\infty(\mathbf{R}^d)$ one can find a positive constant $C_{N,m,\sigma}$ such that

$$(3.17) \qquad \|\chi_k\psi(2^{-j}D)(1 - \widetilde{\chi}_k)u\|_{H^m(\mathbf{R}^d)} \leq C_{N,m,\sigma}2^{-jN}\|u\|_{H^\sigma_{ul}(\mathbf{R}^d)}$$

for every $j \in \mathbf{N}$ and every $k \in \mathbf{Z}^d$.

We introduce now the para-differential calculus.

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Definition 3.15. *Given $m \in \mathbf{R}, \rho \in [0,1], \Gamma_\rho^m(\mathbf{R}^d)$ denotes the space of locally bounded functions on $\mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}$ which are C^∞ with respect to ξ, such that for all $\alpha \in \mathbf{N}^d$ the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbf{R}^d)$ and there exists a constant $C_\alpha > 0$ such that*

$$\|\partial_\xi^\alpha a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}, \quad \forall |\xi| \geq \frac{1}{2}.$$

For such a we set

(3.18) $$M_\rho^m(a) = \sup_{|\alpha| \leq 2d+2} \sup_{|\xi| \geq \frac{1}{2}} \|(1+|\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)}.$$

Then $\dot{\Gamma}_\rho^m(\mathbf{R}^d)$ denotes the subspace of $\Gamma_\rho^m(\mathbf{R}^d)$ which consists of symbols $a(x,\xi)$ which are homogeneous of degree m with respect to ξ.

Given a symbol a we denote by T_a the associated para-differential operator which is given by the formula

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int_{\mathbf{R}^d} \theta(\xi-\eta,\eta) \hat{a}(\xi-\eta,\eta) \psi(\eta) \hat{u}(\eta) d\eta$$

where $\hat{a}(\zeta,\eta) = \int_{\mathbf{R}^d} e^{-ix\cdot\zeta} a(x,\eta) dx$ is the Fourier transform of a with respect to the first variable, ψ, θ are two fixed C^∞ functions on $\mathbf{R}^d$ such that for $0 < \varepsilon_1 < \varepsilon_2$ small enough

(3.19) $$\psi(\eta) = 1 \text{ if } |\eta| \geq 1, \quad \psi(\eta) = 0 \text{ if } |\eta| \leq \frac{1}{2}$$

(3.20) $$\theta(\zeta,\eta) = 1 \text{ if } |\zeta| \leq \varepsilon_1|\eta|, \quad \theta(\zeta,\eta) = 0 \text{ if } |\zeta| \geq \varepsilon_2|\eta|.$$

Notice that if the symbol a is independent of ξ the associated operator T_a is called a paraproduct.

Theorem 3.16. *Let $m, m' \in \mathbf{R}, \rho \in [0,1]$.*

(i) If $a \in \Gamma_0^m(\mathbf{R}^d)$, then for all $\mu \in \mathbf{R}$ T_a is continuous from $H_{ul}^\mu(\mathbf{R}^d)$ to $H_{ul}^{\mu-m}(\mathbf{R}^d)$ with norm bounded by $C M_0^m(a)$.

(ii) If $a \in \Gamma_\rho^m(\mathbf{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$ then, for all $\mu \in \mathbf{R}$, $T_a T_b - T_{ab}$ is continuous from $H_{ul}^\mu(\mathbf{R}^d)$ to $H_{ul}^{\mu-m-m'+\rho}(\mathbf{R}^d)$ with norm bounded by

$$C(M_\rho^m(a) M_0^{m'}(b) + M_0^m(a) M_\rho^{m'}(b)).$$

(iii) Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of T_a and by $\bar{a}$ the complex conjugate of a. Then for all $\mu \in \mathbf{R}$ $(T_a)^* - T_{\bar{a}}$ is continuous from $H_{ul}^\mu(\mathbf{R}^d)$ to $H_{ul}^{\mu-m+\rho}(\mathbf{R}^d)$ with norm bounded by $C M_\rho^m(a)$.

PROOF. All these points are proved along the same lines. We shall only prove the first one and for simplicity we shall consider symbols in $\dot{\Gamma}_\rho^m(\mathbf{R}^d)$. We begin by the case where a is a bounded function. Then we write

$$\chi_k T_a u = \chi_k T_a(\widetilde{\chi}_k u) + \chi_k T_a((1-\widetilde{\chi}_k)u)$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d), \widetilde{\chi} = 1$ on the support of χ. By the classical theory we have

$$\|\chi_k T_a(\widetilde{\chi}_k u)\|_{H^\mu} \leq C\|a\|_{L^\infty} \|\widetilde{\chi}_k u\|_{H^\mu} \leq C\|a\|_{L^\infty} \|u\|_{H_{ul}^\mu}.$$

Now we write

$$\chi_k T_a((1 - \widetilde{\chi}_k)u) = \sum_j \chi_k \{\psi(2^{-j}D)a\}\{\varphi(2^{-j}D)((1 - \widetilde{\chi}_k)u)\}.$$

and the desired estimate follows immediately from the first inequality in Lemma 3.13.

We now assume that $a(x, \xi) = b(x)h(\xi)$ where $h(\xi) = |\xi|^m \widetilde{h}\left(\frac{\xi}{|\xi|}\right)$ with $\widetilde{h} \in C^\infty(\mathbf{S}^{d-1})$. Then directly from the definition we have $T_a = T_b \psi(D_x)h(D_x)$ and our estimate in (i) follows from the first step and from the estimate proved in Proposition 3.9

$$\|h(D)v\|_{H^\mu} \le C\|\widetilde{h}\|_{H^{d+1}(\mathbf{S}^{d-1})}\|u\|_{H^{\mu+m}}.$$

In the last step we introduce $(\widetilde{h}_\nu)_{\nu \in \mathbf{N}^*}$ an orthonormal basis of $L^2(\mathbf{S}^{d-1})$ consisting of eigenfunctions of the (self adjoint) Laplace Beltrami operator $\Delta_\omega = \Delta_{\mathbf{S}^{d-1}}$ on $L^2(\mathbf{S}^{d-1})$ i.e. $\Delta_\omega \widetilde{h}_\nu = \lambda_\nu^2 \widetilde{h}_\nu$. By the Weyl formula we know that $\lambda_\nu \sim c\nu^{\frac{1}{d}}$. Setting $h_\nu = |\xi|^m \widetilde{h}(\omega)$, $\omega = \frac{\xi}{|\xi|}$ when $\xi \ne 0$, we can write

$$a(x, \xi) = \sum_{\nu \in \mathbf{N}^*} b_\nu(x)h_\nu(\xi) \quad \text{where} \quad b_\nu(x) = \int_{\mathbf{S}^{d-1}} a(x, \omega)\overline{\widetilde{h}_\nu(\omega)}d\omega.$$

Since

$$\lambda_\nu^{2d+2} b_\nu(x) = \int_{\mathbf{S}^{d-1}} \Delta_\omega^{d+1} a(x, \omega)\overline{\widetilde{h}_\nu(\omega)}d\omega$$

we deduce that

$$(3.21) \qquad\qquad \|b_\nu\|_{L^\infty(\mathbf{R}^d)} \le C\lambda_\nu^{-(2d+2)} M_0^m(a).$$

Moreover there exists a positive constant K such that for all $\nu \ge 1$

$$(3.22) \qquad\qquad \|\widetilde{h}_\nu\|_{H^{d+1}(\mathbf{S}^{d-1})} \le K\lambda_\nu^{d+1}.$$

Now using the steps above and Proposition 3.9 we obtain

$$\|T_a u\|_{H_{ul}^\mu} \le \sum_{\nu \ge 1} \|T_{b_\nu} \psi(D_x)h_\nu(D_x)u\|_{H_{ul}^\mu}$$

$$\le C \sum_{\nu \ge 1} \|b_\nu\|_{L^\infty(\mathbf{R}^d)}\|\widetilde{h}_\nu\|_{H^{d+1}(\mathbf{S}^{d-1})}\|u\|_{H_{ul}^{\mu+m}}$$

$$\le M_0^m(a)\|u\|_{H_{ul}^{\mu+m}} \sum_{\nu \ge 1} \lambda_\nu^{-(d+1)}$$

and $\lambda_\nu^{-(d+1)} \sim c\nu^{-(1+\frac{1}{d})}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

3.5.2. *Paraproducts.* We have the following result of paralinearization of a product.

Proposition 3.17. *Given two functions* $a \in H_{ul}^\alpha(\mathbf{R}^d), u \in H_{ul}^\beta(\mathbf{R}^d)$ *with* $\alpha + \beta > 0$ *we can write*

$$au = T_a u + T_u a + R(a, u)$$

with

$$(3.23) \qquad\qquad \|R(a, u)\|_{H_{ul}^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)} \le C\|a\|_{H_{ul}^\alpha(\mathbf{R}^d)}\|u\|_{H_{ul}^\beta(\mathbf{R}^d)}.$$

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PROOF. We have

$$R(a, u) = \sum_{j \geq -1} \sum_{|k-j| \leq 1} \varphi(2^{-j}D)a \cdot \varphi(2^{-k}D)u.$$

We take $\chi \in C_0^\infty(\mathbf{R}^d)$ satisfying (2.3), $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ equal to one on the support of χ and we write $a = \widetilde{\chi}_k a + (1 - \widetilde{\chi}_k)a, u = \widetilde{\chi}_k u + (1 - \widetilde{\chi}_k)u$. It follows that

$$\chi_k R(a, u) = \chi_k R(\widetilde{\chi}_k a, \widetilde{\chi}_k u) + \chi_k S_k(a, u).$$

The term $\chi_k R(\widetilde{\chi}_k a, \widetilde{\chi}_k u)$ is estimated by the right hand side of (3.23) using Theorem 2.11 in [3]. The remainder $\chi_k S_k(a, u)$ is estimated using (3.16). $\square$

Proposition 3.18. *Let γ, r, μ be real numbers such that*

$$r + \mu > 0, \quad \gamma \leq r, \quad \gamma < r + \mu - \frac{d}{2}.$$

There exists a constant $C > 0$ such that

(3.24) $$\|(a - T_a)u\|_{H_{ul}^\gamma(\mathbf{R}^d)} \leq C\|a\|_{H_{ul}^r(\mathbf{R}^d)}\|u\|_{H_{ul}^\mu(\mathbf{R}^d)}$$

whenever the right hand side is finite.

PROOF. We write

(3.25) $$\chi_k(a - T_a)u = \chi_k(\widetilde{\chi}_k a - T_{\widetilde{\chi}_k a})\widetilde{\chi}_k u + R_{1,k}u + R_{2,k}u$$

(3.26) $$R_{1,k}u = \chi_k(\widetilde{\chi}_k a - T_{\widetilde{\chi}_k a})(1 - \widetilde{\chi}_k)u$$

(3.27) $$R_{2,k} = -\chi_k T_{(1-\widetilde{\chi}_k)a}u = -\chi_k \sum_j S_j((1 - \widetilde{\chi}_k)a)\Delta_j(u)$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ is equal to one on the support of χ. According to Proposition 2.12 in [3] we have

(3.28) $$\|\chi_k(\widetilde{\chi}_k a - T_{\widetilde{\chi}_k a})\widetilde{\chi}_k u\|_{H^\gamma} \leq \|a\|_{H_{ul}^r}\|u\|_{H_{ul}^\mu}.$$

Now

$$R_{1,k} = \chi_k T_{(1-\widetilde{\chi}_k u)}\widetilde{\chi}_k a + \chi_k \mathcal{R}((1 - \widetilde{\chi}_k u, \widetilde{\chi}_k a)$$
$$= \chi_k \sum_j S_j((1 - \widetilde{\chi}_k)u)\Delta_j(\widetilde{\chi}_k a) + \chi_k \sum_{|i-j| \leq 1} \Delta_i((1 - \widetilde{\chi}_k)u)\Delta_j(\widetilde{\chi}_k a).$$

Therefore we can apply (3.16) in Lemma 3.13 to $R_{1,k}$ and $R_{2,k}$ to conclude that the estimate (3.24) holds for these terms. $\square$

3.6. On transport equations. We will be using the following result about solutions of vector fields.

Lemma 3.19. *Let $I = [0, T]$, $s_0 > 1 + \frac{d}{2}$ and $\mu > 0$. Then there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for $V_j \in L^\infty(I, H^{s_0}(\mathbf{R}^d))_{ul} \cap L^\infty(I, H^\mu(\mathbf{R}^d))_{ul}$ $j = 1, \ldots, d$, $f \in L^1(I, H^\mu(\mathbf{R}^d))_{ul}$, $u_0 \in H_{ul}^\mu(\mathbf{R}^d)$ and any solution $u \in L^\infty(I, H^{s_0}(\mathbf{R}^d))_{ul}$ of the problem*

$$(\partial_t + V \cdot \nabla)u = f, \quad u|_{t=0} = u_0$$

we have,

$$\|u\|_{L^\infty(I,H^\mu)_{ul}} \leq \mathcal{F}\big(T\|V\|_{L^\infty(I,H^{s_0})_{ul}}\big)\Big\{\|u_0\|_{H^\mu_{ul}} + \|f\|_{L^1(I,H^\mu)_{ul}}$$

$$+ \sup_{k\in\mathbf{Z}^d}\Big(\int_0^T \|u(\sigma)\|_{H^{s_0}_{ul}}\|\widetilde{\chi}_k V(\sigma)\|_{H^\mu}\,d\sigma\Big)\Big\}$$

where $\widetilde{\chi}\in C_0^\infty(\mathbf{R}^d)$ is equal to one on the support of χ.

PROOF. Set $V_k = \widetilde{\chi}_k V$. We have

$$(3.29) \qquad (\partial_t + T_{V_k}\cdot\nabla)(\chi_k u) = \chi_k f + V_k\cdot(\nabla\chi_k)\widetilde{\chi}_k u + (T_{V_k} - V_k)\cdot\nabla(\chi_k u) := g_k.$$

Now computing the quantity $\frac{d}{dt}\|\chi_k u(t)\|^2_{L^2}$, using the above equation, the fact that $\|(T_V\cdot\nabla + (T_V\cdot\nabla)^*\|_{L^2\to L^2} \leq C\|V(t)\|_{W^{1,\infty}}$ and the Gronwall inequality we obtain

$$(3.30) \qquad \|\chi_k u(t)\|_{L^2} \leq \mathcal{F}\big(\|V\|_{L^1(I,W^{1,\infty})}\big)\Big\{\|\chi_k u_0\|_{L^2} + \int_0^t \|g_k(\sigma)\|_{L^2}\,d\sigma\Big\}.$$

Now we can write

$$(\partial_t + T_{V_k}\cdot\nabla)\langle D\rangle^\mu(\chi_k u) = \langle D\rangle^\mu g_k + [T_{V_k}, \langle D\rangle^\mu]\cdot\nabla(\chi_k u).$$

By the symbolic calculus (see Theorem 3.16, (ii)) we have

$$\|[T_{V_k}, \langle D\rangle^\mu]\cdot\nabla(\chi_k u)(t)\|_{L^2} \leq C\|V(t)\|_{W^{1,\infty}}\|\chi_k u(t)\|_{H^\mu}.$$

Therefore using (3.30) and Gronwall inequality we obtain

$$(3.31) \qquad \|\chi_k u(t)\|_{H^\mu} \leq \mathcal{F}\big(\|V\|_{L^1(I,W^{1,\infty})}\big)\Big\{\|\chi_k u_0\|_{H^\mu} + \int_0^t \|g_k(\sigma)\|_{H^\mu}\,d\sigma\Big\}.$$

Coming back to the definition of g given in (3.29) we have

$$\|V_k\cdot(\nabla\chi_k)\widetilde{\chi}_k u(t)\|_{H^\mu} \leq C(\|V(t)\|_{L^\infty}\|\widetilde{\chi}_k u(t)\|_{H^\mu} + \|u(t)\|_{L^\infty}\|V_k(t)\|_{H^\mu}).$$

On the other hand we have

$$(V_k - T_{V_k})\cdot\nabla(\chi_k u) = T_{\nabla(\chi_k u)}\cdot V_k + R(V_k, \nabla(\chi_k u)).$$

By Theorem 3.16 (i) and an easy computation we see that

$$\|T_{\nabla(\chi_k u)(t)}\cdot V_k(t)\|_{H^\mu} + \|R(V_k, \nabla(\chi_k u))\| \leq C\|u(t)\|_{W^{1,\infty}}\|V_k(t)\|_{H^\mu}.$$

Using (3.31), the Gronwall inequality, the embedding of $H^{s_0}_{ul}$ in $W^{1,\infty}$ and the above estimates we obtain the desired conclusion. $\qquad\square$

3.7. Commutation with a vector field.

Lemma 3.20. *Let $I = [0,T], V \in C^0(I, W^{1+\varepsilon}(\mathbf{R}^d))$ for some $\varepsilon > 0$ and consider a symbol $p = p(t,x,\xi)$ which is homogeneous of order m. Then there exists a positive constant K (independent of p, V) such that for any $t \in I$ and any $u \in C^0(I, H^m_{ul}(\mathbf{R}^d))$ we have*

$$(3.32) \qquad \|[T_p, \partial_t + T_V\cdot\nabla_x]u(t,\cdot)\|_{L^2_{ul}(\mathbf{R}^d)} \leq KC(p,V)\|u(t,\cdot)\|_{H^m_{ul}(\mathbf{R}^d)}$$

where

$$C(p,V) := M_0^m(p)\|V\|_{C^0(I,W^{1+\varepsilon}(\mathbf{R}^d))} + M_0^m(\partial_t p + V\cdot\nabla_x p)\|V\|_{L^\infty(I\times\mathbf{R}^d)}.$$

PROOF. We proceed as in the proof of Theorem 3.16 and we begin by the case where $m = 0$ and p is a function. We denote by $\mathcal{R}$ the set of continuous operators $R(t)$ from $L^2_{ul}(\mathbf{R}^d)$ to $L^2(\mathbf{R}^d)$ such that $\sup_{t \in I} \|R(t)u(t)\|_{L^2(\mathbf{R}^d)}$ is bounded by the right hand side of (3.32). We write

$$\chi_k[T_p, \partial_t + T_V \cdot \nabla_x] = \chi_k[T_p, \partial_t + T_V \cdot \nabla_x]\widetilde{\chi}_k + [\chi_k T_p, \partial_t + T_V \cdot \nabla_x](1 - \widetilde{\chi}_k)$$

where $\tilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ is equal to one on the support of χ. By Lemma 2.17 in [3] the first operator in the right hand side of the above equality belongs to $\mathcal{R}$. Let us look at the second term. It is equal to

$$-\chi_k T_{\partial_t p}(1 - \widetilde{\chi}_k) + \chi_k T_p T_V \cdot \nabla_x(1 - \widetilde{\chi}_k) - \chi_k T_V \cdot \nabla_x T_p(1 - \widetilde{\chi}_k) =: A + B + C.$$

We can write

$$A = -\chi_k T_{\partial_t p + V \cdot \nabla_x p}(1 - \widetilde{\chi}_k) + \chi_k T_{V \cdot \nabla_x p}(1 - \widetilde{\chi}_k) := A_1 + A_2.$$

By Theorem 3.16 (i) the term A_1 belongs to $\mathcal{R}$. Now

$$\begin{aligned} A_2 u &= \chi_k T_{\mathrm{div}(pV) - p\,\mathrm{div}\,V}(1 - \widetilde{\chi}_k)u \\ &= \sum_{j \geq -1} \psi(2^{-j}D)(\mathrm{div}(pV) - p\,\mathrm{div}\,V)\chi_k\varphi(2^{-j}D)((1 - \widetilde{\chi}_k)u). \end{aligned}$$

Since

$$\|\psi(2^{-j}D)(\mathrm{div}(pV) - p\,\mathrm{div}\,V)\|_{L^\infty} \leq C2^j \|p\|_{L^\infty}\|V\|_{W^{1,\infty}}$$

we deduce from Remark 3.17 that $A_2 \in \mathcal{R}$.

Let $\underline{\chi} \in C_0^\infty(\mathbf{R}^d)$ such that $\widetilde{\chi} = 1$ on the support of $\underline{\chi}$ and $\underline{\chi} = 1$ on the support of χ. We write

$$B = \chi_k T_p \underline{\chi}_k T_V \cdot \nabla_x(1 - \widetilde{\chi}_k) + \chi_k T_p(1 - \underline{\chi}_k)T_V \cdot \nabla_x(1 - \widetilde{\chi}_k) := B_1 + B_2.$$

By Theorem 3.16 (i) we have

$$\begin{aligned} \|B_1 u\|_{L^2} &\leq C\|p\|_{L^\infty}\big\|\underline{\chi}_k T_V \cdot \nabla_x(1 - \widetilde{\chi}_k)u\big\|_{L^2} \\ &\leq C\|p\|_{L^\infty} \sum_{j \geq -1} \big\|(\psi(2^{-j}D)V)2^j\underline{\chi}_k\varphi_1(2^{-j}D)(1 - \widetilde{\chi}_k)u\big\|_{L^2} \\ &\leq C\|p\|_{L^\infty}\|V\|_{L^\infty} \sum_{j \geq -1} 2^j\big\|\underline{\chi}_k\varphi_1(2^{-j}D)(1 - \widetilde{\chi}_k)u\big\|_{L^2} \end{aligned}$$

and Remark 3.17 shows that $B_1 \in \mathcal{R}$.

Now by (3.15) and Theorem 3.16 we can write

$$\begin{aligned} \|B_2 u\|_{L^2} &\leq C\|p\|_{L^\infty}\big\|T_V \cdot \nabla_x(1 - \widetilde{\chi}_k)u\big\|_{H^{-1}_{ul}} \\ &\leq C\|p\|_{L^\infty}\|V\|_{L^\infty}\|u\|_{L^2_{ul}} \end{aligned}$$

so $B_2 \in \mathcal{R}$. The term C is estimated exactly by the same way, introducing a cut-off $\underline{\chi}_k$ after the operator $T_V \cdot \nabla_x$. Thus $C \in \mathcal{R}$.

The case where $p = a(x)h(\xi)$ and then were p is a general homogeneous symbol of order m is handled as in the proof of Theorem 3.16. $\qquad\square$

4. The Dirichlet-Neumann operator

4.1. Definition of the Dirichlet-Neumann operator. For $d \geq 1$ we set

$$(4.1) \qquad \begin{cases} \Omega = \{(x,y) \in \mathbf{R}^{d+1} : \eta_*(x) < y < \eta(x)\}, \\ \Sigma = \{(x,y) \in \mathbf{R}^{d+1} : y = \eta(x)\} \end{cases}$$

where η_* is a fixed bounded continuous function on $\mathbf{R}^d$ and $\eta \in W^{1,\infty}(\mathbf{R}^d)$. We shall assume that there exists $h > 0$ such that

$$(4.2) \qquad \{(x,y) \in \mathbf{R}^{d+1} : \eta(x) - h \leq y < \eta(x)\} \subset \Omega.$$

In [**3**] the Dirichlet-Neumann operator $G(\eta)$ associated to Ω has been defined as a continuous operator from $H^{\frac{1}{2}}(\mathbf{R}^d)$ to $H^{-\frac{1}{2}}(\mathbf{R}^d)$. Our aim here is to prove that it has a unique extension to the space $H^{\frac{1}{2}}_{ul}(\mathbf{R}^d)$ (see Theorem 4.8 below). Define first the space $H^1_{ul}(\Omega)$ by

$$u \in H^1_{ul}(\Omega) \Leftrightarrow \|u\|_{H^1_{ul}(\Omega)} := \sup_{q \in \mathbf{Z}^d} \|\chi_q u\|_{H^1(\Omega)} < +\infty.$$

Each element $u \in H^1_{ul}(\Omega)$ have a trace on Σ (see below) which will be denoted by $\gamma_0 u$. We introduce the subspace $H^{1,0}_{ul}(\Omega) \subset H^1_{ul}(\Omega)$ defined by

$$H^{1,0}_{ul}(\Omega) = \{u \in H^1_{ul}(\Omega) : \gamma_0 u = 0\}.$$

Then we have the following Poincaré inequality.

Lemma 4.1. *There exists $C > 0$ depending on $\|\eta\|_{L^\infty(\mathbf{R}^d)} + \|\eta_*\|_{L^\infty(\mathbf{R}^d)}$ such that for $\alpha \in C_0^\infty(\mathbf{R}^d)$ non negative and $u \in H^{1,0}_{ul}(\Omega)$ we have*

$$\iint_\Omega \alpha(x) |u(x,y)|^2 \, dxdy \leq C \iint_\Omega \alpha(x) |\partial_y u(x,y)|^2 \, dxdy.$$

PROOF. Let $u \in H^{1,0}_{ul}(\Omega)$. It is easy to see that there exists a sequence (u_n) of functions which are C^1 in Ω and vanish near the top boundary $y = \eta(x)$ such that

$$\lim_{n \to +\infty} \|u_n - u\|_{H^1(\Omega \cap \{|x| \leq K\})} = 0.$$

As a consequence, it is enough to prove the result for such functions. Let $\alpha \in C_0^\infty(\mathbf{R}^d), \alpha \geq 0$. We can write

$$u(x,y) = \int_{\eta(x)}^y \partial_s u(x,s) ds$$

from which we deduce

$$\alpha(x) |u(x,y)|^2 \leq \|\eta - \eta_*\|_{L^\infty} \alpha(x) \int_{\eta_*(x)}^{\eta(x)} |\partial_s u(x,s)|^2 ds.$$

Integrating this inequality on Ω we obtain,

$$\iint_\Omega \alpha(x) |u(x,y)|^2 dx \, dy \leq \|\eta - \eta_*\|_{L^\infty} \iint_\Omega \alpha(x) |\partial_y u(x,y)|^2 \, dxdy.$$

$\square$

Remark 4.2. Let
$$H^{1,0}(\Omega) = \{u \in L^2(\Omega) : \nabla_{x,y} u \in L^2(\Omega), \text{ and } u|_{y=\eta(x)} = 0\},$$
then we also have the Poincaré inequality

(4.3)
$$\iint_\Omega \alpha(x)|u(x,y)|^2 \leq C \iint_\Omega \alpha(x)|\partial_y u(x,y)|^2 \, dxdy$$

for all $u \in H^{1,0}(\Omega)$, $\alpha \in C_b^\infty(\mathbf{R}^d)$, $\alpha \geq 0$, with a constant C independent of α. Indeed, this follows from the same computation as above using the fact that any $u \in H^{1,0}(\Omega)$ can be approximated by a sequence of functions which are C^∞ in Ω and vanish near $y = \eta(x)$.

