On the Cauchy problem for gravity water waves

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ABSTRACT. We are interested in the system of gravity water waves equations without surface tension. Our purpose is to study the optimal regularity thresholds for the initial conditions. In terms of Sobolev embeddings, the initial surfaces we consider turn out to be only of $C^{3/2}$ class and consequently have unbounded curvature, while the initial velocities are only Lipschitz. We reduce the system using a paradifferential approach.

1. Introduction

We are interested in this work in the study of the Cauchy problem for the water waves system in arbitrary dimension, without surface tension.

An important question in the theory is the possible emergence of singularities (see [**15, 16, 24, 55, 19**]) and as emphasized by Craig and Wayne [**29**], it is important to decide whether some physical or geometric quantities control the equation. In terms of the velocity field, a natural criterium (in view of Cauchy-Lipschitz theorem) is given by the Lipschitz regularity threshold. Indeed, this is necessary for the "fluid particles" motion (i.e. the integral curves of the velocity field) to be well-defined.

In terms of the free boundary, there is no such natural criterium. In fact, the systematic use of the Lagrangian formulation in most previous works [**8, 51, 52**], and the intensive use of Riemannian geometry tools (parallel transport, vector fields,...) by Shatah-Zeng [**46, 47, 48**], Christodoulou–Lindblad [**20**] or Lindblad [**39**] seem to at least require *bounded curvature assumptions* (see also [**23**] where a logarithmic divergence is allowed). In this direction, the beautiful work by Christodoulou–Lindblad [**20**], gives *a priori* bounds as long as the second fundamental form of the free surface is bounded, and the first-order derivatives of the velocity are bounded. This could lead to the natural conjecture that the regularity threshold for the water waves system is indeed given by Christodoulou–Lindblad's result and that the domain has to be assumed to be essentially C^2. Our main contribution in this work is that this is *not* the case and that the relevant threshold is actually only the Lipschitz regularity of the velocity field. Indeed (see Theorem 1.2), our local existence result involves assumptions which, in view of Sobolev embeddings, require only (in terms of Hölder regularity) the initial free domain to be $C^{3/2}$.

As an illustration of the relevance of the analysis of low regularity solutions in a domain with a rough boundary, let us mention that in a forthcoming paper, we shall give an application of our analysis to the local Cauchy theory of three-dimensional

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gravity water waves in a canal. This question goes back to the work by Boussinesq at the beginning of the 20^{th} century (see [14]).

Our analysis require the introduction of new techniques and new tools. In [1, 2] we started a para-differential study of the water waves system in the presence of surface tension and were able to prove that the equations can be reduced to a simple form

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = f, \tag{1.1}$$

where T_V is a para-product and T_γ is a para-differential operator of order $3/2$. Here the main step in the proof is to perform the same task without surface tension, with T_γ of order $1/2$. It has to be noticed however that performing our reduction is considerably more difficult here than in our previous papers ([1, 2]). Indeed, in the case with non vanishing surface tension, the natural regularity threshold forces the velocity field to be Lipschitz while the domain is actually much smoother ($C^{5/2}$). In the present work, the velocity field is also Lipschitz, but the domain is merely $C^{3/2}$. To overcome these difficulties, we had to give a micro-local description (and contraction estimates) of the Dirichlet-Neumann operator which is non trivial in the whole range of C^s domains, $s > 1$ (see the work by Dahlberg-Kenig [30] and Craig-Schanz-Sulem [27] for results on the Dirichlet-Neumann operator in Lipschitz domains). We think that this analysis is of independent interest.

Finally, let us mention that, as we proceed by energy estimates, our results are proved in L^2-based Sobolev spaces and our initial data (η, V) which describe respectively the initial domain as the graph of the function η and the trace of the initial velocity on the free surface, are assumed to be in $H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d), s > 1 + \frac{d}{2}$. The gravity water waves system enjoys a scaling invariance for which the critical threshold is $s_c = \frac{1}{2} + \frac{d}{2}$ (in other terms our well-posedness result is $1/2$ above the scaling critical index).

1.1. Assumptions on the domain.

Hereafter, $d \geq 1$, t denotes the time variable and $x \in \mathbf{R}^d$ and $y \in \mathbf{R}$ denote the horizontal and vertical spatial variables. We work in a time-dependent fluid domain Ω located underneath a free surface Σ and moving in a fixed container denoted by $\mathcal{O}$. This fluid domain

$$\Omega = \{ (t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{R} \,:\, (x,y) \in \Omega(t) \},$$

is such that, for each time t, one has

$$\Omega(t) = \{(x,y) \in \mathcal{O} \,:\, y < \eta(t,x)\},$$

where η is an unknown function and $\mathcal{O}$ is a given open domain which contains a fixed strip around the free surface

$$\Sigma = \{(t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{R} \,:\, y = \eta(t,x)\}.$$

This implies that there exists $h > 0$ such that, for all $t \in [0,T]$,

$$\Omega_h(t) := \Big\{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, \eta(t,x) - h < y < \eta(t,x)\Big\} \subset \Omega(t). \tag{1.2}$$

We also assume that the domain $\mathcal{O}$ (and hence the domain $\Omega(t)$) is connected.

Remark 1.1.
(i) Two classical examples are given by $\mathcal{O} = \mathbf{R}^d \times \mathbf{R}$ (infinite depth case) or $\mathcal{O} = \mathbf{R}^d \times [-1, +\infty)$ (flat bottom). Notice that, in the following, no regularity assumption is made on the bottom $\Gamma := \partial\mathcal{O}$.
(ii) Notice that Γ does not depend on time. However, our method applies in the case where the bottom is time dependent (with the additional assumption in this case that the bottom is Lipschitz).

1.2. The equations. Below we use the following notations

$$\nabla=(\partial_{x_i})_{1\le i\le d},\quad \nabla_{x,y}=(\nabla,\partial_y),\quad \Delta=\sum\nolimits_{1\le i\le d}\partial_{x_i}^2,\quad \Delta_{x,y}=\Delta+\partial_y^2.$$

We consider an incompressible inviscid liquid, having unit density. The equations by which the motion is to be determined are well known. Firstly, the eulerian velocity field $v\colon\Omega\to\mathbf{R}^{d+1}$ solves the incompressible Euler equation

$$\partial_t v+v\cdot\nabla_{x,y}v+\nabla_{x,y}P=-ge_y,\quad \operatorname{div}_{x,y}v=0\quad\text{in }\Omega,\tag{1.3}$$

where $-ge_y$ is the acceleration of gravity ($g>0$) and where the pressure term P can be recovered from the velocity by solving an elliptic equation. The problem is then given by three boundary conditions. They are

$$\begin{cases} v\cdot n=0 & \text{on }\Gamma,\\ \partial_t\eta=\sqrt{1+|\nabla\eta|^2}\,v\cdot\nu & \text{on }\Sigma,\\ P=0 & \text{on }\Sigma,\end{cases}\tag{1.4}$$

where n and ν are the exterior unit normals to the bottom Γ and the free surface $\Sigma(t)$. The first condition in (1.4) expresses the fact that the particles in contact with the rigid bottom remain in contact with it. Notice that to fully make sense, this condition requires some smoothness on Γ, but in general, it has a weak variational meaning (see Section 3). The second condition in (1.4) states that the free surface moves with the fluid and the last condition is a balance of forces across the free surface. Notice that the pressure at the upper surface of the fluid may be indeed supposed to be zero, provided we afterwards add the atmospheric pressure to the pressure so determined. The fluid motion is supposed to be irrotational. The velocity field is therefore given by $v=\nabla_{x,y}\phi$ for some potential $\phi\colon\Omega\to\mathbf{R}$ satisfying

$$\Delta_{x,y}\phi=0\quad\text{in }\Omega,\qquad \partial_n\phi=0\quad\text{on }\Gamma.$$

Using the Bernoulli integral of the dynamical equations to express the pressure, the condition $P=0$ on the free surface implies that

$$\begin{cases} \partial_t\eta=\partial_y\phi-\nabla\eta\cdot\nabla\phi & \text{on }\Sigma,\\ \partial_t\phi+\dfrac{1}{2}\left|\nabla_{x,y}\phi\right|^2+gy=0 & \text{on }\Sigma,\\ \partial_n\phi=0 & \text{on }\Gamma,\end{cases}\tag{1.5}$$

where recall that $\nabla=\nabla_x$. Many results have been obtained on the Cauchy theory for System (1.5), starting from the pioneering works of Nalimov [43], Shinbrot [49], Yoshihara [56], Craig [25]. In the framework of Sobolev spaces and without smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Wu for the case without surface tension (see [51, 52]) and by Beyer-Günther in [12] in the case with surface tension. Several extensions of their results have been obtained by different methods (see [22, 31, 32, 33, 35, 40, 53, 54, 58] for recent results and the surveys [11, 29, 36] for more references). Here we shall use the Eulerian formulation. Following Zakharov [57] and Craig–Sulem [28], we reduce the analysis to a system on the free surface $\Sigma(t)=\{y=\eta(t,x)\}$. If ψ is defined by

$$\psi(t,x)=\phi(t,x,\eta(t,x)),$$

then ϕ is the unique variational solution of

$$\Delta_{x,y}\phi=0\text{ in }\Omega,\quad \phi|_{y=\eta}=\psi,\quad \partial_n\phi=0\text{ on }\Gamma.$$

Define the Dirichlet-Neumann operator by

$$\begin{aligned}(G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta(t,x)}\\ &= (\partial_y\phi)(t,x,\eta(t,x)) - \nabla\eta(t,x)\cdot(\nabla\phi)(t,x,\eta(t,x)).\end{aligned}$$

For the case with a rough bottom, we recall the precise construction later on (see §3.1). Now (η,ψ) solves (see [28] or [36, chapter 1] for instance)

$$\left\{\begin{aligned}&\partial_t\eta - G(\eta)\psi = 0,\\ &\partial_t\psi + g\eta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{\big(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\big)^2}{1+|\nabla\eta|^2} = 0.\end{aligned}\right. \tag{1.6}$$

1.3. The Taylor condition. Introduce the so-called Taylor coefficient

$$a(t,x) = -(\partial_y P)(t,x,\eta(t,x)). \tag{1.7}$$

The stability of the waves is dictated by the Taylor sign condition, which is the assumption that there exists a positive constant c such that

$$a(t,x) \geq c > 0. \tag{1.8}$$

This assumption is now classical and we refer to [11, 20, 21, 37, 51, 52] for various comments. Here we only recall some basic facts. First of all, as proved by Wu ([51, 52]), this assumption is automatically satisfied in the infinite depth case (that is when $\Gamma = \emptyset$) or for flat bottoms (when $\Gamma = \{y = -k\}$). Notice that the proof remains valid for any $C^{1,\alpha}$-domain, $0 < \alpha < 1$ (by using the fact that the Hopf Lemma is true for such domains, see [44] and the references therein). There are two other cases where this assumption is known to be satisfied. For instance under a smallness assumption. Indeed, if $\partial_t\phi = O(\varepsilon^2)$ and $\nabla_{x,y}\phi = O(\varepsilon)$ then directly from the definition of the pressure we have $P + gy = O(\varepsilon^2)$. Secondly, it was proved by Lannes ([37]) that the Taylor's assumption is satisfied under a smallness assumption on the curvature of the bottom (provided that the bottom is at least C^2). However, for general bottom we will assume that (1.8) is satisfied at time $t = 0$.

1.4. Main result. We work below with the vertical and horizontal traces of the velocity on the free boundary, namely

$$B := (\partial_y\phi)|_{y=\eta}, \quad V := (\nabla_x\phi)|_{y=\eta}.$$

These can be defined only in terms of η and ψ by means of the formulas

$$B = \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \qquad V = \nabla\psi - B\nabla\eta. \tag{1.9}$$

Also, recall that the Taylor coefficient a defined in (1.7) can be defined in terms of η, V, B, ψ only (see Section 1.5 below).

Theorem 1.2. *Let $d \geq 1$, $s > 1 + d/2$ and consider (η_0,ψ_0) such that*

(1) $\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $\quad \psi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $\quad V_0 \in H^s(\mathbf{R}^d)$, $\quad B_0 \in H^s(\mathbf{R}^d)$,
(2) there exists $h > 0$ such that condition (1.2) holds initially for $t = 0$,
(3) there exists a positive constant c such that, for all $x \in \mathbf{R}^d$, $a_0(x) \geq c$.

Then there exists $T > 0$ such that the Cauchy problem for (1.6) with initial data (η_0,ψ_0) has a unique solution $(\eta,\psi) \in C^0\big([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d)\big)$, such that

(1) we have $(V,B) \in C^0\big([0,T]; H^s(\mathbf{R}^d)\times H^s(\mathbf{R}^d)\big)$,
(2) the condition (1.2) holds for $0 \leq t \leq T$, with h replaced by $h/2$,
(3) for all $0 \leq t \leq T$ and for all $x \in \mathbf{R}^d$, $a(t,x) \geq c/2$.

Remark 1.3. The main novelty is that, in view of Sobolev embeddings, the initial surfaces we consider turn out to be only of $C^{3/2}$ class and consequently have unbounded curvature.

Remark 1.4. Assumption 1 in the above theorem is automatically satisfied if

$$\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad \psi_0 \in H^{\frac{1}{2}}(\mathbf{R}^d), \quad V_0 \in H^s(\mathbf{R}^d), \quad B_0 \in H^{\frac{1}{2}}(\mathbf{R}^d).$$

The only point where the estimates depend on ψ (and not only on η, V, B) come from the fact that we consider a general domain without assumption on the bottom. Otherwise, we shall prove *a priori* estimates for the fluid velocity and not for the fluid potential (notice that the fluid potential is defined up to a constant).

1.5. The pressure. The purpose of this paragraph is to clarify, for low regularity solutions of the water waves system in rough domains, the definition of the pressure which is required if one wants to come back from solutions to the Zakharov system to solutions to the free boundary Euler equation. This definition will also provide the basic *a priori* estimates which will be later the starting point when establishing higher order elliptic regularity estimates required when studying the Taylor coefficient $a = -\partial_y P \mid_\Sigma$. On a physics point of view, the pressure is the Lagrange multiplier which is required by the incompressibility of the fluid (preservation of the null divergence condition). As a consequence, taking the divergence in (1.3), it is natural to define the pressure as a solution of

$$\Delta_{x,y} P = -\operatorname{div}_{x,y}(v \cdot \nabla_{x,y} v), \qquad P \mid_{y=\eta} = 0. \tag{1.10}$$

Notice however that the solution of such problem may not be unique as can be seen in the simple case when $\Omega = (-\infty, 0) \times \mathbf{R}^d$. Indeed, if P is a solution, then $P + cy$ is another. Notice also that if P satisfies (1.10), then

$$\Delta_{x,y}\Big(P + gy + \frac{1}{2}|v|^2\Big) = 0.$$

Definition 1.5. *Let $(\eta, \psi) \in (W^{1,\infty} \cap H^{1/2}(\mathbf{R}^d)) \times H^{1/2}(\mathbf{R}^d)$. Assume that the variational solution (as defined in §3.1) of the equation*

$$\Delta_{x,y}\phi = 0, \quad \phi \mid_{y=\eta} = \psi, \tag{1.11}$$

satisfies

$$|\nabla_{x,y}\phi|^2(x, \eta(x)) \in H^{1/2}(\mathbf{R}^d).$$

Let R be the variational solution of

$$\Delta_{x,y} R = 0 \text{ in } \Omega, \qquad R \mid_{y=\eta} = g\eta + \frac{1}{2}|\nabla_{x,y}\phi|^2 \mid_{y=\eta}.$$

We define the pressure P in the domain Ω by

$$P(x,y) := R(x,y) - gy - \frac{1}{2}|\nabla_{x,y}\phi(x,y)|^2.$$

Remark 1.6. The main advantage of defining the pressure as the solution of a variational problem is that it will satisfy automatically an *a priori* estimate (the estimate given by the variational theory).

It remains to link the solutions to the Zakharov system to solutions of the free boundary Euler system (1.3) with boundary conditions (1.4). To do so, we proved in [3] that if (η, ψ) is a solution of the Zakharov system, if we consider the variational solution to (1.11), then the velocity field $v = \nabla\phi$ satisfies (1.3), which is of course equivalent to

$$P = -\partial_t \phi - gy - \frac{1}{2}|\nabla_{x,y}\phi|^2. \tag{1.12}$$

Theorem 1.7 (from [3]). *Assume that $(\eta,\psi)\in C([0,T];H^{s+\frac12}(\mathbf{R}^d)\times H^{s+\frac12}(\mathbf{R}^d))$, with $s>1+d/2$, is a solution of the Zakharov/Craig-Sulem system (1.6). Then the assumptions required to define the pressure are satisfied, and (1.12) is satisfied, and the distribution $\partial_t\phi$ is well defined for fixed t and belongs to the space $H^{1,0}(\Omega(t))$ (see Definition 3.3).*

1.6. Plan of the paper. At first glance, Theorem 1.2 looks very similar to our previous result in presence of surface tension [1, Theorem 1.1]. Indeed, the regularity threshold exhibited by the velocity field (namely $V,B\in H^s(\mathbf{R}^d), s>1+d/2$) is the same in both results and (as explained above) appears to be the natural one. However, an important difference between both cases is that the algebraic nature of (1.6) (and its counter-part in presence of surface tension) requires that the free domain is $3/2$ smoother than the velocity field in presence of surface tension and only $1/2$ smoother without surface tension. This algebraic rigidity of the system implies that in order to lower the regularity threshold to the natural one (Lipschitz velocities), we are forced to work with $C^{3/2}$ domains (compared to the much smoother $C^{5/2}$ regularity in [1]). This in turn poses new challenging questions in the study of the Dirichlet–Neumann operator. Indeed, at this level of regularity the regularity of the remainder term in the paradifferential description of the Dirichlet-Neumann operator $G(\eta)\psi$ is not given by the regularity of the function ψ itself, but rather by the regularity of the domain. This is this phenomenon which forces us to work with the new unknowns V,B rather than with ψ.

In Section 2, we wrote a review of paradifferential calculus and proved various technical results useful in the article. In Section 3 we study the Dirichlet-Neumann operator. In Section 4, we symmetrize the system and prove *a priori* estimates. In Section 5 we prove the contraction estimates required to show uniqueness and stability of solutions. In particular we prove a contraction estimate for the difference of two Dirichlet Neumann operators, involving only the $C^{\frac12}$ norm of the difference of the functions defining the domains (see Theorem 5.2), while in Section 6 we prove the existence of solutions by a regularization process.

2. Paradifferential calculus

Let us review notations and results about Bony's paradifferential calculus. We refer to [13, 34, 41, 42] for the general theory. Here we follow the presentation by Métivier in [41].

2.1. Paradifferential operators. For $k\in\mathbf{N}$, we denote by $W^{k,\infty}(\mathbf{R}^d)$ the usual Sobolev spaces. For $\rho=k+\sigma$, $k\in\mathbf{N},\sigma\in(0,1)$ denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the space of functions whose derivatives up to order k are bounded and uniformly Hölder continuous with exponent σ.

Definition 2.1. *Given $\rho\in[0,1]$ and $m\in\mathbf{R}$, $\Gamma^m_\rho(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbf{R}^d\times(\mathbf{R}^d\setminus 0)$, which are C^∞ with respect to ξ for $\xi\neq0$ and such that, for all $\alpha\in\mathbf{N}^d$ and all $\xi\neq 0$, the function $x\mapsto\partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbf{R}^d)$ and there exists a constant C_α such that,*

$$\forall\,|\xi|\geq\frac12,\quad \left\lVert\partial_\xi^\alpha a(\cdot,\xi)\right\rVert_{W^{\rho,\infty}(\mathbf{R}^d)}\leq C_\alpha(1+|\xi|)^{m-|\alpha|}.$$

Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \psi(\eta) \widehat{u}(\eta)\, d\eta, \tag{2.1}$$

where $\widehat{a}(\theta, \xi) = \int e^{-ix\cdot\theta} a(x, \xi)\, dx$ is the Fourier transform of a with respect to the first variable; χ and ψ are two fixed C^∞ functions such that:

$$\psi(\eta) = 0 \quad \text{for } |\eta| \leq 1, \qquad \psi(\eta) = 1 \quad \text{for } |\eta| \geq 2, \tag{2.2}$$

and $\chi(\theta, \eta)$ satisfies, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta, \eta) = 1 \quad \text{if} \quad |\theta| \leq \varepsilon_1 |\eta|, \qquad \chi(\theta, \eta) = 0 \quad \text{if} \quad |\theta| \geq \varepsilon_2 |\eta|,$$

and such that

$$\forall (\theta, \eta) : \qquad \left| \partial_\theta^\alpha \partial_\eta^\beta \chi(\theta, \eta) \right| \leq C_{\alpha,\beta} (1 + |\eta|)^{-|\alpha| - |\beta|}.$$

The function χ can be constructed as follows. Let $\kappa \in C_0^\infty(\mathbf{R}^d)$ be such that

$$\kappa(\theta) = 1 \quad \text{for } |\theta| \leq 1.1, \qquad \kappa(\theta) = 0 \quad \text{for } |\theta| \geq 1.9.$$

Then we define $\chi(\theta, \eta) = \sum_{k=0}^{+\infty} \kappa_{k-3}(\theta) \varphi_k(\eta)$, where

$$\kappa_k(\theta) = \kappa(2^{-k}\theta) \quad \text{for } k \in \mathbf{Z}, \qquad \varphi_0 = \kappa_0, \qquad \text{and} \qquad \varphi_k = \kappa_k - \kappa_{k-1} \quad \text{for } k \geq 1.$$

2.2. Symbolic calculus. We shall use quantitative results from [41] about operator norms estimates in symbolic calculus. Introduce the following semi-norms.

Definition 2.2. *For $m \in \mathbf{R}$, $\rho \in [0, 1]$ and $a \in \Gamma_\rho^m(\mathbf{R}^d)$, we set*

$$M_\rho^m(a) = \sup_{|\alpha| \leq \frac{3d}{2} + 1 + \rho} \ \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(\cdot, \xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)}. \tag{2.3}$$

Definition 2.3 (Zygmund spaces)**.** *Consider a dyadic decomposition of the identity: $I = \Delta_{-1} + \sum_{q=0}^\infty \Delta_q$. If s is any real number, we define the Zygmund class $C_*^s(\mathbf{R}^d)$ as the space of tempered distributions u such that*

$$\|u\|_{C_*^s} := \sup_q 2^{qs} \|\Delta_q u\|_{L^\infty} < +\infty.$$

Remark 2.4. Recall that $C_*^s(\mathbf{R}^d)$ is the Hölder space $W^{s,\infty}(\mathbf{R}^d)$ if $s \in (0, +\infty) \setminus \mathbf{N}$.

Definition 2.5. *Let $m \in \mathbf{R}$. An operator T is said to be of order m if, for all $\mu \in \mathbf{R}$, it is bounded from H^μ to $H^{\mu - m}$.*

The main features of symbolic calculus for paradifferential operators are given by the following theorem.

Theorem 2.6. *Let $m \in \mathbf{R}$ and $\rho \in [0, 1]$.*

(i) If $a \in \Gamma_0^m(\mathbf{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_a\|_{H^\mu \to H^{\mu - m}} \leq K M_0^m(a). \tag{2.4}$$

(ii) If $a \in \Gamma_\rho^m(\mathbf{R}^d), b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$ then $T_a T_b - T_{ab}$ is of order $m + m' - \rho$. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_a T_b - T_{ab}\|_{H^\mu \to H^{\mu - m - m' + \rho}} \leq K M_\rho^m(a) M_0^{m'}(b) + K M_0^m(a) M_\rho^{m'}(b). \tag{2.5}$$

(iii) Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of T_a and by $\overline{a}$ the complex conjugate of a. Then $(T_a)^* - T_{\overline{a}}$ is of order $m - \rho$. Moreover, for all μ there exists a constant K such that

$$\|(T_a)^* - T_{\overline{a}}\|_{H^\mu \to H^{\mu - m + \rho}} \leq K M_\rho^m(a). \tag{2.6}$$

We shall need in this article to consider paradifferential operators with negative regularity. As a consequence, we need to extend our previous definition.

Definition 2.7. *For $m \in \mathbf{R}$ and $\rho \in (-\infty, 0)$, $\Gamma^m_\rho(\mathbf{R}^d)$ denotes the space of distributions $a(x,\xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $C^\rho_*(\mathbf{R}^d)$ and there exists a constant C_α such that,*

$$\forall |\xi| \geq \frac{1}{2}, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{C^\rho_*} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}. \tag{2.7}$$

For $a \in \Gamma^m_\rho$, we define

$$M^m_\rho(a) = \sup_{|\alpha| \leq \frac{3d}{2}+\rho+1} \ \sup_{|\xi|\geq 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot,\xi) \right\|_{C^\rho_*(\mathbf{R}^d)}. \tag{2.8}$$

2.3. Paraproducts and product rules. If $a = a(x)$ is a function of x only, the paradifferential operator T_a is called a paraproduct. A key feature of paraproducts is that one can replace nonlinear expressions by paradifferential expressions up to smoothing operators. Also, one can define paraproducts T_a for rough functions a which do not belong to $L^\infty(\mathbf{R}^d)$ but merely to $C^{-m}_*(\mathbf{R}^d)$ with $m > 0$.

Definition 2.8. *Given two functions a, b defined on $\mathbf{R}^d$ we define the remainder*

$$R(a,u) = au - T_a u - T_u a.$$

We record here various estimates about paraproducts (see chapter 2 in [10] or [18]).

Theorem 2.9. *i) Let $\alpha, \beta \in \mathbf{R}$. If $\alpha + \beta > 0$ then*

$$\| R(a,u) \|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)} \leq K \| a \|_{H^\alpha(\mathbf{R}^d)} \| u \|_{H^\beta(\mathbf{R}^d)}, \tag{2.9}$$

$$\| R(a,u) \|_{H^{\alpha+\beta}(\mathbf{R}^d)} \leq K \| a \|_{C^\alpha_*(\mathbf{R}^d)} \| u \|_{H^\beta(\mathbf{R}^d)}. \tag{2.10}$$

ii) Let $m > 0$ and $s \in \mathbf{R}$. Then

$$\| T_a u \|_{H^{s-m}} \leq K \| a \|_{C^{-m}_*} \| u \|_{H^s}. \tag{2.11}$$

iii) Let s_0, s_1, s_2 be such that $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$, then

$$\| T_a u \|_{H^{s_0}} \leq K \| a \|_{H^{s_1}} \| u \|_{H^{s_2}}. \tag{2.12}$$

By combining the two previous points with the embedding $H^\mu(\mathbf{R}^d) \subset C^{\mu-d/2}_*(\mathbf{R}^d)$ (for any $\mu \in \mathbf{R}$) we immediately obtain the following results.