Proposition 4.3. *For every* $\psi \in H_{ul}^{\frac{1}{2}}(\mathbf{R}^d)$ *the problem*

(4.4)
$$\Delta_{x,y}\Phi = 0 \text{ in } \Omega, \quad \Phi|_\Sigma = \psi, \quad \frac{\partial\Phi}{\partial\nu}|_\Gamma = 0,$$

has a unique solution $\Phi \in H_{ul}^1(\Omega)$ *and there exists a function* $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ *independent of* (ψ, η) *such that*
$$\|\Phi\|_{H_{ul}^1(\Omega)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi\|_{H_{ul}^{\frac{1}{2}}(\mathbf{R}^d)}.$$

PROOF. Before giving the proof we have to precise the meaning of the boundary condition $\frac{\partial\Phi}{\partial\nu}|_\Gamma = 0$ since Γ is only C^0. This condition means that

(4.5)
$$\iint_\Omega \nabla_{x,y}\Phi(x,y) \cdot \nabla_{x,y}(\alpha(x)\theta(x,y)) \, dx \, dy = 0$$

for every $\theta \in H^1(\Omega)$ (the usual Sobolev space) with $\text{supp}\,\theta \subset \{(x,y) : \eta_*(x) \leq y \leq \eta_*(x) + \varepsilon\}$ for a small $\varepsilon > 0$ and every $\alpha \in C_0^\infty(\mathbf{R}^d)$.

Notice that if $\eta_* \in W^{2,\infty}(\mathbf{R}^d)$ the Green formula (see [**24**] p.62) shows that (4.5) is equivalent to $\frac{\partial\Phi}{\partial\nu}|_\Gamma = 0$.

Lemma 4.4. *We have*

(4.6)
$$\iint_\Omega \nabla_{x,y}\Phi(x,y) \cdot \nabla_{x,y}(\alpha(x)\theta(x,y)) \, dx \, dy = 0$$

for every $\theta \in H^1(\Omega)$ *with* $\gamma_0\theta = 0$ *and every* $\alpha \in C_0^\infty(\mathbf{R}^d)$.

PROOF. If θ has support in a neighborhood of Γ, $V_\Gamma = \{(x,y) : x \in \mathbf{R}^d, \eta_*(x) < y < \eta_*(x) + \varepsilon\}$, this follows from (4.5). Assume that θ vanishes in a neighborhood of Γ. Let $\Omega_0 = \{(x,y) : \eta_*(x) + \frac{\varepsilon}{2} < y < \eta(x)\}$ for $\varepsilon > 0$ small enough. Then $\theta \in H^1(\Omega_0)$ and $\theta|_{\partial\Omega_0} = 0$. Thus $\theta \in H_0^1(\Omega_0)$. Since Ω_0 has a Lipschitz upper boundary there exists a sequence $\theta_n \in C_0^\infty(\Omega_0)$ which converges to θ in $H^1(\Omega_0)$ (see [**24**] Corollary 1.5.1.6). Now by the equation we have
$$0 = \langle\Delta_{x,y}\Phi, \alpha\theta_n\rangle = \iint_\Omega \nabla_{x,y}\Phi(x,y) \cdot \nabla_{x,y}(\alpha(x)\theta_n(x,y)) \, dx \, dy.$$
Moreover
$$\left|\iint_\Omega \nabla_{x,y}\Phi(x,y) \cdot \nabla_{x,y}\Big(\alpha(x)(\theta_n - \theta)(x,y)\Big) \, dx \, dy\right| \leq C\|\Phi\|_{H_{ul}^1(\Omega)}\|\theta_n - \theta\|_{H^1(\Omega_0)}.$$
Therefore, passing to the limit, we obtain (4.6) for such θ. $\qquad\square$

Part 1. Uniqueness: Let us denote by Φ the difference of two solutions in $H^1_{ul}(\Omega)$ of (4.4). Then $\gamma_0\Phi = 0$. Now we take in (4.6) $\alpha(x) = e^{-\frac{\langle x \rangle}{A}}\zeta(\frac{\langle x \rangle}{B})$ where A, B are large constants to be chosen, $\zeta \in C^\infty(\mathbf{R})$ $\zeta(t) = 1$ when $|t| \leq \frac{1}{2}$, $\operatorname{supp}\zeta \subset \{t \in \mathbf{R} : |t| \leq 1\}, 0 \leq \zeta \leq 1$ and $\theta = \alpha_1(x)\Phi$ where $\alpha_1 \in C_0^\infty(\mathbf{R}^d)$ is equal to one on the support of α. Then $\theta \in H^1(\Omega)$ and $\gamma_0\theta = 0$. We can therefore use Lemma 4.4 and we obtain

(4.7)
$$\begin{aligned} I := & \iint_\Omega \alpha(x)|\nabla_{x,y}\Phi(x,y)|^2\,dx\,dy \\ = & -\frac{1}{A}\iint_\Omega (\nabla_x\langle x \rangle)\alpha(x)\Phi(x,y) \cdot \nabla_x\Phi(x,y)\,dx\,dy \\ & + \frac{1}{B}\iint_\Omega e^{-\frac{\langle x \rangle}{A}}\zeta'(\frac{\langle x \rangle}{B})\Phi(x,y)(\nabla_x\langle x \rangle) \cdot \nabla_x\Phi(x,y)\,dx\,dy \\ = & (1) + (2). \end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$|(1)| \leq \frac{C_1}{A}\Big(\iint_\Omega \alpha(x)|\nabla_x\Phi(x,y)|^2\,dx\,dy\Big)^{\frac{1}{2}}\Big(\iint_\Omega \alpha(x)|\Phi(x,y)|^2\,dx\,dy\Big)^{\frac{1}{2}}.$$

Using Lemma 4.1 we deduce that

$$|(1)| \leq \frac{C_2}{A}\Big(\iint_\Omega \alpha(x)|\nabla_x\Phi(x,y)|^2\,dx\,dy + \iint_\Omega \alpha(x)|\partial_y\Phi(x,y)|^2\,dx\,dy\Big).$$

Taking A large enough we see that the term (1) can be absorbed by the left hand side of (4.7). We then fix A. It follows that

$$I \leq \frac{C_3}{B}\sum_{q\in\mathbf{Z}^d}\sum_{k\in\mathbf{Z}^d}\iint_\Omega e^{-\frac{\langle x \rangle}{A}}|\zeta'(\frac{\langle x \rangle}{B})||\chi_q(x)\Phi(x,y)|\big|\nabla_x\big(\chi_k(x)\Phi(x,y)\big)\big|\,dx\,dy.$$

If $|k - q| \geq 2$ we have $\operatorname{supp}\chi_q \cap \operatorname{supp}\chi_k = \emptyset$. Therefore we have $|k - q| \leq 1$ (essentially $k = q$.) Moreover in the integral in the right hand side we have $|x - q| \leq 1$ and $1 \leq \frac{\langle x \rangle}{B} \leq 2$. If B is large enough we have therefore $\frac{B}{3} \leq |q| \leq 3B$ and $|x| \geq \frac{1}{2}|q|$. It follows that

$$I \leq \frac{C_4}{B}\sum_{\frac{B}{3}\leq|q|\leq 3B} I_q$$

where

$$I_q = e^{-\frac{\langle q \rangle}{C_5 A}}\Big(\iint_\Omega |\chi_q(x)\Phi(x,y)|^2\,dx\,dy\Big)^{\frac{1}{2}}\Big(\iint_\Omega \big|\nabla_x\big(\chi_q(x)\Phi(x,y)\big)\big|^2\,dx\,dy\Big)^{\frac{1}{2}}$$

so using again the Poincaré inequality we obtain

$$\begin{aligned} I &\leq \frac{C_6}{B}\sum_{\frac{B}{3}\leq|q|\leq 3B} e^{-\frac{\langle q \rangle}{C_5 A}}\iint_\Omega \big|\nabla_{x,y}\big(\chi_q(x)\Phi(x,y)\big)\big|^2\,dx\,dy \\ &\leq \frac{C_7}{B}\Big(\sum_{\frac{B}{3}\leq|q|\leq 3B} e^{-\frac{\langle q \rangle}{C_5 A}}\Big)\|\Phi\|^2_{H^1_{ul}(\Omega)}. \end{aligned}$$

Since the cardinal of the set $\{q \in \mathbf{Z}^d : \frac{B}{3} \le |q| \le 3B\}$ is bounded by CB^d we obtain eventually

$$\iint_\Omega e^{-\frac{\langle x \rangle}{A}} \theta(\frac{\langle x \rangle}{B}) |\nabla_{x,y} \Phi(x,y)|^2 \, dx \, dy \le C_8 B^{d-1} e^{-c\frac{\langle B \rangle}{A}} \|\Phi\|^2_{H^1_{ul}(\Omega)}.$$

Letting B go to $+\infty$ and applying Fatou's Lemma we obtain

$$\iint_\Omega e^{-\frac{\langle x \rangle}{A}} |\nabla_{x,y} \Phi(x,y)|^2 \, dx \, dy = 0,$$

which implies that $\nabla_{x,y} \Phi(x,y) = 0$ in Ω thus $\Phi = 0$ since $\Phi|_\Sigma = 0$.

Part 2: Existence. We first recall the situation when $\psi \in H^{\frac{1}{2}}(\mathbf{R}^d)$. In the following lemma, whose proof is given below, in section 4.2, we construct a suitable extension of ψ to Ω.

Lemma 4.5. *Let* $\psi \in H^{\frac{1}{2}}(\mathbf{R}^d)$. *One can find* $\underline{\psi}$ *such that*

(1) $\underline{\psi} \in H^1(\Omega)$, $\quad \operatorname{supp} \underline{\psi} \subset \{(x,y) : \eta(x) - h \le y \le \eta(x)\}$,
(2) $\underline{\psi}_{|y=\eta(x)} = \psi(x)$,
(3) $\|\underline{\psi}\|_{H^1(\Omega)} \le \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$.

Then (see [3] for more details) the problem

$$\Delta_{x,y} u = -\Delta_{x,y} \underline{\psi}$$

has a unique solution $u \in H^{1,0}(\Omega)$. This solution, which is the variationnal one, is characterized by

(4.8) $$\iint_\Omega \nabla_{x,y} u(x,y) \cdot \nabla_{x,y} \theta(x,y) dx dy = -\iint_\Omega \nabla_{x,y} \underline{\psi}(x,y) \cdot \nabla_{x,y} \theta(x,y) dx dy$$

for every $\theta \in H^{1,0}(\Omega)$. It satisfies

$$\|\nabla_{x,y} u\|_{L^2(\Omega)} \le C \|\psi\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

Then $\Phi = u + \underline{\psi}$ solves the problem (4.4).

Let us consider now the case where $\psi \in H^{\frac{1}{2}}_{ul}(\mathbf{R}^d)$. If $q \in \mathbf{Z}^d$ and χ_q is defined in (2.3) we set

$$\psi_q = \chi_q \psi \in H^{\frac{1}{2}}(\mathbf{R}^d).$$

By Lemma 4.5 one can find $\underline{\psi}_q \in H^1(\Omega)$ such that $\underline{\psi}_q|_{y=\eta(x)} = \psi_q(x)$ and

(i) $\quad \operatorname{supp} \underline{\psi}_q \subset \{(x,y) : |x - q| \le 2, \eta(x) - h \le y \le \eta(x)\}$

(ii) $\quad \|\underline{\psi}_q\|_{H^1(\Omega)} \le \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$.

To achieve (i) we multiply the function constructed in the lemma by $\widetilde{\chi}_q(x)$, where $\operatorname{supp} \widetilde{\chi}$ is contained in $\{x : |x| \le 2\}$ and $\widetilde{\chi} = 1$ on the support of χ.

Let u_q be the variational solution, described above, of the equation $\Delta_{x,y} u_q = -\Delta_{x,y} \underline{\psi}_q$. Our aim is to prove that the series $\sum_{q \in \mathbf{Z}^d} u_q$ is convergent in the space $H^{1,0}_{ul}(\Omega)$. This will be a consequence of the following lemma.

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Lemma 4.6. *There exists $\delta > 0$ and $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for all $q \in \mathbf{Z}^d$ we have*

$$(4.9) \qquad \|e^{\delta\langle x-q\rangle}\nabla_{x,y}u_q\|_{L^2(\Omega)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)})\|\psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

Assuming that this lemma has been proved, one can write

$$\|\chi_k\nabla_{x,y}u_q\|_{L^2(\Omega)} = \|\chi_k e^{-\delta\langle x-q\rangle}e^{\delta\langle x-q\rangle}\nabla_{x,y}u_q\|_{L^2(\Omega)}$$

$$(4.10) \qquad \leq Ce^{-\delta\langle k-q\rangle}\mathcal{F}(\|\eta\|_{W^{1,\infty}})\|\psi_q\|_{H^{\frac{1}{2}}}$$

$$\leq C'e^{-\delta\langle k-q\rangle}\mathcal{F}(\|\eta\|_{W^{1,\infty}})\|\psi\|_{H^{\frac{1}{2}}_{ul}}.$$

Let us set $S^Q = \sum_{|q|\leq Q} u_q$. First of all (S^Q) converges to $u = \sum_{q\in\mathbf{Z}^d} u_q$ in $\mathcal{D}'(\Omega)$. Indeed if $\varphi \in C_0^\infty(\Omega)$ there exists a finite set $A \subset \mathbf{Z}^d$ such that $\varphi = \sum_{k\in A}\chi_k\varphi$. Then using Lemma 4.1 and (4.10) we can write

$$|\langle u_q, \varphi\rangle| \leq \sum_{k\in A}|\langle\chi_k u_q, \varphi\rangle| \leq C\sum_{k\in A}\|\chi_k\partial_y u_q\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}$$

$$\leq Ce^{-\delta\langle q\rangle}\mathcal{F}(\|\eta\|_{W^{1,\infty}})\|\psi\|_{H^{\frac{1}{2}}_{ul}}\|\varphi\|_{L^2(\Omega)}$$

for large $|q|$.

On the other hand (4.10) shows that for fixed k the series $\sum_{q\in\mathbf{Z}^d}\chi_k u_q$ is absolutely convergent in $H^{1,0}_{ul}(\Omega)$. Therefore $(\chi_k S^Q)$ converges to $(\chi_k u)$ in $H^{1,0}_{ul}(\Omega)$ and we can write using (4.10),

$$\|\chi_k\nabla_{x,y}u\|_{L^2(\Omega)} = \lim_{Q\to+\infty}\|\chi_k\nabla S^Q\|_{L^2(\Omega)}$$

$$\leq \sum_{q\in\mathbf{Z}^d}e^{-\delta\langle k-q\rangle}\mathcal{F}(\|\eta\|_{W^{1,\infty}})\|\psi\|_{H^{\frac{1}{2}}_{ul}}.$$

Therefore $u \in H^{1,0}_{ul}(\Omega)$ and

$$(4.11) \qquad \|\nabla_{x,y}u\|_{L^2_{ul}(\Omega)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}})\|\psi\|_{H^{\frac{1}{2}}_{ul}}.$$

Finally $\Phi = u + \underline{\psi}$ solves the problem (4.4) and we have

$$\|\Phi\|_{H^1_{ul}(\Omega)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}})\|\psi\|_{H^{\frac{1}{2}}_{ul}}$$

which completes the proof of Proposition 4.3 assuming Lemma 4.6. $\qquad\square$

PROOF OF LEMMA 4.6. We set

$$w_\varepsilon(x) = \frac{\langle x-q\rangle}{1 + \varepsilon\langle x-q\rangle}.$$

Let u_q be the variational solution in $H^{1,0}(\Omega)$ of $\Delta u_q = -\Delta\underline{\psi}_q$. According to the variational formulation (4.8), with $\theta = e^{\delta w_\varepsilon(x)}u_q$, we have

$$\iint_\Omega \nabla_{x,y}u_q \cdot \nabla_{x,y}(e^{2\delta w_\varepsilon(x)}u_q)\,dx\,dy = -\iint_\Omega \nabla_{x,y}\underline{\psi}_q \cdot \nabla_{x,y}(e^{2\delta w_\varepsilon(x)}u_q)\,dx\,dy.$$

Therefore

$$
(4.12) \quad
\begin{aligned}
&\iint_\Omega e^{2\delta w_\varepsilon(x)} \nabla_{x,y} u_q \cdot \nabla_{x,y} u_q \, dx \, dy = -\iint_\Omega e^{2\delta w_\varepsilon(x)} \nabla_{x,y} \underline{\psi}_q \cdot \nabla_{x,y} u_q \, dx \, dy \\
&\quad - 2\delta \iint_\Omega e^{2\delta w_\varepsilon(x)} u_q \nabla_{x,y} \underline{\psi}_q \cdot \nabla_x w_\varepsilon \, dx \, dy - 2\delta \iint_\Omega e^{2\delta w_\varepsilon(x)} u_q \nabla_{x,y} u_q \cdot \nabla_x w_\varepsilon \, dx \, dy.
\end{aligned}
$$

Now $\nabla_x w_\varepsilon$ is uniformly bounded in L^∞ with respect to ε and x and, on the support of $\underline{\psi}_q$, we have $e^{\delta w_\varepsilon(x)} \leq e^{C\delta}$. Consequently, using the Cauchy-Schwarz inequality, the inequality (4.3) with $\alpha = e^{2\delta w_\varepsilon(x)}$ and taking δ small enough we obtain

$$
(4.13) \quad \iint_\Omega e^{2\delta w_\varepsilon(x)} |\nabla_{x,y} u_q|^2 \, dx \, dy \leq C \|\underline{\psi}_q\|^2_{H^1(\Omega)}.
$$

We deduce when ε goes to 0, using the Fatou Lemma, that

$$
\|e^{\delta\langle x-q\rangle} \nabla_{x,y} u_q\|_{L^2(\Omega)} \leq C \|\underline{\psi}_q\|_{H^1(\Omega)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi_q\|_{H^{\frac12}(\mathbf{R}^d)}.
$$

This completes the proof. $\qquad\square$

4.2. Straightening the boundary. Before studying more precisely the properties of the Dirichlet-Neumann operator, we first straighten the boundaries of

$$
(4.14) \quad \Omega_h = \{(x,y) \in \mathbf{R}^{d+1} : \eta(x) - h < y < \eta(x)\}.
$$

Lemma 4.7. *There is an absolute constant $C > 0$ such that if we take $\delta > 0$ so small that*

$$
\delta \|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \leq \frac{h}{2C}
$$

then the map $(x,z) \mapsto (x, \rho(x,z))$ where

$$
(4.15) \quad \rho(x,z) = (1+z) e^{\delta z \langle D_x\rangle} \eta(x) - z\{e^{-\delta(1+z)\langle D_x\rangle} \eta(x) - h\}
$$

is a diffeomorphism from $\widetilde{\Omega} = \{(x,z) : x \in \mathbf{R}^d, -1 < z < 0\}$ to Ω_h.

PROOF. First of all we have $\rho(x,0) = \eta(x), \rho(x,-1) = \eta(x) - h$. Moreover we have $\partial_z \rho \geq \frac{h}{2}$. Indeed we have

$$
\begin{aligned}
\partial_z \rho = {}& \eta + (e^{\delta z \langle D_x\rangle} \eta - \eta) + (1+z)\delta e^{\delta z \langle D_x\rangle} \langle D_x\rangle \eta \\
&- \{\eta - h + (e^{-\delta(1+z)\langle D_x\rangle} \eta - \eta)\} + z\delta e^{-\delta(1+z)\langle D_x\rangle} \langle D_x\rangle \eta.
\end{aligned}
$$

Now for any $\lambda < 0$ the symbol $a(\xi) = e^{\lambda\langle\xi\rangle}$ satisfies the estimate $|\partial_\xi^\alpha a(\xi)| \leq C_\alpha \langle\xi\rangle^{-|\alpha|}$ where C_α is independent on λ. Therefore its Fourier transform in an $L^1(\mathbf{R}^d)$ function whose norm is uniformly bounded. This implies that $\|a(D)f\|_{L^\infty(\mathbf{R}^d)} \leq K \|f\|_{L^\infty(\mathbf{R}^d)}$ with K independent of λ. Since $e^{\delta\lambda\langle D_x\rangle} \eta - \eta = \delta\lambda \int_0^1 e^{\delta t \lambda\langle D_x\rangle} \langle D_x\rangle \eta \, dt$ we can write

$$
\begin{aligned}
&\|e^{\delta z\langle D_x\rangle} \eta - \eta\|_{L^\infty(\mathbf{R}^d)} + \delta \|e^{\delta z\langle D_x\rangle} \langle D_x\rangle \eta\|_{L^\infty(\mathbf{R}^d)} + \|e^{-\delta(1+z)\langle D_x\rangle} \eta - \eta\|_{L^\infty(\mathbf{R}^d)} \\
&\qquad + \delta \|e^{-\delta(1+z)\langle D_x\rangle} \langle D_x\rangle \eta\|_{L^\infty(\mathbf{R}^d)} \leq C\delta \|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \leq \frac{h}{2}.
\end{aligned}
$$

This completes the proof. $\qquad\square$

From the above computation we deduce that

$$(4.16) \qquad \partial_z \rho(x,z) \geq \frac{h}{2} \quad \text{and} \quad \|\nabla_{x,z}\rho\|_{L^\infty(\mathbf{R}^d)} \leq C\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}.$$

We shall denote by κ the inverse of ρ,

$$\rho(x,z) = y \iff z = \kappa(x,y).$$

If we set

$$\widetilde{f}(x,z) = f(x, \rho(x,z))$$

we have

$$(4.17) \qquad \begin{cases} \dfrac{\partial f}{\partial y}(x, \rho(x,z)) = \dfrac{1}{\partial_z \rho}\partial_z \widetilde{f}(x,z) := \Lambda_1 \widetilde{f}(x,z) \\[2mm] \nabla_x f(x, \rho(x,z)) = \big(\nabla_x \widetilde{f} - \dfrac{\nabla_x \rho}{\partial_z \rho}\partial_z \widetilde{f}\big)(x,z) := \Lambda_2 \widetilde{f}(x,z). \end{cases}$$

We introduce the space

$$\mathcal{H}^1_{ul}(\widetilde{\Omega}) = \big\{\widetilde{u} \in L^2_{ul}(\widetilde{\Omega}) : \Lambda_j \widetilde{u} \in L^2_{ul}(\widetilde{\Omega}),\ j = 1,2\big\},$$

endowed with the norm

$$\|\widetilde{u}\|_{\mathcal{H}^1_{ul}(\widetilde{\Omega})} = \sup_{q \in \mathbf{Z}^d} \|\chi_q \widetilde{u}\|_{L^2(\widetilde{\Omega})} + \sum_{j=1}^2 \sup_{q \in \mathbf{Z}^d} \|\chi_q \Lambda_j \widetilde{u}\|_{L^2(\widetilde{\Omega})}.$$

Then according to Lemma 3.12 we see that the elements of $\mathcal{H}^1_{ul}(\widetilde{\Omega})$ have a trace on $z = 0$ belonging to the space $\mathcal{H}^{\frac{1}{2}}_{ul}(\mathbf{R}^d)$. Then we introduce the subspace $\mathcal{H}^{1,0}_{ul}(\widetilde{\Omega}) \subset \mathcal{H}^1_{ul}(\widetilde{\Omega})$ defined as follows

$$\mathcal{H}^{1,0}_{ul}(\widetilde{\Omega}) = \{\widetilde{u} \in \mathcal{H}^1_{ul}(\widetilde{\Omega}) : \widetilde{u}|_{z=0} = 0\}.$$

It follows that we have

$$u \in H^1_{ul}(\Omega) \Leftrightarrow \widetilde{u} \in \mathcal{H}^1_{ul}(\widetilde{\Omega}), \quad u \in H^{1,0}_{ul}(\Omega) \Leftrightarrow \widetilde{u} \in \mathcal{H}^{1,0}_{ul}(\widetilde{\Omega}).$$

4.3. Definition of the Dirichlet-Neumann operator. We can now define the Dirichlet-Neumann operator. Formally we set for $\psi \in H^{\frac{1}{2}}_{ul}(\mathbf{R}^d)$

$$(4.18) \qquad \begin{aligned} G(\eta)\psi(x) &= (1 + |\nabla_x \eta|^2)^{\frac{1}{2}}\frac{\partial \Phi}{\partial n}\big|_\Sigma = \big(\frac{\partial \Phi}{\partial y} - \nabla_x \eta \cdot \nabla_x \Phi\big)\big|_\Sigma \\ &= \big(\Lambda_1 \widetilde{\Phi} - \nabla_x \eta \cdot \Lambda_2 \widetilde{\Phi}\big)\big|_{z=0} = \big(\Lambda_1 \widetilde{\Phi} - \nabla_x \rho \cdot \Lambda_2 \widetilde{\Phi}\big)\big|_{z=0} \end{aligned}$$

where Φ is the solution described in Lemma 4.3 and Λ_j is as defined by (4.17). Our aim is to prove the following theorem.