Proposition 2.10. *Let $r, \mu \in \mathbf{R}$ be such that $r + \mu > 0$. If $\gamma \in \mathbf{R}$ satisfies*

$$\gamma \leq r \quad \text{and} \quad \gamma < r + \mu - \frac{d}{2},$$

then there exists a constant K such that, for all $a \in H^r(\mathbf{R}^d)$ and all $u \in H^\mu(\mathbf{R}^d)$,

$$\| au - T_a u \|_{H^\gamma} \leq K \| a \|_{H^r} \| u \|_{H^\mu}.$$

Corollary 2.11. *i) If $u_j \in H^{s_j}(\mathbf{R}^d)$ $(j = 1,2)$ with $s_1 + s_2 > 0$ then*

$$\| u_1 u_2 \|_{H^{s_0}} \leq K \| u_1 \|_{H^{s_1}} \| u_2 \|_{H^{s_2}}, \tag{2.13}$$

if $s_0 \leq s_j$, $j = 1,2$, and $s_0 < s_1 + s_2 - d/2$.

ii) (Tame estimate in Sobolev spaces) If $s \geq 0$ then

$$\| u_1 u_2 \|_{H^s} \leq K \big(\| u_1 \|_{H^s} \| u_2 \|_{L^\infty} + \| u_1 \|_{L^\infty} \| u_2 \|_{H^s} \big). \tag{2.14}$$

iii) *Let $\mu, m \in \mathbf{R}$ be such that $\mu, m > 0$ and $m \notin \mathbf{N}$. Then*

$$\|u_1 u_2\|_{H^\mu} \le K\big(\|u_1\|_{L^\infty} \|u_2\|_{H^\mu} + \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}}\big). \tag{2.15}$$

iv) *Let $s > d/2$ and consider $F \in C^\infty(\mathbf{C}^N)$ such that $F(0) = 0$. Then there exists a non-decreasing function $\mathcal{F} \colon \mathbf{R}_+ \to \mathbf{R}_+$ such that, for any $U \in H^s(\mathbf{R}^d)^N$,*

$$\|F(U)\|_{H^s} \le \mathcal{F}\big(\|U\|_{L^\infty}\big) \|U\|_{H^s}. \tag{2.16}$$

Proof. The first two estimates are well-known, see Hörmander [34] or Chemin [18]. To prove *iii*) we write $u_1 u_2 = T_{u_1} u_2 + T_{u_2} u_1 + R(u_1, u_2)$ and use that

$$\begin{aligned} &\|T_{u_1} u_2\|_{H^\mu} \lesssim \|u_1\|_{L^\infty} \|u_2\|_{H^\mu} && \text{(see (2.4))},\\ &\|T_{u_2} u_1\|_{H^\mu} \lesssim \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}} && \text{(see (2.11))},\\ &\|R(u_1, u_2)\|_{H^\mu} \lesssim \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}} && \text{(see (2.10))}. \end{aligned}$$

Finally, *iv*) is due to Meyer [42, Théorème 2.5 and remarque]. □

Finally, let us finish this section with a generalization of (2.11)

Proposition 2.12. *Let $\rho < 0$, $m \in \mathbf{R}$ and $a \in \dot\Gamma^m_\rho$. Then the operator T_a is of order $m - \rho$:*

$$\|T_a\|_{H^s \to H^{s-(m-\rho)}} \le C M^m_\rho(a), \qquad \|T_a\|_{C_*^s \to C_*^{s-(m-\rho)}} \le C M^m_\rho(a). \tag{2.17}$$

Proof. Let us prove the first estimate. The proof of the second is similar. Notice that if $m = 0$ and $a(x,\xi) = a(x)$, then (2.17) is simply (2.11). Furthermore, if $a(x,\xi) = b(x)p(\xi)$, then $T_a = T_b(\theta p)(|D_x|)$, where θ is a cutoff function vanishing near 0 and equal to 1 for $|\xi| \ge 1$. As a consequence, in this particular case, we get

$$\|T_a\|_{H^s \to H^{s-(m-\rho)}} \le C \|b\|_{C_*^\rho} \|p\,|_{\mathbf{S}^{d-1}}\|_{L^\infty}.$$

In the general case, we can expand, for fixed x, $a(x,\xi)$ in terms of spherical harmonics. Let $(\tilde h_\nu)_{\nu \in \mathbf{N}^*}$ be an orthonormal basis of $L^2(\mathbf{S}^{d-1})$ consisting of eigenfunctions of the (self-adjoint) Laplace–Beltrami operator, $\Delta_\omega = \Delta_{\mathbf{S}^{d-1}}$ on $L^2(\mathbf{S}^{d-1})$, *i.e.* $\Delta_\omega \tilde h_\nu = \lambda_\nu^2 \tilde h_\nu$. By the Weyl formula, we know that $\lambda_\nu \sim c\nu^{\frac{1}{d}}$. Setting

$$h_\nu(\xi) = |\xi|^m \, \tilde h_\nu(\omega), \quad \omega = \frac{\xi}{|\xi|}, \ \xi \neq 0,$$

we can write

$$a(x,\xi) = \sum_{\nu \in \mathbf{N}^*} a_\nu(x) h_\nu(\xi) \quad \text{where} \quad a_\nu(x) = \int_{\mathbf{S}^{d-1}} a(x,\omega) \overline{\tilde h_\nu(\omega)}\, d\omega.$$

Since

$$\lambda_\nu^{2k} a_\nu(t,x) = \int_{\mathbf{S}^{d-1}} \Delta_w^k a(x,\omega) \overline{\tilde h_\nu(\omega)}\, d\omega,$$

we have, for all $\nu \ge 1$,

$$\|a_\nu(\cdot)\|_{C_*^\rho} \le C \lambda_\nu^{-\frac{3d}{2}+1} \le \nu^{-\frac{3}{2}-\frac{1}{d}} M^m_\rho(p). \tag{2.18}$$

Moreover,

$$\big\|\tilde h_\nu\big\|_{L^\infty} \le C \lambda_\nu^{\frac{(d-1)}{2}} \le C\nu^{\frac{1}{2}-\frac{1}{2d}}. \tag{2.19}$$

and the result follows because

$$\|T_a\|_{H^s \to H^{s-(m-\rho)}} \le C \sum_\nu \nu^{-1-\frac{1}{2d}} M^m_\rho(p).$$

This completes the proof. □

We shall also need the following technical result.

Proposition 2.13. *Set* $\langle D_x\rangle = (I-\Delta)^{1/2}$.

i) Let $s > \frac{1}{2}+\frac{d}{2}$ *and* $\sigma\in\mathbf{R}$ *be such that* $\sigma\le s$. *Then there exists* $K>0$ *such that for all* $V\in W^{1,\infty}(\mathbf{R}^d)\cap H^s(\mathbf{R}^d)$ *and* $u\in H^{\sigma-\frac{1}{2}}(\mathbf{R}^d)$ *one has*

$$\|[\langle D_x\rangle^\sigma, V]u\|_{L^2(\mathbf{R}^d)} \le K\big\{\|V\|_{W^{1,\infty}(\mathbf{R}^d)}+\|V\|_{H^s(\mathbf{R}^d)}\big\}\|u\|_{H^{\sigma-\frac{1}{2}}(\mathbf{R}^d)}.$$

ii) Let $s>1+\frac{d}{2}$ *and* $\sigma\in\mathbf{R}$ *be such that* $\sigma\le s$. *Then there exists* $K>0$ *such that for all* $V\in H^s(\mathbf{R}^d)$ *and* $u\in H^{\sigma-1}(\mathbf{R}^d)$ *one has*

$$\|[\langle D_x\rangle^\sigma, V]u\|_{L^2(\mathbf{R}^d)}\le K\|V\|_{H^s(\mathbf{R}^d)}\|u\|_{H^{\sigma-1}(\mathbf{R}^d)}.$$

Proof. To prove *i*) we write

$$\|[\langle D_x\rangle^\sigma, V]u\|_{L^2}\le A+B,\qquad A=\|[\langle D_x\rangle^\sigma, T_V]u\|_{L^2},\quad B=\|[\langle D_x\rangle^\sigma, V-T_V]u\|_{L^2}.$$

By (2.5) we have $A\le K\|V\|_{W^{1,\infty}}\|u\|_{H^{\sigma-1}}$. On the other hand one can write

$$B\le \|\langle D_x\rangle^\sigma (V-T_V)u\|_{L^2}+\|(V-T_V)\langle D_x\rangle^\sigma u\|_{L^2}=B_1+B_2.$$

We use Proposition 2.10 two times. To estimate B_1 we take $\gamma=\sigma, r=s, \mu=\sigma-\frac{1}{2}$. To estimate B_2 we take $\gamma=0, r=s, \mu=-\frac{1}{2}$ and we obtain,

$$B\le K\|V\|_{H^s}\|u\|_{H^{\sigma-\frac{1}{2}}}.$$

To prove *ii*), to estimate B_1 (resp. B_2) we use again Proposition 2.10 with $\gamma=\sigma, r=s, \mu=\sigma-1$ (resp. $\gamma=0, r=s, \mu=-1$). □

We shall need well-known estimates on the solutions of transport equations.

Proposition 2.14. *Let* $I=[0,T]$, $s>1+\frac{d}{2}$ *and consider the Cauchy problem*

$$\begin{cases}\partial_t u+V\cdot\nabla u=f, \quad t\in I,\\ u|_{t=0}=u_0.\end{cases}\tag{2.20}$$

There exists a non decreasing function $\mathcal{F}:\mathbf{R}^+\to\mathbf{R}^+$ *such that*

$$\|u(t)\|_{L^2(\mathbf{R}^d)}\le \mathcal{F}\big(\|V\|_{L^1(I;W^{1,\infty}(\mathbf{R}^d))}\big)\big(\|u_0\|_{L^2(\mathbf{R}^d)}+\int_0^t\|f(t',\cdot)\|_{L^2(\mathbf{R}^d)}\,dt'\big).\tag{2.21}$$

and for any $\sigma\in[0,s]$ *there exists a non decreasing function* $\mathcal{F}:\mathbf{R}^+\to\mathbf{R}^+$ *such that*

$$\|u(t)\|_{H^\sigma(\mathbf{R}^d)}\le \mathcal{F}\big(\|V\|_{L^1(I;H^s(\mathbf{R}^d))}\big)\big(\|u_0\|_{H^\sigma(\mathbf{R}^d)}+\int_0^t\|f(t',\cdot)\|_{H^\sigma(\mathbf{R}^d)}\,dt'\big).\tag{2.22}$$

2.4. Commutation with a vector field. We prove in this paragraph a commutator estimate between a paradifferential operator T_p and the convective derivative $\partial_t+V\cdot\nabla$. Inspired by Chemin [17] and Alinhac [6], we prove an estimate which depends on estimates on $\partial_t p+V\cdot\nabla p$ and not on $\nabla_{t,x}p$.

When a and u are symbols and functions depending on $t\in I$, we still denote by $T_a u$ the spatial paradifferential operator (or paraproduct) such that for all $t\in I$, $(T_a u)(t)=T_{a(t)}u(t)$. Given a symbol $a=a(t;x,\xi)$ depending on time, we use the notation

$$\mathcal{M}_0^m(a):=\sup_{t\in[0,T]}\ \sup_{|\alpha|\le\frac{3d}{2}+1+\rho}\ \sup_{|\xi|\ge 1/2}\Big\|(1+|\xi|)^{|\alpha|-m}\partial_\xi^\alpha a(t;\cdot,\xi)\Big\|_{L^\infty(\mathbf{R}^d)}.$$

Given a scalar symbol $p=p(t,x,\xi)$ of order m, it follows directly from the symbolic calculus rules for paradifferential operators (see (2.4) and (2.5)) that,

$$\left\| \left[T_p, \partial_t + T_V\cdot\nabla \right] u \right\|_{H^\mu} \le K \left\{ \mathcal{M}_0^m(\partial_t p) + \mathcal{M}_0^m(\nabla p) \left\| V \right\|_{W^{1,\infty}} \right\} \left\| u \right\|_{H^{\mu+m}}.$$

A technical key point in our analysis is that one can replace this estimate by a tame estimate which does not involve the first order derivatives of p, but instead $\partial_t p + V\cdot\nabla p$.

Lemma 2.15. *Let $V\in C^0([0,T];C^{1+\varepsilon}_*(\mathbf{R}^d))$ for some $\varepsilon>0$ and consider a symbol $p=p(t,x,\xi)$ which is homogeneous in ξ of order m. Then there exists $K>0$ (independent of p,V) such that for any $t\in[0,T]$ and any $u\in C^0([0,T];H^m(\mathbf{R}^d))$.*

$$\begin{aligned}(2.23)\qquad &\left\| \left[T_p, \partial_t + T_V\cdot\nabla \right] u(t) \right\|_{L^2(\mathbf{R}^d)} \\ &\qquad \le K\left\{ \mathcal{M}_0^m(p) \left\| V(t) \right\|_{C^{1+\varepsilon}_*} + \mathcal{M}_0^m(\partial_t p + V\cdot\nabla p) \right\} \left\| u(t) \right\|_{H^m(\mathbf{R}^d)}.\end{aligned}$$

Proof. Set $I=[0,T]$ and denote by $\mathcal{R}$ the set of continuous operators $R(t)$ from $H^m(\mathbf{R}^d)$ to $L^2(\mathbf{R}^d)$ with norm satisfying

$$\left\| R(t) \right\|_{\mathcal{L}(H^m(\mathbf{R}^d),L^2(\mathbf{R}^d))} \le K\left\{ \mathcal{M}_0^m(p) \left\| V(t) \right\|_{C^{1+\varepsilon}_*} + \mathcal{M}_0^m(\partial_t p + V\cdot\nabla p) \right\}.$$

We begin by noticing that it is sufficient to prove that

$$(2.24)\qquad \big(\partial_t + V\cdot\nabla\big) T_p = T_p\big(\partial_t + T_V\cdot\nabla\big) + R, \quad R\in\mathcal{R}.$$

Indeed, by Theorem 5.2.9 in [41], we have (for fixed t)

$$\left\| (V-T_V)\cdot\nabla T_p u \right\|_{L^2} \lesssim \left\| V \right\|_{W^{1,\infty}} \left\| T_p u \right\|_{L^2} \lesssim \left\| V \right\|_{W^{1,\infty}} \mathcal{M}_0^m(p) \left\| u \right\|_{H^m}$$

by using the operator norm estimate (2.4). This implies that $\big(V-T_V\big)\cdot\nabla T_p\in\mathcal{R}$.

We split the proof of (2.24) into three steps. By decomposing p into a sum of spherical harmonic, we shall reduce the analysis to establishing (2.24) for the special case when T_p is a paraproduct. In the first step we prove (2.24) for $m=0$ and $p=p(t,x)$. In the second step we prove (2.24) for $p=a(t,x)h(\xi)$ where h is homogeneous in ξ of order m. Then we consider the general case.

Step 1: Paraproduct, $m=0$, $p=p(t,x)$. In this case $\mathcal{M}_0^0(p)=\left\| p \right\|_{L^\infty}$. We have

$$(2.25)\qquad \begin{cases} \partial_t T_p u = T_{\partial_t p} u + T_p \partial_t u, \\ V\cdot\nabla T_p u = V\cdot T_{\nabla p} u + V T_p\cdot\nabla u =: A+B. \end{cases}$$

Decompose $V=S_{j-3}(V)+S^{j-3}(V)$, with

$$S_{j-3}(V)=\sum_{k\le j-2}\Delta_k V, \qquad S^{j-3}(V)=\sum_{k\ge j-3}\Delta_k V,$$

to obtain

$$(2.26)\qquad \begin{cases} A=A_1+A_2, \\ A_1 := \sum_j S_{j-3}(V) S_{j-3}(\nabla p)\Delta_j u, \qquad A_2 := \sum_j S^{j-3}(V) S_{j-3}(\nabla p)\Delta_j u. \end{cases}$$

Let us consider the term A_2. Since

$$\left\| S^{j-3}(V) \right\|_{L^\infty} \le \sum_{k\ge j-3} \left\| \Delta_k V \right\|_{L^\infty} \lesssim \sum_{k\ge j-3} 2^{-k(1+\varepsilon)} \left\| V \right\|_{C^{1+\varepsilon}_*} \lesssim 2^{-j(1+\varepsilon)} \left\| V \right\|_{C^{1+\varepsilon}_*}$$

and $\left\| S_{j-3}(\nabla p) \right\|_{L^\infty} \lesssim 2^j \left\| p \right\|_{L^\infty}$, we obtain

$$(2.27)\qquad \left\| A_2 \right\|_{L^2} \lesssim \sum_j 2^{-j\varepsilon} \left\| V \right\|_{C^{1+\varepsilon}_*} \left\| p \right\|_{L^\infty} \left\| u \right\|_{L^2} \lesssim \mathcal{M}_0^0(p) \left\| V \right\|_{C^{1+\varepsilon}_*} \left\| u \right\|_{L^2}.$$

We now estimate $A_1 = A_{11} + A_{12}$, with

$$
\left\{
\begin{aligned}
A_{11} &:= \sum_j S_{j-3}\big\{ S_{j-3}(V)\cdot\nabla p\big\}\Delta_j u,\\
A_{12} &:= \sum_j \big\{ \big[S^{j-3}(V), S_{j-3}\big]\nabla p\big\}\Delta_j u.
\end{aligned}
\right.
\tag{2.28}
$$

Write $S_{j-3}(V) = V - S^{j-3}(V)$, to obtain

$$
A_{11} = \sum_j S_{j-3}(V\cdot\nabla p)\Delta_j u - \sum_j S_{j-3}\big\{S^{j-3}(V)\cdot\nabla p\big\}\Delta_j u = T_{V\cdot\nabla p}u + I + II
$$

where

$$
I = -\sum_j \big(\nabla\cdot S_{j-3}\big\{S^{j-3}(V)p\big\}\big)\Delta_j u, \quad II = \sum_j S_{j-3}\big\{S^{j-3}(\nabla\cdot V)p\big\}\Delta_j u.
$$

Then

$$
\begin{aligned}
\left\| I\right\|_{L^2} &\lesssim \sum_j 2^j \left\| S^{j-3}(V)p\right\|_{L^\infty}\left\|\Delta_j u\right\|_{L^2}\\
&\lesssim \sum_j 2^j 2^{-j(1+\varepsilon)}\left\| V\right\|_{C^{1+\varepsilon}_*}\left\| p\right\|_{L^\infty}\left\| u\right\|_{L^2} \lesssim \left\| V\right\|_{C^{1+\varepsilon}_*}\left\| p\right\|_{L^\infty}\left\| u\right\|_{L^2}.
\end{aligned}
$$

Moreover,

$$
\left\| II\right\|_{L^2} \lesssim \sum_j \left\| S^{j-3}(\nabla V)\right\|_{L^\infty}\left\| p\right\|_{L^\infty}\left\|\Delta_j u\right\|_{L^2} \lesssim \left\| V\right\|_{C^{1+\varepsilon}_*}\left\| p\right\|_{L^\infty}\left\| u\right\|_{L^2}.
$$

Therefore

$$
A_{11} = T_{V\cdot\nabla p}u + Ru, \quad R\in\mathcal{R}. \tag{2.29}
$$

In order to estimate A_{12}, note that one can replace ∇p by $\widetilde{S}_{j-3}(\nabla p)$ where $\widetilde{S}_{j-3} = \widetilde{\psi}(2^{-(j-3)}D)$ for some function $\widetilde{\psi}\in C^\infty_0(\mathbf{R}^d)$ such that $\widetilde{\psi}(\xi) = 1$ for $|\xi|\le 2$. Next, observe that

$$
A_{12} = \sum_j \big\{\big[S^{j-3}(V), S_{j-3}\big]\nabla\widetilde{S}_{j-3}(p)\big\}\Delta_j u = \sum_j w_j,
$$

where w_j is spectrally supported in an annulus $\{c_1 2^j\le|\xi|\le c_2 2^j\}$, $c_j>0$. These annuli have only finite overlap, thus by Plancherel we have

$$
\begin{aligned}
\left\| A_{12}\right\|_{L^2}^2 &\lesssim \sum_j \left\| \big\{\big[S^{j-3}(V), S_{j-3}\big]\nabla\widetilde{S}_{j-3}(p)\big\}\Delta_j u\right\|_{L^2}^2\\
&\lesssim \sum_j 2^{-2j}\left\| V\right\|_{C^{1+\varepsilon}_*}^2 2^{2j}\left\| p\right\|_{L^\infty}^2\left\|\Delta_j u\right\|_{L^2}^2 \lesssim \left\| V\right\|_{C^{1+\varepsilon}_*}\left\| p\right\|_{L^\infty}\left\| u\right\|_{L^2},
\end{aligned}
$$

where we used the fact that the commutator $\big[S^{j-3}(V), S_{j-3}\big]$ is of order -1 (uniformly in j), since $V\in C^0([0,T];W^{1,\infty})$. It follows that $A_{12} = Ru$ with $R\in\mathcal{R}$. Consequently, we deduce from (2.28) and (2.29) that $A_1 = T_{V\cdot\nabla p}u + Ru$ for some R in $\mathcal{R}$. It thus follows from (2.26) and (2.27) that $A = T_{V\cdot\nabla p}u + Ru$ with $R\in\mathcal{R}$.

We estimate now the term B introduced in (2.25). We split this term as follows:

$$
\begin{aligned}
B &= V\cdot(T_p\nabla u) = V\cdot\sum_j S_{j-3}(p)\nabla\Delta_j u\\
&= \sum_j S_{j-3}(V)S_{j-3}(p)\Delta_j\nabla u + \sum_j S^{j-3}(V)S_{j-3}(\nabla p)\Delta_j\nabla u =: B_1 + B_2.
\end{aligned}
$$

We have

$$\|B_2\|_{L^2} \le \sum_j \left\|S^{j-3}(V)\right\|_{L^\infty} \|S_{j-3}(p)\|_{L^\infty} \|\Delta_j \nabla u\|_{L^2}$$
$$\lesssim \sum_j 2^{-j(1+\varepsilon)} \|V\|_{C^{1+\varepsilon}_*} 2^j \|p\|_{L^\infty} \|u\|_{L^2} .$$

and hence $B_2 = Ru$ with $R \in \mathcal{R}$. To deal with the term B_1, let us introduce

(2.30) $$C := T_p T_V \cdot \nabla u = \sum_j S_{j-3}(p)\Delta_j \sum_k S_{k-3}(V) \cdot \nabla \Delta_k u.$$

Since the spectrum of $S_{k-3}(V)\cdot\nabla\Delta_k u$ is contained in $\{(3/8)2^k \le |\xi| \le (2+1/8)2^k\}$, the term $\Delta_j(S_{k-3}(V) \cdot \nabla\Delta_k u)$ vanishes unless $|k-j| \le 3$. On the other hand, for $|k-j| \le 3$, $S_{k-3}(V) - S_{j-3}(V) = \pm \sum_{\ell=-N_0}^{N_0} \Delta_{\ell+j} V$, and hence we can write C under the form

$$C = C_1 + C_2 = C_1 + \sum_j S_{j-3}(p)\Delta_j \big\{ S_{j-3}(V) \cdot \sum_{|k-j|\le 3} \nabla\Delta_k u \big\}$$

where C_1 is given by

$$C_1 = \sum_j S_{j-3}(p)\Delta_j \sum_{i=1}^{3} \sum_{\ell=-1}^{i-2} \big\{ \Delta_{\ell+j}(V)\nabla\Delta_{i+j}(u) - \Delta_{\ell+j-i}(V)\nabla\Delta_{j-i}(u) \big\},$$

so that

$$\|C_1\|_{L^2} \lesssim \sum_j \|p\|_{L^\infty} 2^{-j(1+\varepsilon)} \|V\|_{C^{1+\varepsilon}_*} 2^j \|u\|_{L^2} ,$$

which implies that $C_1 = Ru$ with $R \in \mathcal{R}$. To estimate C_2, as before we write $C_2 = C_{21} + C_{22}$ where

$$C_{21} := \sum_j S_{j-3}(p) \big[\Delta_j, S_{j-3}(V)\big] \cdot \sum_{|k-j|\le 3} \nabla\Delta_k u,$$
$$C_{22} := \sum_j S_{j-3}(p) S_{j-3}(V) \cdot \Delta_j \sum_{|k-j|\le 3} \nabla\Delta_k u,$$

where (using frequency localization in dyadic annuli and Plancherel formula)

$$\|C_{21}\|_{L^2}^2 \lesssim \sum_j \|p\|_{L^\infty}^2 2^{-2j} \|V\|_{W^{1,\infty}}^2 2^{2j} \sum_{|k-j|\le 3} \|\Delta_k u\|_{L^2}^2 \lesssim \|p\|_{L^\infty} \|V\|_{C^{1+\varepsilon}_*} \|u\|_{L^2} .$$

On the other hand, since $\Delta_j \sum_{|k-j|\le 3} \Delta_k = \Delta_j$, we have

$$C_{22} = \sum_j S_{j-3}(V) S_{j-3}(p) \nabla\Delta_j u = B_1.$$

We thus end up with

(2.31) $$II = T_p T_V \cdot \nabla u + Ru, \quad R \in \mathcal{R}.$$

It follows from (2.25) and (2.31) that

(2.32) $$(\partial_t + V\cdot\nabla) T_p u = T_p(\partial_t + T_V \cdot \nabla) u + T_{\partial_t p + V\cdot\nabla p} u + Ru, \quad R \in \mathcal{R}.$$

The symbolic calculus shows that $T_{\partial_t p + V\cdot\nabla p} \in \mathcal{R}$, which proves (2.24) and concludes the proof of the first step.

Step 2 : Higher order paraproducts. We now assume that $p(t,x,\xi)=a(t,x)h(\xi)$ where $h(\xi)=|\xi|^m\,\tilde{h}(\xi/|\xi|)$ with $\tilde{h}\in C^\infty(\mathbf{S}^{d-1})$. Then, directly from the definition (2.1), we have $T_p=T_a\psi(D_x)h(D_x)$ where ψ satisfies (2.2). We have

$$\big[T_p,\partial_t+T_V\cdot\nabla\big]=\big[T_a,\partial_t+T_V\cdot\nabla\big]\psi(D_x)h(D_x)+T_a\big[\psi(D_x)h(D_x),\partial_t+T_V\cdot\nabla\big].$$

The norm from H^m to L^2 of the first term in the right-hand side is estimated by means of the previous step by

$$K\|a\|_{L^\infty}\|V\|_{C^{1+\varepsilon}_*}+\|\partial_t a+V\cdot\nabla a\|_{L^\infty}\|V\|_{L^\infty}\,,$$

while the norm of the second term simplifies to $T_a\big[\psi(D_x)h(D_x),T_V\cdot\nabla\big]$ and is easily estimated using (2.4) and (2.5) by

$$\|a\|_{L^\infty}\|V\|_{C^{1+\varepsilon}_*}(\|\tilde{h}\|_{L^\infty}+\|\nabla_\xi h\,|_{\mathbf{S}^{d-1}}\|_{L^\infty}).$$

Step 3 : Paradifferential operators. Consider an orthonormal basis $(\tilde{h}_\nu)_{\nu\in\mathbf{N}^*}$ of $L^2(\mathbf{S}^{d-1})$ consisting of eigenfunctions of the (self-adjoint) Laplace–Beltrami operator, $\Delta_\omega=\Delta_{\mathbf{S}^{d-1}}$ on $L^2(\mathbf{S}^{d-1})$, *i.e.* $\Delta_\omega\tilde{h}_\nu=\lambda_\nu^2\tilde{h}_\nu$. By the Weyl formula, we know that $\lambda_\nu\sim c\nu^{\frac{1}{d}}$. Setting $h_\nu(\xi)=|\xi|^m\,\tilde{h}_\nu(\omega)$, $\omega=\xi/|\xi|$, $\xi\neq 0$, we can write

$$p(t,x,\xi)=\sum_{\nu\in\mathbf{N}^*}a_\nu(t,x)h_\nu(\xi)\quad\text{where}\quad a_\nu(t,x)=\int_{\mathbf{S}^{d-1}}p(t,x,\omega)\overline{\tilde{h}_\nu(\omega)}\,d\omega.$$

Since

$$\lambda_\nu^{2k}a_\nu(t,x)=\int_{\mathbf{S}^{d-1}}\Delta_w^k p(t,x,\omega)\overline{\tilde{h}_\nu(\omega)}\,d\omega,$$

we deduce

$$\sup_{t\in I}\|a_\nu(t)\|_{L^\infty}\le C\lambda_\nu^{-\frac{3d}{2}-1}\mathcal{M}_0^m(p). \tag{2.33}$$

Moreover, there exists a positive constant K such that, for all $\nu\ge 1$,

$$\big\|\tilde{h}_\nu\big\|_{L^\infty}\le C\lambda_\nu^{\frac{d-1}{2}}. \tag{2.34}$$

Now we can write

$$\|[\partial_t+V\cdot\nabla,T_p]\,u\|_{L^2}\le\sum_{\nu\in\mathbf{N}^*}\|[\partial_t+V\cdot\nabla,T_{a_\nu h_\nu}]\,u\|_{L^2}\,.$$

So using the estimates obtained in the previous steps for every $\nu\ge 1$ and the estimates (2.33)–(2.34), we obtain (2.24), since the sum

$$\sum_\nu\lambda_\nu^{\frac{(d-1)}{2}+1}\lambda_\nu^{-(\frac{3d}{2}+1)}\sim\sum_\nu\nu^{-1-\frac{1}{2d}}$$

is finite. This completes the proof of the lemma. □

We have also a Sobolev analogue of Lemma 2.15 which can be proved similarly.