Theorem 4.8. *Let $d \geq 1$ and $\eta \in W^{1,\infty}(\mathbf{R}^d)$. Then the Dirichlet-Neumann operator is well defined on $H^{\frac{1}{2}}_{ul}(\mathbf{R}^d)$ by (4.18). Moreover there exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that for all $\eta \in W^{1,\infty}(\mathbf{R}^d)$*

$$\|G(\eta)\psi\|_{H^{-\frac{1}{2}}_{ul}(\mathbf{R}^d)} \leq \mathcal{F}\big(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}\big)\|\psi\|_{H^{\frac{1}{2}}_{ul}(\mathbf{R}^d)}.$$

PROOF. Set $U = \Lambda_1\widetilde{\Phi} - \nabla_x\rho \cdot \Lambda_2\widetilde{\Phi}$ and $J = (-1,0)$. We shall prove that for all $q \in \mathbf{Z}^d$,

$$\|\chi_q U\|_{L^2(J,L^2)} \leq \mathcal{F}\big(\|\eta\|_{W^{1,\infty}}\big)\|\psi\|_{H^{\frac{1}{2}}_{ul}}, \tag{4.19}$$

$$\|\chi_q \partial_z U\|_{L^2(J,H^{-1})} \leq \mathcal{F}\big(\|\eta\|_{W^{1,\infty}}\big)\|\psi\|_{H^{\frac{1}{2}}_{ul}}, \tag{4.20}$$

where $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ is independent of q and η. Then Theorem 4.8 will follow from (4.19), (4.20) and Lemma 3.12. Recall that $\widetilde{\Phi} = \widetilde{u} + \underline{\widetilde{\psi}}$. Now the estimate (4.19) follows from (4.11), (4.17) and Corollary 3.11 with $\sigma = 0$ and $m = 1$. To prove (4.20) we observe that

$$\partial_z U = -\nabla_x\big((\partial_z\rho)\Lambda_2\widetilde{\Phi}\big). \tag{4.21}$$

Indeed we have

$$\begin{aligned}
\partial_z U &= \partial_z\Lambda_1\widetilde{\Phi} - \nabla_x\partial_z\rho \cdot \Lambda_2\widetilde{\Phi} - \nabla_x\rho \cdot \partial_z\Lambda_2\widetilde{\Phi} \\
&= (\partial_z\rho)\Lambda_1^2\widetilde{\Phi} - \nabla_x\partial_z\rho \cdot \Lambda_2\widetilde{\Phi} + (\partial_z\rho)(\Lambda_2 - \nabla_x)\Lambda_2\widetilde{\Phi} \\
&= (\partial_z\rho)(\Lambda_1^2 + \Lambda_2^2)\widetilde{\Phi} - \nabla_x\big((\partial_z\rho)\Lambda_2\widetilde{\Phi}\big)
\end{aligned}$$

so (4.21) follows from the fact that $(\Lambda_1^2 + \Lambda_2^2)\widetilde{\Phi} = 0$. Then (4.20) follows from the estimates used to bound (4.19) and the Poincaré inequality (4.3). The proof of Theorem 4.8 is complete. $\qquad\square$

We state now a consequence of the previous estimates which will be used in the sequel. Notice first that the equation $(\Lambda_1^2 + \Lambda_2^2)\widetilde{\Phi} = 0$ is equivalent to the equation

$$(\partial_z^2 + \alpha\Delta_x + \beta \cdot \nabla_x\partial_z - \gamma\partial_z)\widetilde{\Phi} = 0, \tag{4.22}$$

where

$$\alpha := \frac{(\partial_z\rho)^2}{1 + |\nabla\rho|^2}, \quad \beta := -2\frac{\partial_z\rho\nabla_x\rho}{1 + |\nabla_x\rho|^2}, \quad \gamma := \frac{1}{\partial_z\rho}\big(\partial_z^2\rho + \alpha\Delta_x\rho + \beta \cdot \nabla_x\partial_z\rho\big). \tag{4.23}$$

Corollary 4.9. *Let $s_0 > 1 + \frac{d}{2}$ and $J = (-1,0)$. There exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\|\nabla_{x,z}\widetilde{\Phi}\|_{X^{-\frac{1}{2}}_{ul}(J)} \leq \mathcal{F}\big(\|\eta\|_{H^{s_0+\frac{1}{2}}_{ul}(\mathbf{R}^d)}\big)\|\psi\|_{H^{\frac{1}{2}}_{ul}(\mathbf{R}^d)}. \tag{4.24}$$

PROOF. First of all the estimate

$$\|\nabla_{x,z}\underline{\widetilde{\psi}}\|_{X^{-\frac{1}{2}}_{ul}(J)} \leq C\|\psi\|_{H^{\frac{1}{2}}_{ul}}$$

follows from Corollary 3.11 with $\delta = 1, m = 0, 1$ and $\sigma = 0$.

On the other hand we notice that $\partial_z = (\partial_z\rho)\Lambda_1$ and $\nabla_x = \Lambda_2 + (\nabla_x\rho)\Lambda_1$. Let $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d), \widetilde{\chi} = 1$ on the support of χ. Since $s > 1 + \frac{d}{2}$, using Corollary 3.11 with $\sigma = s$ we can write

$$\begin{aligned}
\|\widetilde{\chi}_k\nabla_{x,z}\rho\|_{L^\infty(J\times\mathbf{R}^d)} &\leq C\|\widetilde{\chi}_k\nabla_{x,z}\rho\|_{L^\infty(J,H^{s-\frac{1}{2}})} \leq C'\|\nabla_{x,z}\rho\|_{L^\infty(J,H^{s-\frac{1}{2}})_{ul}} \\
&\leq C''(1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}}).
\end{aligned}$$

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It follows from (4.11) that

$$(4.25) \qquad \|\nabla_{x,z}\widetilde{u}\|_{L^2(J,L^2)_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}})\|\psi\|_{H_{ul}^{\frac{1}{2}}}.$$

Now using Lemma 3.12 we have

$$\|\chi_k\nabla_x\widetilde{u}\|_{L^\infty(J,H^{-\frac{1}{2}})} \leq C(\|\chi_k\nabla_x\widetilde{u}\|_{L^2(J,L^2)} + \|\chi_k\partial_z\nabla_x\widetilde{u}\|_{L^2(J,H^{-1})}).$$

The first term in the right hand side is estimated using (4.25). For the second term using (4.25) we have

$$(4.26) \qquad \begin{aligned} \|\chi_k\partial_z\nabla_x\widetilde{u}\|_{L^2(J,H^{-1})} &\leq \|(\nabla_x\chi_k)\partial_z\widetilde{u}\|_{L^2(J,H^{-1})} + \|\chi_k\partial_z\widetilde{u}\|_{L^2(J,L^2)} \\ &\leq C\|\partial_z\widetilde{u}\|_{L^2(J,L^2)_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}})\|\psi\|_{H_{ul}^{\frac{1}{2}}}. \end{aligned}$$

Therefore

$$(4.27) \qquad \|\nabla_x\widetilde{u}\|_{L^\infty(J,H^{-\frac{1}{2}})_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}})\|\psi\|_{H_{ul}^{\frac{1}{2}}}.$$

Eventually

$$(4.28) \qquad \|\chi_k\partial_z\widetilde{u}\|_{L^\infty(J,H^{-\frac{1}{2}})} \leq C(\|\chi_k\partial_z\widetilde{u}\|_{L^2(J,L^2)} + \|\chi_k\partial_z^2\widetilde{u}\|_{L^2(J,H^{-1})}).$$

The first term in the right hand side is estimated using (4.25). For the second term using (4.23) we have

$$\begin{cases} \|\chi_k\partial_z^2\widetilde{u}\|_{L^2(J,H^{-1})} \leq A_1 + A_2 + A_3 \\ A_1 = \|\chi_k\alpha\Delta\widetilde{u}\|_{L^2(J,H^{-1})} \\ A_2 = \|\chi_k\beta\partial_z\nabla_x\widetilde{u}\|_{L^2(J,H^{-1})} \\ A_3 = \|\chi_k\gamma\partial_z\widetilde{u}\|_{L^2(J,H^{-1})}. \end{cases}$$

Now using (3.5), (4.25) and (4.26) we obtain

$$A_1 \leq \|\alpha\|_{L^\infty(J,H^{s-\frac{1}{2}})_{ul}}\|\Delta\widetilde{u}\|_{L^2(J,H^{-1})_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}})\|\psi\|_{H_{ul}^{\frac{1}{2}}},$$

$$A_2 \leq \|\beta\|_{L^\infty(J,H^{s-\frac{1}{2}})_{ul}}\|\partial_z\nabla_x\widetilde{u}\|_{L^2(J,H^{-1})_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}})\|\psi\|_{H_{ul}^{\frac{1}{2}}},$$

$$A_3 \leq \|\gamma\|_{L^\infty(J,H^{s-\frac{3}{2}})_{ul}}\|\partial_z\widetilde{u}\|_{L^2(J,L^2)_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}\|\psi\|_{H_{ul}^{\frac{1}{2}}}.$$

Therefore using (4.28) we obtain

$$\|\partial_z\widetilde{u}\|_{L^\infty(J,H^{-\frac{1}{2}})_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}})\|\psi\|_{H_{ul}^{\frac{1}{2}}}$$

which completes the proof of Corollary 4.9. $\qquad\qquad\square$

4.4. Higher estimates for the Dirichlet–Neumann operator. In this section we prove the following results.

Theorem 4.10. *Let $d \geq 1$ and $s_0 > 1 + \frac{d}{2}$.*

Case 1. There exists $\mathcal{F}: \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for $-\frac{1}{2} \leq \sigma \leq s_0 - 1$, every $\eta \in H_{ul}^{s_0+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (4.2) and every $\psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d)$ we have

$$\|G(\eta)\psi\|_{H_{ul}^\sigma} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\|\psi\|_{H_{ul}^{\sigma+1}}.$$

Case 2. For every $s \geq s_0$, there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for every $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (4.2) every $s_0 - 1 \leq \sigma \leq s - \frac{1}{2}$ and every $\psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d)$ we have

$$\|G(\eta)\psi\|_{H_{ul}^{\sigma}} \leq \mathcal{F}\big(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0}}\big)\big\{\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + \|\psi\|_{H_{ul}^{\sigma+1}} + 1\big\}.$$

We set

(4.29)
$$R(\eta)\psi := G(\eta)\psi - T_\lambda \psi$$

where

$$\lambda = \big((1 + |\nabla_x\eta|^2)|\xi|^2 - (\nabla_x\eta \cdot \xi)^2\big)^{\frac{1}{2}}.$$

Theorem 4.11. *Let* $d \geq 1$ *and* $s_0 > 1 + \frac{d}{2}$.

Case 1. There exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for $0 \leq t \leq s_0 - \frac{1}{2}$, $\eta \in H_{ul}^{s_0+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (4.2) we have

$$\|R(\eta)\psi\|_{H_{ul}^t} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\|\psi\|_{H_{ul}^{t+\frac{1}{2}}}$$

for every $\psi \in H_{ul}^{t+\frac{1}{2}}(\mathbf{R}^d)$.

Case 2. For all $s \geq s_0$ there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for every $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (4.2), every $s_0 - \frac{1}{2} \leq t \leq s - \frac{1}{2}$ and every $\psi \in H_{ul}^{t+\frac{1}{2}}(\mathbf{R}^d)$ we have

$$\|R(\eta)\psi\|_{H_{ul}^t} \leq \mathcal{F}\big(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0}}\big)\big\{\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + \|\psi\|_{H_{ul}^{t+\frac{1}{2}}} + 1\big\}.$$

The main step in the proof of the above theorems is the following elliptic regularity result.

Theorem 4.12. *Let* $d \geq 1, J = (-1,0), s_0 > 1 + \frac{d}{2}$. *Let* $\tilde{v}$ *be a solution of the problem*

(4.30)
$$\begin{cases} (\partial_z^2 + \alpha\Delta_x + \beta \cdot \nabla_x\partial_z - \gamma\partial_z)\tilde{v} = F & \text{in } \mathbf{R}^d \times J, \\ \tilde{v}|_{z=0} = \psi. \end{cases}$$

Case 1. For $-\frac{1}{2} \leq \sigma \leq s_0 - 1$ let $\eta \in H_{ul}^{s_0+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (4.2), $\psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d), F \in Y_{ul}^{\sigma}(J)$ and

(4.31)
$$\|\nabla_{x,z}\tilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)} < +\infty.$$

Then for every $z_0 \in \,]-1,0[$ *one has* $\nabla_{x,z}\tilde{v} \in X_{ul}^{\sigma}(z_0,0)$ *and one can find* $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ *non decreasing, depending only on* (s_0, d) *such that*

$$\|\nabla_{x,z}\tilde{v}\|_{X_{ul}^{\sigma}(z_0,0)} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\Big\{\|\psi\|_{H_{ul}^{\sigma+1}} + \|F\|_{Y_{ul}^{\sigma}(J)} + \|\nabla_{x,z}\tilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)}\Big\}.$$

Case 2. For $s \geq s_0$, and $s_0 - 1 \leq \sigma \leq s - \frac{1}{2}$ let $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (4.2), $\psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d), F \in Y_{ul}^{\sigma}(J)$ and

(4.32)
$$\|\nabla_{x,z}\tilde{v}\|_{X_{ul}^{s_0-1}(J)} < +\infty.$$

Then for every $z_0 \in]-1, 0[$ one has $\nabla_{x,z}\widetilde{v} \in X_{ul}^{\sigma}(z_0, 0)$ and one can find $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing, depending only on (s_0, s, d) such that

$$\|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{\sigma}(z_0,0)}$$
$$\leq \mathcal{F}\big(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0}}\big)\Big\{\|\psi\|_{H_{ul}^{\sigma+1}} + \|F\|_{Y_{ul}^{\sigma}(J)} + (\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + 1)\|\nabla_{x,z}\tilde{v}\|_{X_{ul}^{s_0-1}(J)}\Big\}.$$

Corollary 4.13. *Let $s_0 > 1 + \frac{d}{2}$. Let Φ be defined in Proposition 4.3.*

Case 1. For $-\frac{1}{2} \leq \sigma \leq s_0 - 1$ assume that $\eta \in H_{ul}^{s_0+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (4.2) and $\psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d)$. Then there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing depending only on (s_0, d) such that

$$\|\widetilde{\Phi}\|_{X_{ul}^{\sigma+1}(z_0,0)} + \|\nabla_{x,z}\widetilde{\Phi}\|_{X_{ul}^{\sigma}(z_0,0)} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\|\psi\|_{H_{ul}^{\sigma+1}}.$$

Case 2. For $s \geq s_0$, $s_0 - 1 \leq \sigma \leq s - \frac{1}{2}$ assume that $\widetilde{\eta} \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$, $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (4.2) and $\psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d)$. Then there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing depending only on (s_0, s, d) such that

$$\|\widetilde{\Phi}\|_{X_{ul}^{\sigma+1}(z_0,0)} + \|\nabla_{x,z}\widetilde{\Phi}\|_{X_{ul}^{\sigma}(z_0,0)}$$
$$\leq \mathcal{F}\big(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0}}\big)\big\{\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + \|\psi\|_{H_{ul}^{\sigma+1}} + 1\big\}.$$

PROOF. Indeed $\widetilde{\Phi}$ satisfies (4.30) with $F = 0$ and it is proved in Corollary 4.9 that

$$\|\nabla_{x,z}\widetilde{\Phi}\|_{X_{ul}^{-\frac{1}{2}}(z_0,0)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\|\psi\|_{H_{ul}^{\frac{1}{2}}} < +\infty.$$

Moreover the estimate of $\widetilde{\Phi}$ in $X_{ul}^{\sigma}(z_0, 0)$ is obtained by the Poincaré inequality from the estimate of $\partial_z\widetilde{\Phi}$. $\qquad\square$

PROOF OF THEOREM 4.10 GIVEN COROLLARY 4.13. Let us set

$$(4.33) \qquad\qquad H = \Lambda_1\widetilde{\Phi} - \nabla_x\rho \cdot \Lambda_2\widetilde{\Phi}.$$

By (4.18) we have $H|_{z=0} = G(\eta)\psi$ and by (4.21)

$$(4.34) \qquad \partial_z H = -\nabla_x\big((\partial_z\rho)\Lambda_2\widetilde{\Phi}\big) = -\nabla_x\big((\partial_z\rho)\nabla_x - (\nabla_x\rho)\partial_z\big)\widetilde{\Phi}.$$

Using Lemma 3.12 with $f = \chi_q H$, (4.34) we deduce that

$$\|\chi_q G(\eta)\psi\|_{H^{\sigma}} \leq C\Big[\|\chi_q H\|_{L^2(J,H^{\sigma+\frac{1}{2}})} + \|\chi_q\partial_z H\|_{L^2(J,H^{\sigma+\frac{1}{2}})}\Big].$$

Moreover by (4.34) we have

$$\|\chi_q\partial_z H\|_{L^2(J,H^{\sigma+\frac{1}{2}})} \leq C'\Big[\|(\partial_z\rho)\nabla_x\widetilde{\Phi}\|_{L^2(J,H^{\sigma+\frac{1}{2}})_{ul}} + \|(\nabla_x\rho)\partial_z\widetilde{\Phi}\|_{L^2(J,H^{\sigma+\frac{1}{2}})_{ul}}\Big].$$

Case 1. If $-\frac{1}{2} \leq \sigma \leq s_0 - 1$ we use the estimate

$$\|fg\|_{L^2(J,H^{\sigma+\frac{1}{2}}))_{ul}} \leq C\|f\|_{L^{\infty}(J,H^{s_0-\frac{1}{2}})_{ul}}\|g\|_{L^2(J,H^{\sigma+\frac{1}{2}})_{ul}}$$

which follows from Proposition 3.3 with $\sigma_0 = \sigma + \frac{1}{2}, \sigma_1 = s_0 - \frac{1}{2}, \sigma_2 = \sigma + \frac{1}{2}$, the estimates on ρ and Corollary 4.13.

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Case 2. If $s_0 - 1 \leq \sigma \leq s - \frac{1}{2}$ we use the inequality

$$\|fg\|_{L^2(J, H^{\sigma+\frac{1}{2}})_{ul}} \leq \|f\|_{L^\infty(J, H^{s_0-1})_{ul}} \|g\|_{X^\sigma_{ul}(J)} + \|g\|_{L^\infty(J, H^{s_0-1})_{ul}} \|f\|_{X^\sigma_{ul}(J)}$$

the estimates on ρ and again Corollary 4.13 to obtain Theorem 4.10. $\qquad\square$

Theorem 4.12 will be a consequence of the following two results.

Proposition 4.14. *Let $s_0 > 1 + \frac{d}{2}$. There exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for $-1 < z_0 < z_1 < 0$, $-\frac{1}{2} \leq \sigma \leq s_0 - 1$ and $k \in \mathbf{Z}^d$ we have*

$$(\mathcal{H}_\sigma) \qquad \|\nabla_{x,z}\widetilde{v}_k\|_{X^\sigma(z_1,0)} \leq \mathcal{F}\big(\|\eta\|_{H^{s_0+\frac{1}{2}}_{ul}}\big) \Big\{ \|\psi\|_{H^{\sigma+1}_{ul}} + \|F\|_{Y^\sigma_{ul}(J)} + \|\nabla_{x,z}\widetilde{v}_k\|_{X^{-\frac{1}{2}}(z_0,0)} \Big\},$$

where $\widetilde{v}_k = \chi_k \widetilde{v}$.

Proposition 4.15. *Let $s_0 > 1 + \frac{d}{2}$, and $s \geq s_0$. Then there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for $-1 < z_0 < z_1 < 0$, $s_0 - 1 \leq \sigma \leq s - \frac{1}{2}$ and $k \in \mathbf{Z}^d$ we have*

$$(\mathcal{K}_\sigma) \quad \|\nabla_{x,z}\widetilde{v}_k\|_{X^\sigma(z_1,0)} \leq \mathcal{F}\big(\|\eta\|_{H^{s_0+\frac{1}{2}}_{ul}}\big) \Big\{ \|\psi\|_{H^{\sigma+1}_{ul}} + \|F\|_{Y^\sigma_{ul}(J)}$$
$$+ (1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}}) \|\nabla_{x,z}\widetilde{v}_k\|_{X^{s_0-1}_{ul}(z_0,0)} \Big\},$$

where $\widetilde{v}_k = \chi_k \widetilde{v}$.

We shall prove these two results by induction on σ and by the same method. However we have to distinguish them since we want the right hand side of these estimates to be linear with respect to the higher norms of (ψ, η). Since $(\mathcal{H}_{-\frac{1}{2}})$ and $(\mathcal{K}_{s_0-1})$ are trivially satisfied if $\mathcal{F} \geq 1$ these propositions will be a consequence of the following one.

Proposition 4.16. *Case 1.   Let $s_0 > 1 + \frac{d}{2}$. If $(\mathcal{H}_\sigma)$ is satisfied for some $-\frac{1}{2} \leq \sigma \leq s_0 - 1$ then $(\mathcal{H}_{\sigma+\frac{1}{2}})$ is true as long as $\sigma + \frac{1}{2} \leq s_0 - 1$.*

Case 2.   Let $s_0 > 1 + \frac{d}{2}$ and $s \geq s_0$. If $(\mathcal{K}_\sigma)$ is satisfied for some $s_0 - 1 \leq \sigma \leq s - \frac{1}{2}$ then $(\mathcal{K}_{\sigma+\frac{1}{2}})$ is true as long as $\sigma + \frac{1}{2} \leq s - \frac{1}{2}$.

In the sequel, case 1 will refer to Proposition 4.14 and case 2 to Proposition 4.15.

4.5. Non linear estimates. We begin by estimating the coefficients α, β, γ, defined in (4.23). We set $J = (z_0, 0)$.

Lemma 4.17. *Case 1.   Let $s_0 > 1 + \frac{d}{2}$. Then there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that*

$$\|\alpha\|_{X^{s_0-\frac{1}{2}}_{ul}(J)} + \|\beta\|_{X^{s_0-\frac{1}{2}}_{ul}(J)} + \|\gamma\|_{X^{s_0-\frac{3}{2}}_{ul}(J)} \leq \mathcal{F}\big(\|\eta\|_{H^{s_0+\frac{1}{2}}_{ul}}\big).$$

Case 2.   Let $s_0 > 1 + \frac{d}{2}$. Then for $s \geq s_0$ there exists $\mathcal{F}_1 : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that

$$\|\alpha\|_{X^{s-\frac{1}{2}}_{ul}(J)} + \|\beta\|_{X^{s-\frac{1}{2}}_{ul}(J)} + \|\gamma\|_{X^{s-\frac{3}{2}}_{ul}(J)} \leq \mathcal{F}_1\big(\|\eta\|_{H^{s_0+\frac{1}{2}}_{ul}}\big) \big(\|\eta\|_{H^{s+\frac{1}{2}}_{ul}} + 1\big).$$

PROOF. Since
$$\rho = (1+z)e^{\delta z \langle D_x \rangle}\eta - z(e^{-\delta(1+z)\langle D_x \rangle}\eta - h)$$
Lemma 3.11 shows that that for all $t \in \mathbf{R}, k \in \mathbf{N}$ and all $a \in S_{1,0}^m$ we have

(4.35)
$$\|\partial_z^k a(D)\rho\|_{X_{ul}^t(J)} \le C(1 + \|\eta\|_{H_{ul}^{t+m+k} \times H_{ul}^{t+m+k}}).$$

Then according to (4.23) we write
$$\alpha = (\partial_z \rho)^2 - (\partial_z \rho)^2 G_1(\nabla_x \rho), \quad G_1(\xi) = \frac{|\xi|^2}{1 + |\xi|^2}.$$

Case 1: The estimate for α follows from (3.3) and (3.5) with $\mu = s_0 - \frac{1}{2}$. The estimate for β is similar. Now we can write
$$\gamma = \frac{\partial_z^2 \rho}{\partial_z \rho} + \left(\partial_z \rho - (\partial_z \rho)G_1(\nabla_x \rho)\right)\Delta_x \rho + G_2(\nabla_x \rho) \cdot \nabla_x \partial_z \rho, \quad G_2(\xi) = \frac{\xi}{1 + |\xi|^2}.$$

To estimate γ we first use the embedding
$$X_{ul}^{s_0 - \frac{1}{2}} \times X_{ul}^{s_0 - \frac{3}{2}} \subset X_{ul}^{s_0 - \frac{3}{2}}$$

which is a consequence of Lemma 3.5 with $p = +\infty, \sigma_0 = \sigma_1 = s_0 - \frac{3}{2}, \sigma_2 = s_0 - \frac{1}{2}$ and $p = 2, \sigma_0 = \sigma_2 = s_0 - 1, \sigma_1 = s_0 - \frac{1}{2}$. Then we use (3.5) and (4.35). Case 2: The estimates of α and β follow from (3.3) and (3.5) with $\mu = s - \frac{1}{2}$ and from (4.35). The estimate of γ follow from (3.4) with $\mu = s - \frac{1}{2}$ and (4.35) with $t = s_0 - \frac{3}{2}, m + k = 2$. $\qquad\square$

According to (4.30) we have

(4.36)
$$\begin{cases} (\partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z)(\chi_k \widetilde{v}) = \chi_k F + F_0 + F_1 \\ F_0 := \alpha \nabla_x \chi_k \cdot \nabla_x \widetilde{v} + \beta \cdot \nabla_x \chi_k \cdot \partial_z \widetilde{v} \\ F_1 := \gamma \chi_k \partial_z \widetilde{v}. \end{cases}$$

Lemma 4.18. *Case 1. Let $s_0 > 1 + \frac{d}{2}$. There exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for $-\frac{1}{2} \le \sigma \le s_0 - 1$ with $\sigma + \frac{1}{2} \le s_0 - 1$*

$$\sum_{j=0}^1 \|F_j\|_{Y^{\sigma+\frac{1}{2}}(J)} \le \mathcal{F}\left(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\right)\|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^\sigma(J)}.$$

Case 2. Assume $s_0 > 1 + \frac{d}{2}$. Then for all $s \ge s_0$ there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for $s_0 - 1 \le \sigma \le s - \frac{1}{2}$ with $\sigma + \frac{1}{2} \le s - \frac{1}{2}$ we have

$$\sum_{j=0}^1 \|F_j\|_{Y^{\sigma+\frac{1}{2}}(J)}$$
$$\le \mathcal{F}\left(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\right)\left\{\|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^\sigma(J)} + \left(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + 1\right)\|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{s_0-1}(J)}\right\}.$$

PROOF. Case 1: The terms F_0 and F_1 have the same structure but F_1 is worse since, according to Lemma 4.17, γ is bounded in a weaker norm.

Let us look at the term F_1. We can use Proposition 3.5 with $p = 2, \sigma_0 = \sigma, \sigma_1 = s_0 - 1, \sigma_2 = \sigma$. Indeed we have $\sigma_1 + \sigma_2 > 0$ since $s_0 > 1 + \frac{d}{2}, \sigma_0 = \sigma_1, \sigma_0 \leq \sigma_2$ due to the definition of σ, eventually $\sigma_0 < \sigma_1 + \sigma_2 - \frac{d}{2}$ since $s_0 > 1 + \frac{d}{2}$. We obtain

$$\|\gamma \chi_k \partial_z \widetilde{v}\|_{L^2(J,H^\sigma)} \leq \|\widetilde{\chi}_k \gamma\|_{L^2(J,H^{s_0-1})} \|\chi_k \partial_z \widetilde{v}\|_{L^\infty(J,H^\sigma)}$$

and we use Lemma 4.17 to conclude.

Case 2: Using (3.4) with $\mu = \sigma - \frac{1}{2}$ we obtain

$$\|\gamma \chi_k \partial_z \widetilde{v}\|_{L^2(J,H^\sigma)} \leq C\big(\|\gamma\|_{X_{ul}^{\sigma-\frac{1}{2}}} \|\partial_z \widetilde{v}\|_{X_{ul}^{s_0-1}} + \|\gamma\|_{X_{ul}^{s_0-\frac{3}{2}}} \|\partial_z \widetilde{v}\|_{X_{ul}^\sigma}\big).$$

Since $\sigma - \frac{1}{2} \leq s - \frac{3}{2}$ we can use Lemma 4.17 to conclude. $\qquad\square$

Our next step is to replace the multiplication by α (resp. β) by the paramultiplication by T_α (resp. T_β). Recall that we see that the equation (4.36) can be written as

$$(4.37) \qquad (\partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z)\widetilde{v}_k = F + F_0 + F_1.$$

Then we have the following result.