Lemma 2.16. *Let $s>1+d/2$ and $V\in C^0([0,T];H^s(\mathbf{R}^d))$. There exists a positive constant K such that for any symbol $p=p(t,x,\xi)$ which is homogeneous in ξ of order $m\in\mathbf{R}$ and all $u\in C^0([0,T];H^{s+m}(\mathbf{R}^d))$,*

$$\begin{aligned}&\big\|\big[T_p,\partial_t+T_V\cdot\nabla\big]u(t)\big\|_{H^s(\mathbf{R}^d)}\\&\qquad\le K\left\{\mathcal{M}_0^m(p)\|V(t)\|_{H^s}+\mathcal{M}_0^m(\partial_t p+V\cdot\nabla p)\|V(t)\|_{L^\infty}\right\}\|u(t)\|_{H^{s+m}(\mathbf{R}^d)}\,.\end{aligned}$$

2.5. Parabolic evolution equation. Consider the evolution equation

$$\partial_z w + |D_x|\, w = 0,$$

where $z \in \mathbf{R}$ and $x \in \mathbf{R}^d$. By using the Fourier transform, one easily checks that

$$\sup_{z\in[0,1]} \|w(z)\|_{H^r} + \Big(\int_0^1 \|w(z)\|_{H^{r+\frac12}}^2\, dz\Big)^{\frac12} \le K \|w(0)\|_{H^r}. \tag{2.35}$$

The purpose of this section is to prove similar results when the constant coefficient operator $|D_x|$ is replaced by an elliptic paradifferential operator.

Given $I \subset \mathbf{R}$, $z_0 \in I$ and a function $\varphi = \varphi(x,z)$ defined on $\mathbf{R}^d \times I$, we denote by $\varphi(z_0)$ the function $x \mapsto \varphi(x,z_0)$. For $I \subset \mathbf{R}$ and a normed space E, $\varphi \in C^0_z(I;E)$ means that $z \mapsto \varphi(z)$ is a continuous function from I to E. Similarly, for $1 \le p \le +\infty$, $\varphi \in L^p_z(I;E)$ means that $z \mapsto \|\varphi(z)\|_E$ belongs to the Lebesgue space $L^p(I)$.

In this section, when a and u are symbols and functions depending on z, we still denote by $T_a u$ the function defined by $(T_a u)(z) = T_{a(z)} u(z)$ where $z \in I$ is seen as a parameter. We denote by $\Gamma^m_\rho(\mathbf{R}^d \times I)$ the space of symbols $a = a(z;x,\xi)$ such that $z \mapsto a(z;\cdot)$ is bounded from I into $\Gamma^m_\rho(\mathbf{R}^d)$ (see Definition 2.2), with the semi-norm

$$\mathcal{M}^m_\rho(a) = \sup_{z\in I} \sup_{|\alpha|\le \frac{3d}{2}+\rho+1} \sup_{|\xi|\ge 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(z;\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)}. \tag{2.36}$$

Given $\mu \in \mathbf{R}$ we define the spaces

$$\begin{aligned} X^\mu(I) &= C^0_z(I; H^\mu(\mathbf{R}^d)) \cap L^2_z(I; H^{\mu+\frac12}(\mathbf{R}^d)),\\ Y^\mu(I) &= L^1_z(I; H^\mu(\mathbf{R}^d)) + L^2_z(I; H^{\mu-\frac12}(\mathbf{R}^d)). \end{aligned} \tag{2.37}$$

Proposition 2.17. *Let $r \in \mathbf{R}$, $\rho \in (0,1)$, $J = [z_0, z_1] \subset \mathbf{R}$ and let $p \in \Gamma^1_\rho(\mathbf{R}^d \times J)$ satisfying*

$$\operatorname{Re} p(z;x,\xi) \ge c\,|\xi|,$$

for some positive constant c. Then for any $f \in Y^r(J)$ and $w_0 \in H^r(\mathbf{R}^d)$, there exists $w \in X^r(J)$ solution of the parabolic evolution equation

$$\partial_z w + T_p w = f, \quad w|_{z=z_0} = w_0, \tag{2.38}$$

satisfying

$$\|w\|_{X^r(J)} \le K \left\{ \|w_0\|_{H^r} + \|f\|_{Y^r(J)} \right\},$$

for some positive constant K depending only on r, ρ, c and $\mathcal{M}^1_\rho(p)$. Furthermore, this solution is unique in $X^s(J)$ for any $s \in \mathbf{R}$.

Proof. Let $r \in \mathbf{R}$. Denote by $\langle\cdot,\cdot\rangle_{H^r}$ the scalar product in $H^r(\mathbf{R}^d)$ and chose F_1 and F_2 such that $f = F_1 + F_2$ with

$$\|F_1\|_{L^1_z(J;H^r)} + \|F_2\|_{L^2(J;H^{r-\frac12})} \le \|f\|_{Y^r(J)} + \delta, \qquad \delta > 0.$$

Let us consider for $\varepsilon > 0$ the equation

$$\partial_z w_\varepsilon + \varepsilon(-\Delta + \mathrm{Id}) w_\varepsilon + T_p w_\varepsilon = f, \quad w_\varepsilon|_{z=z_0} = w_0. \tag{2.39}$$

Then standard methods in parabolic equations show that for any $z_1 > z_0$, this equation have a unique solution in

$$C^0([z_0,z_1]; H^r(\mathbf{R}^d)) \cap L^2((z_0,z_1); H^{r+2}(\mathbf{R}^d))$$

(here we only used that T_p is a Sobolev first order operator). To pass to the limit $\varepsilon \to 0$, we need to establish uniform estimates with respect to ε.

Taking the scalar product in H^r, directly from (2.39), we obtain

$$\frac{1}{2}\frac{d}{dz}\|w_\varepsilon(z)\|_{H^r}^2 + \varepsilon\langle(-\Delta+\mathrm{Id})w_\varepsilon(z), w_\varepsilon(z)\rangle_{H^r} + \mathrm{Re}\langle T_{p(z)}w_\varepsilon(z), w_\varepsilon(z)\rangle_{H^r} \\ \leq \|F_1(z)\|_{H^r}\|w_\varepsilon(z)\|_{H^r} + \|F_2(z)\|_{H^{r-\frac{1}{2}}}\|w_\varepsilon(z)\|_{H^{r+\frac{1}{2}}}.$$

It follows from Gårding's inequality (see [41, Section 6.3.2]) that there exist two constants $C_1, C_2 > 0$ depending only on $\mathcal{M}^1_\rho(p)$ such that for any $u \in H^r$,

$$\mathrm{Re}\langle T_{p(z)}u(z), u(z)\rangle_{H^r} \geq C_1\|u(z)\|^2_{H^{r+\frac{1}{2}}} - C_2\|u(z)\|^2_{H^{r+\frac{1-\rho}{2}}},$$

for each fixed $z \in J$. Therefore, we obtain

$$\frac{1}{2}\frac{d}{dz}\|w_\varepsilon(z)\|_{H^r}^2 + \varepsilon\langle(-\Delta+\mathrm{Id})w_\varepsilon(z), w_\varepsilon(z)\rangle_{H^r} + C_1\|w_\varepsilon(z)\|^2_{H^{r+\frac{1}{2}}} \\ \leq \|F_1(z)\|_{H^r}\|w_\varepsilon(z)\|_{H^r} + \|F_2(z)\|_{H^{r-\frac{1}{2}}}\|w_\varepsilon(z)\|_{H^{r+\frac{1}{2}}} + C_2\|w_\varepsilon(z)\|^2_{H^{r+\frac{1-\rho}{2}}}.$$

Integrating in z we obtain that, for all $z \in [z_0, z_1]$,

$$A(z) := \frac{1}{2}\Big\{\|w_\varepsilon(z)\|^2_{H^r} - \|w_\varepsilon(z_0)\|^2_{H^r}\Big\} + \varepsilon\int_{z_0}^{z}\|w_\varepsilon(z')\|^2_{H^{r+1}}dz' \\ + C_1\int_{z_0}^{z}\|w_\varepsilon(z')\|^2_{H^{r+\frac{1}{2}}}dz'$$

is bounded by

$$B := \|F_1\|_{L^1(J;H^r)}\|w_\varepsilon\|_{L^\infty(J;H^r)} + \|F_2\|_{L^2(J;H^{r-\frac{1}{2}})}\|w_\varepsilon\|_{L^2(J;H^{r+\frac{1}{2}})} \\ + C_2\|w_\varepsilon\|^2_{L^2(J;H^{r+\frac{1-\rho}{2}})}.$$

By standard arguments, it follows that

$$\|w_\varepsilon\|^2_{L^\infty(J;H^r)} + C_1\|w_\varepsilon\|^2_{L^2(J;H^{r+\frac{1}{2}})} \leq \|w_0\|^2_{H^r} \\ + C\big(\|F_1\|^2_{L^1(J;H^r)} + \|F_2\|^2_{L^2(J;H^{r-\frac{1}{2}})} + \|w_\varepsilon\|^2_{L^2(J;H^{r+\frac{1-\rho}{2}})}\big). \tag{2.40}$$

Finally, to eliminate the last term in the right hand side of (2.40), one notices that the left hand side controls by interpolation $c\|w_\varepsilon\|^2_{L^p(J;H^{r+\frac{1-\rho}{2}})}$, for some $p > 2$, hence by Hölder in the z variable, there exists $\kappa > 0$ (depending only on p) such that if $|z_0 - z_1| \leq \kappa$, we have

$$C\|w_\varepsilon\|^2_{L^2(J;H^{r+\frac{1-\rho}{2}})} \leq \frac{1}{2}\big(\|w_\varepsilon\|^2_{L^\infty(J;H^r)} + C_1\|w_\varepsilon\|^2_{L^2(J;H^{r+\frac{1}{2}})}\big).$$

We consequently obtain

$$\|w_\varepsilon\|^2_{L^\infty(J;H^r)} + C_1\|w_\varepsilon\|^2_{L^2(J;H^{r+\frac{1}{2}})} \leq 2\|w_0\|^2_{H^r} + C\big(\|F_1\|^2_{L^1(J;H^r)} + \|F_2\|^2_{L^2(J;H^{r-\frac{1}{2}})}\big).$$

We can now iterate the estimate between $z_0+\kappa$ and $z_0+2\kappa$, ... to get rid of the assumption $|z_1 - z_0| \leq \kappa$ (and of course the constants will depend on z_1). By using the equation, we obtain now that (w_ε) is bounded in $X^r(J) \cap C^1(J; H^{r-2})$. It follows from the Banach–Alaoglu theorem that, up to a subsequence, (w_ε) converges in the sense of distributions to $w \in X^r(J)$, which satisfies the equation $\partial_z w + T_p w = f$. Then $\partial_z w$ belongs to $Y^r(J)$ which implies that w belongs to $C^0([z_0, z_1]; H^r(\mathbf{R}^d))$. Moreover, by the Ascoli theorem, up to a subsequence, (w_ε) converges in $C^0([z_0, z_1]; H^{r-\mu}_{loc})$ for some $\mu > 0$. Since $w_\varepsilon|_{z=0} = w_0$ we obtain that $w|_{z=0} = w_0$, which completes the existence part in Proposition 2.17. The proof of uniqueness follows the same steps and we omit it. □

3. The Dirichlet-Neumann operator

We shall prove some results about elliptic regularity which complement previous works. To do this we shall use a paradifferential approach.

3.1. Definition and continuity. We begin by recalling from [1] the definition of the Dirichlet–Neumann operator under general assumptions on the bottom. One of the novelty with respect to our previous work is that we clarify the regularity assumptions: assuming only that $\eta \in W^{1,\infty}(\mathbf{R}^d)$ and $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$, we show how to define $G(\eta)\psi$ and prove that the map

$$\psi \in H^{\frac{1}{2}}(\mathbf{R}^d) \mapsto G(\eta)\psi \in H^{-\frac{1}{2}}(\mathbf{R}^d)$$

is continuous. Our second contribution is to prove that the map $\eta \mapsto G(\eta)$ is Lipschitz (in a proper topology). Finally, we prove also that in some weak sense, the Dirichlet-Neumann operator thus defined is a local operator (see Theorem 3.9).

The goal is to study the boundary value problem

$$\Delta_{x,y}\phi = 0 \text{ in } \Omega, \quad \phi|_{\Sigma} = f, \quad \partial_n \phi|_{\Gamma} = 0. \tag{3.1}$$

See §1.1 for the definitions of Ω, Σ, Γ. Since we make no assumption on Γ, the definition of ϕ requires some care. We recall here the definition of ϕ as given in [1].

Notation 3.1. Denote by $\mathscr{D}$ the space of functions $u \in C^\infty(\Omega)$ such that $\nabla_{x,y}u \in L^2(\Omega)$. We then define $\mathscr{D}_0$ as the subspace of functions $u \in \mathscr{D}$ such that u is equal to 0 in a neighborhood of the top boundary Σ.

Proposition 3.2 ([1, Proposition 2.2])**.** *There exists a positive weight $g \in L^\infty_{loc}(\Omega)$, equal to 1 near the top boundary of Ω and a constant $C > 0$ such that for all $u \in \mathscr{D}_0$,*

$$\int_\Omega g(x,y)|u(x,y)|^2\,dxdy \le C\int_\Omega |\nabla_{x,y}u(x,y)|^2\,dxdy. \tag{3.2}$$

Definition 3.3. *Denote by $H^{1,0}(\Omega)$ the space of functions u on Ω such that there exists a sequence $(u_n) \in \mathscr{D}_0$ such that,*

$$\nabla_{x,y}u_n \to \nabla_{x,y}u \text{ in } L^2(\Omega, dxdy), \qquad u_n \to u \text{ in } L^2(\Omega, g(x,y)dxdy).$$

We endow the space $H^{1,0}$ with the norm $\|u\| = \|\nabla_{x,y}u\|_{L^2(\Omega)}$.

Let us recall that the space $H^{1,0}(\Omega)$ is a Hilbert space (see [1]). For later purpose, we need to ensure that the functions having compact support in the x variable (at least near the surface Σ) are dense in $H^{1,0}(\Omega)$. By regularizing the function η (see Remark 3.7) it is easy to see that there is $\eta_* \in C_b^\infty(\mathbf{R}^d)$ such that $\eta - \frac{h}{20} > \eta_*$ and

$$\{(x,y) \in \mathbf{R}^d \times \mathbf{R}\,;\, \eta_*(x) < y < \eta(x)\} \subset \Omega.$$

Lemma 3.4. *The set*

$$\widetilde{\mathcal{D}}_0 = \Big\{u \in \mathcal{D}_0\,;\, \text{supp}(u) \cap \{(x,y); -\eta_* + \frac{h}{30} < y < \eta\} \text{ is compact}\Big\}$$

is dense in $H^{1,0}(\Omega_n)$.

Proof. Let $u \in \mathcal{D}_0$, and $\zeta \in C^\infty(\mathbf{R})$ equal to 0 for $z < 0$ and to 1 for $z > h/30$. Then according to Proposition 3.2, we have $\zeta(y - \eta_*)u \in \mathcal{D}_0$ and $(1 - \zeta(y - \eta_*))u \in \mathcal{D}_0 \cap H_0^1(\Omega)$ (where $H_0^1(\Omega)$ is the usual Sobolev space). Let $v_n \in C_0^\infty(\Omega)$ which converges to $(1 - \zeta(y - \eta_*))u$ in $H_0^1(\Omega)$ (and hence in $H^{1,0}(\Omega)$). We get that $\zeta(y - \eta_*)u + u_n \in \widetilde{\mathcal{D}}_0$ and converges to u in $H^{1,0}(\Omega)$. □

We are able now to define the Dirichlet-Neumann operator. Let $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$. We first define an H^1 lifting of f in Ω. To do so let $\chi_0 \in C^\infty(\mathbf{R})$ be such that $\chi_0(z) = 1$ if $z \geq -\frac{1}{2}$ and $\chi_0(z) = 0$ if $z \leq -1$. We set

$$\psi_1(x,z) = \chi_0(z) e^{z\langle D_x\rangle} f(x), \quad x \in \mathbf{R}^d, z \leq 0.$$

By the usual property of the Poisson kernel we have

$$\|\nabla_{x,z}\psi_1\|_{L^2([-1,0]\times\mathbf{R}^d)} \leq C \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

Then we set

$$\underline{\psi}(x,y) = \psi_1\big(x, \frac{y-\eta(x)}{h}\big), \quad (x,y) \in \Omega.$$

This is well defined since $\Omega \subset \{(x,y) : y < \eta(x)\}$. Moreover since the bottom Γ is contained in $\{(x,y) : y < \eta(x) - h\}$, we see that $\underline{\psi}$ vanishes identically near Γ.

Now we have obviously $\underline{\psi}|_\Sigma = f$ and since $\nabla\eta \in L^\infty(\mathbf{R}^d)$, an easy computation shows that $\underline{\psi} \in H^1(\Omega)$ and

$$\|\underline{\psi}\|_{H^1(\Omega)} \leq K(1 + \|\eta\|_{W^{1,\infty}})\|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}. \tag{3.3}$$

Then the map

$$v \mapsto -\int_\Omega \nabla_{x,y}\underline{\psi} \cdot \nabla_{x,y} v \, dxdy$$

is a bounded linear form on $H^{1,0}(\Omega)$. It follows from the Riesz theorem that there exists a unique $u \in H^{1,0}(\Omega)$ such that

$$\forall v \in H^{1,0}(\Omega), \qquad \int_\Omega \nabla_{x,y} u \cdot \nabla_{x,y} v \, dxdy = -\int_\Omega \nabla_{x,y}\underline{\psi} \cdot \nabla_{x,y} v \, dxdy. \tag{3.4}$$

Then u is the variational solution to the problem

$$-\Delta_{x,y} u = \Delta_{x,y}\underline{\psi} \quad \text{in } \mathcal{D}'(\Omega), \qquad u\,|_\Sigma = 0, \qquad \partial_n u\,|_\Gamma = 0,$$

the latter condition being justified as soon as the bottom Γ is regular enough.

Lemma 3.5. *The function $\phi = u + \underline{\psi}$ constructed by this procedure is independent on the choice of the lifting function $\underline{\psi}$ as long as it remains bounded in $H^1(\Omega)$ and vanishes near the bottom.*

Proof. Consider two functions constructed by this procedure, $\phi_k = u_k + \underline{\psi_k}$, $k = 1, 2$. Then, by standard density arguments, since $\underline{\psi_1} - \underline{\psi_2}$ vanishes at the top boundary Σ and in a neighborhood of the bottom Γ, there exists a sequence of functions $\psi_n \in C_0^\infty(\Omega)$ supported in a fixed Lipschitz domain $\widetilde{\Omega} \subset \Omega$ tending to $\underline{\psi_1} - \underline{\psi_2}$ in $H_0^1(\widetilde{\Omega})$ and hence also in $H^{1,0}(\Omega)$. As a consequence, $\underline{\psi_1} - \underline{\psi_2} \in H^{1,0}(\Omega)$ and the function $\phi = \phi_1 - \phi_2$ is the unique (trivial) solution in $H^{1,0}(\Omega)$ of the equation $\Delta_{x,y}\phi = 0$ given by the Riesz Theorem. □

Definition 3.6. *We shall say that the function $\phi = u + \underline{\psi}$ constructed by the above procedure is the variational solution of (3.1). It satisfies*

$$\int_\Omega |\nabla_{x,y}\phi|^2 \, dxdy \leq K \|f\|^2_{H^{\frac{1}{2}}(\mathbf{R}^d)}, \tag{3.5}$$

for some constant K depending only on the Lipschitz norm of η.

Formally the Dirichlet-Neumann operator is defined by

$$G(\eta)\psi = \sqrt{1 + |\nabla\eta|^2}\, \partial_n\phi\,\big|_{y=\eta(x)} = \big[\partial_y\phi - \nabla\eta \cdot \nabla\phi\big]\,\big|_{y=\eta(x)}. \tag{3.6}$$

3.1.1. *Straightenning the free boundary.* We begin by straightening the boundary. In this paragraph, we fix $s > \frac{1}{2} + \frac{d}{2}$.

We shall assume here that one can find a function η_* such that

$$(3.7)\qquad \begin{cases} (i) \quad \eta_* + \dfrac{h}{4} \in H^\infty(\mathbf{R}^d), \\ (ii) \quad \eta(x) - \eta_*(x) = \dfrac{h}{4} + g, \quad \|g\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)} \leq \dfrac{h}{5}, \\ (iii) \quad \Gamma \subset \{(x,y) \in \mathcal{O} : y < \eta_*(x)\}. \end{cases}$$

Remark 3.7. Assume that we have a function depending smoothly on the time, $\eta \in C^0([0,T], H^{s+\frac{1}{2}}(\mathbf{R}^d))$, such that $\eta|_{t=0} = \eta_0$ and satisfying condition (1.2) with $\frac{h}{2}$ (such as a solution in Theorem 1.2). Then one can construct $\eta_* = \eta_*(x)$ satisfying $(i), (iii)$ in (3.7) and for some $T' \leq T$

$$(ii)' \qquad \eta(t,x) - \eta_*(x) = \frac{h}{4} + g(t,x), \quad \|g\|_{L^\infty([0,T'], H^{s-\frac{1}{2}}(\mathbf{R}^d))} \leq \frac{h}{5}.$$

Indeed set

$$\eta_*(x) = -\frac{h}{4} + e^{-\nu\langle D_x\rangle}\eta_0(x)$$

where $\nu > 0$ is chosen such that $\nu\|\eta_0\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \leq \frac{h}{10}$. Then chose T' such that

$$\|\eta(t,\cdot) - \eta_0\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)} \leq \frac{h}{10}, \quad \forall t \in [0,T'],$$

and write

$$\eta(t,x) - \eta_*(x) = \eta(t,x) - \eta_0(x) + \eta_0(x) - e^{-\nu\langle D_x\rangle}\eta_0 + \frac{h}{4}.$$

Then $(ii)'$ follows from the estimate

$$\|\eta_0(x) - e^{-\nu\langle D_x\rangle}\eta_0\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)} \leq \nu\|\eta_0\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \leq \frac{h}{10},$$

and (iii) is a consequence of $(ii)'$. Indeed for $t \in [0,T']$ we have

$$\eta(t,x) - \eta_*(x) \leq \frac{h}{4} + \|g\|_{L^\infty([0,T']\times\mathbf{R}^d)} \leq \frac{h}{4} + \|g\|_{L^\infty([0,T'],H^{s-\frac{1}{2}}\mathbf{R}^d))} \leq \frac{h}{2},$$

therefore

$$\Gamma \subset \{(x,y) : y < \eta(t,x) - \frac{h}{2}\} \subset \{(x,y) : y < \eta_*(x)\}.$$

In what follows we shall set

$$(3.8)\qquad \begin{cases} \Omega_1 = \{(x,y) : x \in \mathbf{R}^d, \eta_*(x) < y < \eta(x)\}, \\ \Omega_2 = \{(x,y) \in \mathcal{O} : y \leq \eta_*(x)\}, \\ \Omega = \Omega_1 \cup \Omega_2. \end{cases}$$

and

$$(3.9)\qquad \begin{cases} \widetilde{\Omega}_1 = \{(x,z) : x \in \mathbf{R}^d, z \in I\}, \quad I = (-1,0), \\ \widetilde{\Omega}_2 = \{(x,z) \in \mathbf{R}^d \times (-\infty,-1] : (x, z+1+\eta_*(x)) \in \Omega_2\}, \\ \widetilde{\Omega} = \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2. \end{cases}$$

Following Lannes ([37]), consider the map $(x,z) \mapsto \rho(x,z)$ from $\widetilde{\Omega}$ to $\mathbf{R}$ defined by

$$(3.10)\qquad \begin{cases} \rho(x,z) = (1+z)e^{\delta z\langle D_x\rangle}\eta(x) - z\eta_*(x) \quad \text{if } (x,z) \in \widetilde{\Omega}_1, \\ \rho(x,z) = z + 1 + \eta_*(x) \quad \text{if } (x,z) \in \widetilde{\Omega}_2, \end{cases}$$

where δ is chosen such that

$$\delta \|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} := \delta_0 \quad \text{is small enough.}$$

Notice that ρ is Lipschitz on $\widetilde{\Omega}$. Moreover since $s > \frac{1}{2} + \frac{d}{2}$, there exists a constant $C > 0$ such that (recall that the spaces $X^s(I)$ are defined in (2.37))

$$\begin{aligned} \Big\| \partial_z \rho - \frac{h}{4} \Big\|_{X^{s-\frac{1}{2}}(I)} &\leq C\delta \left(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} + \|\eta_*\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \right) \\ \|\nabla_x \rho\|_{X^{s-\frac{1}{2}}(I)} &\leq C \left(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} + \|\eta_*\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \right) \end{aligned} \tag{3.11}$$

from which, taking δ_0 small enough, we deduce

$$\begin{cases} (i) \quad \partial_z \rho(x,z) \geq \min\big(1, \dfrac{h}{5}\big), \quad \forall (x,z) \in \widetilde{\Omega}, \\ (ii) \quad \|\nabla_{x,z}\rho\|_{L^\infty(\widetilde{\Omega})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}). \end{cases} \tag{3.12}$$

It follows from (3.12) (i) that the map $(x,z) \mapsto (x, \rho(x,z))$ is a C^1-diffeomorphism from $\widetilde{\Omega}$ to Ω. We denote by κ the inverse map of ρ:

$$(x,z) \in \widetilde{\Omega},\ \ (x, \rho(x,z)) = (x,y) \Leftrightarrow (x,z) = (x, \kappa(x,y)),\ \ (x,y) \in \Omega.$$

Let $\widetilde{\phi}(x,z) = \phi(x, \rho(x,z))$. Then we have

$$\begin{cases} (\partial_y \phi)(x, \rho(x,z)) = (\Lambda_1 \widetilde{\phi})(x,z), \quad (\nabla_x \phi)(x, \rho(x,z)) = (\Lambda_2 \widetilde{\phi})(x,z), \\ \Lambda_1 = \dfrac{1}{\partial_z \rho} \partial_z \quad \Lambda_2 = \nabla_x - \dfrac{\nabla_x \rho}{\partial_z \rho} \partial_z. \end{cases} \tag{3.13}$$

If ϕ is a solution of $\Delta_{x,y}\phi = 0$ in Ω then $\widetilde{\phi}$ satifies

$$(\Lambda_1^2 + \Lambda_2^2)\widetilde{\phi} = 0 \quad \text{in } \widetilde{\Omega}$$

This yields

$$(a\partial_z^2 + \Delta_x + b \cdot \nabla_x \partial_z - c\partial_z)\widetilde{\phi} = 0, \tag{3.14}$$

where

$$a := \frac{1 + |\nabla_x \rho|^2}{(\partial_z \rho)^2}, \quad b := -2\frac{\nabla_x \rho}{\partial_z \rho}, \quad c := \frac{1}{\partial_z \rho}\big(a\partial_z^2 \rho + \Delta_x \rho + b \cdot \nabla_x \partial_z \rho\big). \tag{3.15}$$

It will be convenient to have a constant coefficient in front of $\partial_z^2 \widetilde{\phi}$. Dividing (3.14) by a we obtain

$$(\partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z - \gamma \partial_z)\widetilde{\phi} = 0, \tag{3.16}$$

where

$$\alpha := \frac{(\partial_z \rho)^2}{1 + |\nabla \rho|^2}, \quad \beta := -2\frac{\partial_z \rho \nabla_x \rho}{1 + |\nabla_x \rho|^2}, \quad \gamma := \frac{1}{\partial_z \rho}\big(\partial_z^2 \rho + \alpha \Delta_x \rho + \beta \cdot \nabla_x \partial_z \rho\big). \tag{3.17}$$

In the coordinates (x,z), according to (3.6) we have

$$G(\eta)\psi = U|_{z=0}, \quad U = \Lambda_1 \widetilde{\phi} - \nabla_x \rho \cdot \Lambda_2 \widetilde{\phi}. \tag{3.18}$$

The following remark will be useful in the sequel. We have

$$\partial_z U = -\nabla_x((\partial_z \rho)\Lambda_2 \widetilde{\phi}). \tag{3.19}$$

Indeed we can write

$$\begin{aligned}\partial_z U &= \partial_z \Lambda_1 \widetilde{\phi} - \nabla_x \partial_z \rho \cdot \Lambda_2 \widetilde{\phi} - \nabla_x \rho \cdot \partial_z \Lambda_2 \widetilde{\phi} \\ &= (\partial_z \rho) \Lambda_1^2 \widetilde{\phi} - \nabla_x \partial_z \rho \cdot \Lambda_2 \widetilde{\phi} + (\partial_z \rho)(\Lambda_2 - \nabla_x) \Lambda_2 \widetilde{\phi} \\ &= (\partial_z \rho)(\Lambda_1^2 + \Lambda_2^2) \widetilde{\phi} - \nabla_x \big((\partial_z \rho) \Lambda_2 \widetilde{\phi} \big).\end{aligned}$$

Since $(\Lambda_1^2 + \Lambda_2^2)\widetilde{\phi} = 0$ we obtain (3.19).

3.1.2. *Continuity of the Dirichlet-Neumann operator.*

Theorem 3.8. *Let $\eta \in W^{1,\infty}(\mathbf{R}^d)$, $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$. In the system of coordinates (x, z) defined above, the variational solution of (3.1), $\widetilde{\phi}$, satisfies*

$$\widetilde{\phi} \in C_z^0([-1, 0]; H^{\frac{1}{2}}(\mathbf{R}^d)) \cap C_z^1([-1, 0]; H^{-\frac{1}{2}}(\mathbf{R}^d)). \tag{3.20}$$

As a consequence, the map

$$\begin{aligned}\psi \in H^{\frac{1}{2}}(\mathbf{R}^d) \mapsto G(\eta)\psi &= \sqrt{1 + |\nabla \eta|^2}\, \partial_n \phi \,\big|_{y=\eta(x)} = \big[\partial_y \phi - \nabla \eta \cdot \nabla \phi\big] \,\big|_{y=\eta(x)} \\ &= \big((1 + |\nabla \eta|^2) \partial_z \widetilde{\phi} - \nabla \eta \cdot \nabla \widetilde{\phi}\big)|_{z=0}\end{aligned} \tag{3.21}$$

is well defined. It furthermore satisfies

$$\|G(\eta) f\|_{H^{-\frac{1}{2}}(\mathbf{R}^d)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

The Dirichlet–Neumann operator is also weakly continuous. This fact will be used in Section 6 to prove existence of solutions when passing to the weak limits on weakly convergent sequences of suitably regularized systems.

Theorem 3.9. *Assume that $(\eta_n)_{n \in \mathbf{N}}$ and $(\psi_n)_{n \in \mathbf{N}}$ are two sequences such that*

i) the sequence $(\eta_n, \psi_n)_{n \in \mathbf{N}}$ is bounded in $W^{1,\infty}(\mathbf{R}^d) \times H^{\frac{1}{2}}(\mathbf{R}^d)$,
ii) there exists $\eta \in W^{1,\infty}(\mathbf{R}^d)$ such that η_n converges strongly to η in $W^{1,\infty}_{loc}(\mathbf{R}^d)$,
iii) there exists $\psi \in H^{\frac{1}{2}}(\mathbf{R}^d)$ such that $(\psi_n)_{n \in \mathbf{N}}$ converges weakly to ψ in $H^{\frac{1}{2}}(\mathbf{R}^d)$,
iv) there exists $\eta_* \in W^{1,\infty}(\mathbf{R}^d)$, $h > 0$ such that

$$\eta(x) - \frac{h}{2} > \eta_*(x) > \eta(x) - h, \eta_n(x) - \frac{h}{2} > \eta_*(x) > \eta_n(x) - h \quad \forall x \in \mathbf{R}^d.$$

Then the sequence $(G(\eta_n)\psi_n)$ is bounded in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ and converge weakly to $G(\eta)\psi$.

Let us also state the second basic strong continuity of the Dirichlet Neumann operator. Notice that the map $\eta \mapsto G(\eta)$ is non linear and hence continuity do not imply weak continuity.

Theorem 3.10. *There exists a non decreasing function $\mathcal{F} \colon \mathbf{R}_+ \to \mathbf{R}_+$ such that, for all $\eta_j \in W^{1,\infty}(\mathbf{R}^d)$, $j = 1, 2$ and all $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$,*

$$\big\| \big(G(\eta_1) - G(\eta_2)\big) f \big\|_{H^{-\frac{1}{2}}} \leq \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}\big) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}.$$

Remark 3.11. We shall only prove Theorems 3.9 and 3.10 as the choice $\eta_n = \eta, \psi_n = \psi$ in Theorem 3.9 implies Theorem 3.8. On the other hand, the fact that we can pass to the limit in $G(\eta_n)\psi_n$ under convergence assumptions on (η_n, ψ_n) which are only local in space, shows that, in a very weak sense, the Dirichlet–Neumann operator is a local operator.

Proof of Theorem 3.9. Let $M_0 > 0$ be such that for all $n \in \mathbf{N}$,

(3.22)
$$\|\psi_n\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} + \|\psi\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} + \|\eta_n\|_{W^{1,\infty}(\mathbf{R}^d)} + \|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} + \|\eta_*\|_{W^{1,\infty}(\mathbf{R}^d)} \leq M_0.$$

Our purpose is to prove that $G(\eta_n)\psi_n$ is well defined and bounded in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ by

$$\mathcal{F}(\|\eta_n\|_{W^{1,\infty}})\|\psi_n\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$$

(hence uniformly bounded) and converges weakly to $G(\eta)\psi$ in $H^{-\frac{1}{2}}(\mathbf{R}^d)$. We shall proceed in several steps.

Step 1: preliminaries. We start by straightening the boundaries of the domains Ω_n and Ω using the previous section. We recall that

$$\Omega_n = \{(x,y) \in \mathcal{O} : y < \eta_n(x)\}, \qquad \Omega = \{(x,y) \in \mathcal{O} : y < \eta(x)\}.$$

For this purpose we use the diffeomorphisms given by ρ_n (constructed with η_n) and ρ given by (3.10) and we shall use the vector fields $\Lambda_j^n, \Lambda_j, j = 1,2$ described in (3.13) and we set

$$\Lambda^n = (\Lambda_1^n, \Lambda_2^n) \quad \Lambda = (\Lambda_1, \Lambda_2).$$

We construct now a H^1- extension of ψ_n. Let $\chi \in C^\infty(\mathbf{R}), \chi(z) = 1$ if $z \geq -\frac{1}{2}$ and $\chi(z) = 0$ if $z \leq -1$ and set

$$\underline{\widetilde{\psi}_n}(x,z) = \chi(z)e^{z\langle D_x\rangle}\psi_n(x), \quad \underline{\widetilde{\psi}}(x,z) = \chi(z)e^{z\langle D_x\rangle}\psi(x). \tag{3.23}$$

Then $\underline{\widetilde{\psi}_n}(x,z) \in H^1(\mathbf{R}^d \times I)$ and $\|\underline{\widetilde{\psi}_n}\|_{H^1(\mathbf{R}^d\times I)} \leq C\|\psi_n\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} \leq CM_0$

We make the same construction for ψ. Then it is easy to see that the sequence $(\underline{\widetilde{\psi}_n})$ converges in $H^1(\widetilde{\Omega})$ to $\underline{\widetilde{\psi}}$, Then we set

$$\widetilde{\phi}_n = \widetilde{u}_n + \underline{\widetilde{\psi}_n}, \qquad \widetilde{\phi} = \widetilde{u} + \underline{\widetilde{\psi}}. \tag{3.24}$$

According to (3.5) and the assumptions on η_n and η we see easily that this implies

$$\|\nabla_{x,z}\widetilde{u}_n\|_{L^2(\widetilde{\Omega})} \leq M_2, \quad \forall n \in \mathbf{N}. \tag{3.25}$$

Then $(\widetilde{u}_n)$ is a bounded sequence in $H^{1,0}(\widetilde{\Omega}))$ and therefore that, up to a subsequence, it converges weakly in this space to $\widetilde{u}$.

Step 2: passing to the limit for the variational solutions. Setting $X = (x,z) \in \widetilde{\Omega}$ the variational formulation for $\widetilde{u}_n$ reads

$$\int_{\widetilde{\Omega}} \Lambda^n\widetilde{u}_n(X)\cdot\Lambda^n\zeta(X)J_n(X)dX = \int_{\widetilde{\Omega}} \Lambda^n\underline{\widetilde{\psi}_n}(X)\cdot\Lambda^n\zeta(X)J_n(X)dX \tag{3.26}$$

for all $\zeta \in C_0^\infty(\widetilde{\Omega})$, where $J_n(X) = |\partial_z\rho_n(X)|$. We now want to identify the limit. We have the following Lemma.

Lemma 3.12. *For all $\zeta \in \mathcal{D}_0(\Omega)$ we have*

$$\lim_{n\to+\infty}\int_{\widetilde{\Omega}} \Lambda^n\widetilde{u}_n(X)\cdot\Lambda^n\zeta(X)J_n(X)dX = \int_{\widetilde{\Omega}} \Lambda\widetilde{u}(X)\cdot\Lambda\zeta(X)J(X)dX,$$
$$\lim_{n\to+\infty}\int_{\widetilde{\Omega}} \Lambda^n\underline{\widetilde{\psi}_n}(X)\cdot\Lambda^n\zeta(X)J_n(X)dX = \int_{\widetilde{\Omega}} \Lambda\underline{\widetilde{\psi}}(X)\cdot\Lambda\zeta(X)J(X)dX.$$

Corollary 3.13. *The function $u(x,y) = \widetilde{u}(x,\kappa(x,y))$ is the variational solution in $H^{1,0}(\Omega)$ of the problem $-\Delta_{x,y}u = \Delta_{x,y}\underline{\psi}$ and u_n converges weakly in this space to u.*

Proof of Lemma 3.12. Notice that

$$\begin{cases} \Lambda^n - \Lambda = \beta_n \partial_z, \quad \operatorname{supp} \beta_n \subset \{(x,z) : x \in \mathbf{R}^d, z \in (-1,0)\} \text{ and} \\ \|\beta_n\|_{L^\infty(K)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\eta_n - \eta\|_{W^{1,\infty}(K)} \end{cases} \tag{3.27}$$

Then we can write

$$\begin{aligned} &\Lambda^n \widetilde{u}_n \cdot \Lambda^n \zeta \cdot J_n - \Lambda \widetilde{u} \cdot \Lambda \zeta \cdot J = A_1 + A_2 + A_3 + A_4, \\ &A_1 = (\Lambda^n - \Lambda) \widetilde{u}^n \cdot \Lambda^n \zeta \cdot J_n, \quad A_2 = \Lambda \widetilde{u}_n \cdot \Lambda^n \zeta \cdot (J_n - J), \\ &A_3 = \Lambda \widetilde{u}_n \cdot (\Lambda^n - \Lambda) \zeta \cdot J, \quad A_4 = \Lambda(\widetilde{u}_n - \widetilde{u}) \cdot \Lambda \zeta \cdot J. \end{aligned}$$

It follows from (3.27) that we have

$$\left| \int_{\widetilde{\Omega}} A_1(X) dX \right| \leq C(M_0) \|\eta_n - \eta\|_{W^{1,\infty}(K)} \|\partial_z \widetilde{u}_n\|_{L^2(\widetilde{\Omega})} \|\nabla_X \zeta\|_{L^2(\widetilde{\Omega}))}. \tag{3.28}$$

The same estimate holds for the term coming from A_3. Moreover since $\|J_n - J\|_{L^\infty(K)} \leq \mathcal{F}(M_0) \|\eta_n - \eta\|_{W^{1,\infty}(K)}$ we have for A_2 the same estimate as (3.28).

Eventually since $(\widetilde{u}_n)$ converges to $\widetilde{u}$ in the weak topology of $H^{1,0}(\widetilde{\Omega})$ we obtain

$$\lim_{n \to +\infty} \int_{\Omega} A_4(x,y) dx dy = 0.$$

□

Step 3: taking traces. Notice that we have

$$((\Lambda_1^n)^2 + (\Lambda_2^n)^2) \widetilde{u}_n = 0, \quad ((\Lambda_1)^2 + (\Lambda_2)^2) \widetilde{u} = 0, \tag{3.29}$$

and

$$\begin{cases} G(\eta_n)\psi_n = \big(\Lambda_1^n - \nabla_x \rho_n \cdot \Lambda_2^n\big) \widetilde{u}_n|_{z=0} =: U_n|_{z=0}, \\ \quad G(\eta)\psi = \big(\Lambda_1 - \nabla_x \rho \cdot \Lambda_2\big) \widetilde{u}|_{z=0} =: U|_{z=0}. \end{cases} \tag{3.30}$$

Since ρ_n converges to ρ in $W^{1,\infty}_{loc}(\widetilde{\Omega})$ the sequence (U_n) converges weakly to U in $L^2(\widetilde{\Omega})$. Now using (3.19) we obtain

$$\partial_z U_n = -\nabla_x \big((\partial_z \rho_n) \Lambda_2^n \widetilde{u}_n \big).$$

By the same way we have

$$\partial_z U = -\nabla_x \big((\partial_z \rho) \Lambda_2 v \big).$$

Since $\nabla_{x,z} \rho_n \to \nabla_{x,z} \rho$ in $L^\infty(\mathbf{R}^d \times I)$ and $\widetilde{u}_n \to \widetilde{u}$ weakly in $H^1(\mathbf{R}^d \times I)$, the sequence $(\partial_z U_n)$ converges to $\partial_z U$ weakly in $L^2(I, H^{-1}(\mathbf{R}^d))$.

Now we use the following well known interpolation lemma.

Lemma 3.14. *Let $I = (-1,0)$ and consider $u \in L^2(I, L^2(\mathbf{R}^d))$ such that $\partial_z u \in L^2(I, H^{-1}(\mathbf{R}^d))$. Then $u \in C^0([-1,0], H^{-\frac{1}{2}}(\mathbf{R}^d))$ and there exists an absolute constant $K > 0$ such that*

$$\|u\|_{C^0([-1,0]; H^{-\frac{1}{2}}(\mathbf{R}^d))} \leq K \big(\|u\|_{L^2(I; L^2(\mathbf{R}^d))} + \|\partial_z u\|_{L^2(I; H^{-1}(\mathbf{R}^d))} \big).$$

It follows from this lemma that the sequence $(U_n|_{z=0})$ is bounded in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ by $\mathcal{F}(\|\eta_n\|_{W^{1,\infty}}) \|\psi_n\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$ and converges weakly in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ to $U|_{z=0}$, which completes the proof of Theorem 3.9. □

Proof of Theorem 3.10. We use the notations introduced in §3.1.1. Namely, for $j=1,2$, we introduce $\rho_j(x,z)$ and $v_j(x,z)$ defined by (3.10),

$$\rho_j(x,z) = (1+z)(e^{\delta z\langle D_x\rangle}\eta_j)(x) - z\eta_*, \quad \text{if } x\in \mathbf{R}^d, z\in I := (-1,0)$$
$$\rho_j(x,z) = z+1+\eta_*, \quad \text{if } (x,z)\in \widetilde{\Omega}_2$$

Notice that we have the following estimates

$$\text{(3.31)} \qquad \begin{cases} (i) \quad \partial_z\rho_j \geq \min(1, \frac{h}{5}), \quad (x,z)\in \widetilde{\Omega}, \\ (ii) \quad \Vert \nabla_{x,z}\rho_j\Vert_{L^\infty(\widetilde{\Omega})} \leq C(1+\Vert \eta_i\Vert_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}) \\ (iii) \quad \Vert \nabla_{x,z}(\rho_1-\rho_2)\Vert_{L^\infty(I,L^\infty(\mathbf{R}^d))} \leq C\Vert \eta_1-\eta_2\Vert_{W^{1,\infty}(\mathbf{R}^d)}. \end{cases}$$

Recall also that we have set

$$\text{(3.32)} \qquad \Lambda_1^i = \frac{1}{\partial_z\rho_i}\partial_z, \quad \Lambda_2^i = \nabla_x - \frac{\nabla_x\rho_i}{\partial_z\rho_i}\partial_z.$$

It follows from (3.31) that for $k=1,2$ we have with $W^{1,\infty}=W^{1,\infty}(\mathbf{R}^d)$,

$$\text{(3.33)} \qquad \begin{cases} (i) \quad \Lambda_k^1-\Lambda_k^2 = \beta_j\partial_z, \quad \text{with } \operatorname{supp}\beta_k \subset \mathbf{R}^d\times I, \text{where } I=(-1,0), \\ (ii) \quad \Vert \beta_k\Vert_{L^\infty(I\times\mathbf{R}^d)} \leq \mathcal{F}(\Vert(\eta_1,\eta_2)\Vert_{W^{1,\infty}\times W^{1,\infty}})\Vert \eta_1-\eta_2\Vert_{W^{1,\infty}} \end{cases}$$

Then we set $\widetilde{\phi}_j(x,z) = \phi_j(x,\rho_j(x,z))$ (where $\Delta_{x,y}\phi_j=0$ in $\Omega_j, \phi_j|_{\Sigma_j}=f$) and we recall (see (3.18)) that

$$\text{(3.34)} \qquad G(\eta_j)f = U_j|_{z=0}, \quad U_j = \Lambda_1^j\widetilde{\phi}_j - \nabla_x\rho_j\cdot\Lambda_2^j\widetilde{\phi}_j.$$

Lemma 3.15. *Set $I=(-1,0)$, $v=\widetilde{\phi}_1-\widetilde{\phi}_2$, and $\Lambda^j=(\Lambda_1^j,\Lambda_2^j)$. There exists a non decreasing function $\mathcal{F}:\mathbf{R}^+\to\mathbf{R}^+$ such that*

$$\text{(3.35)} \qquad \Vert \Lambda^j v\Vert_{L^2(I;L^2(\mathbf{R}^d))} \leq \mathcal{F}(\Vert(\eta_1,\eta_2)\Vert_{W^{1,\infty}\times W^{1,\infty}})\Vert \eta_1-\eta_2\Vert_{W^{1,\infty}}\Vert f\Vert_{H^{\frac{1}{2}}}.$$

Let us show how this Lemma implies Theorem 3.10. According to (3.34) we have

$$\text{(3.36)} \qquad \begin{aligned} &U_1-U_2 = (1)+(2)+(3)+(4)+(5) \quad \text{where} \\ &\quad (1) = \Lambda_1^1 v, \quad (2) = (\Lambda_1^1-\Lambda_1^2)\widetilde{\phi}_2, \quad (3) = -\nabla_x(\rho_1-\rho_2)\Lambda_2^1\widetilde{\phi}_1 \\ &\quad (4) = -(\nabla_x\rho_2)\Lambda_2^1 v, \quad (5) = -(\nabla_x\rho_2)(\Lambda_2^1-\Lambda_2^2)\widetilde{\phi}_2. \end{aligned}$$

The $L^2(I,L^2(\mathbf{R}^d))$ norms of (1) and (4) are estimated using (3.35). Also, the $L^2(I,L^2(\mathbf{R}^d))$ norms of (2) and (5) are estimated by the right hand side of (3.35) using (3.33) and (3.5). Eventually the $L^2(I,L^2(\mathbf{R}^d))$ norm of (3) is also estimated by the right hand side of (3.35) using (3.31) (iii) and (3.5). It follows that

$$\text{(3.37)} \qquad \Vert U_1-U_2\Vert_{L^2(I,L^2)} \leq \mathcal{F}(\Vert(\eta_1,\eta_2)\Vert_{W^{1,\infty}\times W^{1,\infty}})\Vert \eta_1-\eta_2\Vert_{W^{1,\infty}}\Vert f\Vert_{H^{\frac{1}{2}}}.$$

Now according to (3.19) we have

$$\text{(3.38)} \qquad \partial_z(U_1-U_2) = -\nabla_x\big(\partial_z(\rho_1-\rho_2)\Lambda_2^1\widetilde{\phi}_1 + (\partial_z\rho_2)(\Lambda_2^1-\Lambda_2^2)\widetilde{\phi}_1 + (\partial_z\rho_2)\Lambda_2^2 v\big).$$

Therefore using the same estimates as above we see easily that

$$\text{(3.39)} \qquad \Vert \partial_z(U_1-U_2)\Vert_{L^2(I,H^{-1})} \leq \mathcal{F}(\Vert(\eta_1,\eta_2)\Vert_{W^{1,\infty}\times W^{1,\infty}})\Vert \eta_1-\eta_2\Vert_{W^{1,\infty}}\Vert f\Vert_{H^{\frac{1}{2}}}.$$

Then Theorem 3.10 follows from (3.37), (3.39) and Lemma 3.14. □

Proof of Lemma 3.15. We use the variational characterization of the solutions u_i. First of all we notice that $\widetilde{\phi}_1 - \widetilde{\phi}_2 = \widetilde{u}_1 - \widetilde{u}_2 =: v$. Now setting $X = (x, z)$ we have

$$\int_{\widetilde{\Omega}} \Lambda^i \widetilde{u}_i \cdot \Lambda^i \theta \, J_i \, dX = - \int_{\widetilde{\Omega}} \Lambda^i \underline{\widetilde{f}} \cdot \Lambda^i \theta \, J_i \, dX \tag{3.40}$$

for all $\theta \in H^{1,0}(\widetilde{\Omega})$, where $J_i = |\partial_z \rho_i|$.

Taking the difference between the two equations (3.40), using (3.31) and setting $\theta = v = \widetilde{u}_1 - \widetilde{u}_2$ one can find a positive constant C such that

$$\int_{\widetilde{\Omega}} |\Lambda^1 v|^2 \, dX \leq C \sum_{k=1}^{6} A_k,$$

where

$$\begin{cases} A_1 = \int_{\widetilde{\Omega}} |(\Lambda^1 - \Lambda^2)\widetilde{u}_2| |\Lambda^1 v| \, J_1 \, dX, & A_2 = \int_{\widetilde{\Omega}} |(\Lambda^1 - \Lambda^2) v| |\Lambda^2 \widetilde{u}_2| \, J_1 dX, \\ A_3 = \int_{\widetilde{\Omega}} |\Lambda^2 \widetilde{u}_2| |\Lambda^2 v| \, |J_1 - J_2| \, dX, & A_4 = \int_{\widetilde{\Omega}} |(\Lambda^1 - \Lambda^2) \underline{\widetilde{f}}| |\Lambda^1 v| \, J_1 \, dX, \\ A_5 = \int_{\widetilde{\Omega}} |(\Lambda^1 - \Lambda^2) v| |\Lambda^2 \underline{\widetilde{f}}| \, J_1 \, dX, & A_6 = \int_{\widetilde{\Omega}} |\Lambda^2 \widetilde{f}| |\Lambda^2 v| \, |J_1 - J_2| \, dX. \end{cases}$$

Using (3.33), (3.5), (3.31) we can write

$$\begin{aligned} |A_1| &\leq \|\beta\|_{L^\infty(I \times \mathbf{R}^d)} \|J_1\|_{L^\infty(I \times \mathbf{R}^d)} \|\partial_z \widetilde{u}_2\|_{L^2(I \times \mathbf{R}^d)} \|\Lambda^1 v\|_{L^2(\widetilde{\Omega})} \\ &\leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}} \|\Lambda^1 v\|_{L^2(\widetilde{\Omega})}. \end{aligned} \tag{3.41}$$

Since $\Lambda_j^1 - \Lambda_j^2 = \frac{\beta_j}{\partial_z \rho_1} \Lambda_1^1$ the term A_2 can be bounded by the right hand side of (3.41).

Now we have $\|J_1 - J_2\|_{L^\infty(I \times \mathbf{R}^d)} \leq C \|\eta_1 - \eta_2\|_{W^{1,\infty}(\mathbf{R}^d)}$ and

$$\|\Lambda^2 v\|_{L^2(\widetilde{\Omega})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}) \|\Lambda^1 v\|_{L^2(\widetilde{\Omega})}.$$

So using (3.5) we see that the term A_3 can be also estimated by the right hand side of (3.41). To estimate the terms A_4 to A_6 we use the same arguments and also (3.3). This completes the proof. □

Let us finish this definition section by recalling also the following result which is a consequence of [**1**, Lemma 2.9].

Lemma 3.16. *Assume that* $-\frac{1}{2} \leq a < b \leq -\frac{1}{5}$ *then the strip* $S_{a,b} = \{(x, y) \in \mathbf{R}^{d+1} : ah < y - \eta(x) < bh\}$ *is included in* Ω *and for any* $k \geq 1$, *there exists* $C > 0$ *such that*

$$\|\phi\|_{H^k(S_{a,b})} \leq C \|\psi\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

3.2. Paralinearization of the Dirichlet-Neumann operator. In the case of smooth domains, it is known that, modulo a smoothing operator, $G(\eta)$ is a pseudo-differential operator with principal symbol given by

$$\lambda(x, \xi) := \sqrt{(1 + |\nabla \eta(x)|^2) \, |\xi|^2 - (\nabla \eta(x) \cdot \xi)^2}. \tag{3.42}$$

Notice that λ is well-defined for any C^1 function η. The main result of this section allow to compare $G(\eta)$ to the paradifferential operator T_λ when η has limited regularity. Namely we want to estimate the operator

$$R(\eta) = G(\eta) - T_\lambda.$$

Such an analysis was at the heart of our previous work [4] [1, Proposition 3.14] for "smooth domains" ($\eta \in H^{s+\frac{1}{2}}, s > 2 + \frac{d}{2}$). Here we are able to lower the regularity thresholdsup to $s > 1 + \frac{d}{2}$. The following results, which we think are of independent interest, complement previous estimates about the Dirichlet-Neumann operator by Craig-Schanz-Sulem [27], Beyer–Günther [12], Wu [51, 52], Lannes [37].

Theorem 3.17. *Let $d \geq 1$, $s > \frac{1}{2} + \frac{d}{2}$ and $\frac{1}{2} \leq \sigma \leq s + \frac{1}{2}$. Then there exists a non-decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that, for all $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ and all $f \in H^\sigma(\mathbf{R}^d)$, we have $G(\eta)f \in H^{\sigma-1}(\mathbf{R}^d)$, together with the estimate*

$$\|G(\eta)f\|_{H^{\sigma-1}(\mathbf{R}^d)} \leq \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}\big) \|f\|_{H^\sigma(\mathbf{R}^d)} . \tag{3.43}$$

We also prove error estimates.

Proposition 3.18. *Let $d \geq 1$ and $s > \frac{1}{2} + \frac{d}{2}$. For any $\frac{1}{2} \leq \sigma \leq s$ and any*

$$0 < \varepsilon \leq \frac{1}{2}, \qquad \varepsilon < s - \frac{1}{2} - \frac{d}{2},$$

there exists a non-decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that $R(\eta)f := G(\eta)f - T_\lambda f$ satisfies

$$\|R(\eta)f\|_{H^{\sigma-1+\varepsilon}(\mathbf{R}^d)} \leq \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}\big) \|f\|_{H^\sigma(\mathbf{R}^d)} .$$

To prove Theorem 3.17 and Proposition 3.18, we shall use the diffeomorphism ρ defined by (3.10) which satisfies the estimates (3.12). Then recall from (3.16) that the function $\widetilde{\phi}(x,z) = \phi(x, \rho(x,z))$ satisfies

$$(\partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z - \gamma \partial_z)\widetilde{\phi} = 0, \qquad \widetilde{\phi}|_{z=0} = \phi|_{y=\eta(x)} = f, \tag{3.44}$$

and

$$G(\eta)f = (\Lambda_1 \widetilde{\phi} - \nabla\rho \cdot \Lambda_2 \widetilde{\phi})|_{z=0} = \big(\frac{1 + |\nabla\rho|^2}{\partial_z \rho} \partial_z \widetilde{\phi} - \nabla\rho \cdot \nabla \widetilde{\phi}\big)\,\big|_{z=0}.$$

We conclude this paragraph by stating elliptic estimates for the solutions of (3.44). For later purpose, we will consider the non-homogeneous case. This yields no new difficulty and will be useful later to estimate the pressure (see Section 4.2). We thus consider the problem

$$\partial_z^2 v + \alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v = F_0, \quad v|_{z=0} = f, \tag{3.45}$$

where $f = f(x)$ and $F_0 = F_0(x,z)$ are given functions. Recall that for $\mu \in \mathbf{R}$, the spaces $X^\mu(I), Y^\mu(I)$ are defined by (see (2.37)):

$$\begin{aligned} X^\mu(I) &= C_z^0(I; H^\mu(\mathbf{R}^d)) \cap L_z^2(I; H^{\mu+\frac{1}{2}}(\mathbf{R}^d)), \\ Y^\mu(I) &= L_z^1(I; H^\mu(\mathbf{R}^d)) + L_z^2(I; H^{\mu-\frac{1}{2}}(\mathbf{R}^d)). \end{aligned} \tag{3.46}$$

Recall that $H^\sigma(\mathbf{R}^d)$ is an algebra for $\sigma > d/2$ and so is $C^0(I; H^\sigma(\mathbf{R}^d))$. Also, using the tame estimate (2.14), we obtain the following

Lemma 3.19. *Assume that $\sigma > \frac{d}{2}$. Then the space $X^\sigma(I)$ is an algebra. Moreover if $F : \mathbf{C}^N \to \mathbf{C}$ is a C^∞-bounded function such that $F(0) = 0$ one can find non decreasing functions $\mathcal{F}, \mathcal{F}_1$ from $\mathbf{R}^+$ to $\mathbf{R}^+$ such that*

$$\|F(U)\|_{X^\sigma(I)} \leq \mathcal{F}(\|U\|_{L^\infty(I\times\mathbf{R}^d)}) \|U\|_{X^\sigma(I)} \leq \mathcal{F}_1(\|U\|_{X^\sigma(I)}).$$

With these notations, we want to estimate the X^σ-norm of $\nabla_{x,z}v$ in terms of the $H^{\sigma+1}$-norm of the data and the Y^σ-norm of the source term. An important point is that we need to consider the case of rough coefficients. In this section we only assume that $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ for some $s > 1/2 + d/2$. An interesting point is that we shall prove elliptic estimates as well as elliptic regularity (in other words, we do not prove only *a priori* estimates). Our only assumption is that v is given by a variational problem, so that

$$\left\| \nabla_{x,z} v \right\|_{X^{-\frac{1}{2}}([-1,0])} < +\infty. \tag{3.47}$$

Remark 3.20. In the case where $v(x,z) = \widetilde{\phi}(x,z) = \phi(x,\rho(x,z))$ with ϕ the variational solution of

$$\Delta_{x,y}\phi = 0, \quad \phi|_{y=\eta} = f, \quad \partial_n \phi = 0 \text{ on } \Gamma,$$

then (3.20) shows that v satisfies this assumption (3.47).