Lemma 4.19. Let $J = (z_0, 0), s_0 > 1 + \frac{d}{2}$ and $s \geq s_0$. There exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that, for all $I \subset J$, $\tilde{v}_k$ satisfies the paradifferential equation

$$(4.38) \qquad (\partial_z^2 + T_\alpha \Delta_x + T_\beta \cdot \nabla_x \partial_z)\tilde{v}_k = F + F_0 + F_1 + F_2$$

for some remainder F_2 satisfying

Case1: if $\quad 0 \leq \sigma \leq s_0 - 1$ with $\sigma + \frac{1}{2} \leq s_0 - 1$

$$(4.39) \qquad \|F_2\|_{Y^{\sigma+\frac{1}{2}}(I)} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big) \|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^\sigma(I)}.$$

Case 2. if $\quad s_0 - 1 \leq \sigma \leq s - \frac{1}{2}$ with $\sigma + \frac{1}{2} \leq s - \frac{1}{2}$

$$(4.40) \qquad \|F_2\|_{Y^{\sigma+\frac{1}{2}}(I)} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\big(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + 1\big)\|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{s_0-1}(J)}.$$

PROOF. Case 1: Using Proposition 3.18 with $\gamma = \sigma, r = s_0 - \frac{1}{2}, \mu = \sigma - \frac{1}{2}$ which satisfy all the conditions we obtain

$$\|(\alpha - T_\alpha)\Delta_x \widetilde{v}_k\|_{L^2(J,H^\sigma)} \leq C\|\alpha\|_{L^\infty(J,H^{s_0-\frac{1}{2}})_{ul}} \|\nabla_{x,z}\widetilde{v}_k\|_{L^2(J,H^{\sigma+\frac{1}{2}})},$$

$$\|(\beta - T_\beta) \cdot \nabla_x \partial_z \widetilde{v}_k\|_{L^2(J,H^\sigma)} \leq C\|\beta\|_{L^\infty(J,H^{s_0-\frac{1}{2}})_{ul}} \|\nabla_{x,z}\widetilde{v}_k\|_{L^2(J,H^{\sigma+\frac{1}{2}})}.$$

The result follows then from Lemma 4.17.

Case 2. By Theorem 2.10 in [3] we have the following estimate for $\sigma > 0$

$$\|(\alpha - T_\alpha)u\|_{H^\sigma} \leq C\|u\|_{C_*^{-\frac{1}{2}}} \|\alpha\|_{H^{\sigma+\frac{1}{2}}}.$$

Let $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ equal to one on the support of χ. We write

$$(\alpha - T_\alpha)\Delta_x \widetilde{v}_k = (\widetilde{\chi}_k \alpha - T_{\widetilde{\chi}_k \alpha})\Delta_x \widetilde{v}_k + T_{(\widetilde{\chi}_k - 1)\alpha}\Delta_x \widetilde{v}_k.$$

It follows from Lemma (3.13) and the above inequality that

$$\|(\alpha - T_\alpha)\Delta_x \widetilde{v}_k\|_{H^\sigma} \leq C\|\Delta_x \widetilde{v}_k\|_{C_*^{-\frac{1}{2}}} \|\widetilde{\chi}_k \alpha\|_{H^{\sigma+\frac{1}{2}}}.$$

Since $H^{s_0-\frac{1}{2}} \subset C_*^{\frac{1}{2}}$ and $\sigma \leq s - 1$ we obtain

$$\|(\alpha - T_\alpha)\Delta_x \widetilde{v}_k\|_{L^2(J, H^\sigma)} \leq C\|\nabla_x \widetilde{v}\|_{X_{ul}^{s_0-1}(J)}\|\alpha\|_{X_{ul}^{s-\frac{1}{2}}(J)}$$

wich in view of Lemma 4.17 Case 2, proves (4.40). $\qquad\square$

Then as in [3] we perform a decoupling in a forward and a backward parabolic evolution equations. Recall that $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$, in particular $\eta \in W^{\frac{3}{2},\infty}(\mathbf{R}^d)$. We can apply Lemma 3.29 in [3] to obtain the following result.

Lemma 4.20. *Let $s_0 > 1 + \frac{d}{2}$. There exist two symbols a, A in $\Gamma_{\frac{1}{2}}^1(\mathbf{R}^d \times J)$, $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing and a remainder term F_3 such that*

$$(4.41) \qquad (\partial_z - T_{a(z)})(\partial_z - T_{A(z)})\widetilde{v}_k = F + F_0 + F_1 + F_2 + F_3$$

with

$$(4.42) \qquad \sup_{z \in (-1,0)} \left(\mathcal{M}_{\frac{1}{2}}^1(a(z)) + \mathcal{M}_{\frac{1}{2}}^1(A(z))\right) \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}(\mathbf{R}^d)})$$

and

$$(4.43) \qquad \|F_3\|_{Y^{\sigma+\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}(\mathbf{R}^d)})\|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^\sigma(J)}$$

for all $\sigma \in \mathbf{R}$.

PROOF. We follow closely the proof of Lemma 3.29 in [3]. We set

$$(4.44) \qquad \begin{aligned} a &= \frac{1}{2}\left(-i\beta \cdot \xi - \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}\right) \\ A &= \frac{1}{2}\left(-i\beta \cdot \xi + \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}\right). \end{aligned}$$

We claim the there exists $c > 0$ depending only on $\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}(\mathbf{R}^d)}$ such that

$$(4.45) \qquad \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2} \geq c|\xi|.$$

Indeed according to (4.23) we see by an elementary computation that

$$4\alpha|\xi|^2 - (\beta \cdot \xi)^2 \geq \frac{(\partial_z \rho)^2}{(1 + |\nabla_x \rho|^2)^2}|\xi|^2.$$

Then our claim follows from (4.16).

Since we have $s_0 > 1 + \frac{d}{2}$ we deduce from the paradifferential symbolic calculus that

$$(\partial_z - T_a)(\partial_z - T_A) = \partial_z^2 + T_\alpha \Delta_x + T_\beta \cdot \nabla_x \partial_z + R_0 + R_1$$

where

$$R_0(z) := T_{a(z)}T_{A(z)} - T_\alpha \Delta_x \quad \text{is of order } \frac{3}{2}$$

$$R_1(z) := -T_{\partial_z A(z)} \quad \text{is of order } \frac{3}{2}$$

together with the estimates

$$\sup_{z \in (-1,0)} \left(\|R_0(z)\|_{H^{t+\frac{3}{2}} \to H^t} + \|R_1(z)\|_{H^{t+\frac{3}{2}} \to H^t}\right) \leq \mathcal{F}(\mathcal{M}_{\frac{1}{2}}^1(A) + \mathcal{M}_{\frac{1}{2}}^1(a)).$$

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Now the seminorms $\mathcal{M}_{\frac{1}{2}}^1(A)$ and $\mathcal{M}_{\frac{1}{2}}^1(a)$ are bounded by the $W^{\frac{1}{2},\infty}(\mathbf{R}^d)$ norms of α and β. Since for $f = \alpha, \beta$ we have

$$\|f(z)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)} \le C\|f(z)\|_{H_{ul}^{s_0-\frac{1}{2}}(\mathbf{R}^d)}$$

we deduce from Lemma 4.17 and the fact that the symbols of R_j vanish near the origin that for $j = 0, 1$

$$\|R_j(z)\widetilde{v}_k\|_{H^\sigma} \le \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\|\nabla_x\widetilde{v}_k\|_{H^{\sigma+\frac{1}{2}}}.$$

The proof is complete. $\qquad\qquad\square$

4.6. Proof of Proposition 4.16. Case 1.   Assume that $(\mathcal{H}_\sigma)$ is satisfied, which means that there exists $I_0 = (z_0, 0)$ such that

$$(4.46) \qquad \|\nabla_{x,z}\widetilde{v}_k\|_{X^\sigma(I_0)} \le \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\left\{\|\psi\|_{H_{ul}^{\sigma+1}} + \|F\|_{Y_{ul}^\sigma(J)} + \|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)}\right\}.$$

From this estimate and the Poincaré inequality we deduce that

$$(4.47) \qquad \|\nabla_{x,z}\widetilde{v}\|_{X^\sigma(I_0)} \le \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\left\{\|\psi\|_{H_{ul}^{\sigma+1}} + \|F\|_{Y_{ul}^\sigma(J)} + \|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)}\right\}.$$

We want to prove that

$$(4.48) \quad \|\nabla_{x,z}\widetilde{v}_k\|_{X^{\sigma+\frac{1}{2}}(I_1)} \le \mathcal{F}_1(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\left\{\|\psi\|_{H_{ul}^{\sigma+\frac{3}{2}}} + \|F\|_{Y_{ul}^{\sigma+\frac{1}{2}}(J)} + \|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)}\right\}.$$

Introduce a cutoff function θ such that $\theta(z_0) = 0$, $\quad\theta(z) = 1$ for $z \ge z_1$. Set

$$(4.49) \qquad\qquad\qquad \widetilde{w}_k(z,\cdot) := \theta(z)(\partial_z - T_A)\widetilde{v}_k(z,\cdot).$$

It follows from Lemma 4.20 for $z \ge z_0$ that

$$(4.50) \qquad\qquad\qquad \partial_z\widetilde{w}_k - T_a\widetilde{w}_k = \theta(z)\left(F + \sum_{j=0}^3 F_j\right) + F_4$$

where

$$F_4 = \theta'(z)(\partial_z - T_A)\widetilde{v}_k.$$

We deduce from Lemma 4.18, Lemma 4.19, Lemma 4.20 and (4.47) that

$$(4.51) \qquad \sum_{j=0}^3 \|\theta F_j\|_{Y^{\sigma+\frac{1}{2}}(I_0)} \le \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\left\{\|F\|_{Y_{ul}^\sigma(J)} + \|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)}\right\}$$

and we see easily using (4.46) that

$$(4.52) \qquad \|F_4\|_{Y^{\sigma+\frac{1}{2}}(I_0)} \le \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\left\{\|\psi\|_{H_{ul}^{\sigma+1}} + \|F\|_{Y_{ul}^\sigma(J)} + \|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)}\right\}.$$

Now using Proposition 2.18 in [3] and (4.50), (4.51) and (4.52) we see, since $\widetilde{w}_k|_{z=z_0} = 0$, that

$$(4.53) \qquad \|\widetilde{w}_k\|_{X^{\sigma+\frac{1}{2}}(I_0)} \le \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\left\{\|\psi\|_{H_{ul}^{\sigma+1}} + \|F\|_{Y_{ul}^\sigma(J)} + \|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)}\right\}.$$

Now notice that on $I_1 := (z_1, 0)$ we have $\theta(z) = 1$ so that

$$(4.54) \qquad\qquad\qquad\qquad (\partial_z - T_A)\widetilde{v}_k = w_k.$$

We may apply again Proposition 2.19 in [**3**] and write

$$\|\nabla_x \widetilde{v}_k\|_{X^{\sigma+\frac{1}{2}}(I_1)} \leq \|\widetilde{v}_k\|_{X^{\sigma+1+\frac{1}{2}}(I_1)}$$

$$\leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\big(\|\widetilde{w}_k\|_{Y^{\sigma+\frac{3}{2}}(I_1)} + \|\psi\|_{H_{ul}^{\sigma+\frac{3}{2}}}\big)$$

$$\leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\big(\|\widetilde{w}_k\|_{X^{\sigma+\frac{1}{2}}(I_1)} + \|\psi\|_{H_{ul}^{\sigma+\frac{3}{2}}}\big)$$

$$\leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\big(\|\psi\|_{H_{ul}^{\sigma+\frac{3}{2}}} + \|F\|_{Y_{ul}^{\sigma+\frac{1}{2}}(J)} + \|\nabla_{x,z}\widetilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)}\big).$$

The same estimate for $\partial_z \widetilde{v}_k$ follows then from (4.54) and (4.53). Thus we have proved (4.48) which completes the induction.

Case 2. Assuming that $(\mathcal{K}_\sigma)$ is true, the exact same method shows that $(\mathcal{K}_{\sigma+\frac{1}{2}})$ holds as long as $\sigma + \frac{1}{2} \leq s - \frac{1}{2}$. Details are left to the reader.

4.7. Proof of Theorem 4.11. Let $0 \leq t \leq s - \frac{1}{2}$. Recall that

$$G(\eta)\psi = g_1 \partial_z \widetilde{\Phi} - g_2 \cdot \nabla_x \widetilde{\Phi}\big|_{z=0}, \quad g_1 = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho}, \quad g_2 = \nabla_x \rho.$$

We shall set

$$g_j|_{z=0} = g_j^0, \quad j = 1, 2, \quad A|_{z=0} = A_0, \quad a|_{z=0} = a_0.$$

We recall that we have set $\chi_k \widetilde{\Phi} = \widetilde{\Phi}_k$ where $\widetilde{\Phi}|_{z=0} = \psi$ and $\widetilde{w}_k = (\partial_z - T_A)\widetilde{\Phi}_k$ (see (4.49)) for $z \in I_1$. It follows that we can write

(4.55)
$$\chi_k G(\eta)\psi = g_1^0 (\partial_z \widetilde{\Phi}_k)|_{z=0} - \chi_k g_2^0 \nabla_x \psi$$
$$= g_1^0 \widetilde{w}_k|_{z=0} + g_1^0 [T_{A_0}, \chi_k]\psi + \chi_k\big(g_1^0 T_{A_0} - g_2^0 \cdot \nabla\big)\psi.$$

We shall set

(4.56)
$$R_1 = (\widetilde{\chi}_k g_1^0)[T_{A_0}, \chi_k]\psi, \quad R_2 = (\widetilde{\chi}_k g_1^0)\widetilde{w}_k|_{z=0}$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ is equal to one on the support of χ so

(4.57)
$$\chi_k G(\eta)\psi = \chi_k\big(g_1^0 T_{A_0} - g_2^0 \cdot \nabla\big)\psi + R_1 + R_2.$$

Let us set $U = [T_{A_0}, \chi_k]\psi$. By the symbolic calculus, since $H_{ul}^{s_0+\frac{1}{2}} \subset W^{\frac{1}{2},\infty}$ we have for all $\sigma \in \mathbf{R}$

(4.58)
$$\|U\|_{H^\sigma} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\|\psi\|_{H_{ul}^{\sigma+\frac{1}{2}}}.$$

If $0 \leq t \leq s_0 - \frac{1}{2}$ the product law in Proposition 3.3 gives

$$\|\widetilde{\chi}_k g_1^0 U\|_{H^t} \leq \|\widetilde{\chi}_k g_1^0\|_{H^{s_0-\frac{1}{2}}} \|U\|_{H^t} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\|U\|_{H^t}.$$

If $s_0 - \frac{1}{2} \leq t \leq s - \frac{1}{2}$ we use the estimation

$$\|\widetilde{\chi}_k g_1^0 U\|_{H^t} \leq C\big(\|\widetilde{\chi}_k g_1^0\|_{H^{s_0-1}}\|U\|_{H^t} + \|\widetilde{\chi}_k g_1^0\|_{H^t}\|U\|_{H^{s_0-1}}\big).$$

Therefore using (4.58) and (3.5) we obtain

$$\|\widetilde{\chi}_k g_1^0 U\|_{H^t} \leq \mathcal{F}\big(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}}\big)\big\{\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + \|\psi\|_{H_{ul}^{t+\frac{1}{2}}} + 1\big\}.$$

It follows that we have

(4.59)
$$\|R_1\|_{H^t} \leq \mathcal{F}\big(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}}\big)\big\{\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + \|\psi\|_{H_{ul}^{t+\frac{1}{2}}} + 1\big\}.$$

By the same argument as above we have

(4.60)
$$\|R_2\|_{H^t} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\big\{\|\widetilde{w}_k|_{z=0}\|_{H_{ul}^t} + \|\eta\|_{H_{ul}^{s+\frac{1}{2}}}\|\widetilde{w}_k|_{z=0}\|_{H_{ul}^{s_0-1}}\big\}.$$

Now using (4.53) with $\sigma = t - \frac{1}{2}$ we obtain in particular

(4.61)
$$\|\widetilde{w}_k\|_{L^2(I_1,H^{t+\frac{1}{2}})} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)(\|\psi\|_{H_{ul}^{t+\frac{1}{2}}} + 1).$$

Moreover we deduce from (4.50) that

$$\|\partial_z \widetilde{w}_k\|_{L^2(I_1,H^{t-\frac{1}{2}})} \leq \|T_a \widetilde{w}_k\|_{L^2(I_1,H^{t-\frac{1}{2}})} + \sum_{j=0}^{4}\|F_j\|_{L^2(I_1,H^{t-\frac{1}{2}})}.$$

It follows from (4.61) and the estimates already obtained on the $F_j's$ that

(4.62)
$$\|\partial_z \widetilde{w}_k\|_{L^2(I_1,H^{t-\frac{1}{2}})} \leq \mathcal{F}\big(\|\eta\|_{H^{s_0+\frac{1}{2}}}\big)(\|\psi\|_{H^{t+\frac{1}{2}}} + 1).$$

Applying Lemma 3.12 to $\chi_q \widetilde{w}_k$ we obtain, for $0 \leq t \leq s - \frac{1}{2}$

(4.63)
$$\|\widetilde{w}_k|_{z=0}\|_{H_{ul}^t} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)(\|\psi\|_{H_{ul}^{t+\frac{1}{2}}} + 1).$$

Combining with (4.60) we obtain

(4.64)
$$\|R_2\|_{H^t} \leq \mathcal{F}\big(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}}\big)\big\{\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + \|\psi\|_{H_{ul}^{t+\frac{1}{2}}} + 1\big\}.$$

Now we have

(4.65)
$$\chi_k\big(g_1^0 T_{A_0} - g_2^0 \cdot \nabla\big)\psi = \chi_k T_{g_1^0 A_0 - i\xi \cdot g_2^0}\psi + R_3 + R_4,$$
$$R_3 = \chi_k\big\{(g_1^0 - T_{g_1^0})T_{A_0}\psi - (g_2^0 - T_{g_2^0}) \cdot \nabla\psi\big\}$$
$$R_4 = \chi_k\big\{T_{g_1^0}T_{A_0} - T_{g_1^0 A_0}\big\}.$$

If $0 \leq t \leq s_0 - \frac{1}{2}$ we use Proposition 3.18 with $\gamma = t, r = s_0 - \frac{1}{2}, \mu = t - \frac{1}{2}$ which satisfy the conditions and we obtain

$$\|\chi_k\big\{(g_1^0 - T_{g_1^0})T_{A_0}\psi\|_{H^t} \leq \|g_1^0\|_{H_{ul}^{s_0-\frac{1}{2}}}\|T_{A_0}\psi\|_{H_{ul}^{t-\frac{1}{2}}}$$

and an analogue estimate for the term containing g_2^0, from which we deduce

(4.66)
$$\|R_3\|_{H^t} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\|\psi\|_{H_{ul}^{t+\frac{1}{2}}}.$$

From Theorem 3.16, (ii) with $\rho = \frac{1}{2}$ we have

(4.67)
$$\|R_4\|_{H^t} \leq \mathcal{F}\big(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\big)\|\psi\|_{H_{ul}^{t+\frac{1}{2}}}.$$

Summing up, using (4.57), (4.59), (4.64), (4.66), and (4.67), we obtain

$$\chi_k G(\eta)\psi = \chi_k T_{g_1^0 A_0 - i\xi \cdot g_2^0}\psi + R_5$$

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with
$$\|R_5\|_{H^t} \leq \mathcal{F}\big(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}}\big)\big\{\|\eta\|_{H_{ul}^{s+\frac{1}{2}}} + \|\psi\|_{H_{ul}^{t+\frac{1}{2}}} + 1\big\}.$$
So Theorem 4.11 follows from the fact that
$$g_1^0 A_0 - i\xi \cdot g_2^0 = \sqrt{(1 + |\nabla_x \eta|^2)|\xi|^2 - (\nabla_x \eta \cdot \xi)^2}.$$

5. A priori estimates in the uniformly local Sobolev space

5.1. Reformulation of the equations. We introduce the following unknowns

(5.1) $\qquad \zeta = \nabla_x \eta, \quad B = (\partial_y \Phi)|_{y=\eta}, \quad V = (\nabla_x \Phi)|_{y=\eta}, \quad a = -(\partial_y P)|_{y=\eta}$

where Φ is the velocity potential and the pressure P is given by
$$P = Q - \frac{1}{2}|\nabla_{x,y}\Phi|^2 - gy.$$
where Q is obtained from B, V, η by solving a variational problem (see §7.2 below for details).

We begin by a useful formula.

Lemma 5.1. *Let $I = [0,T]$ and $s_0 > 1 + \frac{d}{2}$. For all $s \geq s_0$ one can find $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that*
$$G(\eta)B = -\operatorname{div} V + \gamma$$
with
$$\|\gamma\|_{L^\infty(I,H^{s-\frac{1}{2}}(\mathbf{R}^d))_{ul}} \leq \mathcal{F}\big(\|\eta\|_{L^\infty(I,H^{s_0+\frac{1}{2}}(\mathbf{R}^d))_{ul}}\big)\Big\{1 + \|\eta\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}^d))_{ul}}\Big\}.$$

PROOF. The estimate of the lemma will be proved first with fixed t which therefore will be skipped. Let θ be the variational solution of the problem
$$\Delta_{x,y}\theta = 0 \quad \text{in } \Omega, \quad \theta|_{y=\eta(x)} = B.$$
Then $G(\eta)B = (\partial_y\theta - \nabla_x\eta \cdot \nabla_x\theta)|_{y=\eta(x)}$. On the other hand since $V_i(x) = \partial_i\Phi(x,\eta(x))$ we have
$$\begin{aligned}
\operatorname{div} V &= (\Delta_x\Phi(x,\eta) + \nabla_x\eta \cdot \nabla_x\partial_y\Phi)(x,\eta(x)) \\
&= (-\partial_y^2\Phi(x,\eta) + \nabla_x\eta \cdot \nabla_x\partial_y\Phi)(x,\eta(x)) \\
&= -(\partial_y - \nabla_x\eta \cdot \nabla_x)\partial_y\Phi(x,\eta(x)).
\end{aligned}$$
It follows that
$$G(\eta)B + \operatorname{div} V = \gamma, \quad \gamma = (\partial_y - \nabla_x\eta \cdot \nabla_x)(\theta - \partial_y\Phi)(x,\eta(x)).$$
Setting $\Theta = \theta - \partial_y\Phi$ we see that $\Delta_{x,y}\Theta = 0$ in Ω, and $\Theta|_{y=\eta} = 0$. Therefore we may apply Theorem 4.12 with $\sigma = s - \frac{1}{2}$ to deduce that
$$\|\nabla_{x,z}\widetilde{\Theta}\|_{X_{ul}^{s-\frac{1}{2}}(z_0,0)} \leq \mathcal{F}(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}})(1 + \|\eta\|_{H_{ul}^{s+\frac{1}{2}}})$$
where $\widetilde{\Theta}(x,z) = \Theta(x,\rho(x,z))$. Using (3.3) with $\mu = s - \frac{1}{2}$ we deduce that
$$\|\nabla_{x,z}\widetilde{\gamma}\|_{X_{ul}^{s-\frac{1}{2}}(z_0,0)} \leq \mathcal{F}(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}})(1 + \|\eta\|_{H_{ul}^{s+\frac{1}{2}}}).$$
Now using the equation satified by Θ, Lemma 4.17 and Lemma 3.12 we obtain the desired conclusion. $\qquad\square$

Proposition 5.2. *Let $s_0 > 1 + \frac{d}{2}$. Then for all $s \geq s_0$ we have*

(5.2)
$$(\partial_t + V \cdot \nabla_x)B = a - g,$$

(5.3)
$$(\partial_t + V \cdot \nabla_x)V + a\zeta = 0,$$

(5.4)
$$(\partial_t + V \cdot \nabla_x)\zeta = G(\eta)V + \zeta G(\eta)B + R,$$

where the remainder term $R = R(\eta, \psi, V, B)$ satisfies the estimate

(5.5)
$$\|R\|_{L^\infty(I, H_{ul}^{s-\frac{1}{2}})} \leq \mathcal{F}(\|(\eta, \psi, V)\|_{L^\infty(I, H^{s_0+\frac{1}{2}} \times H^{s_0+\frac{1}{2}} \times H^{s_0})_{ul}})(1 + \|\eta\|_{L^\infty(I, H^{s+\frac{1}{2}})_{ul}})$$

where $H^\sigma = H^\sigma(\mathbf{R}^d)$.