Proposition 3.21. *Let $d \geq 1$ and*

$$s > \frac{1}{2} + \frac{d}{2}, \quad -\frac{1}{2} \leq \sigma \leq s - \frac{1}{2}.$$

Consider $f \in H^{\sigma+1}(\mathbf{R}^d)$, $F_0 \in Y^\sigma([-1,0])$ and v satisfying the assumption (3.47) solution to (3.45). Then for any $z_0 \in (-1,0)$, $\nabla_{x,z}v \in X^\sigma([z_0,0])$, and

$$\left\| \nabla_{x,z}v \right\|_{X^\sigma([z_0,0])} \leq \mathcal{F}(\left\|\eta\right\|_{H^{s+\frac{1}{2}}}) \left\{ \left\|f\right\|_{H^{\sigma+1}} + \left\|F_0\right\|_{Y^\sigma([-1,0])} + \left\|\nabla_{x,z}v\right\|_{X^{-\frac{1}{2}}([-1,0])} \right\},$$

for some non-decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ depending only on σ.

To prove Proposition 3.21 we shall proceed by induction on the regularity σ.

Definition 3.22. *Given σ such that $-1/2 \leq \sigma \leq s-1/2$, we say that the property $\mathcal{H}_\sigma$ is satisfied if for any interval $I \Subset (-1,0]$,*

$$\left\| \nabla_{x,z}v \right\|_{X^\sigma(I)} \leq \mathcal{F}(\left\|\eta\right\|_{H^{s+\frac{1}{2}}}) \left\{ \left\|f\right\|_{H^{\sigma+1}} + \left\|F_0\right\|_{Y^\sigma([-1,0])} + \left\|\nabla_{x,z}v\right\|_{X^{-\frac{1}{2}}([-1,0])} \right\}, \tag{3.48}$$

for some non-decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ depending only on I and σ.

With this definition, note that Assumption (3.47) means that property $\mathcal{H}_{-1/2}$ is satisfied. Consequently, Proposition 3.21 is an immediate consequence of the following proposition which will be proved in Sections 3.3 and 3.3.1.

Proposition 3.23. *Let $s > \frac{1}{2} + \frac{d}{2}$. For any ε such that*

$$0 < \varepsilon \leq \frac{1}{2}, \qquad \varepsilon < s - \frac{1}{2} - \frac{d}{2}, \tag{3.49}$$

if $\mathcal{H}_\sigma$ is satisfied for some $-1/2 \leq \sigma \leq s - 1/2 - \varepsilon$, then $\mathcal{H}_{\sigma+\varepsilon}$ is satisfied.

3.3. Nonlinear estimates.

Let us fix ε satisfying (3.49), σ such that

$$-\frac{1}{2} \leq \sigma \leq s - \frac{1}{2} - \varepsilon$$

and assume that $\mathcal{H}_\sigma$ is satisfied. We begin by estimating the coefficients α, β, γ in (3.17) in terms of $\left\|\eta\right\|_{H^{s+\frac{1}{2}}}$,

Lemma 3.24. *Let $J = [-1,0]$ and $s > \frac{1}{2} + \frac{d}{2}$. We have*

$$\left\| \alpha - \frac{h^2}{16} \right\|_{X^{s-\frac{1}{2}}(J)} + \left\|\beta\right\|_{X^{s-\frac{1}{2}}(J)} + \left\|\gamma\right\|_{X^{s-\frac{3}{2}}(J)} \leq \mathcal{F}(\left\|\eta\right\|_{H^{s+\frac{1}{2}}}). \tag{3.50}$$

Proof. According to (3.11), we can write

$$(\partial_z\rho)^2=\frac{h^2}{16}+G\quad\text{with } \|G\|_{X^{s-\frac12}(I)}\le \mathcal{F}(\|\eta\|_{H^{s+\frac12}(\mathbf{R}^d)}).$$

Noticing that $\frac{1}{1+|\nabla\rho|^2}=1-\frac{|\nabla\rho|^2}{1+|\nabla\rho|^2}$ we obtain

$$\alpha-\frac{h^2}{16}=-\Big(\frac{h^2}{16}\Big)\frac{|\nabla\rho|^2}{1+|\nabla\rho|^2}+G-G\frac{|\nabla\rho|^2}{1+|\nabla\rho|^2}$$

and we use Lemma 3.19 with $\sigma=s-\frac12$ together with (3.11). The estimates for β and γ are proved along the same lines. □

Lemma 3.25. *There exists a constant K such that for all $I\subset[-1,0]$,*

$$\|F_1\|_{Y^{\sigma+\varepsilon}(I)}\le K\,\|\gamma\|_{X^{s-\frac32}(I)}\,\|\partial_z v\|_{X^\sigma(I)}\,,\tag{3.51}$$

where $F_1=\gamma\partial_z v$.

Proof. We shall prove that, on the one hand, if $-1/2\le\sigma\le s-1-\varepsilon$ then

$$\|\gamma\partial_z v\|_{L^1(I;H^{\sigma+\varepsilon})}\lesssim\|\gamma\|_{L^2(I;H^{s-1})}\,\|\partial_z v\|_{L^2(I;H^{\sigma+\frac12})}\,,\tag{3.52}$$

and on the other hand, if $-\varepsilon\le\sigma\le s-\frac12-\varepsilon$ then

$$\|\gamma\partial_z v\|_{L^2(I;H^{\sigma-\frac12+\varepsilon})}\lesssim\|\gamma\|_{L^2(I;H^{s-1})}\,\|\partial_z v\|_{L^\infty(I;H^\sigma)}\,.\tag{3.53}$$

Since $s>\varepsilon+\frac12+d/2$, if $-1/2\le\sigma\le s-1-\varepsilon$ then

$$s-1+\sigma+\frac12>0,\ \sigma+\varepsilon\le\sigma+\frac12,\ \sigma+\varepsilon\le s-1,\ \sigma+\varepsilon<s-1+\sigma+\frac12-\frac d2.$$

and hence the product rule in Sobolev spaces (2.13) implies that

$$\|\gamma(z)\partial_z v(z)\|_{H^{\sigma+\varepsilon}}\lesssim\|\gamma(z)\|_{H^{s-1}}\,\|\partial_z v(z)\|_{H^{\sigma+\frac12}}\,.$$

Integrating in z and using the Cauchy-Schwarz inequality, we obtain (3.52). On the other hand, if $-\varepsilon\le\sigma\le s-\frac12-\varepsilon$ then one easily checks that

$$s-1+\sigma>0,\quad \sigma-\frac12+\varepsilon\le\sigma,\quad \sigma-\frac12+\varepsilon\le s-1,\quad \sigma-\frac12+\varepsilon<s-1+\sigma-\frac d2,$$

and hence the product rule (2.13) implies that

$$\|\gamma(z)\partial_z v(z)\|_{H^{\sigma-\frac12+\varepsilon}}\lesssim\|\gamma(z)\|_{H^{s-1}}\,\|\partial_z v(z)\|_{H^\sigma}\,.$$

Taking the L^2-norm in z, we obtain (3.53). □

Our next step is to replace the multiplication by α (resp. β) by the paramultiplication by T_α (resp. T_β).

Lemma 3.26. *There exists a constant K such that for all $I\subset[-1,0]$, v satisfies the paradifferential equation*

$$\partial_z^2 v+T_\alpha\Delta v+T_\beta\cdot\nabla\partial_z v=F_0+F_1+F_2,\tag{3.54}$$

for some remainder

$$F_2=(T_\alpha-\alpha)\Delta v+(T_\beta-\beta)\cdot\nabla\partial_z v\tag{3.55}$$

satisfying

(3.56)

$$\|F_2\|_{Y^{\sigma+\varepsilon}(I)}\le K\left\{1+\left\|\alpha-\frac{h^2}{16}\right\|_{X^{s-\frac12}([-1,0])}+\|\beta\|_{X^{s-\frac12}([-1,0])}\right\}\|\nabla_{x,z}v\|_{X^\sigma(I)}\,.$$

Proof. According to Proposition 2.10, we have

$$\|au - T_a u\|_{H^\gamma} \lesssim \|a\|_{H^r} \|u\|_{H^\mu},$$

provided that $r, \mu, \gamma \in \mathbf{R}$ satisfy

$$r + \mu > 0, \quad \gamma \le r \quad \text{and } \gamma < r + \mu - \frac{d}{2}. \tag{3.57}$$

Since $s > \varepsilon + 1/2 + d/2$, if $-1/2 \le \sigma \le s - \frac{1}{2} - \varepsilon$ then

$$s + \sigma - \frac{1}{2} > 0, \quad \sigma + \varepsilon \le s, \quad \sigma + \varepsilon < s + \sigma - \frac{1}{2} - \frac{d}{2},$$

and hence (3.57) applies with

$$\gamma = \sigma + \varepsilon, \quad r = s, \quad \mu = \sigma - \frac{1}{2}.$$

This implies that if $-1/2 \le \sigma \le s - \frac{1}{2} - \varepsilon$ then

$$\begin{aligned} &\|(T_\alpha - \alpha)\Delta v\|_{L^1(I;H^{\sigma+\varepsilon})} \lesssim \left(1 + \left\|\alpha - \frac{h^2}{16}\right\|_{L^2(I;H^s)}\right) \|\Delta v\|_{L^2(I;H^{\sigma-\frac{1}{2}})}, \\ &\|(T_\beta - \beta)\nabla\partial_z v\|_{L^1(I;H^{\sigma+\varepsilon})} \lesssim \|\beta\|_{L^2(I;H^s)} \|\nabla\partial_z v\|_{L^2(I;H^{\sigma-\frac{1}{2}})}, \end{aligned} \tag{3.58}$$

which yields

$$\|F_2\|_{Y^{\sigma+\varepsilon}(I)} \le \|F_2\|_{L^1(I;H^{\sigma+\varepsilon})} \lesssim \left\{1 + \left\|\alpha - \frac{h^2}{16}\right\|_{X^{s-\frac{1}{2}}(I)} + \|\beta\|_{X^{s-\frac{1}{2}}(I)}\right\} \|\nabla_{x,z} v\|_{X^\sigma(I)}.$$

This concludes the proof. □

Our next task is to perform a decoupling into a forward and a backward parabolic evolution equations. Recall that by assumption $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ with $s > \varepsilon + 1/2 + d/2$. In particular, $\eta \in C_*^{1+\varepsilon}(\mathbf{R}^d)$.

Lemma 3.27. *There exist two symbols a, A in $\Gamma^1_\varepsilon(\mathbf{R}^d \times [-1, 0])$ and a remainder F_3 such that,*

$$(\partial_z - T_a)(\partial_z - T_A)v = F_0 + F_1 + F_2 + F_3, \tag{3.59}$$

with

$$\mathcal{M}^1_\varepsilon(a) + \mathcal{M}^1_\varepsilon(A) \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}), \tag{3.60}$$

and

$$\|F_3\|_{L^2(I;H^{\sigma-\frac{1}{2}+\varepsilon})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{L^2(I;H^{\sigma+\frac{1}{2}})},$$

for some non-decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$.

Proof. We seek a, A satisfying

$$a(z; x, \xi)A(z; x, \xi) = -\alpha(x, z)|\xi|^2, \quad a(z; x, \xi) + A(z; x, \xi) = -i\beta(x, z) \cdot \xi.$$

We thus set

$$a = \frac{1}{2}\left(-i\beta\cdot\xi - \sqrt{4\alpha|\xi|^2 - (\beta\cdot\xi)^2}\right), \qquad A = \frac{1}{2}\left(-i\beta\cdot\xi + \sqrt{4\alpha|\xi|^2 - (\beta\cdot\xi)^2}\right). \tag{3.61}$$

Directly from the definition of α, β (3.17), note that

$$\exists c > 0; \quad \sqrt{4\alpha|\xi|^2 - (\beta\cdot\xi)^2} \ge c|\xi|.$$

According to (3.50) the symbols a, A belong to $\Gamma^1_\varepsilon(\mathbf{R}^d \times [-1,0])$ and they satisfy the bound (3.60). Therefore, we have

$$(3.62)\quad (\partial_z - T_a)(\partial_z - T_A)v = \partial_z^2 v - T_\beta \nabla \partial_z v + T_\alpha \Delta v + F_3, \qquad F_3 = R_0 v + R_1 v,$$

where

$$R_0(z) := T_{a(z)} T_{A(z)} - T_\alpha \Delta, \quad R_1(z) := -T_{\partial_z A(z)}.$$

According to Theorem 2.6, applied with $\rho = \varepsilon$, $R_0(z)$ is of order $2-\varepsilon$, uniformly in $z \in [-1,0]$. On the other hand, since

$$\partial_z \rho \in L^\infty((-1,0); H^{s-\frac{1}{2}}), \qquad \partial_z^2 \rho \in L^\infty((-1,0); H^{s-\frac{3}{2}}),$$

according to (2.13) we have

$$\partial_z \alpha, \partial_z \beta \in L^\infty((-1,0); H^{s-\frac{3}{2}}) \subset L^\infty((-1,0); C_*^{\varepsilon-1}).$$

Therefore $\partial_z A \in \Gamma^1_{\varepsilon-1}(\mathbf{R}^d \times [-1,0])$. As a consequence, using Proposition 2.12, we get that $R_1(z)$ is also of order $2-\varepsilon$. We end up with

$$\sup_{z\in[-1,0]} \|R_0(z)\|_{H^{\mu+2-\varepsilon}\to H^\mu} + \|R_1(z)\|_{H^{\mu+2-\varepsilon}\to H^\mu} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}}\big).$$

Now we notice that, given any symbol p and any function u, by definition of paradifferential operators we have $T_p u = T_p(1-\Psi(D_x))u$ for any Fourier multiplier $(I - \Psi(D_x))$ such that $\Psi(\xi) = 0$ for $|\xi| \ge 1/2$. This means that we can replace $\|v(z)\|_{H^{\sigma+\frac{3}{2}}}$ by $\|\nabla v(z)\|_{H^{\sigma+\frac{1}{2}}}$. We thus obtain the desired result from Lemma 3.26. □

3.3.1. *Proof of Proposition 3.23.* We shall apply Proposition 2.17 twice. At first we apply it to the *forward* parabolic evolution equation $\partial_z u - T_a u = F$ (by definition $\mathrm{Re}(-a) \ge c|\xi|$). This requires an initial data on $z=-1$ that might be chosen to be 0 by using a cut-off function, up to shrinking the interval I. Next we apply it to the *backward* parabolic evolution equation $\partial_z u - T_A u = F$ (by definition $\mathrm{Re}\, A \ge c|\xi|$). This requires an initial data on $z = 0$ (which is given by our assumption on f) and this requires also an estimate for the remainder term F which is given by means of the first step.

Suppose that $\mathcal{H}_\sigma$ is satisfied and let $I_0 = [\zeta_0, 0]$ with $\zeta_0 \in (-1,0)$. Then

$$\|\nabla_{x,z} v\|_{X^\sigma(I_0)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\Big\{ \|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma([-1,0])} + \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}([-1,0])} \Big\}.$$

We shall prove that, for any $0 > \zeta_1 > \zeta_0$,

$$(3.63)\quad \|\nabla_{x,z} v\|_{X^{\sigma+\varepsilon}([\zeta_1,0])} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\Big\{ \|f\|_{H^{\sigma+1+\varepsilon}} + \|F_0\|_{Y^{\sigma+\varepsilon}([-1,0])} + \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}([-1,0])} \Big\}.$$

Introduce a cutoff function χ such that $\chi\,|_{\zeta<\zeta_0} = 0$, $\chi\,|_{\zeta>\zeta_1} = 1$. Set $w := \chi(z)(\partial_z - T_A)v$. It follows from (3.59) for v that $\partial_z w - T_a w = F'$,

where

$$F' = \chi(z)(F_0 + F_1 + F_2 + F_3) + \chi'(z)(\partial_z - T_A)v.$$

We have already estimated F_1, F_2, F_3 and F_0 is given. We now turn to an estimate for $(\partial_z - T_A)v$. According to (2.4) and (3.60), we have

$$\|T_A v\|_{L^2(I_0;H^{\sigma+\frac{1}{2}})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla v\|_{L^2(I_0;H^{\sigma+\frac{1}{2}})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)},$$

and similarly

$$\|T_A v\|_{L^\infty(I_0;H^\sigma)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla v\|_{L^\infty(I_0;H^\sigma)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)},$$

Consequently

$$\|(\partial_z - T_A)v\|_{X^\sigma(I_0)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)}.$$

This implies that

$$\|w\|_{X^\sigma(I_0)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)}, \tag{3.64}$$

$$\|F'\|_{Y^{\sigma+\varepsilon}(I_0)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)} + \|F_0\|_{Y^{\sigma+\varepsilon}(I_0)}. \tag{3.65}$$

Since $w(x, z_0) = 0$ and since $a \in \Gamma^1_\varepsilon$ satisfies $\mathrm{Re}(-a(x,\xi)) \ge c|\xi|$, by using Proposition 2.17 applied with $J = I_0$, $\rho = \varepsilon$ and $r = \sigma + \varepsilon$, we have

$$\|w\|_{X^{\sigma+\varepsilon}(I_0)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|F'\|_{Y^{\sigma+\varepsilon}(I_0)},$$

and hence, using (3.64) and (3.65)

$$\|w\|_{X^{\sigma+\varepsilon}(I_0)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \big\{ \|\nabla_{x,z} v\|_{X^\sigma(I_0)} + \|F_0\|_{Y^{\sigma+\varepsilon}(I_0)} \big\}. \tag{3.66}$$

Now notice that on $I_1 := [\zeta_1, 0]$ we have $\chi = 1$ so that

$$\partial_z v - T_A v = w \quad \text{for } z \in I_1.$$

Therefore the function $\widetilde{v}$ defined by $\widetilde{v}(x,z) = v(x,-z)$ satisfies

$$\partial_z \widetilde{v} + T_{\widetilde{A}} \widetilde{v} = -\widetilde{w} \quad \text{for } z \in \widetilde{I}_1 = [0, -\zeta_1],$$

with obvious notations for $\widetilde{w}$ and $\widetilde{A}$. By using Proposition 2.17 with $J = \widetilde{I}_1$, noticing that $\widetilde{v}|_{z=0} = v|_{z=0} = f$, we obtain that

$$\|\widetilde{v}\|_{X^{\sigma+1+\varepsilon}(\widetilde{I}_1)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \big(\|f\|_{H^{\sigma+1+\varepsilon}} + \|\widetilde{w}\|_{Y^{\sigma+1+\varepsilon}(\widetilde{I}_1)} \big).$$

Using the obvious estimate

$$\|\widetilde{w}\|_{Y^{\sigma+1+\varepsilon}(\widetilde{I}_1)} = \|w\|_{Y^{\sigma+1+\varepsilon}(I_1)} \le \|w\|_{L^2_z(I_1; H^{\sigma+\frac12+\varepsilon})} \le \|w\|_{X^{\sigma+\varepsilon}(I_1)},$$

it follows from (3.66) that

$$\|v\|_{X^{\sigma+1+\varepsilon}(I_1)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \big(\|f\|_{H^{\sigma+1+\varepsilon}} + \|\nabla_{x,z} v\|_{X^\sigma(I_0)} + \|F_0\|_{Y^{\sigma+\varepsilon}(I_0)} \big).$$

We easily estimate $\partial_z v$ directly from $\partial_z v = T_A v + w$ (by using (3.66) and the fact that T_A is an operator of order 1). This completes the proof of (3.63). This proves that if $\mathcal{H}_\sigma$ is satisfied then $\mathcal{H}_{\sigma+\varepsilon}$ is satisfied and hence concludes the proof of Proposition 3.23 (and hence the proof of Proposition 3.21).

3.3.2. *Proof of Theorem 3.17.* Let v be the solution of (3.16) with data $v|_{z=0} = f$. By definition of the Dirichlet–Neumann operator we have

$$G(\eta) f = \frac{1 + |\nabla \rho|^2}{\partial_z \rho} \partial_z v - \nabla \rho \cdot \nabla v \,\big|_{z=0}. \tag{3.67}$$

Now, by applying Proposition 3.21 with $F_0 = 0$ and Remark 3.20, we find that if v solves (3.16), then for any $I \Subset (-1, 0]$,

$$\|\nabla_{x,z} v\|_{X^{\sigma-1}(I)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|f\|_{H^\sigma}. \tag{3.68}$$

According to (3.11) and (2.14), we obtain that

$$\left\| \frac{1 + |\nabla \rho|^2}{\partial_z \rho} \partial_z v - \nabla \rho \cdot \nabla v \right\|_{C^0([z_0,0]; H^{\sigma-1})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|f\|_{H^\sigma}.$$

As a result, taking the trace on $z = 0$ immediately implies the desired result (3.43).

3.3.3. *Proof of Proposition 3.18.* Let $1/2 \le \sigma_0 \le s$. It follows from (3.66) applied with $\sigma = \sigma_0 - 1$ and $F_0 = 0$ that

$$\|\chi(z)(\partial_z v - T_A v)\|_{X^{\sigma_0-1+\varepsilon}(I_0)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})\big\{ \|\nabla_{x,z} v\|_{X^{\sigma_0-1}(I_0)} \big\},$$

for some cut-off function χ such that $\chi(0) = 1$. By using Proposition 3.21, we thus obtain

$$\|\partial_z v - T_A v|_{z=0}\|_{H^{\sigma_0-1+\varepsilon}} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big) \|f\|_{H^{\sigma_0}}. \tag{3.69}$$

The previous estimate allows us to express the "normal" derivative $\partial_z v$ in terms of the tangential derivatives. Which is the main step to paralinearize the Dirichlet-Neumann operator.

Now, as mentioned above, by definition of v,

$$G(\eta)f = \frac{1 + |\nabla\rho|^2}{\partial_z\rho}\partial_z v - \nabla\rho\cdot\nabla v\,\big|_{z=0}.$$

Set

$$\zeta_1 := \frac{1+|\nabla\rho|^2}{\partial_z\rho}, \quad \zeta_2 := \nabla\rho.$$

According to (3.11),

$$\Big\|\zeta_1 - \frac{4}{h}\Big\|_{C^0_z([-1,0];H^{s-\frac12}_x)} + \|\zeta_2\|_{C^0_z([-1,0];H^{s-\frac12}_x)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}). \tag{3.70}$$

Let

$$R' = \zeta_1\partial_z v - \zeta_2\cdot\nabla v - (T_{\zeta_1}\partial_z v - T_{\zeta_2}\nabla v).$$

Since $\varepsilon \le \frac12$ and $\varepsilon < s - \frac12 - \frac{d}{2}$, we verify that Proposition 2.10 applies with

$$\gamma = \sigma_0 - 1 + \varepsilon, \quad r = s - \frac12, \quad \mu = \sigma_0 - 1,$$

which, according to (3.70) and (3.68), implies

$$\|R'\|_{C^0(I;H^{\sigma_0-1+\varepsilon})} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big) \|f\|_{H^{\sigma_0}}.$$

Furthermore, according to (3.69) and (3.70), we obtain

$$T_{\zeta_1}\partial_z v - T_{\zeta_2}\nabla v\,\big|_{z=0} - (T_{\zeta_1}T_A v - T_{i\zeta_2\cdot\xi} v\,\big|_{z=0}) = R'',$$

with

$$\|R''\|_{H^{\sigma_0-1+\varepsilon}} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big) \|f\|_{H^{\sigma_0}}.$$

Finally, thanks to (2.5), (3.70) and (3.60), we have

$$\big\|T_{\zeta_1(z)}T_{A(z)} - T_{\zeta_1(z)A(z)}\big\|_{H^{\sigma_0}\to H^{\sigma_0-\frac12}} \lesssim \|\zeta_1(z)\|_{W^{\varepsilon,\infty}} \mathcal{M}^1_\varepsilon(A) \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}),$$

and hence

$$G(\eta)f = T_{\zeta_1 A} v - T_{i\zeta_2\cdot\xi} v\,\big|_{z=0} + R(\eta)f$$

where

$$\|R(\eta)f\|_{H^{\sigma_0-1+\varepsilon}} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|f\|_{H^{\sigma_0}}.$$

Let

$$\lambda = \frac{1+|\nabla\rho|^2}{\partial_z\rho}A - i\nabla\rho\cdot\xi\,\big|_{z=0} = \sqrt{(1+|\nabla\eta(x)|^2)|\xi|^2 - \big(\nabla\eta(x)\cdot\xi\big)^2}.$$

Then

$$G(\eta)f = T_\lambda f + R(\eta)f,$$

which concludes the proof of Proposition 3.18.

4. A priori estimates in Sobolev spaces

In this section, we shall prove *a priori* estimates on smooth solutions on a fixed time interval $[0,T]$. Recall that the system reads

$$\left\{\begin{aligned}&\partial_t\eta - G(\eta)\psi = 0,\\&\partial_t\psi + g\eta + \frac{1}{2}\left|\nabla\psi\right|^2 - \frac{1}{2}\frac{\big(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\big)^2}{1+|\nabla\eta|^2} = 0.\end{aligned}\right.\tag{4.1}$$

As already mentioned, we work with the unknowns $B = B(\eta,\psi)$ and $V = V(\eta,\psi)$ defined by

$$B := \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2},\qquad V := \nabla\psi - B\nabla\eta.$$

It follows from Theorem 3.17 that, for all $s > 1 + d/2$ and all $(\eta,\psi)\in H^{s+\frac{1}{2}}$, B and V are well defined and belong to $H^{s-\frac{1}{2}}$. Moreover, we shall prove that if they belong initially to H^s then this regularity is propagated by the equation. We shall prove estimates in terms of

$$\begin{aligned}M_s(T) &:= \sup_{\tau\in[0,T]}\left\|(\psi(\tau),B(\tau),V(\tau),\eta(\tau))\right\|_{H^{s+\frac{1}{2}}\times H^s\times H^s\times H^{s+\frac{1}{2}}},\\M_{s,0} &:= \left\|(\psi(0),B(0),V(0),\eta(0))\right\|_{H^{s+\frac{1}{2}}\times H^s\times H^s\times H^{s+\frac{1}{2}}}.\end{aligned}\tag{4.2}$$

The main result of this section is the following proposition.

Proposition 4.1. *Let $d\geq 1$ and consider $s > 1+\frac{d}{2}$. Consider a fluid domain such that, there exists $h>0$ such that for all $t\in[0,T]$,*

$$\Big\{(x,y)\in \mathbf{R}^d\times\mathbf{R}\,:\,\eta(t,x)-h<y<\eta(t,x)\Big\}\subset\Omega(t).\tag{4.3}$$

Assume that for any $t\in[0,T]$,

$$a(t,x)\geq c_0,$$

for some given positive constant c_0. Then, there exists a non-decreasing function $\mathcal{F}\colon\mathbf{R}^+\to\mathbf{R}^+$ such that, for all $T\in(0,1]$ and all smooth solution (η,ψ) of (4.1) defined on the time interval $[0,T]$, there holds

$$M_s(T)\leq\mathcal{F}\big(\mathcal{F}(M_{s,0})+T\mathcal{F}\big(M_s(T)\big)\big).\tag{4.4}$$

Remark 4.2. The assumption (4.3) holds provided that it holds initially at time 0 and $\left\|\eta-\eta\,|_{t=0}\right\|_{H^{s+\frac{1}{2}}}\leq\epsilon$, for some small enough positive constant ϵ.

4.1. A new formulation.

Since we consider low regularity solutions, various cancellations have to be used. We found that these cancellations are most easily seen by working with the incompressible Euler equation directly, and hence we do not use the Zakharov formulation. This means that we begin with a new formulation of the water waves system which involves the following unknowns

$$\zeta=\nabla\eta,\quad B=\partial_y\phi|_{y=\eta},\quad V=\nabla_x\phi|_{y=\eta},\quad a=-\partial_yP|_{y=\eta},\tag{4.5}$$

where recall that ϕ is the velocity potential and the pressure $P=P(t,x,y)$ is given by

$$-P=\partial_t\phi+\frac{1}{2}\left|\nabla_{x,y}\phi\right|^2+gy.\tag{4.6}$$

Proposition 4.3. *Let $s > \frac{1}{2} + \frac{d}{2}$. We have*

$$(\partial_t + V \cdot \nabla)B = a - g, \tag{4.7}$$

$$(\partial_t + V \cdot \nabla)V + a\zeta = 0, \tag{4.8}$$

$$(\partial_t + V \cdot \nabla)\zeta = G(\eta)V + \zeta G(\eta)B + \gamma, \tag{4.9}$$

where the remainder term $\gamma = \gamma(\eta, \psi, V)$ satisfies the following estimate :

$$\|\gamma\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}(\|(\eta, \psi, V)\|_{H^{s+\frac{1}{2}} \times H^s \times H^s}). \tag{4.10}$$

Remark 4.4. In the case $\Gamma = \emptyset$, one can see that at least formally $\gamma = 0$.

Proof. For any function $f = f(t, x, y)$, by using the chain rule, we check that, with $\nabla = \nabla_x$,

$$\begin{aligned}(\partial_t + V \cdot \nabla)(f|_{y=\eta(t,x)}) &= (\partial_t + V \cdot \nabla) f(t, x, \eta(t, x)) \\ &= \big[\partial_t f + \nabla\phi \cdot \nabla f + \partial_y f(\partial_t\eta + V \cdot \nabla\eta)\big]\big|_{y=\eta(t,x)} \\ &= \big[(\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y}) f\big]\big|_{y=\eta(t,x)},\end{aligned}$$

since $\partial_t\eta + V \cdot \nabla\eta = B$ (see (3.21)). Applying ∂_y to (4.6), this identity yields (4.7). On the other hand, applying ∂_{x_k} to (4.6), the previous identity gives

$$(\partial_t + V \cdot \nabla)V + (\nabla P)|_{y=\eta} = 0.$$

Since $P|_{y=\eta} = 0$, we have

$$0 = \nabla(P|_{y=\eta}) = (\nabla P)|_{y=\eta} + (\partial_y P)|_{y=\eta}\nabla\eta,$$

which yields (4.8).

To derive equation (4.9) on $\zeta := \nabla\eta$ we start from

$$\partial_t\eta = B - V \cdot \nabla\eta$$

Differentiating with respect to x_i (for $i = 1, \dots, d$) we find that $\partial_i\eta = \partial_{x_i}\eta$ satisfies

$$(\partial_t + V \cdot \nabla)\partial_i\eta = \partial_i B - \sum_{j=1}^{d} \partial_i V_j \partial_j\eta, \tag{4.11}$$

Starting from the definitions of B and V ($B = \partial_y\phi|_{y=\eta}$, $V = \nabla\phi|_{y=\eta}$), and using the chain rule, we compute that

$$\begin{aligned}\partial_i B - \sum_{j=1}^{d} \partial_i V_j \partial_j\eta &= \big[\partial_i\partial_y\phi + \partial_i\eta\partial_y^2\phi\big]\big|_{y=\eta} - \sum_{j=1}^{d} \partial_j\eta\big[\partial_i\partial_j\phi + \partial_i\eta\partial_j\partial_y\phi\big]\big|_{y=\eta} \\ &= \big[\partial_y\partial_i\phi - \sum_{j=1}^{d} \partial_j\eta\partial_i\partial_j\phi\big]\big|_{y=\eta} + \partial_i\eta\big[\partial_y^2\phi - \sum_{j=1}^{d} \partial_j\eta\partial_j\partial_y\phi\big]\big|_{y=\eta}.\end{aligned}$$

Therefore

$$(\partial_t + V \cdot \nabla)\partial_i\eta = \big[\partial_y\partial_i\phi - \nabla\eta \cdot \nabla\partial_i\phi\big]\big|_{y=\eta} + \partial_i\eta\big[\partial_y(\partial_y\phi) - \nabla\eta \cdot \nabla\partial_y\phi\big]\big|_{y=\eta}. \tag{4.12}$$

Let now θ_i be the variational solution of the problem

$$\Delta_{x,y}\theta_i = 0 \text{ in } \Omega, \quad \theta_i|_{y=\eta} = V_i, \quad \partial_n\theta_i = 0 \text{ on } \Gamma.$$

Then

$$G(\eta)V_i = \sqrt{1 + |\nabla\eta|^2}\frac{\partial\theta_i}{\partial n}|_{y=\eta} = (\partial_y\theta_i - \nabla\eta \cdot \nabla\theta_i)|_{y=\eta}.$$

Then we write

$$(\partial_y - \nabla\eta \cdot \nabla)\partial_i\phi|_{y=\eta} = G(\eta)V_i + R_i, \quad \text{where} \quad R_i = (\partial_y - \nabla\eta \cdot \nabla)(\partial_i\phi - \theta_i)|_{y=\eta}. \tag{4.13}$$

Due to the presence of the bottom we have to localize the problem near Σ.
Let $\chi_0 \in C^\infty(\mathbf{R})$, $\eta_1 \in H^\infty(\mathbf{R}^d)$ be such that $\chi_0(z) = 1$ if $z \geq 0, \chi_0(z) = 0$ if $z \leq -\frac{1}{4}$ and

$$\eta(x) - \frac{h}{4} \leq \eta_1(x) \leq \eta(x) - \frac{h}{5}.$$

Set

$$U_i(x,y) = \chi_0\big(\frac{y-\eta_1(x)}{h}\big)(\partial_i\phi - \theta_i)(x,y).$$

We see easily that $R_i = (\partial_y - \nabla\eta\cdot\nabla)U_i|_{y=\eta}$. Moreover U_i satisfies the equation

$$\Delta_{x,y}U_i = \big[\Delta_{x,y}, \chi_0\big(\frac{y-\eta_1(x)}{h}\big)\big](\partial_i\phi - \theta_i) := F_i \tag{4.14}$$

and, with a slight change of notation, we have

$$\operatorname{supp} F_i \subset S_{\frac{1}{2},\frac{1}{5}} := \big\{(x,y) : x\in\mathbf{R}^d, \eta(x) - \frac{h}{2} \leq y \leq \eta(x) - \frac{h}{5}\big\}. \tag{4.15}$$

Moreover by ellipticity (see Lemma 3.16) we have for all $\alpha \in \mathbf{N}^{d+1}$,

$$\|D^\alpha_{x,y}F_i\|_{L^\infty(S_{\frac{1}{2},\frac{1}{5}})\cap L^2(S_{\frac{1}{2},\frac{1}{5}})} \leq C_\alpha \|(V,B)\|_{H^{\frac{1}{2}}\times H^{\frac{1}{2}}}. \tag{4.16}$$

Now we change variables. We set $x = x, y = \rho(x,z) = (1+z)e^{\delta z\langle D_x\rangle}\eta(x) - z\eta_*(x)$ and $\widetilde{g_i}(x,z) = g_i(x,\rho(x,z))$. Since we have taken $\delta\|\eta\|_{H^{s+\frac{1}{2}}} \leq \frac{1}{2}$ it is easy to see that on the image of $S_{\frac{1}{2},\frac{1}{5}}$ one has $-h \leq z \leq -\frac{h}{10}$. Now, according to section 3.1.1, $\widetilde{U_i}$ is a solution of the problem

$$(\partial_z^2 + \alpha\Delta + \beta\cdot\nabla\partial_z - \gamma\partial_z)\widetilde{U_i} = \frac{(\partial_z\rho)^2}{1+|\nabla\rho|^2}\widetilde{F_i}.$$

Due to the exponential smoothing and to (4.16), on the support of $\widetilde{F_i}$ the right hand side of the above equation belongs in fact to $C^0_z((-h,0);H^\infty(\mathbf{R}^d))$. In particular we can apply Proposition 3.21 with $f = 0$. It follows that

$$\|\nabla_{x,z}\widetilde{U_i}\|_{C^0([z_0,0];H^{s-\frac{1}{2}}(\mathbf{R}^d))} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\big(\|\widetilde{F_i}\|_{Y^\sigma([-1,0])} + \|\nabla_{x,z}\widetilde{U_i}\|_{X^{-\frac{1}{2}}([-1,0])}\big)\ .$$

Notice that according to the constructions of variational solutions and (3.20), the norm of $\widetilde{U_i}$ in $X^{-\frac{1}{2}}([-1,0])$ is bounded by

$$\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\big(\|\psi\|_{H^{\frac{1}{2}}} + \|V_i\|_{H^{\frac{1}{2}}}\big).$$

Since

$$R_i = \Big[\big(\frac{1+|\nabla\eta|^2}{1+\delta\langle D_x\rangle\eta}\partial_z - \nabla\eta\cdot\nabla\big)\widetilde{U_i}\Big]\Big|_{z=0},$$

we deduce that

$$\|R_i\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\big(\|\psi\|_{H^{\frac{1}{2}}} + \|V_i\|_{H^{\frac{1}{2}}}\big) \leq \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}}, \|\psi\|_{H^s}, \|V\|_{H^s}\big),$$

since $s > \frac{1}{2} + \frac{d}{2}$. We use exactly the same argument to show that

$$(\partial_y - \nabla\eta\cdot\nabla)\partial_y\phi|_{y=\eta} = G(\eta)B + R_0, \tag{4.17}$$

where R_0 satisfies the same estimate as R_i. This completes the proof. □

Following the same lines, we have the following relation between V and B.

Proposition 4.5. *Let $s > \frac{1}{2} + \frac{d}{2}$. Then we have $G(\eta)B = -\operatorname{div} V + \gamma$ where*

$$\|\gamma\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta,V,B)\|_{H^{s+\frac{1}{2}}\times H^{\frac{1}{2}}\times H^{\frac{1}{2}}}).$$

Proof. Recall that, by definition, $B=\partial_y\phi|_{y=\eta}$ and $V=\nabla\phi|_{y=\eta}$. Let θ be the variational solution to the problem

$$\Delta_{x,y}\theta=0,\quad \theta|_{y=\eta}=B,\quad \partial_n\theta|_\Gamma=0.$$

Then $G(\eta)B=(\partial_y\theta-\nabla\eta\cdot\nabla\theta)\big|_{y=\eta}$. Now let $\widetilde{\theta}=\partial_y\phi$. We claim that

$$(\partial_y\widetilde{\theta}-\nabla\eta\cdot\nabla\widetilde{\theta})\big|_{y=\eta}=-\operatorname{div}V.$$

Indeed, on the one hand we have

$$(\partial_y\widetilde{\theta}-\nabla\eta\cdot\nabla\widetilde{\theta})=\partial_y^2\phi-\nabla\eta\cdot\nabla\partial_y\phi,$$

and on the other hand

$$\operatorname{div}V=\sum_{1\le i\le d}\partial_{x_i}V=\Big(\sum_{1\le i\le d}\partial_i^2\phi+\nabla\eta\cdot\partial_y\phi\Big)\Big|_{y=\eta}.$$

Then our claim follows from the fact that $\Delta_{x,y}\phi=0$. Now we have

$$\Delta_{x,y}(\theta-\widetilde{\theta})=0,\quad (\theta-\widetilde{\theta})|_{y=\eta}=0,$$

so, as in the proof of Proposition 4.3, we deduce from Proposition 3.21 that

$$\left\|(\partial_y-\nabla\eta\cdot\nabla)(\theta-\widetilde{\theta})\right\|_{H^{s-\frac12}}\le\mathcal{F}(\|(\eta,V,B)\|_{H^{s+\frac12}\times H^{\frac12}\times H^{\frac12}}),$$

which is the desired result. □

4.2. Estimates for the Taylor coefficient. In this paragraph, we prove several estimates for the Taylor coefficient.

Proposition 4.6. *Let $d\ge1$ and $s>1+\frac{d}{2}$. There exists a non-decreasing function $\mathcal{F}\colon\mathbf{R}^+\to\mathbf{R}^+$ such that, for all $t\in[0,T]$,*

$$\|a(t)-g\|_{H^{s-\frac12}}\le\mathcal{F}\big(\|(\eta,\psi,V,B)(t)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big),\tag{4.18}$$

For $0<\varepsilon<s-1-d/2$, there exists a non-decreasing function $\mathcal{F}$ such that,

$$\|(\partial_t a+V\cdot\nabla a)(t)\|_{C^\varepsilon}\le\mathcal{F}\big(\|(\eta,\psi,V,B)(t)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big).\tag{4.19}$$

Recall that $a=-\partial_yP|_{y=\eta}$ where

$$P=P(t,x,y)=-\big(\partial_t\phi+\frac12|\nabla_x\phi|^2+\frac12(\partial_y\phi)^2+gy\big).$$

The basic idea is that one should be able to easily estimate P since it satisfies an elliptic equation. Indeed, since $\Delta_{x,y}\phi=0$, we have

$$\Delta_{x,y}P=-\left|\nabla_{x,y}^2\phi\right|^2.$$

Moreover, by assumption we have $P=0$ on the free surface $\{y=\eta(t,x)\}$. Yet, this requires some preparation because, as we shall see, the regularity of P is *not* given by the right-hand side in the elliptic equation above. Instead the regularity of P is limited by the regularity of the domain (i.e. the regularity of the function η).

Hereafter, since the time variable is fixed, we shall skip it. We use the change of variables $(x,z)\mapsto(x,\rho(x,z))$ introduced in §3.1.1. Introduce φ and $\wp$ given by

$$\varphi(x,z)=\phi(x,\rho(x,z)),\quad \wp(x,z)=P(x,\rho(x,z))+g\rho(x,z),$$

and notice that

$$a-g=-\frac{1}{\partial_z\rho}\partial_z\wp\,|_{z=0}\,.$$

The first elementary step is to compute the equation satisfied by the new unknown v in $\{z<0\}$ as well as the boundary conditions on $\{z=0\}$. Set (see (3.13))

$$\Lambda=(\Lambda_1,\Lambda_2),\quad \Lambda_1=\frac{1}{\partial_z\rho}\partial_z,\quad \Lambda_2=\nabla-\frac{\nabla\rho}{\partial_z\rho}\partial_z.$$

We find that

$$\begin{aligned}&(\Lambda_1^2+\Lambda_2^2)\varphi=0 \text{ in } \quad -1<z<0,\\&(\Lambda_1^2+\Lambda_2^2)\wp=-\left|\Lambda^2\varphi\right|^2 \text{ in } \quad -1<z<0,\\&(\Lambda_1^2+\Lambda_2^2)\rho=0 \text{ in } \quad z<0,\end{aligned}$$

together with the boundary conditions

$$\begin{aligned}&\wp=g\eta,\quad \Lambda_1\wp=g-a \quad \text{on } \ z=0,\\&\Lambda_2\varphi=V,\quad \Lambda_1\varphi=B,\quad \text{on } \ z=0.\end{aligned}$$

According to (3.5) and Remark 3.20, we have the *a priori estimate*

$$\|\nabla_{x,z}\varphi\|_{X^{-\frac12}(-\frac{h}{2},0)}\le\mathcal{F}(\|\eta\|_{H^{s+\frac12}})\|\psi\|_{H^{\frac12}},$$

while according to Proposition 3.21

(4.20)

$$\|\nabla_{x,z}\wp\|_{X^{-\frac12}(-1,0)}\le\mathcal{F}\Big(\|\mathcal{R}\|_{X^{\frac12}(-1,0)}+\||\nabla\varphi|^2\|_{X^{\frac12}(-1,0)}\Big)\le\mathcal{F}\big(\|(\eta,\psi)\|_{H^{s+\frac12}}\big).$$

where $\mathcal{R}(x,z)=R(x,\rho(x,z))$ and R is defined in Definition 1.5.

Expanding $\Lambda_1^2+\Lambda_2^2$, we thus find that $\wp$ solves

$$\begin{aligned}&\partial_z^2\wp+\alpha\Delta\wp+\beta\cdot\nabla\partial_z\wp-\gamma\partial_z\wp=F_0(x,z) && \text{for } z<0,\\&\wp=g\eta && \text{on } z=0,\end{aligned}\tag{4.21}$$

where α,β,γ are as above (see (3.15)) and where

$$F_0=-\alpha\left|\Lambda^2\varphi\right|^2.\tag{4.22}$$

Our first task is to estimate the source term F_0.

Lemma 4.7. *Let $d\ge1$ and $s>1+d/2$. Then there exists $z_0<0$ such that*

$$\|F_0\|_{L^1([z_0,0];H^{s-\frac12})}\le\mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big).$$

Proof. Since $[\Lambda_1,\Lambda_2]=0$ we have

$$(\Lambda_1^2+\Lambda_2^2)\Lambda_2\varphi=0,\quad (\Lambda_1^2+\Lambda_2^2)\Lambda_1\varphi=0.$$

Since $\Lambda_2\varphi|_{z=0}=V$ and $\Lambda_1\varphi|_{z=0}=B$, it follows from Proposition 3.21 (and Theorem 3.8 which guarantees that $\nabla_{x,z}\varphi\in X^{-\frac12}(z_0,0)$) that

$$\|\nabla_{x,z}\Lambda_j\varphi\|_{X^{s-1}([z_0,0])}\le\mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big).$$

By using the easy estimate (3.11)

$$\|\nabla_x\rho\|_{C^0([z_0,0];H^{s-\frac12})}+\left\|\partial_z\rho-\frac{h}{4}\right\|_{C^0([z_0,0];H^{s-\frac12})}\lesssim\|\eta\|_{H^{s+\frac12}},$$

and the product rule in Sobolev spaces, we obtain

$$\|\Lambda_j\Lambda_k\varphi\|_{L^2([z_0,0];H^{s-\frac12})}\le\mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big).\tag{4.23}$$

Since $H^{s-\frac12}$ is an algebra, according to Lemma 3.24, we obtain

$$\begin{aligned}(4.24)\qquad \left\Vert F_0\right\Vert_{L^1([z_0,0];H^{s-\frac12})} &\lesssim \left(1+\left\Vert \alpha-\frac{h^2}{16}\right\Vert_{C^0([z_0,0];H^{s-\frac12})}\right)\left\Vert \Lambda_j\Lambda_k\varphi\right\Vert^2_{L^2([z_0,0];H^{s-\frac12})}\\ &\le \mathcal{F}\big(\left\Vert(\eta,\psi,V,B)\right\Vert_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big).\end{aligned}$$

This completes the proof. □

It follows from Lemma 4.7 and Proposition 3.21 applied with $\sigma=s-1/2$ that there exists z_0 such that

$$(4.25)\qquad \left\Vert\nabla_{x,z}\wp\right\Vert_{X^{s-\frac12}([z_0,0])}\le \mathcal{F}\big(\left\Vert(\eta,\psi)\right\Vert_{H^{s+\frac12}},\left\Vert F_0\right\Vert_{L^1([z_0,0];H^{s-\frac12})}\big).$$

where we used the estimate (4.20) According to (4.24), this implies that

$$(4.26)\qquad \left\Vert\nabla_{x,z}\wp\right\Vert_{X^{s-\frac12}([z_0,0])}\le \mathcal{F}\big(\left\Vert(\eta,\psi,V,B)\right\Vert_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big).$$

which in turn implies that $\left\Vert a-g\right\Vert_{H^{s-\frac12}}$ is bounded by a constant depending only on $\left\Vert(\eta,\psi)\right\Vert_{H^{s+\frac12}}$ and $\left\Vert(V,B)\right\Vert_{H^s}$.

4.3. Paralinearization of the system.

Introduce

$$(4.27)\qquad U=V+T_\zeta B.$$

To clarify notations, let us mention that the ith component ($i=1,\dots,d$) of this vector valued unknown satisfies $U_i=V_i+T_{\partial_i\eta}B$. The new unknown U is related to what is called the good-unknown of Alinhac in [4, 1, 5, 7].

To estimate (U,ζ) in Sobolev spaces, we want to estimate $(\langle D_x\rangle^s U,\langle D_x\rangle^{s-\frac12}\zeta)$ in $L^\infty([0,T];L^2\times L^2)$ where $\langle D_x\rangle:=(I-\Delta)^{1/2}$. However, for technical reasons, instead of working with $(\langle D_x\rangle^s U,\langle D_x\rangle^{s-\frac12}\zeta)$, it is more convenient to work with

$$(4.28)\qquad \begin{aligned}U_s&:=\langle D_x\rangle^s V+T_\zeta\langle D_x\rangle^s B,\\ \zeta_s&:=\langle D_x\rangle^s\zeta.\end{aligned}$$

Proposition 4.8. *Under the assumptions of Proposition 4.1, there exists a non decreasing function $\mathcal{F}$ such that*

$$(4.29)\qquad (\partial_t+T_V\cdot\nabla)U_s+T_a\zeta_s=f_1,$$

$$(4.30)\qquad (\partial_t+T_V\cdot\nabla)\zeta_s=T_\lambda U_s+f_2,$$

where recall that λ is the symbol

$$\lambda(t;x,\xi):=\sqrt{(1+\left\vert\nabla\eta(t,x)\right\vert^2)\left\vert\xi\right\vert^2-(\nabla\eta(t,x)\cdot\xi)^2},$$

and where, for each time $t\in[0,T]$,

$$(4.31)\qquad \left\Vert(f_1(t),f_2(t))\right\Vert_{L^2\times H^{-\frac12}}\le\mathcal{F}\big(\left\Vert\eta(t)\right\Vert_{H^{s+\frac12}},\left\Vert(V,B)(t)\right\Vert_{H^s}\big).$$

Proof. The proof is based on the paralinearization of the Dirichlet-Neumann operator (see Proposition 3.18), the Bony's paralinearization formula for a product, some simple computations and the commutator estimate proved in Section 2.4.

Step 1: Paralinearization of the equation

$$(\partial_t+V\cdot\nabla)V+a\zeta=0.$$

We begin by proving

Lemma 4.9. *We have*

$$(4.32)\qquad \begin{aligned}&(\partial_t + T_V\cdot\nabla)V + T_a\zeta + T_\zeta(\partial_t + T_V\cdot\nabla)B = h_1 \quad \text{with}\\ &\|h_1\|_{H^s} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}, \|(V,B)\|_{H^s}\big).\end{aligned}$$

PROOF. Using (2.10) and (2.4) we have $V\cdot\nabla V = T_V\cdot\nabla V + A_1$ where $A_1 = \sum_j T_{\partial_j V} V_j + R(\partial_j V, V_j)$ satisfies

$$\|A_1\|_{H^s} \lesssim \|\nabla V\|_{L^\infty}\|V\|_{H^s}.$$

Similarly, $(a-g)\zeta = T_{a-g}\zeta + T_\zeta(a-g) + R(\zeta, a-g)$ where

$$(4.33)\qquad \|R(\zeta, a-g)\|_{H^s} \lesssim \|\zeta\|_{H^{s-1/2}}\|a-g\|_{H^{s-1/2}}.$$

and where $\|a-g\|_{H^{s-1/2}}$ is estimated by means of (4.18).

Since $T_\zeta g = 0$, by replacing a by $g + (\partial_t B + V\cdot\nabla B)$ we obtain

$$\begin{aligned}T_\zeta a &= T_\zeta(\partial_t B + V\cdot\nabla B)\\ &= T_\zeta(\partial_t B + T_V\cdot\nabla B) + T_\zeta(V - T_V)\cdot\nabla B.\end{aligned}$$

As in the analysis of A_1 above, we have

$$\|(V - T_V)\cdot\nabla B\|_{H^s} \lesssim \|\nabla B\|_{L^\infty}\|V\|_{H^s}.$$

Now we use $\|T_\zeta\|_{H^s\to H^s} \lesssim \|\zeta\|_{L^\infty} \lesssim \|\eta\|_{H^{s+1/2}}$ (since $s+1/2 > 1 + d/2$) to obtain

$$\|T_\zeta(V - T_V)\cdot\nabla B\|_{H^s} \lesssim \|\eta\|_{H^{s+\frac12}}\|\nabla B\|_{L^\infty}\|V\|_{H^s}.$$

By Sobolev injection, this proves (4.32). □

STEP 2. We now commute (4.32) with $\langle D_x\rangle^s = (I-\Delta)^{s/2}$. The paradifferential rule (2.5) implies that

$$\begin{aligned}&\|[T_a, \langle D_x\rangle^s]\|_{H^{s-1/2}\to L^2} \lesssim \|a\|_{W^{1/2,\infty}} \lesssim 1 + \|a-g\|_{H^{s-1/2}},\\ &\|[T_\zeta, \langle D_x\rangle^s]\|_{H^{s-1/2}\to L^2} \lesssim \|\zeta\|_{W^{1/2,\infty}} \lesssim \|\zeta\|_{H^{s-1/2}},\\ &\|[T_V\cdot\nabla, \langle D_x\rangle^s]\|_{H^s\to L^2} \lesssim \|V\|_{W^{1,\infty}} \lesssim \|V\|_{H^s}.\end{aligned}$$

Consequently, it easily follows from (4.19) and (4.32) that

$$(\partial_t + T_V\cdot\nabla)\langle D_x\rangle^s V + T_a\langle D_x\rangle^s\zeta + T_\zeta(\partial_t + T_V\cdot\nabla)\langle D_x\rangle^s B = h_2$$

for some remainder h_2 satisfying $\|h_2\|_{L^2} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}, \|(V,B)\|_{H^s}\big)$.

On the other hand, Lemma 2.15 implies that

$$\|[T_\zeta, \partial_t + T_V\cdot\nabla]\langle D_x\rangle^s B(t)\|_{L^2} \le \mathcal{F}\big(\|\eta(t)\|_{H^{s+\frac12}}, \|(V,B)(t)\|_{H^s}\big).$$

Here we have used the fact that the L^∞ norm of $\partial_t\zeta + V\cdot\nabla\zeta$ is, since $s > \frac12 + \frac d2$, estimated by means of the identity (4.11):

$$\begin{aligned}\|\partial_t\zeta + V\cdot\nabla\zeta\|_{L^\infty} &\lesssim \|\nabla B\|_{L^\infty} + \|\zeta\|_{L^\infty}\|\nabla V\|_{L^\infty}\\ &\lesssim \|\nabla B\|_{L^\infty} + \|\eta\|_{H^{s+\frac12}}\|\nabla V\|_{L^\infty}.\end{aligned}$$

By combining the previous results we obtain

$$(\partial_t + T_V\cdot\nabla)\big(\langle D_x\rangle^s V + T_\zeta\langle D_x\rangle^s B\big) + T_a\langle D_x\rangle^s\zeta = f_1$$

where f_1 satisfies the desired estimate (4.31).