PROOF. According to Proposition 4.3 in [**3**] the only point to be proved is the estimate (5.5). Let us recall how R is defined. Let θ_i, Φ be the variationnal solutions of the problems

(5.6)
$$\Delta_{x,y}\theta_i = 0 \text{ in } \Omega, \quad \theta_i|_{y=\eta} = V_i, \quad i = 1, \ldots, d$$

(5.7)
$$\Delta_{x,y}\theta_{d+1} = 0 \text{ in } \Omega, \quad \theta_{d+1}|_{y=\eta} = B,$$

(5.8)
$$\Delta_{x,y}\Phi = 0 \text{ in } \Omega, \quad \Phi|_{y=\eta} = \psi.$$

Then, (see [**3**] Proposition 4.3)

(5.9)
$$R_i = (\partial_y - \nabla_x \eta \cdot \nabla_x)U_i|_{y=\eta}, \quad U_i = \partial_i \Phi - \theta_i, \quad i = 1 \ldots, d$$

(5.10)
$$R_{d+1} = (\partial_y - \nabla_x \eta \cdot \nabla_x)U_{d+1}|_{y=\eta}, \quad U_{d+1} = \partial_y \Phi - \theta_{d+1}.$$

First of all for $i = 1, \ldots, d$ we have $\Delta_{x,y}U_i = 0$ in Ω and $U_i|_{y=\eta} = 0$ since $\partial_i \Phi|_{y=\eta} = V_i$. Denoting by $\widetilde{U}_i$ the image of U_i by the diffeomorphism (4.15) we see that $\widetilde{U}_i$ satisfies the equation (4.30) with $F = \psi = 0$. It follows from Theorem 4.12 wit $\sigma = s - \frac{1}{2}$ that

(5.11)
$$\|\nabla_{x,z}\widetilde{U}_i\|_{X_{ul}^{s-\frac{1}{2}}(J)} \leq \mathcal{F}(\|(\eta, \psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}})(1 + \|\eta\|_{H_{ul}^{s+\frac{1}{2}}})\|\nabla_{x,z}\widetilde{U}_i\|_{X_{ul}^{-\frac{1}{2}}(J)}.$$

We are left with the condition (4.31), that is,

$$\|\nabla_{x,z}\widetilde{U}_i\|_{X_{ul}^{-\frac{1}{2}}(J)} < +\infty.$$

Indeed, since θ_i is the variationnal solution of (5.6) Corollary 4.13 shows that

$$\|\nabla_{x,z}\widetilde{\theta}_i\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\|V_i\|_{H_{ul}^{\frac{1}{2}}}.$$

Now $\widetilde{\partial_i \Phi} = (\partial_i - \frac{\partial_i \rho}{\partial_z \rho}\partial_z)\widetilde{\Phi}$. It follows that

$$\|\nabla_x\widetilde{\partial_i \Phi}\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \|\widetilde{\partial_i \Phi}\|_{X_{ul}^{\frac{1}{2}}(J)} \leq \|\partial_i \widetilde{\Phi}\|_{X_{ul}^{\frac{1}{2}}(J)} + \left\|\frac{\partial_i \rho}{\partial_z \rho}\partial_z \widetilde{\Phi}\right\|_{X_{ul}^{\frac{1}{2}}(J)}.$$

Now we use the following facts: since $s_0 > 1 + \frac{d}{2}$ one has $X_{ul}^{s_0-\frac{1}{2}}(J) \subset X_{ul}^{\frac{1}{2}}(J)$; moreover $X_{ul}^{s_0-\frac{1}{2}}(J)$ is an algebra and eventually $\|\frac{\partial_i \rho}{\partial_z \rho}\|_{X_{ul}^{s_0-1}(J)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})$. We deduce using Corollary 4.13 that

$$\|\nabla_x\widetilde{\partial_i \Phi}\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}_1(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}})\|\psi\|_{H_{ul}^{s_0+\frac{1}{2}}}.$$

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To estimate the term $\left\|\partial_z \widetilde{\partial_i \Phi}\right\|_{X_{ul}^{-\frac{1}{2}}(J)}$ we follow the same path using furthermore the equation (4.30) with $F = 0$ satisfied by $\widetilde{\Phi}$. We obtain eventually

$$\left\|\nabla_{x,z}\widetilde{U}_i\right\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}_2\Big(\|(\eta,\psi,V)\|_{H_{ul}^{s_0+\frac{1}{2}}\times H_{ul}^{s_0+\frac{1}{2}}\times H_{ul}^{s_0}}\Big).$$

Using (5.11) we obtain

$$(5.12) \qquad \left\|\nabla_{x,z}\widetilde{U}_i\right\|_{X_{ul}^{s-\frac{1}{2}}(J)} \leq \mathcal{F}\Big(\|(\eta,\psi,V)\|_{H_{ul}^{s_0+\frac{1}{2}}\times H_{ul}^{s_0+\frac{1}{2}}\times H_{ul}^{s_0}}\Big)\Big\{1 + \|\eta\|_{H_{ul}^{s+\frac{1}{2}}}\Big\}.$$

Now from (5.9) we have

$$(5.13) \qquad R_i := \mathcal{R}_i|_{z=0}, \quad \mathcal{R}_i = \Big(\frac{1+|\nabla_x\rho|^2}{\partial_z\rho}\partial_z - \nabla_x\rho\cdot\nabla_x\Big)\widetilde{U}_i := (A\partial_z + B\cdot\nabla_x)\widetilde{U}_i.$$

Using (3.3) with $\mu = s - \frac{1}{2}$ and (5.12) with s and s_0 we obtain

$$(5.14) \qquad \left\|\mathcal{R}_i\right\|_{X_{ul}^{s-\frac{1}{2}}(J)} \leq \mathcal{F}\Big(\|(\eta,\psi,V)\|_{H_{ul}^{s_0+\frac{1}{2}}\times H_{ul}^{s_0+\frac{1}{2}}\times H_{ul}^{s_0}}\Big)\Big\{1 + \|\eta\|_{H_{ul}^{s+\frac{1}{2}}}\Big\}.$$

Now we claim that

$$(5.15) \qquad \left\|\partial_z\mathcal{R}_i\right\|_{X_{ul}^{s-\frac{3}{2}}(J)} \leq \mathcal{F}\Big(\|(\eta,\psi,V)\|_{H_{ul}^{s_0+\frac{1}{2}}\times H_{ul}^{s_0+\frac{1}{2}}\times H_{ul}^{s_0}}\Big)\Big\{1 + \|\eta\|_{H_{ul}^{s+\frac{1}{2}}}\Big\}.$$

Indeed we can write

$$\partial_z\mathcal{R}_i = (\partial_z A)\partial_z\widetilde{U}_i + (\partial_z B)\cdot\nabla_x\widetilde{U}_i - (\text{div}B)\partial_z\widetilde{U}_i + \nabla_x\cdot(B\partial_z\widetilde{U}_i) + A\partial_z^2\widetilde{U}_i.$$

The first three terms are bounded using (3.4) with $\mu = s - \frac{3}{2}$ and (5.12), the fourth is estimated using (3.3) with $\mu = s - \frac{1}{2}$ and (5.12), eventually for the last term we use the fact that $\partial_z^2\widetilde{U}_i = -(\alpha\Delta_x + \beta\cdot\nabla\partial_z - \gamma\partial_z)\widetilde{U}_i$ together with (3.3), (3.4), and (5.12). Finally from (5.14) and (5.15), using Lemma 3.12 we obtain (5.5) for R_i.

We use exactly the same argument to estimate $\|R_{d+1}\|_{H_{ul}^{s-\frac{1}{2}}(\mathbf{R}^d)}$. This completes the proof of Proposition 5.2 $\hfill\square$

5.2. Estimate of the Taylor coefficient.

Proposition 5.3. *Let* $I = [0, T], s_0 > 1 + \frac{d}{2}$. *For all* $s \geq s_0$ *there exists* $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ *non decreasing such that, with* $H^\sigma = H^\sigma(\mathbf{R}^d)$

$$(5.16) \quad \|a - g\|_{L^\infty(I, H^{s-\frac{1}{2}})_{ul}} \leq \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{L^\infty(I, H^{s_0+\frac{1}{2}}\times H^{s_0+\frac{1}{2}}\times H^{s_0}\times H^{s_0})_{ul}}\big)$$

$$\times \Big\{1 + \|\eta\|_{L^\infty(I, H^{s+\frac{1}{2}})_{ul}} + \|B\|_{L^\infty(I, H^s)_{ul}} + \|V\|_{L^\infty(I, H^s)_{ul}}\Big\}.$$

For convenience we shall set in what follows

$$(5.17) \qquad \mathcal{F}_0 =: \mathcal{F}\Big(\|(\eta,\psi,V,B)\|_{L^\infty(I, H^{s_0+\frac{1}{2}}\times H^{s_0+\frac{1}{2}}\times H^{s_0}\times H^{s_0})_{ul}}\Big)$$

where $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ is a non decreasing function which may change from line to line. Before giving the proof of this result let us recall how a is defined. As is [3] the pressure is defined as follows. Le Q be the variationnal solution of the problem

$$(5.18) \qquad \Delta_{x,y}Q = 0 \quad \text{in } \Omega, \quad Q|_{y=\eta} = \frac{1}{2}B^2 + \frac{1}{2}|V|^2 + g\eta.$$

Then

(5.19)
$$P = Q - \frac{1}{2}|\nabla_{x,y}\Phi|^2 - gy.$$

It is shown in [3] that $Q = -\partial_t\Phi$. Then

(5.20)
$$a = -\partial_y P|_{y=\eta}.$$

We deduce from (5.18), (5.19) that P is solution of the problem

(5.21)
$$\Delta_{x,y}P = -|\nabla^2_{x,y}\Phi|^2, \quad P|_{y=\eta} = 0.$$

Denoting, as usual, by $\widetilde{P}, \widetilde{Q}, \widetilde{\Phi}$ the images of P, Q, Φ by the diffeomorphism (4.15) we have, using the notation (4.17),

$$\widetilde{P} = \widetilde{Q} - \frac{1}{2}(\Lambda_1\widetilde{\Phi})^2 - \frac{1}{2}|\Lambda_2\widetilde{\Phi}|^2 - g\rho$$

and we see that $\widetilde{P}$ is a solution of the problem in $\mathbf{R}^d \times J$,

(5.22)
$$\left(\partial^2_z + \alpha\Delta_x + \beta \cdot \nabla_x\partial_z - \gamma\partial_z\right)\widetilde{P} = -\alpha\sum_{i,j=1}^{2}\left|\Lambda_i\Lambda_j\widetilde{\Phi}\right|^2, \quad \widetilde{P}|_{z=0} = 0.$$

Notice that we have

(5.23)
$$\Lambda_1\widetilde{\Phi}|_{z=0} = B, \quad \Lambda_2\widetilde{\Phi}|_{z=0} = V.$$

PROOF OF PROPOSITION 5.3. Below the time is fixed and we will skip it. We want to apply Theorem 4.12 with $\sigma = s - \frac{1}{2}$, so we must estimate the source term and show that the condition (4.31) is satisfied. We claim that (see (5.17))

(5.24)
$$\left\|\nabla_{x,z}\widetilde{P}\right\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}_0$$

First of all since Q is a the variationnal solution of (5.18) we have according to Corollary 4.9

$$\left\|\nabla_{x,z}\widetilde{Q}\right\|_{X_{ul}^{-\frac{1}{2}}} \leq \mathcal{F}\left(\|\eta\|_{H_{ul}^{s_0+\frac{1}{2}}}\right)\left(\left\||B^2|\right\|_{H_{ul}^{\frac{1}{2}}} + \left\||V|^2\right\|_{H_{ul}^{\frac{1}{2}}} + \|\eta\|_{H_{ul}^{\frac{1}{2}}}\right).$$

Using the fact that $H_{ul}^{s_0}(\mathbf{R}^d)$ is an algebra contained in $H_{ul}^{\frac{1}{2}}(\mathbf{R}^d)$ we obtain

(5.25)
$$\left\|\nabla_{x,z}\widetilde{Q}\right\|_{X_{ul}^{-\frac{1}{2}}} \leq \mathcal{F}_0.$$

The estimate of $\|\nabla_{x,z}\rho\|_{X_{ul}^{-\frac{1}{2}}(\mathbf{R}^d)}$ by the right hand side of (5.24) is straightforward. Now, for $j = 1, 2$ since $X_{ul}^{s_0-\frac{1}{2}}(J)$ is an algebra contained in $X_{ul}^{\frac{1}{2}}(J)$ we have

$$\left\|\nabla_x|\Lambda_j\widetilde{\Phi}|^2\right\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq C\left\||\Lambda_j\widetilde{\Phi}|^2\right\|_{X_{ul}^{\frac{1}{2}}(J)} \leq C'\left\|\Lambda_j\widetilde{\Phi}\right\|^2_{X_{ul}^{s_0-\frac{1}{2}}(J)}$$

so using Corollary 4.13 with $\sigma = s_0 - \frac{1}{2}$ and the estimates on ρ we obtain

(5.26)
$$\left\|\nabla_x|\Lambda_j\widetilde{\Phi}|^2\right\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}_3\left(\|(\eta,\psi)\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}}\right).$$

The same kind of arguments show that

(5.27)
$$\left\|\partial_z|\Lambda_j\widetilde{\Phi}|^2\right\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}_3\left(\|\eta,\psi\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0+\frac{1}{2}}}\right).$$

Using (5.25), (5.26), and (5.27) we obtain the claim (5.24). Now we estimate the source term $F = -\alpha \sum_{i,j=1}^{2} \left| \Lambda_i \Lambda_j \widetilde{\Phi} \right|^2$ in equation (5.22). Since $s - \frac{1}{2} > \frac{d}{2}$ we can write

$$
\begin{aligned}
\|F\|_{Y^{s-\frac{1}{2}}_{ul}(J)} &\leq \|F\|_{L^1(J, H^{s-\frac{1}{2}})_{ul}} \\
&\leq C \|\alpha\|_{L^\infty(J, H^{s-\frac{1}{2}})_{ul}} \sum_{i,j=1}^{2} \|\Lambda_i \Lambda_j \widetilde{\Phi}\|^2_{L^2(J, H^{s-\frac{1}{2}})_{ul}}.
\end{aligned}
$$

Since $(\Lambda_1^2 + \Lambda_2^2)\widetilde{\Phi} = 0$ an Λ_1, Λ_2 commute, we have for $j = 1, 2$

$$
(\Lambda_1^2 + \Lambda_2^2)\Lambda_j \widetilde{\Phi} = 0, \quad (\Lambda_1 \widetilde{\Phi}, \Lambda_2 \widetilde{\Phi})|_{z=0} = (B, V) \in H^s_{ul} \times H^s_{ul}.
$$

Since we have (see (5.17))

$$
\|\Lambda_j \widetilde{\Phi}\|_{X^{-\frac{1}{2}}_{ul}(J)} \leq \mathcal{F}_0
$$

we can apply Theorem 4.12 with $\sigma = s - 1$ and conclude that

$$
\|\nabla_{x,z} \Lambda_j \widetilde{\Phi}\|_{L^2(J, H^{s-\frac{1}{2}})_{ul}} \leq \mathcal{F}_0 \cdot \left(1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}} + \|B\|_{H^s_{ul}} + \|V\|_{H^s_{ul}} \right).
$$

Since $\Lambda_1 = \frac{1}{\partial_z \rho} \partial_z, \Lambda_2 = \nabla_x - \frac{\nabla_x \rho}{\partial_z \rho} \partial_z$, using (3.3), the estimates on ρ, the above inequality for $s = s_0$ and for s we obtain

$$
(5.28) \qquad \|\Lambda_i \Lambda_j \widetilde{\Phi}\|_{L^2(J, H^{s-\frac{1}{2}}(\mathbf{R}^d))_{ul}} \leq \mathcal{F}_0 \cdot \left(1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}} + \|B\|_{H^s_{ul}} + \|V\|_{H^s_{ul}} \right).
$$

It follows easily that

$$
(5.29) \qquad \|F\|_{Y^{s-\frac{1}{2}}_{ul}(J)} \leq \mathcal{F}_0 \cdot \left(1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}} + \|B\|_{H^s_{ul}} + \|V\|_{H^s_{ul}} \right).
$$

Using (5.22), (5.24), (5.29), Theorem 4.12 and the fact that $X^{s-\frac{1}{2}}_{ul}(z_0, 0) \subset L^2((z_0, 0), H^s)_{ul}$ we obtain, using (5.17)

$$
(5.30) \qquad \|\nabla_{x,z} \widetilde{P}\|_{L^2((z_0,0), H^s)_{ul}} \leq \mathcal{F}_0 \cdot \left(1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}} + \|B\|_{H^s_{ul}} + \|V\|_{H^s_{ul}} \right).
$$

We claim that

$$
\|\partial_z^2 \widetilde{P}\|_{L^2((z_0,0), H^{s-1})_{ul}} \leq \mathcal{F}_0 \cdot \left(1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}} + \|B\|_{H^s_{ul}} + \|V\|_{H^s_{ul}} \right).
$$

Indeed this follows from (5.22), (5.28), (3.3), (3.4), (5.30). Noticing that $a = \frac{1}{\partial_z \rho} \partial_z \widetilde{P}|_{z=0}$ and applying Lemma 3.12 we obtain the conclusion of Proposition 5.3. $\qquad \square$

5.3. Paralinearization of the system. As in [3] for $s > 1 + \frac{d}{2}$ we set

$$
(5.31) \qquad \begin{cases} U_s = \langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B, \\ \zeta_s = \langle D_x \rangle^s \zeta. \end{cases}
$$

and we recall that we have set (see the statement of Theorem 4.11)

$$
(5.32) \qquad \lambda(t, x, \xi) := \sqrt{(1 + |\nabla_x \eta(t, x)|^2)|\xi|^2 - (\nabla_x \eta(t, x) \cdot \xi)^2}.
$$

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Proposition 5.4. *Let $s_0 > 1 + \frac{d}{2}$. For all $s \geq s_0$ there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that*

$$(5.33) \qquad (\partial_t + T_V \cdot \nabla_x)U_s + T_a\zeta_s = f_1,$$

$$(5.34) \qquad (\partial_t + T_V \cdot \nabla_x))\zeta_s - T_\lambda U_s = f_2,$$

where for each time $t \in [0, T]$

$$
(5.35) \quad
\begin{aligned}
\|(f_1(t), f_2(t))\|_{L^2_{ul} \times H^{-\frac{1}{2}}_{ul}} \leq &\mathcal{F}\Big(\|(\eta(t), \psi(t), V(t), B(t))\|_{H^{s_0+\frac{1}{2}}_{ul} \times H^{s_0+\frac{1}{2}}_{ul} \times H^{s_0}_{ul} \times H^{s_0}_{ul}} \Big) \\
&\times \Big\{ 1 + \|\eta(t)\|_{H^{s+\frac{1}{2}}_{ul}} + \|B(t)\|_{H^s_{ul}} + \|V(t)\|_{H^s_{ul}} \Big\}.
\end{aligned}
$$

PROOF. We follow the proof of Proposition 4.9 in [**3**]. First of all we shall say that a positive quantity $A(t)$ is controlled if it is bounded by the right hand side of (5.35). Here t will be fixed so we will skip it, taking care that the estimates are uniform with respect to $t \in [0, T]$. We also set

$$\mathcal{L}_0 = \partial_t + T_V \cdot \nabla_x.$$

5.3.1. *Paralinearization of the first equation.* We begin by proving that

$$(5.36) \qquad \mathcal{L}_0 V + T_a\zeta + T_\zeta\mathcal{L}_0 B = h_1$$

where $\|h_1\|_{H^s_{ul}}$ is controlled. Indeed using (5.2), (5.3) and the fact that $T_\zeta g = 0$ we see that $h_1 = (T_V - V) \cdot \nabla_x V - R(a, \zeta)$. By Proposition 3.18 with $\gamma = s, r = s, \mu = s_0 - 1$ we see that $\|(T_V - V) \cdot \nabla_x V\|_{H^s_{ul}} \leq C\|V\|_{H^s_{ul}}\|V\|_{H^{s_0}_{ul}}$. On the other hand since $s_0 > 1 + \frac{d}{2}$, Proposition 3.17 with $\alpha = s - \frac{1}{2}, \beta = s_0 - \frac{1}{2}$ shows that

$$\|R(a, \zeta)\|_{H^s_{ul}(\mathbf{R}^d)} \leq C\|a\|_{H^{s-\frac{1}{2}}_{ul}(\mathbf{R}^d)}\|\nabla_x\eta\|_{H^{s_0-\frac{1}{2}}_{ul}(\mathbf{R}^d)}.$$

These estimates together with Proposition 5.3 prove that h_1 is controlled.

5.3.2. *Higher order energy estimates.* Now we apply the operator $\langle D_x \rangle^s = (I - \Delta_x)^{\frac{s}{2}}$ to the equation (5.36) and we commute. We claim that we obtain

$$(5.37) \qquad \mathcal{L}_0\langle D_x \rangle^s V + T_a\langle D_x \rangle^s\zeta + T_\zeta\mathcal{L}_0\langle D_x \rangle^s B = h_2$$

where $\|h_2(t)\|_{L^2_{ul}(\mathbf{R}^d)}$ is controlled. Indeed this is a consequence of the following estimates

$$\|[T_V \cdot \nabla_x, \langle D_x \rangle^s]\|_{H^s_{ul} \to L^2_{ul}} \leq C\|V\|_{W^{1,\infty}} \leq C'\|V\|_{H^{s_0}_{ul}},$$

$$\|[T_a, \langle D_x \rangle^s]\|_{H^{s-\frac{1}{2}}_{ul} \to L^2_{ul}} \leq C\|a\|_{W^{\frac{1}{2},\infty}} \leq C'\|a\|_{H^{s_0-\frac{1}{2}}_{ul}},$$

$$\|[T_\zeta, \langle D_x \rangle^s]\|_{H^{s-\frac{1}{2}}_{ul} \to L^2_{ul}} \leq C\|\zeta\|_{W^{\frac{1}{2},\infty}} \leq C'\|\zeta\|_{H^{s_0-\frac{1}{2}}_{ul}},$$

which follow from Theorem 3.16. Now Lemma 3.20 shows that

$$\|[T_\zeta, \mathcal{L}_0]\langle D_x \rangle^s B\|_{L^2_{ul}} \leq C(\|\zeta\|_{L^\infty}\|V\|_{W^{1+\varepsilon,\infty}} + \|\mathcal{L}_0\zeta\|_{L^\infty})\|B\|_{H^s_{ul}}.$$

Since $s_0 > 1 + \frac{d}{2}$ one can find $\varepsilon > 0$ such that $H^{s_0}_{ul}(\mathbf{R}^d)$ is continuously embedded in $W^{1+\varepsilon,\infty}(\mathbf{R}^d)$. Therefore we obtain

$$\|[T_\zeta, \mathcal{L}_0]\langle D_x \rangle^s B\|_{L^2_{ul}} \leq \mathcal{F}(\|(\eta, B, V)\|_{H^{s_0+\frac{1}{2}}_{ul} \times H^{s_0}_{ul} \times H^{s_0}_{ul}})\|B\|_{H^s_{ul}}$$

which shows that $\|[T_\zeta, \mathcal{L}_0]\langle D_x\rangle^s B\|_{L^2_{ul}}$ is controlled. Using (5.37) and (5.31) we obtain (5.33).

5.3.3. *Paralinearization of the second equation.*

$$(5.38) \qquad (\partial_t + V \cdot \nabla)\zeta = G(\eta)V + \zeta G(\eta)B + R.$$

We first replace V by T_V modulo a controlled term. To do this we use Proposition 3.18 with $\gamma = s - \frac{1}{2}, r = s, \mu = s_0 - \frac{3}{2}$ and we obtain

$$(5.39) \qquad \|(V - T_V)\nabla\zeta\|_{H^{s-\frac{1}{2}}_{ul}} \leq \|V\|_{H^s_{ul}}\|\eta\|_{H^{s_0+\frac{1}{2}}_{ul}}.$$

Next we paralinearize the Dirichlet-Neumann part. To achieve this paralinearization we use the analysis performed in Section 2. Using Theorem 4.11 with $t = s - \frac{1}{2}$ we can write

$$(5.40) \qquad G(\eta)V + \zeta G(\eta)B = T_\lambda U + \mathcal{R}$$

where

$$(5.41) \qquad \begin{aligned} U &= V + T_\zeta B \\ \mathcal{R} &= [T_\zeta, T_\lambda]B + R(\eta)V + \zeta R(\eta)B + (\zeta - T_\zeta)T_\lambda B. \end{aligned}$$

and

$$\|R(\eta)V\|_{H^{s-\frac{1}{2}}_{ul}} + \|\zeta R(\eta)B\|_{H^{s-\frac{1}{2}}_{ul}}$$
$$\leq \mathcal{F}(\|(\eta, B, V)\|_{H^{s_0+\frac{1}{2}}_{ul} \times H^{s_0}_{ul} \times H^{s_0}_{ul}})\big(1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}} + \|B\|_{H^s_{ul}} + \|V\|_{H^s_{ul}}\big).$$

Using again Proposition 3.18 with $\gamma = s - \frac{1}{2}, r = s - \frac{1}{2}, \mu = s_0 - 1$ and Theorem 3.16 (i) we can write

$$\|(\zeta - T_\zeta)T_\lambda B\|_{H^{s-\frac{1}{2}}_{ul}} \leq C\|\eta\|_{H^{s+\frac{1}{2}}_{ul}} M^1_0(\lambda)\|B\|_{H^{s_0}_{ul}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}_{ul}})\|B\|_{H^{s_0}_{ul}},$$

which shows that this term is controlled. Eventually, by Theorem 3.16 (ii), the term $\|[T_\zeta, T_\lambda]B\|_{H^{s-\frac{1}{2}}_{ul}}$ is also controlled. Therefore we have the equality (5.40) with $\|\mathcal{R}\|_{H^{s-\frac{1}{2}}_{ul}}$ controlled. It follows from (5.38), (5.39) and (5.40) that

$$\mathcal{L}_0\zeta = T_\lambda U + \mathcal{R}$$

where $\|\mathcal{R}\|_{H^{s-\frac{1}{2}}_{ul}}$ controlled. As in the second step by commuting the equation (5.39) with $\langle D_x\rangle^s$ we obtain equation (5.34). This completes the proof of Proposition 5.4. $\qquad \square$

5.4. Symmetrization of the equations. As in [3] before proving an L^2 estimate for our system we begin by performing a symmetrization of the non diagonal part. Recall that $\mathcal{L}_0 = \partial_t + T_V \cdot \nabla$.

Proposition 5.5. *Introduce the symbols*

$$\gamma = \sqrt{a\lambda}, \quad q = \sqrt{\frac{a}{\lambda}},$$

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where a is the Taylor coefficient and λ is recalled in (5.32). Set $\theta_s = T_q \zeta_s$ and $\mathcal{L}_0 = \partial_t + T_V \cdot \nabla$. Then

$$(5.42) \qquad\qquad \mathcal{L}_0 U_s + T_\gamma \theta_s = F_1$$

$$(5.43) \qquad\qquad \mathcal{L}_0 \theta_s - T_\gamma U_s = F_2$$

where F_1, F_2 satisfy, with $L_{ul}^2 = L_{ul}^2(\mathbf{R}^d)$, $H_{ul}^\sigma = H_{ul}^\sigma(\mathbf{R}^d)$,

$$\|(F_1(t), F_2(t))\|_{L_{ul}^2 \times L_{ul}^2} \leq \mathcal{F}(\|(\eta(t), B(t), V(t))\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0} \times H_{ul}^{s_0}})$$
$$\cdot \left(1 + \|\eta(t)\|_{H_{ul}^{s+\frac{1}{2}}} + \|B(t)\|_{H_{ul}^s} + \|V(t)\|_{H_{ul}^s}\right),$$

for some non decreasing function $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ and all $t \in [0, T]$.

PROOF. We follow [3]. From (5.33) and (5.34) we have

$$F_1 := f_1 + (T_\gamma T_q - T_a)\zeta_s$$
$$F_2 := T_q f_2 + (T_q T_\lambda - T_\gamma)U_s - [T_q, \mathcal{L}_0]\zeta_s.$$

Then the Proposition follows from Lemma 3.20 and from the symbolic calculus. $\qquad\square$

We can now state our L^2 estimate. Let us set with $H_{ul}^\sigma = H_{ul}^\sigma(\mathbf{R}^d)$

$$\begin{cases} M_s(0) = \|(\eta(0), \psi(0), B(0), V(0))\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}} \times H_{ul}^s \times H_{ul}^s}, \\ M_s(T) = \sup_{t \in [0,T]} \|(\eta(t), \psi(t), B(t), V(t))\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}} \times H_{ul}^s \times H_{ul}^s}. \end{cases}$$

Proposition 5.6. *There exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that*

$(i) \quad \|U_s(t)\|_{L_{ul}^2} + \|\theta_s(t)\|_{L_{ul}^2} \leq \mathcal{F}(M_{s_0}(t))M_s(t), \quad t \in I =: [0, T],$

$(ii) \quad \|U_s\|_{L^\infty(I, L^2)_{ul}} + \|\theta_s\|_{L^\infty(I, L^2)_{ul}} \leq \mathcal{F}(T M_{s_0}(T))\left\{M_s(0) + \sqrt{T} M_s(T)\right\}.$

PROOF. (i) This follows easily from the definition of U_s and θ_s given in (5.31) and in Proposition 5.5.