STEP 3. Paralinearization of the equation

$$(\partial_t + V\cdot\nabla)\zeta = G(\eta)V + \zeta G(\eta)B + \gamma.$$

Writing $(V - T_V) \cdot \nabla\zeta = T_{\nabla\zeta} \cdot V + \sum_{j=1}^{d} R(\partial_j \zeta, V_j)$ and using (2.10) and (2.11), we obtain

$$\left\| (V - T_V) \cdot \nabla\zeta \right\|_{H^{s-\frac12}} \lesssim \left\| \nabla\zeta \right\|_{C_*^{-\frac12}} \left\| V \right\|_{H^s} \lesssim \left\| \eta \right\|_{C_*^{\frac32}} \left\| V \right\|_{H^s}. \tag{4.34}$$

The key step is to paralinearize $G(\eta)V + \zeta G(\eta)B$. This is where we use the analysis performed in the previous Section. By definition of $R(\eta) = G(\eta) - T_\lambda$ we have

$$G(\eta)V + \zeta G(\eta)B = T_\lambda U + F_2(\eta, V, B),$$

where

$$F_2 = [T_\zeta, T_\lambda]B + R(\eta)V + \zeta R(\eta)B + (\zeta - T_\zeta)T_\lambda B. \tag{4.35}$$

The commutator $[T_\zeta, T_\lambda]B$ is estimated by means of (2.5) which implies that

$$\left\| [T_\zeta, T_\lambda]B \right\|_{H^{s-\frac12}} \lesssim \left\{ M_0^0(\zeta) M_{1/2}^1(\lambda) + M_{1/2}^0(\zeta) M_0^1(\lambda) \right\} \left\| B \right\|_{H^s}.$$

Since $M_{1/2}^0(\zeta) + M_{1/2}^1(\lambda) \le \mathcal{K}\big(\left\| \eta \right\|_{H^{s+\frac12}}\big)$ we conclude that

$$\left\| [T_\zeta, T_\lambda]B \right\|_{H^{s-\frac12}} \le \mathcal{K}\big(\left\| \eta \right\|_{H^{s+\frac12}}, \left\| B \right\|_{H^s}\big).$$

Moving to the estimate of the second and third terms in the right-hand side of (4.35), we use Proposition 3.18 to obtain that the $H^{s-\frac12}$-norm of $R(\eta)V$ and $R(\eta)B$ satisfy

$$\left\| R(\eta(t))V(t) \right\|_{H^{s-\frac12}} + \left\| R(\eta(t))B(t) \right\|_{H^{s-\frac12}} \le \mathcal{F}\big(\left\| \eta(t) \right\|_{H^{s+\frac12}}, \left\| (V,B)(t) \right\|_{H^s}\big).$$

Since $H^{s-\frac12}$ is an algebra, the term $\zeta R(\eta)B$ satisfies the same estimate as $R(\eta)B$ does. It remains only to estimate $(\zeta - T_\zeta)T_\lambda B$. To do so we write

$$(\zeta - T_\zeta)T_\lambda B = T_{T_\lambda B}\zeta + R(\zeta, T_\lambda B).$$

Thus (2.10) (applied with $\alpha = 0$ and $\beta = s - 1/2$) implies that

$$\left\| (\zeta - T_\zeta)T_\lambda B \right\|_{H^{s-\frac12}} \lesssim \left\| T_\lambda B \right\|_{C_*^0} \left\| \zeta \right\|_{H^{s-\frac12}}.$$

Using (2.4) this yields

$$\left\| (\zeta - T_\zeta)T_\lambda B \right\|_{H^{s-\frac12}} \lesssim M_0^1(\lambda) \left\| B \right\|_{C_*^1} \left\| \zeta \right\|_{H^{s-\frac12}}.$$

We thus end up with $\left\| F_2 \right\|_{H^{s-\frac12}} \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac12}}, \left\| (V,B) \right\|_{H^s}\big)$.

By combining the previous results, we obtain

$$(\partial_t + T_V \cdot \nabla)\zeta = T_\lambda U + h_3, \tag{4.36}$$

where $\left\| h_3 \right\|_{H^{s-\frac12}} \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac12}}, \left\| (V,B) \right\|_{H^s}\big)$. As in the second step, by commuting the equation (4.36) with $\langle D_x \rangle^s$ we obtain the desired result (4.30), which concludes the proof. □

4.4. Symmetrization of the equations. We shall use Proposition 4.8. To prove an L^2 estimate for System (4.29)–(4.30), we begin by performing a symmetrization of the non-diagonal part. Here we use in an essential way the fact that the Taylor coefficient a is a positive function. Again, let us mention that this assumption is automatically satisfied for infinitely deep fluid domain: this result was first proved by Wu (see [**51, 52**]) and one can check that the proof remains valid for any $C^{1,\alpha}$-domain, with $0 < \alpha < 1$.

Proposition 4.10. *Introduce the symbols*

$$\gamma = \sqrt{a\lambda}, \quad q = \sqrt{\frac{a}{\lambda}},$$

and set $\theta_s = T_q \zeta_s$. *Then*

$$\partial_t U_s + T_V \cdot \nabla U_s + T_\gamma \theta_s = F_1, \tag{4.37}$$

$$\partial_t \theta_s + T_V \cdot \nabla \theta_s - T_\gamma U_s = F_2, \tag{4.38}$$

for some source terms F_1, F_2 *satisfying*

$$\|(F_1(t), F_2(t))\|_{L^2 \times L^2} \le \mathcal{F}\big(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|(V,B)(t)\|_{H^s}\big).$$

Proof. Directly from (4.29)–(4.30), we obtain (4.37)–(4.38) with

$$F_1 := f_1 + \big(T_\gamma T_q - T_a\big)\zeta_s,$$
$$F_2 := T_q f_2 + \big(T_q T_\lambda - T_\gamma\big)U_s - [T_q, \partial_t + T_V \cdot \nabla]\, \zeta_s.$$

The commutator between T_q and $\partial_t + T_V \cdot \nabla$ is estimated by means of Lemma 2.15:

$$\big\| [T_q, \partial_t + T_V \cdot \nabla] \zeta_s \big\|_{L^2(\mathbf{R}^d)} \le K \left\{ \mathcal{M}_0^{-\frac{1}{2}}(q)\, \|V\|_{C_*^{1+\varepsilon}} + \mathcal{M}_0^{-\frac{1}{2}}(\partial_t q + V \cdot \nabla q) \right\} \times \|\zeta_s\|_{H^{-\frac{1}{2}}(\mathbf{R}^d)}. \tag{4.39}$$

$T_q f_2$ is estimated by means of (2.4). The key point is to estimate $(T_\gamma T_q - T_a)\zeta_s$ and $(T_q T_\lambda - T_\gamma)U_s$. Since $\gamma q = a$, the operator $T_\gamma T_q - T_a$ if of order $-1/2$ since γ is a symbol of order $1/2$, q is of order $-1/2$, and since these symbols are $C^{1/2}$ in x. Similarly, since $q\lambda = \gamma$, the operator $T_q T_\lambda - T_\gamma$ is of order 0. More precisely, by using the tame estimate for symbolic calculus (see (2.5)), we obtain

$$\|T_\gamma T_q - T_a\|_{H^{-\frac{1}{2}} \to L^2} \lesssim M_{1/2}^{1/2}(\gamma) M_0^{-1/2}(q) + M_0^{1/2}(\gamma) M_{1/2}^{-1/2}(q),$$
$$\|T_q T_\lambda - T_\gamma\|_{L^2 \to L^2} \lesssim M_{1/2}^{-1/2}(q) M_0^1(\lambda) + M_0^{-1/2}(q) M_{1/2}^1(\lambda).$$

The above semi-norms are easily estimated by means of the $C^{1/2}$ norms of $\zeta = \nabla\eta$ and a (given by the Sobolev injection and Proposition 4.6). □

We are now in position to prove an L^2 estimate for (U_s, θ_s).

Lemma 4.11. *There exists a non-decreasing function* $\mathcal{F}$ *such that*

$$\|U_s\|_{L^\infty([0,T];L^2)} + \|\theta_s\|_{L^\infty([0,T];L^2)} \le \mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T)). \tag{4.40}$$

Remark 4.12. The fact that this implies corresponding estimates for the Sobolev norms of η, ψ, V, B is explained below in §4.5.

Proof. Multiply (4.37) by U_s and (4.38) by θ_s and integrate in space to obtain

$$\frac{d}{dt}\left\{ \|U_s(t)\|_{L^2}^2 + \|\theta_s(t)\|_{L^2}^2 \right\} + (I) + (II) = (III),$$

where

$$(I) := \langle T_{V(t)} \cdot \nabla U_s(t), U_s(t)\rangle + \langle T_{V(t)} \cdot \nabla \theta_s(t), \theta_s(t)\rangle,$$
$$(II) := \langle T_{\gamma(t)} \theta_s(t), U_s(t)\rangle - \langle T_{\gamma(t)} U_s(t), \theta_s(t)\rangle,$$
$$(III) := \langle F_1, U_s\rangle + \langle F_2, \theta_s\rangle.$$

Then the key points are that (see point (iii) in Theorem 2.6)

$$\left\|(T_{V(t)}\cdot\nabla)^*+T_{V(t)}\cdot\nabla\right\|_{L^2\to L^2}\lesssim\|V(t)\|_{W^{1,\infty}},$$

and

$$\left\|T_{\gamma(t)}-(T_{\gamma(t)})^*\right\|_{L^2\to L^2}\lesssim M_{1/2}^{1/2}(\gamma(t)).$$

We then easily obtain (4.40). □

4.5. Back to estimates for the original unknowns. Up to now, we only estimated (U_s,θ_s) in $L^\infty([0,T];L^2\times L^2)$. In this section, we shall show how we can recover estimates for the original unknowns (η,ψ,V,B) in $L^\infty([0,T];H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s)$. Recall that the functions U_s and θ_s are obtained from (η,V,B) through:

$$\begin{aligned}U_s&:=\langle D_x\rangle^s V+T_\zeta\langle D_x\rangle^s B,\\ \theta_s&:=T_{\sqrt{a/\lambda}}\langle D_x\rangle^s\nabla\eta.\end{aligned}$$

The analysis is in four steps:

(i) We first prove some estimates for (B,V,η) and the Taylor coefficient a in some low order norms.
(ii) Then, by using the previous estimate of θ_s, we show how to recover an estimate of the $L^\infty([0,T],H^{s+\frac12})$-norm of η.
(iii) Once η is estimated in $L^\infty([0,T],H^{s+\frac12})$, by using the estimate for U_s, we estimate (B,V) in $L^\infty([0,T];H^s)$. Here we make an essential use of our first result on the paralinearization of the Dirichlet-Neumann operator (see Proposition 3.18). Namely, we use the fact that one can paralinearize the Dirichlet-Neumann operator for any domain whose boundary is in H^μ for some $\mu>1+d/2$.
(iv) The desired estimate for ψ follows directly from the previous estimates for η,V,B, the identity $\nabla\psi=V+B\nabla\eta$ and the fact that one easily obtain an $L^\infty([0,T];L^2)$-estimate for ψ.

We begin with the following lemma.

Lemma 4.13. *There exists a non-decreasing function $\mathcal{F}$ such that,*

$$\|\eta\|_{L^\infty([0,T];H^s)}+\|(B,V)\|_{L^\infty([0,T];H^{s-\frac12})}\le\mathcal{F}(M_{s,0})+T\mathcal{F}(M_s(T)).\tag{4.41}$$

and, for any $0<\varepsilon<s-1-d/2$,

$$\|a\|_{L^\infty([0,T];C^\varepsilon_*)}\le\mathcal{F}(M_{s,0})+\sqrt{T}\mathcal{F}.\tag{4.42}$$

Proof. The proof is based on the fact that it is easy to estimate the solution w of a transport equation of the form $\partial_t w+V\cdot\nabla w=F$. Indeed, by using the estimates (4.18)–(4.19) for a, tame product rules in Sobolev or Hölder spaces and the identity $\partial_t\eta+V\cdot\nabla\eta=B$, we readily obtain that there exists a non-decreasing function C (depending only on parameters that are considered fixed) such that

$$\begin{aligned}\|a-g\|_{H^{s-\frac12}}&=\|\partial_t B+V\cdot\nabla B\|_{H^{s-\frac12}}\le\mathcal{C}(t),\\ \|a\zeta\|_{H^{s-\frac12}}&=\|\partial_t V+V\cdot\nabla V\|_{H^{s-\frac12}}\le\mathcal{C}(t),\\ \|\partial_t\eta+V\cdot\nabla\eta\|_{H^s}&\le\mathcal{C}(t),\\ \|\partial_t a+V\cdot\nabla a\|_{C^\varepsilon_*}&\le\mathcal{C}(t),\end{aligned}$$

where $\mathcal{C}(t)=C\big(\|\eta(t)\|_{H^{s+\frac12}},\|(V,B)(t)\|_{H^s}\big)$.

Let us come back to the proof of Lemma 4.13. To fix matters, we prove the estimate for V only (the proofs of the estimates for B, η and a are similar) and we begin by proving the Sobolev estimate. Using the obvious estimate $\|h\|_{L^1([0,T])} \le T\|h\|_{L^\infty([0,T])}$, note that $F_V := \partial_t V + V\cdot\nabla V$ satisfies

$$\|F_V\|_{L^1([0,T];C^{\frac12}_*\cap H^{s-\frac12})} \le T\mathcal{F}(M_s(T)).$$

Using Proposition 2.14 with $\sigma = s-\frac12$, we estimate V in $L^\infty([0,T];H^{s-\frac12})$. □

Lemma 4.14. *There exists a non-decreasing function $\mathcal{F}$ such that*

$$\|\eta\|_{L^\infty([0,T];H^{s+\frac12})} \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T)\big)\big). \tag{4.43}$$

Proof. Chose ε and an integer N such that

$$0<\varepsilon<s-1-\frac{d}{2},\qquad (N+1)\varepsilon>\frac12.$$

Set $R = I - T_{1/q}T_q$ to obtain

$$\zeta_s = T_{1/q}T_q\zeta_s + R\zeta_s,$$

where recall that $\zeta_s = \langle D_x\rangle^s\zeta$. Consequently,

$$\zeta_s = (I+R+\cdots+R^N)T_{1/q}T_q\zeta_s + R^{N+1}\zeta_s.$$

By definition of $q=\sqrt{a/\lambda}$, Theorem 2.6 implies that, for all $\mu\in\mathbf{R}$, there exists a non-decreasing function $\mathcal{F}$ depending only on ε and $\inf_{(t,x)\in[0,T]\times\mathbf{R}^d} a(t,x)>0$ such that,

$$\|R(t)\|_{H^\mu\to H^{\mu+\varepsilon}} \le \mathcal{F}\big(\|a(t)\|_{C^\varepsilon_*}, \|\eta(t)\|_{C^{1+\varepsilon}_*}\big),$$

and

$$\big\|T_{1/q(t)}\big\|_{H^{\mu+1/2}\to H^\mu} \le \mathcal{F}\big(\|\eta(t)\|_{W^{1,\infty}}\big).$$

Therefore

$$\|\nabla\eta\|_{H^{s-\frac12}} = \|\zeta_s\|_{H^{-\frac12}} \le \mathcal{F}\big(\|a\|_{C^\varepsilon_*}, \|\eta\|_{C^{1+\varepsilon}_*}\big)\left\{\|T_q\zeta_s\|_{L^2} + \|\zeta_s\|_{H^{-1}}\right\}.$$

Now it follows from Lemma 4.13 that

$$\|a\|_{L^\infty([0,T];C^\varepsilon_*)} + \|\eta\|_{L^\infty([0,T];C^{1+\varepsilon}_*)} + \|\zeta_s\|_{L^\infty([0,T];H^{-1})} \le \mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T)).$$

On the other hand, it follows from Lemma 4.11 that

$$\|T_q\zeta_s\|_{L^\infty([0,T];L^2)} \le \mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T)).$$

This implies the desired result. □

It remains only to estimate (V,B).

Lemma 4.15. *There exists a non-decreasing function $\mathcal{F}$ such that*

$$\|(V,B)\|_{L^\infty([0,T];H^s)} \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T)\big)\big). \tag{4.44}$$

Proof. The proof is based on the relation between V and B given by Proposition 4.5.

Step 1. Recall that $U = V + T_\zeta B$. We begin by proving that there exists a non-decreasing function $\mathcal{F}$ such that

$$\|U\|_{L^\infty([0,T];H^s)} \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T)\big)\big). \tag{4.45}$$

To see this, write

$$\langle D_x\rangle^s U = U_s + \big[\langle D_x\rangle^s, T_\zeta\big]B$$

and use Theorem 2.6 to obtain

$$\left\| \left[\langle D_x \rangle^s , T_\zeta \right] B \right\|_{L^2} \lesssim \left\| \zeta \right\|_{C_*^{\frac12}} \left\| B \right\|_{H^{s-\frac12}} .$$

Since, by assumption, $s > 1 + d/2$ we have $\left\| \zeta \right\|_{C_*^{\frac12}} \lesssim \left\| \zeta \right\|_{H^{s-\frac12}} \leq \left\| \eta \right\|_{H^{s+\frac12}}$ and hence

$$\left\| U \right\|_{H^s} \lesssim \left\| U_s \right\|_{L^2} + \left\| \eta \right\|_{H^{s+\frac12}} \left\| B \right\|_{H^{s-\frac12}} .$$

The three terms in the right-hand side of the above inequalities have been already estimated (see Lemma 4.11 for U_s, Lemma 4.13 for B and Lemma 4.14 for η). This proves (4.45).

STEP 2. Taking the divergence in $U = V + T_\zeta B$, we get according to Proposition 4.5, Lemma 4.13 and Lemma 4.14:

$$\begin{aligned} \operatorname{div} U &= \operatorname{div} V + \operatorname{div} T_\zeta B = \operatorname{div} V + T_{\operatorname{div}\zeta} B + T_\zeta \cdot \nabla B \\ &= -G(\eta) B + T_{i\zeta\cdot\xi + \operatorname{div}\zeta} B + \gamma \\ &= -T_\lambda B + R(\eta) B + T_{i\zeta\cdot\xi + \operatorname{div}\zeta} B + \gamma \\ &= T_q B + R(\eta) B + T_{\operatorname{div}\zeta} B + \gamma \end{aligned}$$

where, by notation,

$$q := -\lambda + i\zeta \cdot \xi,$$

and

$$\left\| \gamma \right\|_{H^{s-\frac12}} \leq \mathcal{F}(\left\| (\eta, V, B) \right\|_{H^{s+\frac12} \times H^{\frac12} \times H^{\frac12}}).$$

According to Proposition 3.18 (with $\mu = s - \frac12$) and Lemma 4.13, we deduce

$$T_q B = \operatorname{div} U - T_{\operatorname{div}\zeta} B - R(\eta) B - \gamma. \tag{4.46}$$

Now write

$$B = T_{\frac1q} T_q B + \Big(I - T_{\frac1q} T_q \Big) B$$

to obtain from (4.46)

$$B = T_{\frac1q} \operatorname{div} U - T_{\frac1q} \gamma + R_{-\epsilon} B$$

where

$$R_{-\epsilon} := T_{\frac1q} \Big(-T_{\operatorname{div}\zeta} - R(\eta) \Big) + \Big(I - T_{\frac1q} T_q \Big). \tag{4.47}$$

Notice now that according to Lemma 4.14, we control $\operatorname{div}\zeta = \Delta\eta$ in $H^{s-\frac32}$, and since $s > 1 + \frac{d}{2}$, $T_{\operatorname{div}\zeta}$ is an operator of order (both Sobolev and Hölder) $1 - \frac12 = \frac12$. Finally, $q = -\lambda + i\zeta\cdot\xi \in \Gamma^1_{1/2}$ with $M^1_{1/2}(q) \leq C(\left\| \eta \right\|_{H^{s+\frac12}})$. Moreover, q^{-1} is of order -1 and we have

$$M^1_{1/2}(q^{-1}) \leq C(\left\| \eta \right\|_{H^{s+\frac12}}).$$

Consequently, according to (2.4) and (2.5), the operator $R_{-\epsilon}$ given by (4.47) is of order $-\frac12$. applying $T_{(-\lambda + i\zeta\cdot\xi)^{-1}}$ to (4.46), we get

$$B = W + R_{-\epsilon} B$$

where $W := T_{\frac1q} \operatorname{div} U - T_{\frac1q} \gamma$ satisfies

$$\left\| W \right\|_{H^s} \leq \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T)\big)\big).$$

Since $R_{-\epsilon}$ is an operator of order $-\frac12$ and since we have estimated the $H^{s-\frac12}$-norm of B (see Lemma 4.13), we conclude that

$$\left\| B \right\|_{H^s} \leq \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T)\big)\big),$$

and coming back to the relation $U = V + T_\zeta B$ we get that V satisfies the same estimate. □

Lemma 4.16. *There exists a non-decreasing function $\mathcal{F}$ such that*

$$\|\psi\|_{L^\infty([0,T];H^{s+\frac{1}{2}})} \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T)\big)\big).$$

PROOF. Since $\nabla\psi = V + B\nabla\eta$ and since the $L^\infty([0,T];H^{s-\frac{1}{2}})$-norm of $(\nabla\eta, V, B)$ has been previously estimated, it remains only to estimate $\|\psi\|_{L^\infty([0,T];L^2)}$.
Since

$$B := \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2},$$

the equation for ψ can be written under the form

$$\partial_t\psi + g\eta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}(1+|\nabla\eta|^2)B^2 = 0.$$

Therefore, since $V = \nabla\psi - B\nabla\eta$,

$$\begin{aligned}\partial_t\psi + V\cdot\nabla\psi &= \partial_t\psi + |\nabla\psi|^2 - B\nabla\eta\cdot\nabla\psi\\ &= -g\eta + \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}(1+|\nabla\eta|^2)B^2 - B\nabla\eta\cdot\nabla\psi\\ &= -g\eta + \frac{1}{2}|\nabla\psi - B\nabla\eta|^2 - \frac{1}{2}B^2|\nabla\eta|^2 + \frac{1}{2}(1+|\nabla\eta|^2)B^2\\ &= -g\eta + \frac{1}{2}V^2 + \frac{1}{2}B^2.\end{aligned} \tag{4.48}$$

The desired L^2 estimate then follows from classical results (see Proposition 2.14). □

5. Contraction

In this section we prove a contraction estimate for the difference of two solutions which implies the uniqueness of solutions and a Lipschitz property in a lower norm ($\mathcal{H}^{s-1}$, compared to the $\mathcal{H}^s$ norm where the *a priori* estimates are established). This phenomenon is standard for quasi-linear PDE's. This choice of norm to establish the contraction property is the result of a compromise as on the one hand, the highest the norm is chosen the easiest the non linear analysis will be (as the norm controls more quantities), while some loss of derivatives are necessary (in particular as far as the Dirichlet–Neumann operator is concerned), see Remark 5.3.

Theorem 5.1. *Let (η_j, ψ_j), $j = 1, 2$, be two solutions of (1.6) such that*

$$(\eta_j, \psi_j, V_j, B_j) \in C^0([0, T_0]; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s),$$

for some fixed $T_0 > 0$, $d \ge 1$ and $s > 1 + d/2$. We also assume that the condition (1.2) holds for $0 \le t \le T_0$ and that there exists a positive constant c such that for all $0 \le t \le T_0$ and for all $x \in \mathbf{R}^d$, we have $a_j(t,x) \ge c$ for $j = 1, 2$, $t \in [0,T]$. Set

$$M_j := \sup_{t\in[0,T]} \|(\eta_j, \psi_j, V_j, B_j)(t)\|_{H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s},$$

$$\eta := \eta_1 - \eta_2, \quad \psi = \psi_1 - \psi_2, \quad V := V_1 - V_2, \quad B = B_1 - B_2.$$

Then we have

$$\begin{aligned}&\|(\eta,\psi,V,B)\|_{L^\infty((0,T);H^{s-\frac{1}{2}}\times H^{s-\frac{1}{2}}\times H^{s-1}\times H^{s-1})}\\ &\qquad\le \mathcal{K}(M_1, M_2)\|(\eta,\psi,V,B)\,|_{t=0}\,\|_{H^{s-\frac{1}{2}}\times H^{s-\frac{1}{2}}\times H^{s-1}\times H^{s-1}}.\end{aligned} \tag{5.1}$$

Let us recall that

$$
(5.2)\qquad \begin{cases} (\partial_t + V_j\cdot\nabla)B_j = a_j - g,\\ (\partial_t + V_j\cdot\nabla)V_j + a_j\zeta_j = 0,\\ (\partial_t + V_j\cdot\nabla)\zeta_j = G(\eta_j)V_j + \zeta_j G(\eta_j)B_j + \gamma_j, \quad \zeta_j = \nabla\eta_j, \end{cases}
$$

where γ_j is the remainder term given by (4.9). Let

$$
N(T) := \sup_{t\in[0,T]} \left\| (\eta,\psi,V,B)(t)\right\|_{H^{s-\frac12}\times H^{s-\frac12}\times H^{s-1}\times H^{s-1}}.
$$

Our goal is to prove an estimate of the form

$$
(5.3)\qquad N(T) \le \mathcal{K}(M_1,M_2)N(0) + T\,\mathcal{K}(M_1,M_2)N(T),
$$

for some non-decreasing function $\mathcal{K}$ depending only on s and d. Then, by choosing T small enough, this implies $N(T)\le 2\mathcal{K}(M_1,M_2)N(0)$ for T_1 smaller than the minimum of T_0 and $1/2\mathcal{K}(M_1,M_2)$, and iterating the estimate between $[T_1,2T_1]$,..., $[T-T_1,T_1]$ implies Theorem 5.1.

5.1. Contraction for the Dirichlet-Neumann. The first step in the proof of Theorem 5.1 is to prove a Lipschitz property for the Dirichlet-Neumann operator. This was already achieved in a very weak norm in Theorem 3.10, and here we used elliptic theory to improve the result.

Theorem 5.2. *Assume that $s > 1+\frac{d}{2}$. There exists a non-decreasing function $\mathcal{F}$ such that, for all $\eta_1,\eta_2\in H^{s+\frac12}$ and all $f\in H^s$, we have*

$$
(5.4)\qquad \left\| [G(\eta_1)-G(\eta_2)]\,f\right\|_{H^{s-\frac32}} \le \mathcal{F}\big(\left\|(\eta_1,\eta_2)\right\|_{H^{s+\frac12}}\big)\left\|\eta_1-\eta_2\right\|_{H^{s-\frac12}}\left\| f\right\|_{H^s}.
$$

Remark 5.3. We were unable to prove a similar estimate in a higher norm. On the other hand, this estimate is in some sense stronger than Theorem 3.10. Indeed, in view of Sobolev injections, the r.h.s. here does not control the Lipschitz norm of $(\eta_1-\eta_2)$ which appears in Theorem 3.10.

Proof. The proof follows closely that of Theorem 3.10 and we keep the notations $\rho_j,\widetilde{\phi}_j, v=\widetilde{\phi}_1-\widetilde{\phi}_2,\Lambda^j$ introduced there.