(ii) Let χ_k be as in (2.3). Then we have

$$(5.44) \qquad \begin{cases} \mathcal{L}_0(\chi_k U_s) + T_\gamma(\chi_k \theta_s) = G_1 \\ \mathcal{L}_0(\chi_k \theta_s) - T_\gamma(\chi_k U_s) = G_2 \end{cases}$$

where G_1, G_2 are given by

$$G_1 = \chi_k F_1 + V \cdot (\nabla \chi_k)U_s + [T_\gamma, \chi_k]\theta_s$$
$$G_2 = \chi_k F_2 + V \cdot (\nabla \chi_k)\theta_s - [T_\gamma, \chi_k]U_s.$$

We claim that for all $t \in [0, T]$ we have

$$(5.45) \quad \|(G_1(t), G_2(t))\|_{L^2 \times L^2} \leq \mathcal{F}\left(\|(\eta(t), B(t), V(t))\|_{H_{ul}^{s_0+\frac{1}{2}} \times H_{ul}^{s_0} \times H_{ul}^{s_0}}\right)$$
$$\cdot \left(1 + \|\eta(t)\|_{H_{ul}^{s+\frac{1}{2}}} + \|B(t)\|_{H_{ul}^s} + \|V(t)\|_{H_{ul}^s}\right).$$

According to Proposition 5.5 this is true for the terms coming from $\chi_k F_j, j = 1, 2$. Now according to (5.31) we have (for fixed t which is skipped)

$$
\begin{aligned}
\|V \cdot (\nabla \chi_k) U_s\|_{L^2} &\leq \|V\|_{L^\infty} \|U_s\|_{L^2_{ul}} \\
&\leq \|V\|_{H^{s_0}_{ul}} (\|V\|_{H^s_{ul}(\mathbf{R}^d)} + \|\nabla \eta\|_{H^{s_0+\frac{1}{2}}_{ul}} \|B\|_{H^s_{ul}}) \\
&\leq \mathcal{F}\big(\|(\eta, V)\|_{H^{s_0+\frac{1}{2}}_{ul} \times H^{s_0}_{ul}}\big)\big(1 + \|B\|_{H^s_{ul}} + \|V\|_{H^s_{ul}}\big).
\end{aligned}
$$

The same estimate holds for $\|V \cdot (\nabla \chi_k) \theta_s\|_{L^2}$. Eventually we have

$$
\|[T_\gamma, \chi_k]\theta_s\|_{L^2} \leq \mathcal{F}\big(\|(\eta, B, V)\|_{H^{s_0+\frac{1}{2}}_{ul} \times H^{s_0}_{ul} \times H^{s_0}_{ul}}\big) \cdot \big(1 + \|\eta\|_{H^{s+\frac{1}{2}}_{ul}} + \|B\|_{H^s_{ul}} + \|V\|_{H^s_{ul}}\big).
$$

This proves our claim.

Now we compute the quantity

$$
(I) = \frac{d}{dt}\Big\{\|\chi_k U_s(t)\|^2_{L^2} + \|\chi_k \theta_s(t)\|^2_{L^2}\Big\}.
$$

Using the equations (5.42), (5.43), the point (i), the fact that

$$
\begin{aligned}
\|(T_{V(t} \cdot \nabla)^* + T_{V(t)} \cdot \nabla\|_{L^2 \to L^2} &\leq C M_{s_0}(t) \\
\|T_{\gamma(t)} - (T_{\gamma(t)})^*\|_{L^2 \to L^2} &\leq C M_{s_0}(t)
\end{aligned}
$$

and (5.45) we obtain easily (ii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.5. Back to the original unknowns. Recall that

$$
\begin{aligned}
U_s &= \langle D_x \rangle^s V + T_{\nabla \eta} \langle D_x \rangle^s B, \\
\theta_s &= T_{\sqrt{\frac{a}{\lambda}}} \langle D_x \rangle^s \nabla \eta.
\end{aligned}
$$

From the estimate in Proposition 5.6 we would like to recover estimates of the original unknowns ψ, η, V, B. We follow closely [3]. The result is as follows.

Proposition 5.7. Let $s_0 > 1 + \frac{d}{2}$. For all $s \geq s_0$ one can find $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that

$$
M_s(T) \leq \mathcal{F}(M_{s_0}(0) + T M_{s_0}(T))\big\{M_s(0) + T M_s(T)\big\}.
$$

The Proposition will be implied by the following Lemmas.

Lemma 5.8. There exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that with $I = [0, T]$ and $H^\sigma_{ul} = H^\sigma_{ul}(\mathbf{R}^d)$ we have

$$
\begin{aligned}
\|\eta\|_{L^\infty(I, H^s)_{ul}} &+ \|(B, V)\|_{L^\infty(I, H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}})_{ul}} \\
&\leq \mathcal{F}(M_{s_0}(0) + \sqrt{T} M_{s_0}(T))\big\{M_s(0) + \sqrt{T} M_s(T)\big\}.
\end{aligned}
$$

PROOF. Set $\mathcal{L}_1 = \partial_t + V \cdot \nabla$. According to Proposition 5.2 we have $\mathcal{L}_1 B = a - g$, $\mathcal{L}_1 V = -a\nabla \eta$ and from the definition of V, B and the equations we have $\mathcal{L}_1 \eta = B$. Then the lemma follows from Lemma 3.19 with $\mu = s, \mu = s - \frac{1}{2}$ and Proposition 5.3. $\quad\square$

Lemma 5.9. *Let $s_0 > 1 + \frac{d}{2}$. For $s \geq s_0$ one can find $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that, with $H_{ul}^\sigma = H_{ul}^\sigma(\mathbf{R}^d)$, we have*

$$(i) \quad \|\eta\|_{L^\infty(I, H^{s+\frac{1}{2}})_{ul}} \leq \mathcal{F}(M_{s_0}(0) + \sqrt{T} M_{s_0}(T))\{M_s(0) + \sqrt{T} M_s(T)\},$$

$$(ii) \quad \|(V, B)\|_{L^\infty(I, H^s)_{ul}} \leq \mathcal{F}(M_{s_0}(0) + \sqrt{T} M_{s_0}(T))\{M_s(0) + \sqrt{T} M_s(T)\},$$

$$(iii) \quad \|\psi\|_{L^\infty(I, H^{s+\frac{1}{2}})_{ul}} \leq \mathcal{F}(M_{s_0}(0) + \sqrt{T} M_{s_0}(T))\{M_s(0) + \sqrt{T} M_s(T)\}.$$

PROOF. (i) By Lemma 5.8 it is sufficient to bound $A = \|\nabla \eta\|_{L^\infty(I, H_{ul}^{s-\frac{1}{2}})}$. Recall that $q = \sqrt{\frac{a}{\lambda}}$, $\theta_s = T_q \zeta_s$, and $\zeta_s = \langle D \rangle^s \nabla \eta$. By Theorem 3.16 (ii) we can write $\zeta_s = T_{\frac{1}{q}} T_q \zeta_s + R \zeta_s$ where $\|R\|_{L^\infty(I, H^\mu)_{ul} \to L^\infty(I, H^{\mu+\frac{1}{2}})_{ul}} \leq C(\|a\|_{L^\infty(I, H^{s_0 - \frac{1}{2}})_{ul}} + \|\eta\|_{L^\infty(I, H^{s_0 + \frac{1}{2}})_{ul}})$. Then we have

$$A = \|\langle D \rangle^{-\frac{1}{2}} \zeta_s\|_{L^\infty(I, L^2)_{ul}} \leq \|\langle D \rangle^{-\frac{1}{2}} T_{\frac{1}{q}} \theta_s\|_{L^\infty(I, L^2)_{ul}} + \|\langle D \rangle^{-\frac{1}{2}} R \zeta_s\|_{L^\infty(I, L^2)_{ul}}.$$

Using Theorem 3.16, (i), the above estimate on the norm of R with $\mu = -1$, Lemma 5.8 and Proposition 5.3 we deduce that

$$A \leq \mathcal{F}(M_{s_0}(0) + \sqrt{T} M_{s_0}(T))\{\|\theta_s\|_{L^\infty(I, L^2)_{ul}} + \|\eta\|_{L^\infty(I, H^s)_{ul}}\}.$$

Then the conclusion follows from Proposition 5.6 and Lemma 5.8.

(ii) Recall that $U = V + T_\zeta B$. The commutator $[\langle D \rangle^s, T_\zeta]$ is of order $s - \frac{1}{2}$ which norm from $L^\infty(I, L^2)_{ul}$ to $L^\infty(I, L^2)_{ul}$ is bounded by $C\|\eta\|_{L^\infty(I, C_*^{\frac{3}{2}})}$ thus by $C'\|\eta\|_{L^\infty(I, H^{s_0 + \frac{1}{2}})_{ul}}$. Therefore we deduce from Proposition 5.6 and Lemma 5.8 that

$$(5.46) \qquad \|U\|_{L^\infty(I, H^s)_{ul}} \leq \mathcal{F}(M_{s_0}(0) + \sqrt{T} M_{s_0}(T))(M_s(0) + \sqrt{T} M_s(T)).$$

Now by Lemma 5.1 we have

$$\text{div } U = \text{div } V + T_{\text{div } \zeta} B + T_\zeta \cdot \nabla B = -G(\eta)B + T_{i\zeta \cdot \xi + \text{div } \zeta} B + \gamma$$

$$= -T_\lambda B + R(\eta)B + T_{i\zeta \cdot \xi + \text{div } \zeta} B + \gamma = T_e B + T_{\text{div } \zeta} B + \gamma$$

where $e = -\lambda + i\zeta \cdot \xi$. Writing $B = T_{\frac{1}{e}} T_e B + (I - T_{\frac{1}{e}} T_e)B$ we obtain

$$B = T_{\frac{1}{e}} \text{div } U - T_{\frac{1}{e}} \gamma + SB.$$

Then using (5.46), Lemma 5.1 we obtain the desired estimate on B and since $V = U - T_\zeta B$ the estimate on V follows as well.

(iii) We have $\nabla \psi = V + B \nabla \eta$. Since the $L^\infty(I, H^{s-\frac{1}{2}})_{ul}$ norm of $(\nabla \eta, V, B)$ has been already estimated, it remains to bound $\|\psi\|_{L^\infty(I, L^2)_{ul}}$. Now from (2.1) and (2.2) by a simple computation we see that

$$(\partial_t + V \cdot \nabla)\psi = -g\eta + \frac{1}{2}|V|^2 + \frac{1}{2}B^2.$$

Then the conclusion follows from Lemma 3.19 with $\mu = s$ and (ii). $\qquad \square$

6. Contraction and well posedness

6.1. Contraction. In this section we shall prove estimates on the difference of two solutions of the system described in (5.2), (5.3), (5.4) which will prove the uniqueness and also enter in the proof by contraction of the existence. Let $(\eta_j, \psi_j, V_j, B_j), j = 1, 2$ be two solutions of the system

$$(6.1) \qquad \begin{cases} (\partial_t + V_j \cdot \nabla_x)B_j = a_j - g, \\ (\partial_t + V_j \cdot \nabla_x)V_j + a_j\zeta_j = 0, \\ (\partial_t + V_j \cdot \nabla_x)\zeta_j = G(\eta_j)V_j + \zeta_j G(\eta_j)B_j + R_j, \end{cases}$$

on $[0, T_0]$ such that with $H^\sigma_{ul} = H^\sigma_{ul}(\mathbf{R}^d)$ we have

$$M_j = \sup_{t \in [0,T_0]} \|(\eta_j(t), \psi_j(t), V_j(t), B_j(t))\|_{H^{s+\frac{1}{2}}_{ul} \times H^{s+\frac{1}{2}}_{ul} \times H^s_{ul} \times H^s_{ul}} < +\infty.$$

We also assume the Taylor sign condition satisfied that is that there exist $c_j > 0$ for $j = 1, 2$ such that $a_j(t, x) \geq c_j$ for all $t \in [0, T_0]$. We set

$$\eta = \eta_1 - \eta_2, \quad \psi = \psi_1 - \psi_2, \quad V = V_1 - V_2, \quad B = B_1 - B_2,$$

$$N(T) = \sup_{t \in [0,T]} \|(\eta(t), \psi(t), V(t), B(t))\|_{H^{s-\frac{1}{2}}_{ul} \times H^{s-\frac{1}{2}}_{ul} \times H^{s-1}_{ul} \times H^{s-1}_{ul}}.$$

Theorem 6.1. *Let (η_j, ψ_j), $j = 1, 2$, be two solutions of (2.1) such that*

$$(\eta_j, \psi_j, V_j, B_j) \in C^0([0, T_0], H^{s+\frac{1}{2}}_{ul} \times H^{s+\frac{1}{2}}_{ul} \times H^s_{ul} \times H^s_{ul}),$$

for some fixed $T_0 > 0$, $d \geq 1$ and $s > 1 + \frac{d}{2}$. We also assume that the condition (4.2) holds for $0 \leq t \leq T_0$ and that there exists a positive constant c such that for all $0 \leq t \leq T_0$ and for all $x \in \mathbf{R}^d$, we have $a_j(t, x) \geq c$ for $j = 1, 2$, $t \in [0, T]$. Set

$$M_j := \sup_{t \in [0,T]} \|(\eta_j, \psi_j, V_j, B_j)(t)\|_{H^{s+\frac{1}{2}}_{ul} \times H^{s+\frac{1}{2}}_{ul} \times H^s_{ul} \times H^s_{ul}},$$

$$\eta := \eta_1 - \eta_2, \quad \psi = \psi_1 - \psi_2, \quad V := V_1 - V_2, \quad B = B_1 - B_2.$$

Then we have

$$(6.2) \quad \|(\eta, \psi, V, B)\|_{L^\infty((0,T), H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s-1} \times H^{s-1})_{ul}}$$
$$\leq \mathcal{K}(M_1, M_2)\|(\eta, \psi, V, B)|_{t=0}\|_{H^{s-\frac{1}{2}}_{ul} \times H^{s-\frac{1}{2}}_{ul} \times H^{s-1}_{ul} \times H^{s-1}_{ul}}.$$

Let us recall that

$$(6.3) \qquad \begin{cases} (\partial_t + V_j \cdot \nabla)B_j = a_j - g, \\ (\partial_t + V_j \cdot \nabla)V_j + a_j\zeta_j = 0, \\ (\partial_t + V_j \cdot \nabla)\zeta_j = G(\eta_j)V_j + \zeta_j G(\eta_j)B_j + \gamma_j, \quad \zeta_j = \nabla\eta_j, \end{cases}$$

where γ_j is the remainder term given by (5.4). Let

$$N(T) := \sup_{t \in [0,T]} \|(\eta, \psi, V, B)(t)\|_{H^{s-\frac{1}{2}}_{ul} \times H^{s-\frac{1}{2}}_{ul} \times H^{s-1}_{ul} \times H^{s-1}_{ul}}.$$

Our goal is to prove an estimate of the form

$$(6.4) \qquad N(T) \leq \mathcal{K}(M_1, M_2)N(0) + T\,\mathcal{K}(M_1, M_2)N(T),$$

for some non-decreasing function $\mathcal{K}$ depending only on s and d. Then, by choosing T small enough, this implies $N(T) \leq 2\mathcal{K}(M_1, M_2)N(0)$ for T_1 smaller than the minimum of T_0 and $1/2\mathcal{K}(M_1, M_2)$, and iterating the estimate between $[T_1, 2T_1], \ldots, [T - T_1, T_1]$ implies Theorem 6.1.

Remark 6.2. Notice that we prove a Lipschitz property in weak norms. This is a general fact related to the fact that the flow map of a quasi-linear equation is not expected to be Lipschitz in the highest norms (this means that one does not expect to control the difference (η, ψ, V, B) in $L^\infty([0, T_0], H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)_{ul})$.

The proof of Theorem 6.1 follows the same lines as the proof of the similar result [**3**, Theorem 5.1]. Il follows 4 steps: first we prove a Lipschitz estimate for the Dirichlet-Neumann operator. Then we paralinearize the system satisfied by (η, ψ, V, B), symmetrize this system, estimate the good unknowns of the symmetrized system and finally estimate (η, ψ, V, B). The Lipschitz estimate of the Dirichlet-Neumann operator is the crucial one and we shall give some details. Having established the paradifferential calculus in uniformly local spaces, the other steps are identical *mutatis mutandi* as in [**3**] and we shall skip the proofs.

6.2. Contraction for the Dirichlet-Neumann operator. In this section the time being fixed we will skip it.

Lemma 6.3. *Assume $s > 1 + \frac{d}{2}$. Then there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for all $\eta_1, \eta_2 \in H_{ul}^{s+\frac{1}{2}}$ and all $f \in H_{ul}^s$ we have*

$$(6.5) \qquad \|(G(\eta_1) - G(\eta_2))f\|_{H_{ul}^{s-\frac{3}{2}}} \leq \mathcal{F}(\|(\eta_1, \eta_2, f)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}} \times H_{ul}^s})\|\eta_1 - \eta_2\|_{H_{ul}^{s-\frac{1}{2}}}$$

where $H_{ul}^\sigma = H_{ul}^\sigma(\mathbf{R}^d)$.

PROOF. We follow closely [**3**]. As in (4.15), (4.23) we introduce $\rho_j, \widetilde{u}_j, \alpha_j, \beta_j, \gamma_j$ for $j = 1, 2$. Then if $\widetilde{u}_j|_{z=0} = f$ we have

$$(6.6) \qquad G(\eta_j)f = \big(\frac{1 + |\nabla_x \rho_j|^2}{\partial_z \rho_j}\partial_z \widetilde{u}_j - \nabla_x \rho_j \cdot \nabla_x \widetilde{u}_j\big)|_{z=0}.$$

We set $\widetilde{u} = \widetilde{u}_1 - \widetilde{u}_2$. Then

$$(\partial_z^2 + \alpha_1 \Delta_x + \beta_1 \cdot \nabla_x \partial_z - \gamma_1 \partial_z)\widetilde{u} := F$$

where

$$F = \big\{(\alpha_2 - \alpha_1)\Delta_x + (\beta_2 - \beta_1) \cdot \nabla_x \partial_z - (\gamma_2 - \gamma_1)\partial_z\big\}\widetilde{u}_2.$$

Since $s > 1 + \frac{d}{2}$, Lemma 3.5 with $s_0 = s - 2, s_1 = s - 2, s_2 = s - 1, p = 2$ gives (with $J = (z_0, 0)$ and $H_{ul}^\mu = H_{ul}^\mu(\mathbf{R}^d)$)

$$\|F\|_{L^2(J,H^{s-2})_{ul}} \leq K\big\{\|\alpha_2 - \alpha_1\|_{L^2(J,H^{s-1})_{ul}}\|\Delta \widetilde{u}_2\|_{L^\infty(J,H^{s-2})_{ul}}$$
$$+ \|\beta_2 - \beta_1\|_{L^2(J,H^{s-1})_{ul}}\|\nabla \partial_z \widetilde{u}_2\|_{L^\infty(J,H^{s-2})_{ul}}$$
$$+ \|\gamma_2 - \gamma_1\|_{L^2(J,H^{s-2})_{ul}}\|\partial_z \widetilde{u}_2\|_{L^\infty(J,H^{s-1})_{ul}}\big\}.$$

Using (4.23) and Lemma 3.5 we can find a non decreasing function $\mathcal{F}: \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\|\alpha_2 - \alpha_1\|_{L^2(J,H^{s-1})_{ul}} + \|\beta_2 - \beta_1\|_{L^2(J,H^{s-1})_{ul}} + \|\gamma_2 - \gamma_1\|_{L^2(J,H^{s-2})_{ul}}$$
$$\leq \mathcal{F}(\|(\eta_1,\eta_2)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}}})\|\eta_1 - \eta_2\|_{H_{ul}^{s-\frac{1}{2}}}.$$

On the other hand by Theorem 4.12 with $\sigma = s - 1$ we have

$$\|\nabla_{x,z}\widetilde{u}_2\|_{L^\infty((z_0,0),H^{s-1})_{ul}} \leq \mathcal{F}(\|\eta_2\|_{H_{ul}^{s+\frac{1}{2}}})\|f\|_{H_{ul}^s}.$$

Combining these estimates we obtain eventually

$$\|F\|_{L^2(J,H^{s-2})_{ul}} \leq \mathcal{F}(\|(\eta_1,\eta_2,f)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}} \times H_{ul}^s}).$$

Since $\tilde{u}$ vanishes at $z = 0$ Theorem 4.12 with $\sigma = s - \frac{3}{2}$ gives

$$\|\nabla_{x,z}\widetilde{u}\|_{C^0((z_0,0),H^{s-\frac{3}{2}})_{ul}} \leq \mathcal{F}(\|(\eta_1,\eta_2,f)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}} \times H_{ul}^s})\|\eta_1 - \eta_2\|_{H_{ul}^{s-\frac{1}{2}}}.$$

Using (6.6) and Proposition 3.3 (i) we obtain (6.5). $\qquad\square$

6.3. Paralinearization of the equations. Notice that it is enough to estimate η, B, V. Indeed, since $V_j = \nabla\psi_j - B_j\nabla\eta_j$, one can estimate the $L^\infty([0,T], H^{s-\frac{3}{2}})_{ul}$-norm of $\nabla\psi$ from the identity

$$\nabla\psi = V + B\nabla\eta_1 + B_2\nabla\eta.$$

Lemma 6.4 ([3], Lemma 5.6). *The differences ζ, B, V satisfy a system of the form*

(6.7)
$$\begin{cases} (\partial_t + V_1 \cdot \nabla)(V + \zeta_1 B) + a_2\zeta = f_1, \\ (\partial_t + V_2 \cdot \nabla)\zeta - G(\eta_1)V - \zeta_1 G(\eta_1)B = f_2, \end{cases}$$

for some remainders such that

$$\|(f_1, f_2)\|_{L^\infty([0,T],H^{s-1} \times H^{s-\frac{3}{2}})_{ul}} \leq \mathcal{K}(M_1, M_2)N(T).$$

6.4. Estimates for the good unknown. In this section we introduce the good-unknown of Alinhac in [5, 1, 6, 7] and symmetrize the system. Let $I = [0, T]$.

Lemma 6.5 ([3], Lemma 5.7). *Set*

$$\ell := \sqrt{\lambda_1 a_2}, \quad \varphi := T_{\sqrt{\lambda_1}}(V + \zeta_1 B), \quad \vartheta := T_{\sqrt{a_2}}\zeta.$$

Then

(6.8)
$$(\partial_t + T_{V_1} \cdot \nabla)\varphi + T_\ell\vartheta = g_1,$$

(6.9)
$$(\partial_t + T_{V_2} \cdot \nabla)\vartheta - T_\ell\varphi = g_2,$$

where

$$\|(g_1, g_2)\|_{L^\infty(I,H^{s-\frac{3}{2}} \times H^{s-\frac{3}{2}})_{ul}} \leq \mathcal{K}(M_1, M_2)N(T).$$

Once this symmetrization has been performed, simple energy estimates allow to prove

Lemma 6.6 ([**3**, Lemma 5.8]). *Let*
$$N'(T) := \sup_{t \in I} \left\{ \|\vartheta(t)\|_{H^{s-\frac{3}{2}}} + \|\varphi(t)\|_{H^{s-\frac{3}{2}}} \right\}.$$

We have

(6.10)
$$N'(T) \leq \mathcal{K}(M_1, M_2)\big(N(0) + T N(T)\big).$$

6.5. Back to the original unknowns. From the estimates in Lemma 6.6, it is fairly easy to recover estimates for η.

Lemma 6.7 ([**3**, Lemma 5.9]).

(6.11)
$$\|\eta\|_{L^\infty(I; H^{s-\frac{1}{2}})_{ul}} \leq \mathcal{K}(M_1, M_2)\{N(0) + T N(T)\}.$$

We now estimate (V, B).

Proposition 6.8 ([**3**, Proposition 5.10]).

(6.12)
$$\|(V, B)\|_{L^\infty(I, H^{s-1} \times H^{s-1})_{ul}} \leq \mathcal{K}(M_1, M_2)\{N(0) + T N(T)\}.$$

The proof will require several preliminary lemmas. We begin by noticing that it is enough to estimate B. Indeed, if
$$\|B\|_{L^\infty(I, H^{s-1})_{ul}} \leq \mathcal{K}(M_1, M_2)\{N(0) + T N(T)\}.$$
then, the estimate of φ in (6.10) above allows to recover an estimate for $V + \zeta_1 B$ (by applying $T_{\sqrt{\lambda_1}^{-1}}$), which in turn implies the estimate for V.

Let $v = \widetilde{\phi}_1 - \widetilde{\phi}_2$, where $\widetilde{\phi}_j$ is the harmonic extension in $\tilde{\Omega}$ of the function ψ_j and set
$$b_2 := \frac{\partial_z \widetilde{\phi}_2}{\partial_z \rho_2}, \quad w = v - T_{b_2}\rho.$$

We have

(6.13)
$$w|_{z=0} = \psi - T_{B_2}\eta.$$

We first state the following result.

Lemma 6.9 ([**3**, Lemma 5.11]). *We have*

(6.14)
$$\|\psi - T_{B_2}\eta\|_{L^\infty(I, H^s)_{ul}} \leq \mathcal{K}(M_1, M_2)\{N(0) + T N(T)\}$$

We next relate w, ρ and B.

Lemma 6.10 ([**3**, Lemma 5.12]). *We have*
$$B = \left[\frac{1}{\partial_z \rho_1} \Big(\partial_z w - (b_2 - T_{b_2}) \partial_z \rho + T_{\partial_z b_2} \rho \Big) \right] \Big|_{z=0}.$$

Lemma 6.11 ([**3**, Lemma 5.13]). *Recall that* $b_2 := \frac{\partial_z \widetilde{\phi}_2}{\partial_z \rho_2}$. *For* $k = 0, 1, 2$, *we have*
$$\left\| \partial_z^k b_2 \right\|_{C^0([-1,0], L^\infty(I, H^{s-\frac{1}{2}-k})_{ul})} \leq C \|\psi_2\|_{H^{s+\frac{1}{2}}_{ul}}.$$
for some constant C *depending only on* $\|\eta_2\|_{H^{s+\frac{1}{2}}_{ul}}$.

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Notice that η and hence ρ are estimated in $L^\infty(I; H^{s-\frac{1}{2}})$ (see (6.11)). To complete the proof of the Proposition 6.8, it remains only to estimate $\partial_z w|_{z=0}$ in $L^\infty(I, H_{ul}^{s-1})$.

Lemma 6.12 ([**3**, Lemma 5.14]). *For $t \in [0, T]$ we have*

$$(6.15) \qquad \|\nabla_{x,z} w\|_{C^0([-1,0],H^{s-1})_{ul}} \leq \mathcal{K}(M_1, M_2)\{N(0) + TN(T)\}.$$

6.6. Well posedness. The proof goes as follows. In a first step we prove the main theorem for very smooth data, using a parabolic regularization. Then, when the data are rough, we regularize them, thus obtaining a sequence of solutions living on an interval depending on a small parameter ε. In a second step, using the tame estimates proved in Proposition 5.7, we show that this sequence exists on a fixed interval. In the last step, using the results stated in section 7, we prove that it is a Cauchy sequence and we conclude. Let us notice that most of this work has been already done in [**3**] in the case of the classical Sobolev spaces. Therefore we will only sketch here the main points.

6.7. Parabolic regularization. We assume first that $(\eta_0, \psi_0) \in H_{ul}^s \times H_{ul}^s$ for $s \geq n_0 + \frac{d}{2}, n_0$ large enough. and we consider for $\varepsilon > 0$ the problem

$$(6.16) \qquad \begin{cases} \partial_t \eta = G(\eta)\psi + \varepsilon \Delta_x \eta, \\ \partial_t \psi = -\dfrac{1}{2}|\nabla_x \psi|^2 + \dfrac{1}{2}\dfrac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1 + |\nabla_x \eta|^2} - g\eta + \varepsilon \Delta_x \psi \\ (\eta, \psi)|_{t=0} = (\eta_0, \psi_0). \end{cases}$$

Setting $U = (\eta, \psi)$ we can rewrite this problem as

$$(6.17) \qquad U(t) = e^{\varepsilon t \Delta_x}U_0 + \int_0^t e^{\varepsilon(t-\tau)\Delta_x}\big[\mathcal{A}(U(\tau))\big]\,d\tau.$$

We set $I = [0, T]$ and we introduce the space

$$E_s = L^\infty(I, H^s)_{ul} \cap L^2(I, H^{s+1})_{ul}.$$

According to Lemma 3.10 we have $\|e^{\varepsilon t \Delta_x}U_0\|_{E_s} \leq C_\varepsilon \|U_0\|_{H_{ul}^s} := R$. Then using the estimates

$$(6.18) \qquad \begin{aligned} \|\mathcal{A}(U)\|_{L^2(I,H^s)_{ul}} &\leq \mathcal{F}(\|U\|_{L^\infty(I,H^s)_{ul}})\|U\|_{L^2(I,H^{s+1})_{ul}} \\ \|\mathcal{A}(U_1) - \mathcal{A}(U_2)\|_{L^2(I,H^s)_{ul}} &\leq \mathcal{F}(\|(U_1, U_2)\|_{L^\infty(I,H^s \times H^s)_{ul}})\|U_1 - U_2\|_{L^2(I,H^{s+1})_{ul}} \end{aligned}$$

we can show that, if $T = T_\varepsilon$ is small enough, the right hand side of (6.17) maps the ball of radius $2R$ in E_s into itself and is contracting. By the Banach principle the equation (6.17) has a maximal solution on $[0, T_\varepsilon^*)$. Moreover if $T_\varepsilon < +\infty$ then

$$(6.19) \qquad \lim_{t \to T_\varepsilon^*} \|(\eta, \psi)(t)\|_{H_{ul}^s \times H_{ul}^s} = +\infty.$$

Now with this large s we set

$$M_s^\varepsilon(T) = \sup_{t \in [0,T]} \|(\eta^\varepsilon, \psi^\varepsilon, V^\varepsilon, B^\varepsilon)(t)\|_{H_{ul}^s \times H_{ul}^s \times H_{ul}^{s-1} \times H_{ul}^{s-1}}.$$

Using the same computations as in [**3**] and the method of proof of Proposition 5.7 (but in an easier way since here s is large) we deduce that one can find $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ strictly increasing such that

$$M_s^\varepsilon(T) \leq \mathcal{F}(M_s^\varepsilon(0) + \sqrt{T}M_s^\varepsilon(T)).$$

Since $M_s^\varepsilon(0) = M_s(0)$ does not depend on ε, this will imply that there exists $T_0 > 0$ independent of ε such that $M_s^\varepsilon(T) \leq \mathcal{F}(2M_s(0))$ for $T \in [0, T_0]$. Using this uniform bound on this fixed interval and the arguments of [3] we can pass to the limit in the equations 6.16 to obtain a solution (η, ψ) of the water wave system.

6.8. Regularizing the data, *a priori* estimates. . Assume that $(\eta_0, \psi_0, V_0, B_0)$ belong to $H_{ul}^{s_0 + \frac{1}{2}} \times H_{ul}^{s_0 + \frac{1}{2}} \times H_{ul}^{s_0} \times H_{ul}^{s_0}$ where $s_0 > 1 + \frac{d}{2}$. Let $j \in C_0^\infty(\mathbf{R}^d), j(\xi) = 1$ when $|\xi| \leq 1$. We regularize the data in setting $f_0^\varepsilon = j(\varepsilon D) f_0$ if f_0 is one of them. Then the regularized data belong to H_{ul}^s for s large. Therefore we can apply Step 1. to get a maximal solution $(\eta_\varepsilon, \psi_\varepsilon, V_\varepsilon, B_\varepsilon)$ of the water wave system, on an interval $[0, T_\varepsilon^*)$, wich is very regular. Moreover we know that if $T_\varepsilon^* < +\infty$ then

$$(6.20) \qquad \lim_{T \to T_\varepsilon^*} M_s^\varepsilon(T) = +\infty.$$

We first apply Proposition 5.7 with $s = s_0$ and we obtain

$$M_{s_0}^\varepsilon(T) \leq \mathcal{F}(M_{s_0}^\varepsilon(0) + \sqrt{T} M_{s_0}^\varepsilon(T)).$$

Since there exists $A_0 > 0$, independent of ε, such that $M_{s_0}^\varepsilon(0) \leq A_0$, for all $\varepsilon > 0$ small, we deduce that one can find $T_0 > 0$ independent of ε such that $M_{s_0}^\varepsilon(T) \leq \mathcal{F}(2A_0)$ for all $T \leq \min(T_0, T_\varepsilon^*)$. We apply again Proposition 5.7 with s large and we get

$$M_s^\varepsilon(T) \leq \mathcal{F}(A_0 + \sqrt{T_0}\mathcal{F}(2A_0))(M_s^\varepsilon(0) + \sqrt{T} M_s^\varepsilon(T)).$$

Let $T_1 > 0$ be such that $\sqrt{T_1}\mathcal{F}(A_0 + \sqrt{T_0}\mathcal{F}(2A_0)) \leq \frac{1}{2}$. Then

$$M_s^\varepsilon(T) \leq 2\mathcal{F}(A_0 + \sqrt{T_0}\mathcal{F}(2A_0))M_s^\varepsilon(0), \forall 0 < T \leq \min(T_1, T_\varepsilon^*).$$

Using (6.20) we deduce that $T_\varepsilon^* \geq T_1$ for all $\varepsilon > 0$ small. This shows that our solution $(\eta_\varepsilon, \psi_\varepsilon, V_\varepsilon, B_\varepsilon)$ exists on a fixed interval $[0, T_1]$. Moreover, as seen above, $M_{s_0}^\varepsilon(T)$ is uniformly bounded on this interval.

6.9. Passing to the limit. According to Theorem 6.1, the sequence $(\eta_\varepsilon, \psi_\varepsilon, V_\varepsilon, B_\varepsilon)$ which is, according to Section 6.8, bounded in

$$L^\infty((0, T); H_{ul}^{s_0 + \frac{1}{2}} \times H_{ul}^{s_0 + \frac{1}{2}} \times H_{ul}^{s_0} \times H_{ul}^{s_0}),$$

is convergent in

$$L^\infty((0, T); H_{ul}^{s_0 - \frac{1}{2}} \times H_{ul}^{s_0 - \frac{1}{2}} \times H_{ul}^{s_0 - 1} \times H_{ul}^{s_0 - 1}),$$

and hence also for any $\delta > 0$ in

$$L^\infty((0, T); H_{ul}^{s_0 + \frac{1}{2} - \delta} \times H_{ul}^{s_0 + \frac{1}{2} - \delta} \times H_{ul}^{s_0 - \delta} \times H_{ul}^{s_0 - \delta}).$$

To get the existence of solutions, it remains to pass to the limit in the equations (the uniqueness follows once again from Theorem 6.1). For this step, we rewrite the system (2.1), (2.2) as

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$$(6.21) \quad \begin{cases} \partial_t \eta_\varepsilon = G(\eta_\varepsilon)\psi_\varepsilon, \\ \partial_t \psi_\varepsilon + V_\varepsilon \cdot \nabla \psi_\varepsilon = \dfrac{1}{2}(V_\varepsilon^2 + B_\varepsilon^2) - g\eta_\varepsilon. \\ B_\varepsilon = \dfrac{\nabla_x \eta_\varepsilon \cdot \nabla_x \psi_\varepsilon + G(\eta_\varepsilon)\psi_\varepsilon}{1 + |\nabla_x \eta_\varepsilon|^2}, \\ V_\varepsilon = \nabla_x \psi_\varepsilon - B_\varepsilon \nabla_x \eta_\varepsilon. \end{cases}$$

Choosing $\delta > 0$ such that $s - \delta - \frac{1}{2} > \frac{d}{2}$ (so that $H^{s-\delta-\frac{1}{2}}$ is an algebra), we deduce that

$$\partial_t \eta_\varepsilon \rightharpoonup \partial_t \eta \text{ in } \mathcal{D}'((0,T) \times \mathbf{R}^d)$$

$$\partial_t \psi_\varepsilon \rightharpoonup \partial_t \psi \text{ in } \mathcal{D}'((0,T) \times \mathbf{R}^d)$$

$$V_\varepsilon \cdot \nabla \psi_\varepsilon \to V \cdot \nabla \psi \text{ in } L^\infty((0,T); H_{ul}^{s-\delta-\frac{1}{2}})$$

$$(6.22) \quad V_\varepsilon^2 + B_\varepsilon^2 \to V^2 + B^2 \text{ in } L^\infty((0,T); H_{ul}^{s-\delta})$$

$$\nabla_x \eta_\varepsilon \cdot \nabla_x \psi_\varepsilon \to \nabla_x \eta \cdot \nabla_x \psi \text{ in } L^\infty((0,T); H_{ul}^{s-\delta-\frac{1}{2}}) \subset L^\infty((0,T); L_{ul}^2)$$

$$|\nabla_x \eta_\varepsilon|^2 \to |\nabla_x \eta|^2 \text{ in } L^\infty((0,T); H_{ul}^{s-\delta-\frac{1}{2}}) \subset L^\infty((0,T); C^0 \cap L^\infty(\mathbf{R}^d))$$

On the other hand, according to Lemma 6.3, we get

$$G(\eta_\varepsilon)\psi_\varepsilon - G(\eta)\psi = G(\eta_\varepsilon)(\psi_\varepsilon - \psi) + (G(\eta_\varepsilon) - G(\eta))\psi \to 0,$$

in

$$L^\infty((0,T); H_{ul}^{s-\frac{3}{2}-\delta}) \subset L^\infty((0,T); L_{ul}^2),$$

which allows to pass to he limit in (6.21) and show that the same system of equations is satisfied by (η, ψ, V, B) in $\mathcal{D}'((0,T) \times \mathbf{R}^d)$.

6.10. Continuity in time. We now prove that (η, ψ, V, B) is continuous in time with values in $H_{ul}^{s_0+\frac{1}{2}-\delta} \times H_{ul}^{s_0+\frac{1}{2}-\delta} \times H_{ul}^{s_0-\delta} \times H_{ul}^{s_0-\delta}$. From the equation, and product rules, its time derivative is clearly in

$$L^\infty((0,T); H_{ul}^{s_0-\frac{1}{2}} \times H_{ul}^{s_0-\frac{1}{2}} \times H_{ul}^{s_0-\frac{3}{2}} \times H_{ul}^{s_0-\frac{3}{2}})$$

and consequently (interpolating with the a priori estimate), for any $\delta > 0$,

$$(6.23) \qquad (\eta, \psi, V, B) \in C^0((0,T); H_{ul}^{s_0+\frac{1}{2}-\delta} \times H_{ul}^{s_0+\frac{1}{2}-\delta} \times H_{ul}^{s_0-\delta} \times H_{ul}^{s_0-\delta}).$$

7. The canal

We consider now the case of a canal a canal having vertical walls near the free surface or the case of a rectangular basin. The propagation of waves whose crests are orthogonal to the walls is one of the main motivation for the analysis of 2D waves. It was historically at the heart of the analysis of water waves. The study of the propagation of three-dimensional water waves for the linearized equations goes back to Boussinesq (see [**13**]). However, there are no existence results for the nonlinear equations in the general case where the waves can be reflected on the walls of the canals (except the analysis of 3D-periodic travelling waves

which correspond to the reflexion of a 2D-wave off a vertical wall, see Reeder-Shinbrot [34], Craig and Nicholls [17] and Iooss-Plotnikov [26]). We hence consider a fluid domain which at time t is of the form

$$\Omega(t) = \{(x_1, x_2, y) \in M \times \mathbf{R} \,:\, b(x) < y < \eta(t, x),\ x = (x_1, x_2)\},$$

where $M = (0, 1) \times \mathbf{R}$ in the case of the canal and $M = (0, 1) \times (0, L)$ in the case of a rectangular basin, and b is a fixed continuous function on M describing the bottom.

Denote by Σ the free surface and by Γ the fixed boundary of the canal:

$$\Sigma(t) = \{(x_1, x_2, y) \in M \times \mathbf{R} \,:\, y = \eta(t, x)\},$$

and we set $\Gamma = \partial\Omega(t) \setminus \Sigma(t)$ (which does not depend on time). We have

$$\Gamma = \Gamma_1 \cup \Gamma_2,$$

(7.1)
$$\Gamma_1 = \{\,(x_1, x_2, y) \in M \times \mathbf{R}\,;\, b(x) = y\,\}$$
$$\Gamma_2 = \{\,(x_1, x_2, y) \in \partial M \times \mathbf{R}\,;\, b(x) < y < \eta(x_1, x')\,\}.$$

Denote by n the normal to the boundary Γ and denote by ν the normal to the free surface Σ. The irrotational water-waves system is then the following: the Eulerian velocity field $v \colon \Omega \to \mathbf{R}^3$ solves the incompressible Euler equation

(7.2) $\qquad \partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P = -g e_y, \quad \mathrm{div}_{x,y}\, v = 0 \quad \mathrm{curl}_{x,y} v = 0 \quad \text{in } \Omega,$

where $-g e_y$ is the acceleration of gravity $(g > 0)$ and where the pressure term P can be recovered from the velocity by solving an elliptic equation. The problem is then given by three boundary conditions. They are

(7.3)
$$\begin{cases} v \cdot n = 0 & \text{on } \Gamma, \\ \partial_t \eta = \sqrt{1 + |\nabla\eta|^2}\, v \cdot \nu & \text{on } \Sigma, \\ P = 0 & \text{on } \Sigma. \end{cases}$$

We notice that the first condition in (7.3) expresses the fact that the particles in contact with the rigid bottom remain in contact with it. Notice that to fully make sense, this condition requires some smoothness on Γ, but in general it has a weak variational meaning (see [3]).

Finally we impose the initial condition

(7.4) $$(\eta, v)|_{t=0} = (\eta_0, v_0),$$

where v_0 satisfies

$$\mathrm{div}_{x,y}\, v_0 = 0 \quad \mathrm{curl}_{x,y} v_0 = 0 \quad \text{in } \Omega_0, \quad v_0 \cdot n = 0,\, \text{on } \Gamma.$$

It follows that there exists a function $\phi_0 : \Omega_0 \to \mathbf{R}$ such that

$$v_0 = \nabla_{x,y}\phi_0 \quad \text{in } \Omega_0, \quad \text{with } \Delta_{x,y}\phi_0 = 0.$$

We set

$$\psi_0 = \phi_0|_{y=\eta_0(x)}$$

and introduce the trace of the velocity field $v_0 = (v_{0,x_1}, v_{0,x_2}, v_{0,y})$ on $\Sigma_0 = \{(x, \eta_0(x))\}$ in setting

$$v_{0,x_1}|_{y=\eta_0} = V_{0,x_1}, \quad v_{0,x_2}|_{y=\eta_0} = V_{0,x_2}, \quad v_{0,y}|_{y=\eta_0} = B_0,\ V_0 = (V_{0,x_1}, V_{0,x_2}).$$

Similarly, to a solution v of (7.2)-(7.3) we associate ϕ, ψ and $(V, B) = v|_{y=\eta}$ as above.

The stability of the waves is dictated by the Taylor sign condition, which is the assumption that there exists a positive constant c such that

$$(7.5) \qquad a(t,x) := -(\partial_y P)(t, x, \eta(t,x)) \geq c > 0.$$

7.1. A simple observation. We begin with a elementary calculation showing that, at least for regular enough solutions, as soon as the Taylor sign condition (7.5) is satisfied, in the case of vertical walls, it is *necessary* that at the points where the free surface and the boundary of the canal meet $(\Sigma(t) \cap \Gamma)$, the scalar product between the two normals (to the free surface and to the boundary of the canal) vanishes : $\nu \cdot n = 0$ on $\Sigma \cap \Gamma$, which means that the free surface Σ necessarily makes a right-angle with the rigid walls (see Figure 1).
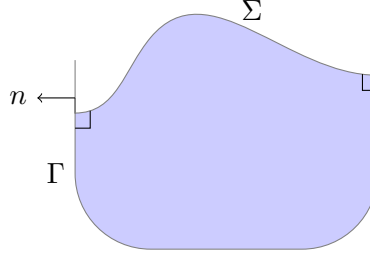

FIGURE 1. Two-dimensional section of the fluid domain, exhibiting the right-angles at the interface $\Sigma \cap \Gamma$

Proposition 7.1. *Let* $(\eta, v) \in C^0([0,T]; C^{\frac{3}{2}}([0,1] \times \mathbf{R}) \times C^1(\overline{\Omega}))$ *be a solution of System* (7.2), (7.3) *such that the Taylor coefficient* a *is continuous and non-vanishing and* $\eta(t,x) \geq b(x) + h$ *for some positive constant* h. *Then the angle between the free surface,* $\Sigma(t)$ *and the boundary of the canal* Γ *is a right angle:*

$$\forall t \in [0,T], \forall x \in \Sigma(t) \cap \Gamma, \quad n \cdot \nu(t,x) = 0,$$

which is equivalent to

$$(7.6) \qquad \partial_{x_1}\eta(t, x_1, x_2)\,|_{x_1=0,1} = 0.$$

PROOF. We give the proof in the case of a canal, the proof for the rectangular basin is similar. Since $\eta_0(x) \geq b(x) + h$ at a point m_0 where $\Sigma(t)$ and Γ meet we have $m_0 = (\varepsilon, x_2^0, y^0)$ where $\varepsilon = 0$ or 1. Let $m = (\varepsilon, x_2, y)$ be a point on Γ near m_0. At m the normal n to Γ is $n = (\pm 1, 0, 0)$. Taking the scalar product of the equation (7.2) with n we obtain, since $e_y \cdot n = 0$,

$$(7.7) \qquad (\nabla_{x,y} P) \cdot n = -(\partial_t v) \cdot n - ((v \cdot \nabla_{x,y})v) \cdot n \quad \text{at } m.$$

Denote by (v_{x_1}, v_{x_2}, v_y) the three components of the velocity field v. The first condition in (7.3) implies that $(v \cdot n)(m) = \pm v_{x_1}(t, \varepsilon, x_2, y) = 0$. It follows that $(\partial_t v) \cdot n = \partial_t(v \cdot n) = 0$

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at m. Moreover on Γ near m_0 we have

$$
\begin{aligned}
\left[((v \cdot \nabla_{x,y})v) \cdot n\right](t, \varepsilon, x_2, y) &= \pm\left[((v \cdot \nabla_{x,y})v_{x_1}\right](t, \varepsilon, x_2, y) \\
&= \pm\left[(v_{x_2}\partial_{x_2} + v_y\partial_y)v_{x_1}\right](t, \varepsilon, x_2, y) \\
&= \pm\left[(v_{x_2}\partial_{x_2} + v_y\partial_y)\right](v_{x_1}(t, \varepsilon, x_2, y)) = 0.
\end{aligned}
$$

It follows from (7.7) that

$$(7.8) \qquad\qquad\qquad (\nabla_{x,y}P) \cdot n = 0 \quad \text{at } m.$$

Now by the third condition in (7.3) we have $P = 0$ on Σ and by (7.5) and our hypothesis on the Taylor coefficient we have $\nabla_{x,y}P \neq 0$ on Σ. It follows that $\nabla_{x,y}P$ is proportional to the normal ν at Σ and by continuity at $\Sigma \cap \Gamma$. We deduce from (7.8) that $\nu \cdot n = 0$ at m_0. $\qquad\square$

Once this *right angle* property is ensured, it is easy to show that some additional compatibility conditions have also to be fulfilled. Namely, for $f = B_0, V_{0,x_2}, \partial_{x_1}V_{0,x_1}$, using (7.3), as soon as the function ϕ is smooth enough so that all terms below are defined, we have with $m = (\varepsilon, x_2)$ ($\varepsilon = 0$ or 1).

$$(7.9) \quad \begin{cases}
\partial_{x_1}\psi_0(m) = \partial_{x_1}\phi_0(m, \eta_0(m)) + \partial_y\phi_0(m, \eta_0(m))\partial_{x_1}\eta_0(m) = 0, \\
\partial_{x_1}B_0(m) = \partial_{x_1}\partial_y\phi_0(m, \eta_0(m)) + \partial_y^2\phi_0(m, \eta_0(m))\partial_{x_1}\eta_0(m) = 0, \\
\partial_{x_1}V_{0,x_2}(m) = \partial_{x_1}\partial_{x_2}\phi_0(m, \eta_0(m)) + \partial_y\partial_{x_2}\phi_0(m, \eta_0(m))\partial_{x_1}\eta_0(m) = 0, \\
\partial_{x_1}^2 V_{0,x_1}(m) = \partial_{x_1}^3\phi_0(m, \eta_0(m)) + 2\partial_y\partial_{x_1}^2\phi_0(m, \eta_0(m))\partial_{x_1}\eta_0(m) \\
\qquad\qquad + \partial_y^2\partial_{x_1}\phi_0(m, \eta_0(m)(\partial_{x_1}\eta_0(m))^2 + \partial_y\partial_{x_1}\phi_0(m, \eta_0(m))\partial_{x_1}^2\eta_0(m) \\
\qquad\quad = 0,
\end{cases}$$

where in the last equality, we used that $\partial_{x_1}^3\phi = -(\partial_{x_2}^2 + \partial_y^2)\partial_{x_1}\phi$, since ϕ is harmonic.

7.2. The result. As before, we denote by x_1 (resp. x_2) the variable in $(0, 1)$ (resp. in $\mathbf{R}$). To state our results we need to introduce the uniformly local Sobolev spaces in the x_2 direction (these spaces are introduced by Kato in [**27**]). Let $1 = \sum_{k\in\mathbf{Z}}\chi(x_2 - k)$ be a partition of unity and define for any $s \in \mathbf{R}$,

$$H_{ul}^s((0,1) \times \mathbf{R}) = \left\{u \in H_{loc}^s((0,1) \times \mathbf{R}) : \sup_k \|\chi(x_2 - k)u\|_{H^s((0,1)\times\mathbf{R})} < +\infty\right\}.$$

These are Banach spaces when endowed with the norm

$$\|u\|_{H_{ul}^s} = \sup_k \|\chi(x_2 - k)u\|_{H^s((0,1)\times\mathbf{R})}.$$

In Section 7.1 we showed that in order to get smooth solutions, a set of compatibility conditions (7.6), (7.9) *have* to be assumed. Here we prove that these conditions are not only *necessary*, but they are *sufficient*.

Theorem 7.2. *Set $M = (0, 1) \times \mathbf{R}$. Let $s \in (2, 3)$, $s \neq \frac{5}{2}$, and*

$$\mathcal{H}^s(M) = H_{ul}^{s+\frac{1}{2}}(M) \times H_{ul}^{s+\frac{1}{2}}(M) \times H_{ul}^s(M) \times H_{ul}^s(M).$$

Consider $(\eta_0, \psi_0, V_0, B_0) \in \mathcal{H}^s(M)$ and assume that, with $\varepsilon = 0, 1$

(H_1) $V_{0,x_1}(\varepsilon, x_2) = 0$ and $\partial_{x_1} f(\varepsilon, x_2) = 0$ when $f = \eta_0, \psi_0, B_0, V_{0,x_2}$. Furthermore, $\partial_{x_1}^2 V_{0,x_1}(\varepsilon, x_2) = 0$ if $s > 5/2$.

(H_2) The Taylor sign condition, $a_0(x) \geq c > 0$ is satisfied at time $t = 0$.

(H_3) $\eta_0(x) \geq b(x) + h$ for some positive constant h.

Then there exists a time $T > 0$ and a unique solution $(\eta, v = \nabla_{x,y} \phi)$ of the system (7.2), (7.3), (7.4) such that

i) $(\eta, \phi|_\Sigma, V, B) \in C([0, T]; \mathcal{H}^s(M))$,

ii) the Taylor sign condition is satisfied at time t and $\eta(t) \geq b + h/2$.

In the case of a rectangular basin we have the following result.

Theorem 7.3. *Set $M = (0, 1) \times (0, L)$. Let $s \in (2, 3)$, $s \neq \frac{5}{2}$, and*

$$\mathcal{H}^s(M) = H^{s+\frac{1}{2}}(M) \times H^{s+\frac{1}{2}}(M) \times H^s(M) \times H^s(M).$$

Consider initial data $(\eta_0, \psi_0, V_0, B_0) \in \mathcal{H}^s(M)$, such that

(C_1) $V_{0,x_1}(\varepsilon, x_2) = 0$ and $\partial_{x_1} f(\varepsilon, x_2) = 0$ when $f = \eta_0, \psi_0, B_0, V_{0,x_2}$. Furthermore, $\partial_{x_1}^2 V_{0,x_1}(\varepsilon, x_2) = 0$ if $s > 5/2$. Here $\varepsilon = 0$ or 1.

(C_2) $V_{0,x_1}(x_1, \delta) = 0$ and $\partial_{x_2} f(x_1, \delta) = 0$ when $f = \eta_0, \psi_0, B_0, V_{0,x_2}$. Furthermore, $\partial_{x_2}^2 V_{0,x_1}(x_1, \delta) = 0$ if $s > 5/2$. Here $\delta = 0$ or L.

(C_3) The Taylor sign condition, $a_0(x) \geq c > 0$ is satisfied at time $t = 0$,

(C_4) $\eta_0(x) \geq b(x) + h$ for some positive constant h.

Then there exists $T > 0$ and a unique solution $(\eta, v = \nabla_{x,y}\phi)$ of the system (7.2) (7.3), (7.4) such that

(1) $(\eta, \phi|_\Sigma, V, B) \in C([0, T]; \mathcal{H}^s(M))$,

(2) the Taylor sign condition is satisfied at time t and $\eta(t) \geq b + h/2$.

Remark 7.4. (i) Our results exclude the case $s = \frac{5}{2}$ for technical reasons. It would be possible (but unnecessarily complicated) to include this case.

(ii) In the case of a flat bottom (say $b(x) = -1$) we do not need assumption (H_2) (and (C_3)) which is in this case always satisfied as proved by Wu ([**36**, **35**]), see also [**29**]. Also, this condition is satisfied under a smallness assumption.

(iii) Condition (H_1), when $f = \eta_0$, says that at $t = 0$ the fluid has to be orthogonal to the fixed vertical walls.

7.3. Proof of the result. Following Boussinesq (see [**13**, page 37]) the strategy of proof is to perform a symmetrization process (following the process which is illustrated on Figure 2 below).

Once this symmetrization process is performed, we will apply our result [**4**, Theorem 2.3] to conclude.

7.3.1. *The periodization process.* Without additional assumptions, the reflection procedure should yield in general a Lipschitz singularity. However, here the possible singularities are weaker according to the physical hypothesis (H_1).
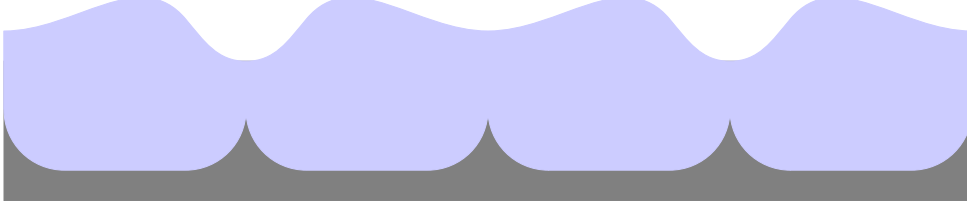
FIGURE 2. Two-dimensional section of the extended fluid domain.

For a function v defined on $(0, +\infty)$, define v^{ev} and v^{od} to be the even and odd extensions of v to $(-\infty, +\infty)$ defined by

(7.10)
$$
v^{\mathrm{ev}}(y) = \begin{cases} v(-y), & \text{if } y < 0 \\ v(y) & \text{if } y \geq 0. \end{cases}
$$
$$
v^{\mathrm{od}}(y) = \begin{cases} -v(-y), & \text{if } y < 0 \\ v(y) & \text{if } y \geq 0. \end{cases}
$$

We have the following result

Proposition 7.5. *We have*

(1) *Assume that $0 \leq s < \frac{3}{2}$. Then the map $v \mapsto v^{ev}$ is continuous from $H^s(0, +\infty)$ to $H^s(\mathbf{R})$.*

(2) *Assume that $\frac{3}{2} < s < \frac{7}{2}$. Then the map $v \mapsto v^{ev}$ is continuous from the space $\{v \in H^s(0, +\infty) : v'(0) = 0\}$ to $H^s(\mathbf{R})$.*

(3) *Assume that $0 \leq s < \frac{1}{2}$. Then the map $v \mapsto v^{od}$ is continuous from $H^s(0, +\infty)$ to $H^s(\mathbf{R})$.*

(4) *Assume that $\frac{1}{2} \leq s < \frac{5}{2}$. Then the map $v \mapsto v^{od}$ is continuous from the space $\{v \in H^s(0, +\infty) : v(0) = 0\}$ to $H^s(\mathbf{R})$.*

(5) *Assume that $\frac{5}{2} < s \leq 4$. Then the map $v \mapsto v^{od}$ is continuous from the space $\{v \in H^s(0, +\infty) : v(0) = v''(0) = 0\}$ to $H^s(\mathbf{R})$.*

PROOF. Let $I = (0, +\infty)$. Then $C_0^\infty(\overline{I})$ is dense in $H^s(I)$ for all $s \in \mathbf{R}$.

(1) The case $s = 0$ is trivial since $\|v^{\mathrm{ev}}\|_{L^2(\mathbf{R})}^2 = 2\|v\|_{L^2(I)}^2$. Consider now the case $0 < s < 1$. Then the square of the $H^s(\mathbf{R})$-norm of v^{ev} is equivalent to

$$
\|v^{\mathrm{ev}}\|_{L^2(\mathbf{R})}^2 + A, \quad A := \iint_{\mathbf{R} \times \mathbf{R}} \frac{|v^{\mathrm{ev}}(x) - v^{\mathrm{ev}}(y)|^2}{|x - y|^{1+2s}} \, dx dy.
$$

Then we can write

$$
A = 2\iint_{I \times I} \frac{|v(x) - v(y)|^2}{|x - y|^{1+2s}} \, dx dy + 2\iint_{I \times I} \frac{|v(x) - v(y)|^2}{|x + y|^{1+2s}} \, dx dy := A_1 + A_2.
$$

We have

$$
A_1 \leq 2\|v\|_{H^s(I)}^2, \quad A_2 \leq 2\|v\|_{H^s(I)}^2
$$

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since $\frac{1}{(x+y)^{1+2s}} \leq \frac{1}{|x-y|^{1+2s}}$. The case $s = 1$ being straightforward consider the case $1 < s < \frac{3}{2}$. Set $\sigma = s - 1 \in (0, \frac{1}{2})$. Then

$$\|v^{\mathrm{ev}}\|_{H^s(\mathbf{R})}^2 = \|v^{\mathrm{ev}}\|_{L^2(\mathbf{R})}^2 + \|\partial_x v^{\mathrm{ev}}\|_{H^\sigma(\mathbf{R})}^2.$$

Since $0 < \sigma < 1$ we have

$$\|\partial_x v^{\mathrm{ev}}\|_{H^\sigma(\mathbf{R})}^2 \leq C(A_0 + A_1 + A_2)$$

$$A_0 = \|\partial_x v^{\mathrm{ev}}\|_{L^2(\mathbf{R})}^2 \leq C_1\|v'\|_{L^2(I)}^2 \leq C_1\|v\|_{H^s(I)}^2$$

$$A_1 = \iint_{I \times I} \frac{|v'(x) - v'(y)|^2}{|x - y|^{1+2\sigma}}\, dxdy \leq C_2\|v'\|_{H^\sigma(I)} \leq C_2\|v\|_{H^s(I)}$$

$$A_2 = \iint_{I \times I} \frac{|v'(x) + v'(y)|^2}{|x + y|^{1+2\sigma}}\, dxdy.$$

Eventually we have

$$A_2 \leq C_3 \int_I \frac{|v'(x)|^2}{|x|^{2\sigma}}\, dx \leq C_4\|v'\|_{H^\sigma(I)} \leq C_4\|v\|_{H^s(I)}$$

by Theorem 11.2 in [**32**], since $0 < \sigma < \frac{1}{2}$. This completes the proof of (1).

(2) If $\frac{3}{2} < s < 2$ let $\sigma = s - 1 \in (\frac{1}{2}, 1)$; arguing as above we see that

$$\|v^{\mathrm{ev}}\|_{H^s(\mathbf{R})}^2 \leq C\Big(\|v\|_{H^s(I)}^2 + \int_I \frac{|v'(x)|^2}{|x|^{2\sigma}}\, dxdy\Big).$$

Now since $v' \in H^\sigma(I)$ and $v'(0) = 0$ we can apply Theorem 11.3 in [**32**] which ensures that the integral in the right hand side can be estimated by $C\|v\|_{H^s(I)}^2$. The case $s = 2$ being straightforward let $2 < s \leq \frac{7}{2}$. Then

$$\|v^{\mathrm{ev}}\|_{H^s(\mathbf{R})}^2 \leq C\big(\|v^{\mathrm{ev}}\|_{L^2(\mathbf{R})}^2 + \|\partial_x^2 v^{\mathrm{ev}}\|_{H^{s-2}(\mathbf{R})}^2\big).$$

Since $0 < s - 2 < \frac{3}{2}$ and $v'(0) = 0$ we may apply the same argument as in the case (1) to ensure that $\|\partial_x^2 v^{\mathrm{ev}}\|_{H^{s-2}(\mathbf{R})}^2 \leq C\|v\|_{H^s(I)}^2$.

The cases (3) to (5) are proved by exactly the same arguments. $\qquad\square$

To state the reflection procedure in higher dimension we need to introduce the uniformly local Sobolev spaces in $\mathbf{R}^n, n \geq 2$.

Let $1 = \sum_{k \in \mathbf{Z}} \chi(x - k)$ be a partition of unity in $\mathbf{R}^n$ and define for any $s \in \mathbf{R}$,

$$H_{ul}^s(\mathbf{R}^n) = \{u \in H_{loc}^s(\mathbf{R}^n) : \sup_k \|\chi(\cdot - k)u\|_{H^s(\mathbf{R}^n)} < +\infty\}.$$

These are Banach spaces when endowed with the norm

$$\|u\|_{H_{ul}^s} = \sup_k \|\chi(\cdot - k)u\|_{H^s(\mathbf{R}^n)}.$$

Now, if v is a function on $M = (0,1) \times \mathbf{R}^d$ we define the even (resp. odd) periodic extensions on $\mathbf{T} \times \mathbf{R}^d$, v^{ev} (resp. v^{od}), by

(7.11)
$$v^{\mathrm{ev}}(x_1, x') = \begin{cases} v(-x_1, x'), & \text{if } -1 < x_1 < 0, \\ v(x_1, x'), & \text{if } 0 \leq x_1 < 1, \\ v(x_1 - 2k, x'). & \text{if } x_1 - 2k \in (-1, 1), \, k \in \mathbf{Z}. \end{cases}$$

$$v^{\mathrm{od}}(x_1, x') = \begin{cases} -v(-x_1, x'), & \text{if } -1 < x_1 < 0, \\ v(x_1, x'), & \text{if } 0 \leq x_1 < 1, \\ v(x_1 - 2k, x'). & \text{if } x_1 - 2k \in (-1, 1), \, k \in \mathbf{Z}. \end{cases}$$

Corollary 7.6. *Let $M = (0,1) \times \mathbf{R}$.*
(1) *Assume $0 \leq s < \frac{3}{2}$ then the map $v \mapsto v^{ev}$ is continuous from $H^s_{ul}(M)$ to $H^s_{ul}(\mathbf{R}^2)$.*
(2) *Assume $\frac{3}{2} < s < \frac{7}{2}$. Let*
$$E_s = \{u \in H^s_{ul}(M) : \partial_{x_1} u(\varepsilon, x_2) = 0, \varepsilon = 0, 1, \forall x_2 \in \mathbf{R}\}.$$
Then the map $v \mapsto v^{ev}$ is continuous from E_s to $H^s_{ul}(\mathbf{R}^2)$.
(3) *Assume $0 \leq s < \frac{1}{2}$ then the map $v \mapsto v^{od}$ is continuous from $H^s_{ul}(M)$ to $H^s_{ul}(\mathbf{R}^2)$.*
(4) *Assume $\frac{1}{2} < s < \frac{5}{2}$. Let*
$$F_s = \{u \in H^s_{ul}(M) : u(\varepsilon, x_2) = 0, \varepsilon = 0, 1, \forall x_2 \in \mathbf{R}\}.$$
Then the map $v \mapsto v^{od}$ is continuous from F_s to $H^s_{ul}(\mathbf{R}^2)$.
(5) *Assume $\frac{5}{2} < s \leq 4$. Let*
$$G_s = \{u \in H^s_{ul}(M) : u(\varepsilon, x_2) = \partial^2_{x_1} u(\varepsilon, x_2) = 0, \varepsilon = 0, 1, \forall x_2 \in \mathbf{R}\}.$$
Then the map $v \mapsto v^{od}$ is continuous from G_s to $H^s_{ul}(\mathbf{R}^2)$.

PROOF. Since $(D^\alpha_{x_2} v)^{\mathrm{ev}} = D^\alpha_{x_2}(v^{\mathrm{ev}})$, $(D^\alpha_{x_2} v)^{\mathrm{od}} = D^\alpha_{x_2}(v^{\mathrm{od}})$ the result is clearly a one dimensional result and it is enough to prove it for the one dimensional case, in which case it is a direct consequence of Proposition 7.5 and a localization argument. □

Consider now an initial data $(\eta_0, \psi_0 = \phi_0|_{\Sigma_0}, V_0, B_0)$ satisfying the assumptions in Theorem 7.2 and define
$$\widetilde{\eta}_0 = \eta_0^{\mathrm{ev}}, \quad \widetilde{\psi}_0 = \psi_0^{\mathrm{ev}}, \quad \widetilde{V}_{0,x_1} = V_{0,x_1}^{\mathrm{od}}, \quad \widetilde{V}_{0,x_2} = V_{0,x_2}^{\mathrm{ev}}, \quad \widetilde{B}_0 = B_0^{\mathrm{ev}}$$
on $\mathbf{T} \times \mathbf{R}$.

Recall (see Theorem 7.2) that, with $M = (0,1) \times \mathbf{R}$, we have set
$$\mathcal{H}^s(M) = H^{s+\frac{1}{2}}_{ul}(M) \times L^2_{ul}(M) \times H^s_{ul}(M) \times H^s_{ul}(M),$$
and introduce
$$\mathcal{H}^s(\mathbf{R}^2) = H^{s+\frac{1}{2}}_{ul}(\mathbf{R}^2) \times L^2_{ul}(\mathbf{R}^2) \times H^s_{ul}(\mathbf{R}^2) \times H^s_{ul}(\mathbf{R}^2).$$
Then we have the following lemma.

Lemma 7.7. *Let $2 < s < 3, s \neq \frac{5}{2}$ and $(\eta_0, \psi_0, V_0, B_0) \in \mathcal{H}^s(M)$ satisfying the hypothesis (H_1) in Theorem 7.2. Then $(\widetilde{\eta}_0, \widetilde{\psi}_0, \widetilde{V}_0, \widetilde{B}_0) \in \mathcal{H}^s(\mathbf{R}^2)$ and are 2-periodic with respect to the x_1 variable.*

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PROOF. This follows immediately from the hypothesis (H_1) and Corollary 7.6. □

In the case of a rectangular basin, performing both reflection and periodizations with respect to the x_1 and the x_2 variables leads similarly to extensions

$$(\widetilde{\eta}_0, \widetilde{\psi}_0, \widetilde{V}_0, \widetilde{B}_0) \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^2) \times L_{ul}^2(\mathbf{R}^2) \times H_{ul}^s(\mathbf{R}^2) \times H_{ul}^s(\mathbf{R}^2)$$

which are 2- periodic with respect to the x_1 variable and $2L$ periodic with respect to the x_2 variable).

7.3.2. *Conclusion.* We are now in position to apply Theorem 2.3. We consider first the case of the canal. Starting from $(\eta_0, \psi_0, V_0, B_0)$, we define $(\widetilde{\eta}_0, \widetilde{\psi}_0, \widetilde{V}_0, \widetilde{B}_0)$ their periodized extensions following the process in Section 7.3.1. Let $(\widetilde{\eta}, \widetilde{v})$ be the solution of the free surface water waves system given by Theorem 2.3. Since the initial data $(\widetilde{\eta}_0, \widetilde{\psi}_0, \widetilde{V}_0', \widetilde{B}_0)$ are even while $\widetilde{V}_{0,x_1}$ is odd, our uniqueness result guaranties that the solution satisfies the same symmetry property (because if we consider our solution, the function obtained by symmetrization is also a solution with same initial data). The same argument shows that as the initial data are 2-periodic with respect to the variable x_1, so is the solution. As a consequence if we define v, η, P as the trace of $\widetilde{v}, \widetilde{\eta}, \widetilde{P}$ on $(0,1) \times \mathbf{R}$, we get that they satisfy trivially the system free boundary Euler equation

$$\partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P = -g e_y, \quad \mathrm{div}_{x,y}\, v = 0 \quad \text{in } \Omega,$$

(7.12)
$$\partial_t \eta = \sqrt{1 + |\nabla \eta|^2}\, v \cdot \nu \quad \text{on } \Sigma,$$

$$P = 0 \quad \text{on } \Sigma,$$

and to conclude on the existence point in Theorem 7.2, it only remains to check that the "solid wall condition"

(7.13)
$$v \cdot n = 0, \text{ on } \Gamma = \Gamma_1 \cup \Gamma_2$$

is satisfied. On Γ_1 it is a straightforward consequence of the condition $\widetilde{v} \cdot \widetilde{\Gamma} = 0$, while on Γ_2 it is simply consequence of the fact that the component of the velocity field along x_1, $\widetilde{v}_{x_1}$ is odd and 2-periodic. To prove the uniqueness part in Theorem 7.2, starting from a a solution of (7.12), (7.13), on the time interval $[-T, T]$, if we define the function $\widetilde{v}, \widetilde{\eta}$ at each time t following the same procedure, we end up with a solution of (7.2), (7.3) in the domain $\{(t, x, y); t \in (-T, T), (x, y) \in \widetilde{\Omega}(t)\}$, at the same level of regularity. Indeed, the jump formula gives

$$\partial_t \widetilde{v} + \widetilde{v} \cdot \nabla_{x,y} \widetilde{v} + \nabla_{x,y} \widetilde{P} = -g e_y + [v_{x_1} \cdot \partial_{x_1} v] \otimes \delta_{\Gamma_2} = -g e_y,$$

where in the last equality we used that the component of the velocity field along x_1 vanishes on Γ_2. The uniqueness part in Theorem 7.2 consequently follows from the uniqueness part in Theorem 2.3. The case of a rectangular basin is similar.

8. Appendix

Let $\alpha \in]0, +\infty[, , \alpha \neq 1$ and $S(t) = e^{-it|D_x|^\alpha}$. Our aim is to prove the following result.

Proposition 8.1. *Let $s, \sigma \in \mathbf{R}$. Assume that there exists $t_0 \neq 0$ such that $S(t_0)$ is continuous from $C_*^\sigma(\mathbf{R}^d)$ to $C_*^s(\mathbf{R}^d)$. Then $s \leq \sigma - \frac{d\alpha}{2}$.*

PROOF. Without loss of generality we can assume that $t_0 = -1$. Our hypothesis reads

$$(8.1) \qquad \exists C > 0 : \|S(-1)u\|_{C^s_*(\mathbf{R}^d)} \leq C\|u\|_{C^\sigma_*(\mathbf{R}^d)}, \quad \forall u \in C^\sigma_*(\mathbf{R}^d).$$

Now if $u \in L^\infty(\mathbf{R}^d)$ we set $\widehat{\Delta_j u}(\xi) = \varphi(2^{-j}\xi)\widehat{u}(\xi)$, where $\varphi \in C_0^\infty(\mathbf{R}^d)$, with supp $\varphi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Then for fixed $j \in \mathbf{N}$ we have $\Delta_j u \in C^\sigma_*(\mathbf{R}^d)$ and

$$\|\Delta_j u\|_{C^\sigma_*(\mathbf{R}^d)} \leq C 2^{j\sigma}\|\Delta_j u\|_{L^\infty(\mathbf{R}^d)} \leq C' 2^{j\sigma}\|u\|_{L^\infty(\mathbf{R}^d)}.$$

This follows from the fact that $\|\Delta_j u\|_{C^\sigma_*(\mathbf{R}^d)} = \sup_{k\in\mathbf{N}} 2^{k\sigma}\|\Delta_k\Delta_j u\|_{L^\infty(\mathbf{R}^d)}$ and $\Delta_k\Delta_j = 0$ if $|j - k| \geq 2$. Since Δ_j commutes with $S(-1)$, we see that

$$2^{js}\|S(-1)\Delta_j\Delta_j u\|_{L^\infty(\mathbf{R}^d)} \leq \|S(-1)\Delta_j u\|_{C^s_*(\mathbf{R}^d)}.$$

It follows from (8.1) applied to $\Delta_j u$ with $u \in L^\infty(\mathbf{R}^d)$ that one can find a positive constant C such that

$$(8.2) \qquad 2^{js}\|S(-1)\Delta_j\Delta_j u\|_{L^\infty(\mathbf{R}^d)} \leq C 2^{j\sigma}\|u\|_{L^\infty(\mathbf{R}^d)} \quad \forall u \in L^\infty(\mathbf{R}^d), \quad \forall j \in \mathbf{N}.$$

Let us set $T = S(-1)\Delta_j\Delta_j$. Then

$$Tu(x) = (2\pi)^{-d} \iint e^{i[(x-y)\cdot\xi + |\xi|^\alpha]}\varphi^2(2^{-j}\xi)u(y)\,dy\,d\xi.$$

We shall set $h = 2^{-j}$ and take j large enough. Then setting $\eta = h\xi$ we obtain

$$Tu(x) = \int_{\mathbf{R}^d} K_h(x-y)u(y)\,dy$$

where

$$K_h(z) = (2\pi h)^{-d} \int_{\mathbf{R}^d} e^{\frac{i}{h}(z\cdot\eta + h^{1-\alpha}|\eta|^{\frac{1}{2}})}\varphi^2(\eta)\,d\eta.$$

We shall use the following well known lemma.

Lemma 8.2. *Let $K \in C^0(\mathbf{R}^d \times \mathbf{R}^d)$ be such that $\sup_{x\in\mathbf{R}^d} \int |K(x,y)|\,dy < +\infty$. Then the operator T defined by $Tu(x) = \int K(x,y)u(y)\,dy$ is continuous from $L^\infty(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}^d)$ and $\|T\|_{L^\infty\to L^\infty} = \sup_{x\in\mathbf{R}^d} \int |K(x,y)|\,dy$.*

It follows from this lemma that in our case we have

$$\|T\|_{L^\infty\to L^\infty} = \int_{\mathbf{R}^d} |K_h(z)|\,dz.$$

Setting $z = h^{1-\alpha}s$ and $\widetilde{K}_h(s) = K_h(h^{1-\alpha}s)$ we find that

$$(8.3) \qquad \|T\|_{L^\infty\to L^\infty} = h^{d(1-\alpha)} \int_{\mathbf{R}^d} |\widetilde{K}_h(s)|\,ds$$

with

$$(8.4) \qquad \widetilde{K}_h(s) = (2\pi h)^{-d} \int_{\mathbf{R}^d} e^{ih^{-\alpha}\phi(s,\eta)}\varphi^2(\eta)\,d\eta, \quad \phi(s,\eta) = s\cdot\eta + |\eta|^\alpha.$$

Recall that supp $\varphi \subset \{\eta : \frac{1}{2} \leq |\eta| \leq 2\}$. We have $\frac{\partial\phi}{\partial\eta} = s + \alpha\frac{\eta}{|\eta|^{2-\alpha}}$.

Case 1: $|s| \leq \frac{1}{2} \frac{\alpha}{2^{|1-\alpha|}}$. Here, on the support of φ, we have

$$\left| \frac{\partial \phi}{\partial \eta}(s, \eta) \right| \geq \frac{\alpha}{|\eta|^{1-\alpha}} - |s| \geq \frac{1}{2} \frac{\alpha}{2^{|1-\alpha|}}.$$

Therefore integrating by parts in the right hand side of (8.4) using the vector field $L = \frac{h^\alpha}{i} \frac{1}{|\partial_\eta \phi|^2} \sum_{k=1}^d \frac{\partial \phi}{\partial \eta_k} \frac{\partial}{\partial \eta_k}$ we obtain

$$(8.5) \qquad\qquad |\widetilde{K}_h(s)| \leq C_N h^N, \quad \forall N \in \mathbf{N}.$$

Case 2: $|s| \geq 2^{1+|\alpha-1|}\alpha$. On the support of φ we have

$$\left| \frac{\partial \phi}{\partial \eta}(s, \eta) \right| \geq |s| - \frac{\alpha}{|\eta|^{1-\alpha}} \geq 2^{|\alpha-1|}\alpha.$$

Then using the same vector field as in the first case and noticing that $\partial_\eta^\alpha \phi$ is independent of s when $|\alpha| \geq 2$ we obtain

$$(8.6) \qquad\qquad |\widetilde{K}_h(s)| \leq C_N |s|^{-N} h^N, \quad \forall N \in \mathbf{N}.$$

Case 3: $\frac{1}{2} \frac{\alpha}{2^{|1-\alpha|}} \leq |s| \leq 2^{1+|\alpha-1|}\alpha$. Here the function ϕ has a critical point given by $\frac{\eta}{|\eta|^{2-\alpha}} = -\frac{s}{\alpha}$. It follows that $\frac{1}{|\eta|^{1-\alpha}} = \frac{|s|}{\alpha}$, which implies that $\eta_c = c_\alpha s |s|^{\frac{2-\alpha}{\alpha-1}}$. Moreover we have

$$\frac{\partial^2 \phi}{\partial \eta_j \partial \eta_k} = \alpha |\eta|^{\alpha-2} m_{jk} \quad m_{jk} = \delta_{jk} - (\alpha-2)\omega_j \omega_k, \quad \omega = \frac{\eta}{|\eta|}.$$

Since $\det(m_{jk}) = c_0 \neq 0$ we obtain $\left(\left| \det \left(\frac{\partial^2 \phi}{\partial \eta_j \partial \eta_k}(s, \eta_c) \right) \right| \right)^{\frac{1}{2}} = c_{\alpha,d} |s|^{\frac{(\alpha-2)d}{2(\alpha-1)}}$. The stationary phase formula implies that there exists $C_d > 0$ such that

$$(8.7) \qquad\qquad \widetilde{K}_h(s) = C_{\alpha,d} h^{-d} h^{\frac{\alpha d}{2}} \left\{ \frac{e^{i\phi(s,\eta_c)}}{|s|^{\frac{(\alpha-2)d}{2(\alpha-1)}}} \varphi^2(\eta_c) + \mathcal{O}(h^\alpha) \right\}.$$

Using (8.3), (8.5), (8.6), (8.7) we can conclude that

$$\|T\|_{L^\infty \to L^\infty} \geq C h^{d(1-\alpha)} h^{-d} h^{\frac{d\alpha}{2}} - C_N h^N \geq C' h^{-\frac{d\alpha}{2}}.$$

Recalling that $h = 2^{-j}$ we obtain

$$\|T\|_{L^\infty \to L^\infty} \geq C 2^{j\frac{d\alpha}{2}}.$$

Thus for any $\varepsilon > 0$ one can find $u_0 \in L^\infty(\mathbf{R}^d)$, non identically zero, such that

$$\|Tu_0\|_{L^\infty(\mathbf{R}^d)} \geq (C 2^{j\frac{d\alpha}{2}} - \varepsilon) \|u_0\|_{L^\infty(\mathbf{R}^d)}.$$

Taking ε small and using (8.2) with u_0 we obtain

$$C_d'' 2^{j(s+\frac{d\alpha}{2})} \|u_0\|_{L^\infty(\mathbf{R}^d)} \leq C 2^{j\sigma} \|u_0\|_{L^\infty(\mathbf{R}^d)}$$

for all $j \geq j_0$ large enough which proves the Proposition. $\qquad\square$

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