Notice that, using the smoothing property of the Poisson kernel, we have

$$
(5.5)\qquad \begin{cases} (i)\quad \Lambda_k^1-\Lambda_k^2 = \beta_k\partial_z, \quad \text{with }\operatorname{supp}\beta_k\subset \mathbf{R}^d\times J, \text{where } J=[-1,0],\\ (ii)\quad \left\|\beta_k\right\|_{L^2(J,H^{s-1}(\mathbf{R}^d))} \le \mathcal{F}(\left\|(\eta_1,\eta_2)\right|_{H^{s+\frac12}\times H^{s+\frac12}})\left\|\eta_1-\eta_2\right\|_{H^{s-\frac12}(\mathbf{R}^d)}. \end{cases}
$$

Recall that

$$
(5.6)\qquad G(\eta_j)f = U_j\vert_{z=0},\quad U_j = \Lambda_1^j\widetilde{\phi}_j - \nabla_x\rho_j\cdot\Lambda_2^j\widetilde{\phi}_j.
$$

Let us set $U=U_1-U_2$. According to Lemma 3.14, Theorem 5.2 will follow from the following estimate

(5.7)

$$
\left\| U\right\|_{L^2(J,H^{s-1})} + \left\|\partial_z U\right\|_{L^2(J,H^{s-2})} \le \mathcal{F}\big(\left\|(\eta_1,\eta_2)\right\|_{H^{s+\frac12}\times H^{s+\frac12}}\big)\left\| f\right\|_{H^s}\left\|\eta_1-\eta_2\right\|_{H^{s-\frac12}}.
$$

According to (3.36) and (3.38) the estimate (5.7) will be a consequence of the following one
(5.8)
$$\sum_{k=1}^{5} \|B_k\|_{L^2(J,H^{s-1})} \leq \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}}\big)\, \|f\|_{H^s}\, \|\eta_1-\eta_2\|_{H^{s-\frac12}} \quad \text{where}$$
$$B_1=\Lambda_1^1 v,\quad B_2=(\nabla_{x,z}\rho_2)\Lambda_2^2 v,\quad B_3=(\Lambda_1^1-\Lambda_1^2)\widetilde{\phi}_2,\quad B_4=\nabla_{x,z}(\rho_1-\rho_2)\Lambda_2^1\widetilde{\phi}_1,$$
$$B_5=(\nabla_{x,z}\rho_2)(\Lambda_1^1-\Lambda_1^2)\widetilde{\phi}_1.$$
Since $\widetilde{\phi}_j$ is a variational solution, Proposition 3.21 with $\sigma=s-1$ show that
$$\|\nabla_{x,z}\widetilde{\phi}_j\|_{L^\infty(J,H^{s-1})}+\|\Lambda_k^j\widetilde{\phi}_j\|_{L^\infty(I,H^{s-1})} \leq \mathcal{F}(\|\eta_j\|_{H^{s+\frac12}})\|f\|_{H^s}.$$
Since $s>1+\frac{d}{2}$, it follows the from (3.33) that
$$\sum_{l=3}^{5}\|B_l\|_{L^2(J,H^{s-1})} \leq \mathcal{F}(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}})\|\eta_1-\eta_2\|_{H^{s-\frac12}}\|f\|_{H^s}. \tag{5.9}$$
Since
$$\|B_1\|_{L^2(J,H^{s-1})}+\|B_2\|_{L^2(J,H^{s-1})} \leq \mathcal{F}(\|(\eta_1,\eta_2)|_{H^{s+\frac12}\times H^{s+\frac12}})\|\nabla_{x,z}v\|_{L^2(I,H^{s-1})}$$
using the estimate (5.9), we see that (5.8) will be a consequence of the following Lemma. Therefore Theorem 5.2 will be proved if we prove the following result.

Lemma 5.4. *We have*
$$\|\nabla_{x,z}v\|_{L^2(J,H^{s-1})} \leq \mathcal{F}(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}})\|\eta_1-\eta_2\|_{H^{s-\frac12}}\|f\|_{H^s}.$$

Proof. Notice that $v=\widetilde{\phi}_1-\widetilde{\phi}_2$ is a solution of the problem
$$\partial_z^2 v+\alpha_1\Delta v+\beta_1\cdot\nabla\partial_z v-\gamma_1\partial_z v=F,\quad v|_{z=0}=0 \tag{5.10}$$
where
$$F=(\alpha_2-\alpha_1)\Delta\widetilde{\phi}_2+(\beta_2-\beta_1)\cdot\nabla\partial_z\widetilde{\phi}_2-(\gamma_2-\gamma_1)\partial_z\widetilde{\phi}_2$$
and α_j are given by (3.17). We would like to apply Proposition 3.21 with $\sigma=s-\frac32$. To this end, according to (2.37), we shall estimate the $L^2(J,H^{s-2}(\mathbf{R}^d))$ norm of F and the $X^{-\frac12}(J)$ norm of $\nabla_{x,z}v$.

Estimate on F: Since $s>1+\frac{d}{2}$ (thus $2s-3>0$) we may apply (2.13) with $s_1=s-2, s_2=s-1, s_0=s-2$. We get
$$\Big\|(\alpha_1-\alpha_2)\Delta\widetilde{\phi}_2\Big\|_{L^2(J,H^{s-2})} \leq K\,\|\alpha_1-\alpha_2\|_{L^2(J,H^{s-1})}\Big\|\Delta\widetilde{\phi}_2\Big\|_{L^\infty(J,H^{s-2})},$$
$$\Big\|(\beta_1-\beta_2)\cdot\nabla\partial_z\widetilde{\phi}_2\Big\|_{L^2(J,H^{s-2})} \leq K\,\|\beta_1-\beta_2\|_{L^2(J,H^{s-1})}\Big\|\nabla\partial_z\widetilde{\phi}_2\Big\|_{L^\infty(J,H^{s-2})},$$
$$\Big\|(\gamma_1-\gamma_2)\partial_z\widetilde{\phi}_2\Big\|_{L^2(J,H^{s-2})} \leq K\,\|\gamma_1-\gamma_2\|_{L^2(J,H^{s-2})}\Big\|\partial_z\widetilde{\phi}_2\Big\|_{L^\infty(J,H^{s-1})}.$$
Then, using the product rule in Sobolev space (2.13), and (3.11), we obtain
$$\begin{aligned}\|\alpha_1-\alpha_2\|_{L^2(J,H^{s-1})}+\|\beta_1-\beta_2\|_{L^2(J,H^{s-1})}+\|\gamma_1-\gamma_2\|_{L^2(J,H^{s-2})}\\ \leq \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}}\big)\,\|\eta_1-\eta_2\|_{H^{s-\frac12}}.\end{aligned} \tag{5.11}$$
Moreover from from Proposition 3.21 with $\sigma=s-1$ we have
$$\Big\|\nabla_{x,z}\widetilde{\phi}_j\Big\|_{L^\infty(J,H^{s-1})} \leq C(\|\eta_j\|_{H^{s+\frac12}})\,\|f\|_{H^s}.$$

It follows that

$$\|F\|_{L^2_z(J,H^{s-2}_x)} \le \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}}\big)\,\|\eta_1-\eta_2\|_{H^{s-\frac12}}\,\|f\|_{H^s} \tag{5.12}$$

Estimate of $\|\nabla_{x,z}v\|_{X^{-\frac12}(J)}$, $J=(-1,0)$.

We claim that

$$\|\nabla_{x,z}v\|_{X^{-\frac12}(J)} \le \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}}\big)\,\|\eta_1-\eta_2\|_{H^{s-\frac12}}\,\|f\|_{H^s}\,. \tag{5.13}$$

Since $\widetilde{\phi}_j=\widetilde{u}_j+\underline{\widetilde{f}}$ we have $v=\widetilde{u}_1-\widetilde{u}_2$. We begin by proving the following estimate. There exists a non decreasing function $\mathcal{F}\colon \mathbf{R}_+\to\mathbf{R}_+$ such that

$$\|\nabla_{x,z}v\|_{L^2(J,L^2)} \le \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}}\big)\,\|\eta_1-\eta_2\|_{H^{s-\frac12}}\,\|f\|_{H^s} \tag{5.14}$$

For this purpose we use the variational characterization of the solutions u_i. Setting $X=(x,z)$ we have

$$\int_{\widetilde{\Omega}}\Lambda^i\widetilde{u}_i\cdot\Lambda^i\theta\,J_i\,dX = -\int_{\widetilde{\Omega}}\Lambda^i\underline{\widetilde{f}}\cdot\Lambda^i\theta\,J_i\,dX \tag{5.15}$$

for all $\theta\in H^{1,0}(\widetilde{\Omega})$, where $J_i=|\partial_z\rho_i|$.

Making the difference between the two equations (5.15), using (3.31) and taking $\theta=v=\widetilde{u}_1-\widetilde{u}_2$ one can find a positive constant C such that

$$\int_{\widetilde{\Omega}}|\Lambda^1 v|^2\,dX \le C(A_1+A_2+A_3+A_4)$$

where

$$\begin{cases} A_1=\displaystyle\int_{\widetilde{\Omega}}|(\Lambda^1-\Lambda^2)\widetilde{u}_2||\Lambda^1 v|\,J_1\,dX, & A_2=\displaystyle\int_{\widetilde{\Omega}}|(\Lambda^1-\Lambda^2)v||\Lambda^2\widetilde{u}_2|\,J_1dX,\\ A_3=\displaystyle\int_{\widetilde{\Omega}}|\Lambda^2\widetilde{u}_2||\Lambda^2 v|\,|J_1-J_2|\,dX, & A_4=\displaystyle\int_{\widetilde{\Omega}}|(\Lambda^1-\Lambda^2)\underline{\widetilde{f}}||\Lambda^1\widetilde{u}|\,J_1\,dX,\\ A_5=\displaystyle\int_{\widetilde{\Omega}}|(\Lambda^1-\Lambda^2)v||\Lambda^2\underline{\widetilde{f}}|\,J_1\,dX, & A_6=\displaystyle\int_{\widetilde{\Omega}}|\Lambda^2\widetilde{f}||\Lambda^2 v|\,|J_1-J_2|\,dX \end{cases}$$

It follows from the elliptic regularity theorem that

$$\begin{aligned} A_1 &\le \|\Lambda^1 v\|_{L^2(\widetilde{\Omega})}\|\beta\|_{L^2(\widetilde{\Omega})}\|\partial_z\widetilde{u}_2\|_{L^\infty(J,L^\infty(\mathbf{R}^d))}\\ &\le \|\Lambda^1 v\|_{L^2(\widetilde{\Omega})}\mathcal{F}(\|\eta_1\|_{H^{s+\frac12}(\mathbf{R}^d)}\|\eta_2\|_{H^{s+\frac12}(\mathbf{R}^d)},\|\psi\|_{H^s(\mathbf{R}^d)})\|\eta_1-\eta_2\|_{H^{\frac12}(\mathbf{R}^d)}. \end{aligned}$$

Noticing that $\Lambda^1-\Lambda^2=\beta(\partial_z\rho_1)\Lambda^1_1$ where β satisfies the estimate in (3.33) we obtain

$$A_2\le\|\partial_z\rho_1\|_{L^\infty(\widetilde{\Omega})}\|\beta\|_{L^2(\widetilde{\Omega})}\|\Lambda^2\widetilde{u}_2\|_{L^\infty(\widetilde{\Omega})}\|\Lambda^1 v\|_{L^2(\widetilde{\Omega})}.$$

Using (3.31), (3.33) and the elliptic regularity we obtain

$$A_2\le\|\Lambda^1 v\|_{L^2(\widetilde{\Omega})}\mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}}\big)\,\|\eta_1-\eta_2\|_{H^{s-\frac12}}\,\|f\|_{H^s}\,.$$

Now we estimate A_3 as follows. We have

$$A_3\le\|\Lambda^2\widetilde{u}_2\|_{L^\infty(\widetilde{\Omega})}\|\Lambda^2 v\|_{L^2(\widetilde{\Omega})}\|J_1-J_2\|_{L^2(\widetilde{\Omega})}.$$

Then we observe that

$$\begin{aligned} &\|J_1-J_2\|_{L^2(\widetilde{\Omega})}\le C\|\eta_1-\eta_2\|_{H^{\frac12}(\mathbf{R}^d)}\\ &\|\Lambda^2 v\|_{L^2(\widetilde{\Omega})}\le C\|\Lambda^1 v\|_{L^2(\widetilde{\Omega})} \end{aligned}$$

and we use the elliptic regularity. To estimate A_4 and A_5 we recall that $\underline{\widetilde{f}} = e^{z\langle D_x\rangle} f$. Then we have

$$\|\beta\partial_z\underline{\widetilde{f}}\|_{L^2(J\times\mathbf{R}^d)} \le \|\beta\|_{L^2(I\times\mathbf{R}^d)}\|\partial_z\underline{\widetilde{f}}\|_{L^\infty(J\times\mathbf{R}^d)}.$$

Since $\|\partial_z\underline{\widetilde{f}}\|_{L^\infty(J\times\mathbf{R}^d)} \le \|\partial_z\underline{\widetilde{f}}\|_{L^\infty(J,H^{s-1}(\mathbf{R}^d))} \le \|f\|_{H^s(\mathbf{R}^d)}$, using (3.33) we obtain

$$A_4 + A_5 \le \|\Lambda^1 v\|_{L^2(\widetilde{\Omega})}\mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}}\big)\,\|\eta_1-\eta_2\|_{H^{s-\frac12}}\,\|f\|_{H^s}\,.$$

The term A_6 is estimated like A_3. Since $\frac12 < s-\frac12$ this proves (5.14) .

To complete the proof of (5.13) we have to estimate $\|\nabla_{x,z}v\|_{L^\infty(I,H^{-\frac12})}$. The estimate of $\|\nabla_x v\|_{L^\infty(J,H^{-\frac12})}$ follows from (5.14)and from Lemma 3.14. To estimate $\|\partial_z v\|_{L^\infty(J,H^{-\frac12})}$ we have to use (5.14) and the equation satisfied by v. If we prove that

$$\|\partial_z^2 v\|_{L^2(J,H^{-1})} \le \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}}\big)\,\|\eta_1-\eta_2\|_{H^{s-\frac12}}\,\|f\|_{H^s}\,. \tag{5.16}$$

the result will follow again from Lemma 3.14. Recall that v satifies the equation (5.10).

It follows that we have

$$\begin{aligned}\|\partial_z^2 v\|_{L^2(J,H^{-1})} \le{}& \|\alpha_1\Delta v\|_{L^2(J,H^{-1})} + \|\beta_1\cdot\nabla\partial_z v\|_{L^2(J,H^{-1})}\\ &+\|\gamma_1\partial_z v\|_{L^2(J,H^{-1})} + \|F\|_{L^2(J,H^{-1})}.\end{aligned} \tag{5.17}$$

Since $-1 < s-2$ (5.12) yields

$$\|F\|_{L^2(J,H^{-1})} \le \|F\|_{L^2(J,H^{s-2})} \le \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}\times H^{s+\frac12}}\big)\,\|\eta_1-\eta_2\|_{H^{s-\frac12}}\,\|f\|_{H^s}\,.$$

On the other hand, since $s-\frac12-1>0$ and $-1 < s-\frac12-1-\frac{d}{2}$ (2.13) show that we have

$$\begin{aligned}\|\alpha_1\Delta v\|_{L^2(J,H^{-1})} &\le \|\alpha_1\|_{L^\infty(J,H^{s-\frac12})}\|\nabla_x v\|_{L^2(J,L^2)}\\ \|\beta_1\cdot\nabla\partial_z v\|_{L^2(J,H^{-1})} &\le \|\beta_1\|_{L^\infty(J,H^{s-\frac12})}\|\partial_z v\|_{L^2(J,L^2)}\\ \|\gamma_1\partial_z v\|_{L^2(J,H^{-1})} &\le \|\gamma_1\|_{L^\infty(J,H^{s-\frac32})}\|\partial_z v\|_{L^2(J,L^2)}.\end{aligned}$$

Using Lemma 3.24 and (5.14) we obtain eventually (5.16).

Now Lemma 5.4 follows from (5.12), (5.13) and Proposition 3.21 with $\sigma = s-\frac32$. □

Lemma 5.4 together with (5.8) prove (5.7) which in turn proves Proposition5.2. □

5.2. Paralinearization of the equations. We begin by noticing that, as in the proof of Lemma 4.16, it is enough to estimate η, B, V. Indeed, the estimate of the $L^\infty([0,T];H^{s-1/2}_x)$-norm of ψ is in two elementary steps. Firstly, since $V_j = \nabla\psi_j - B_j\nabla\eta_j$, one can estimate the $L^\infty([0,T];H^{s-3/2})$-norm of $\nabla\psi$ from the identity

$$\nabla\psi = V + B\nabla\eta_1 + B_2\nabla\eta.$$

On the other hand, the estimate of the $L^\infty([0,T];L^2_x)$-norm of ψ follows from the equation (4.48).

An elementary calculation shows that the functions

$$\zeta = \zeta_1-\zeta_2,\quad V = V_1 - V_2,\quad B = B_1 - B_2$$

satisfy the system of equations

$$(5.18)\qquad \begin{cases} \partial_t B + V_1\cdot\nabla B + V\cdot\nabla B_2 = a,\\ \partial_t V + V_1\cdot\nabla V + V\cdot\nabla V_2 + a_2\zeta + a\zeta_1 = 0,\\ \partial_t\zeta + V_2\cdot\nabla\zeta + V\cdot\nabla\zeta_1 = G(\eta_1)V + \zeta_1 G(\eta_1)B + \zeta G(\eta_2)B_2 + R + \gamma, \end{cases}$$

where

$$(5.19)\qquad R = \left[G(\eta_1) - G(\eta_2)\right]V_2 + \zeta_1\left[G(\eta_1) - G(\eta_2)\right]B_2,$$

and $\gamma = \gamma_1 - \gamma_2$, γ_j are given by (4.9)

Lemma 5.5. *The differences ζ, B, V satisfy a system of the form*

$$(5.20)\qquad \begin{cases} (\partial_t + V_1\cdot\nabla)(V + \zeta_1 B) + a_2\zeta = f_1,\\ (\partial_t + V_2\cdot\nabla)\zeta - G(\eta_1)V - \zeta_1 G(\eta_1)B = f_2, \end{cases}$$

for some remainders such that

$$\left\|(f_1,f_2)\right\|_{L^\infty([0,T];H^{s-1}\times H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).$$

Proof. We begin by rewriting System (5.18) under the form

$$\begin{cases} \partial_t B + V_1\cdot\nabla B = a + R_1,\\ \partial_t V + V_1\cdot\nabla V + a_2\zeta + a\zeta_1 = R_2,\\ \partial_t\zeta + V_2\cdot\nabla\zeta = G(\eta_1)V + \zeta_1 G(\eta_1)B + R + \gamma + R_3, \end{cases}$$

where R is given by (5.19), $\gamma = \gamma_1 - \gamma_2$ and

$$R_1 = -V\cdot\nabla B_2,\quad R_2 = -V\cdot\nabla V_2,\quad R_3 = V\cdot\nabla\zeta_1 + \zeta G(\eta_2)B_2.$$

¿From Theorem 5.2 one has

$$\left\|R\right\|_{L^\infty(0,T;H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).$$

Similarly, proceeding as in the end of the proof of Proposition 4.3, we have

$$\left\|\gamma\right\|_{L^\infty(0,T;H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).$$

On the other hand, since $s-1>d/2$, H^{s-1} is an algebra and

$$\left\|V\cdot\nabla B_2\right\|_{H^{s-1}} \le K\left\|V\right\|_{H^{s-1}}\left\|\nabla B_2\right\|_{H^{s-1}} \le K\left\|V\right\|_{H^{s-1}}\left\|B_2\right\|_{H^s}$$

and similarly

$$\left\|V\cdot\nabla V_2\right\|_{H^{s-1}} \le K\left\|V\right\|_{H^{s-1}}\left\|V_2\right\|_{H^s}.$$

On the other hand, according to Theorem 3.17 we have

$$\left\|G(\eta_2)B_2\right\|_{H^{s-1}} \le C(\left\|\eta_2\right\|_{H^{s+1/2}})\left\|B_2\right\|_{H^s},$$

and hence

$$\left\|\zeta G(\eta_2)B_2\right\|_{H^{s-\frac{3}{2}}} \le C(\left\|\eta_2\right\|_{H^{s+1/2}})\left\|B_2\right\|_{H^s}\left\|\zeta\right\|_{H^{s-\frac{3}{2}}}.$$

To estimate $V\cdot\nabla\zeta_1$ we use the product rule (2.13) to deduce

$$\left\|V\cdot\nabla\zeta_1\right\|_{H^{s-\frac{3}{2}}} \le K\left\|V\right\|_{H^{s-1}}\left\|\nabla\zeta_1\right\|_{H^{s-\frac{3}{2}}} \le K\left\|V\right\|_{H^{s-1}}\left\|\eta_1\right\|_{H^{s+\frac{1}{2}}}.$$

Therefore we have,

$$\left\|R_1\right\|_{H^{s-1}} + \left\|R_2\right\|_{H^{s-1}} + \left\|R_3\right\|_{H^{s-\frac{3}{2}}} \le C\left\{\left\|\eta\right\|_{H^{s-\frac{1}{2}}} + \left\|B\right\|_{H^{s-1}} + \left\|V\right\|_{H^{s-1}}\right\},$$

for some constant C depending only on $\left\|\eta_j\right\|_{H^{s+\frac{1}{2}}}, \left\|B_j\right\|_{H^s}, \left\|V_j\right\|_{H^s}$. The next step consists in transforming again the equation. We want to replace $a\zeta_1$ in the second equation by

$$(\partial_t B + V_1\cdot\nabla B - R_1)\zeta_1.$$

The idea is that this allows to factor out the convective derivative $\partial_t + V_1 \cdot \nabla$. Writing

$$(\partial_t B + V_1 \cdot \nabla B)\zeta_1 = (\partial_t + V_1 \cdot \nabla)(B\zeta_1) - B(\partial_t + V_1 \cdot \nabla)\zeta_1$$

we thus end up with

$$(\partial_t + V_1 \cdot \nabla)(V + \zeta_1 B) + a_2\zeta = R_1\zeta_1 + B(\partial_t + V_1 \cdot \nabla)\zeta_1 + R_2. \tag{5.21}$$

Since

$$(\partial_t + V_1 \cdot \nabla)\zeta_1 = G(\eta_1)V_1 + \zeta_1 G(\eta_1)B_1 + \gamma_1,$$

we have

$$\|(\partial_t + V_1 \cdot \nabla)\zeta_1\|_{H^{s-1}} \le \mathcal{F}\big(\|(\eta_1, B_1, V_1)\|_{H^{s+\frac12}\times H^s \times H^s}\big).$$

By using this estimate and our previous bounds for R_1, R_2, we find

$$\|R_1\zeta_1 + B(\partial_t + V_1 \cdot \nabla)\zeta_1 + R_2\|_{H^{s-1}} \le C\left\{\|\eta\|_{H^{s-\frac12}} + \|B\|_{H^{s-1}} + \|V\|_{H^{s-1}}\right\},$$

for some constant C depending only on $\|\eta_j\|_{H^{s+\frac12}}, \|B_j\|_{H^s}, \|V_j\|_{H^s}$. Notice that here, as we used the equation satisfied by ζ_1, it was important to have $(\partial_t + V_1 \cdot \nabla)$ in the l.h.s. of (5.21) and not $(\partial_t + V_2 \cdot \nabla)$, and this algebraic reduction required some care in the previous step. □

5.3. Estimates for the good unknown. We now symmetrize System (5.20). We set $I = [0, T]$.

Lemma 5.6. *Set*

$$\ell := \sqrt{\lambda_1 a_2}, \quad \varphi := T_{\sqrt{\lambda_1}}(V + \zeta_1 B), \quad \vartheta := T_{\sqrt{a_2}}\zeta.$$

Then

$$(\partial_t + T_{V_1} \cdot \nabla)\varphi + T_\ell \vartheta = g_1, \tag{5.22}$$

$$(\partial_t + T_{V_2} \cdot \nabla)\vartheta - T_\ell \varphi = g_2, \tag{5.23}$$

where

$$\|(g_1, g_2)\|_{L^\infty(I; H^{s-\frac32}\times H^{s-\frac32})} \le \mathcal{K}(M_1, M_2)N(T).$$

Proof. We start from Lemma 5.5. By using Proposition 2.10, one can replace $V_1 \cdot \nabla$ by $T_{V_1} \cdot \nabla$ and $a_2\zeta$ by $T_{a_2}\zeta$, modulo admissible remainders. It is found that

$$(\partial_t + T_{V_1} \cdot \nabla)(V + \zeta_1 B) + T_{a_2}\zeta = f_1', \tag{5.24}$$

for some remainder f_1' such that

$$\|f_1'\|_{L^\infty(I; H^{s-1})} \le \mathcal{K}(M_1, M_2)N(T).$$

Similarly, one can replace $V_2 \cdot \nabla$ by $T_{V_2} \cdot \nabla$. According to Proposition 3.18, with $\varepsilon = \frac12$, we have

$$\|G(\eta_1)V - T_{\lambda_1}V\|_{L^\infty(I; H^{s-\frac32})} + \|G(\eta_1)B - T_{\lambda_1}B\|_{L^\infty(I; H^{s-\frac32})} \le \mathcal{K}(M_1)N(T),$$

and according to Proposition 2.10, with $\gamma = r = s - \frac32, \mu = s - \frac12$,

$$\|\zeta_1 G(\eta_1)B - T_{\zeta_1}T_{\lambda_1}B\|_{L^\infty(I; H^{s-\frac32})} \le \mathcal{K}(M_1)N(T).$$

We deduce

$$(\partial_t + T_{V_2} \cdot \nabla)\zeta - T_{\lambda_1}V - T_{\zeta_1}T_{\lambda_1}B = f_2', \tag{5.25}$$

where

$$\lambda_1 := \sqrt{(1 + |\nabla \eta_1|^2)|\xi|^2 - (\nabla \eta_1 \cdot \xi)^2},$$

and

$$\|f_2'\|_{L^\infty(I;H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).$$

Now, according to Lemma 2.16, (3.42) and (4.9) and we find that

$$(5.26)\quad \|[T_{\sqrt{\lambda_1}},(\partial_t+T_{V_1}\cdot\nabla)]\|_{H^{s-1}\to H^{s-\frac{3}{2}}} \le \mathcal{K}(M_1)\big(\mathcal{M}_0^{\frac{1}{2}}(\sqrt{\lambda_1})+\mathcal{M}_0^{\frac{1}{2}}((\partial_t+V_1\cdot\nabla)\sqrt{\lambda_1})\big)\le \mathcal{K}'(M_1)$$

and similarly, according to Lemma 2.16 and (4.19),

$$(5.27)\quad \|[T_{\sqrt{a_2}},(\partial_t+T_{V_2}\cdot\nabla)]\|_{H^{s-1}\to H^{s-\frac{3}{2}}} \le \mathcal{K}(M_2)\big(\mathcal{M}_0^{0}(\sqrt{a_2})+\mathcal{M}_0^{0}((\partial_t+V_2\cdot\nabla)\sqrt{a_2})\big)\le \mathcal{K}'(M_2),$$

which implies

$$(5.28)\quad (\partial_t+T_{V_1}\cdot\nabla)T_{\sqrt{\lambda_1}}(V+\zeta_1 B)+T_{\sqrt{\lambda_1}}T_{a_2}\zeta=f_1'',$$

$$(5.29)\quad (\partial_t+T_{V_2}\cdot\nabla)T_{\sqrt{a_2}}\zeta-T_{\sqrt{a_2}}\big(T_{\lambda_1}V-T_{\zeta_1}T_{\lambda_1}B\big)=f_2'',$$

where

$$\|(f_1'',f_2'')\|_{L^\infty(I;H^{s-\frac{3}{2}})}\le \mathcal{K}(M_1,M_2)N(T).$$

According to (2.5), (3.42) and (4.18), since $s>1+\frac{d}{2}$,

$$T_{\sqrt{\lambda_1}}T_{a_2}-T_{\sqrt{\lambda_1 a_2}}T_{\sqrt{a_2}}\quad \text{is of order } 0,$$

which implies (5.22). On the other hand, according to (2.5) and (3.42) the operators $T_{\zeta_1}T_{\lambda_1}-T_{\lambda_1\zeta_1}$ and $T_{\lambda_1}T_{\zeta_1}-T_{\lambda_1\zeta_1}$ are of order $1/2$ (with norm controlled by $\mathcal{K}(M_1)$, which allows to commute $T_{\sqrt{\lambda_1}}$ and T_{a_2} in (5.29)). Now, according to Proposition 2.10 (with $\gamma=r=s-\frac{1}{2},\mu=s-1$)

$$\|T_{\zeta_1}B-\zeta_1 B\|_{H^{s-\frac{1}{2}}}\le \mathcal{K}(M_1)\|B\|_{H^{s-1}}.$$

Which implies (5.23) (using again (2.5)). □

Recall that we have set

$$(5.30)\quad N(T):=\sup_{t\in I}\|(\eta,\psi,V,B)(t)\|_{H^{s-\frac{1}{2}}\times H^{s-\frac{1}{2}}\times H^{s-1}\times H^{s-1}}.$$

Lemma 5.7. *Set*

$$N'(T):=\sup_{t\in I}\big\{\|\vartheta(t)\|_{H^{s-\frac{3}{2}}}+\|\varphi(t)\|_{H^{s-\frac{3}{2}}}\big\}.$$

We have

$$(5.31)\quad N'(T)\le \mathcal{K}(M_1,M_2)\big(N(0)+TN(T)\big).$$

Proof. We first prove that

$$(5.32)\quad N'(T)\le \mathcal{K}(M_1,M_2)\big(N'(0)+TN(T)+TN'(T)\big).$$

The desired estimate (5.31) then follows from the fact that

$$(5.33)\quad N'(0)\le \mathcal{K}(M_1,M_2)N(0),\qquad N'(T)\le \mathcal{K}(M_1,M_2)N(T).$$

which follows from the continuity of paradifferential operators in the Sobolev spaces (see Theorem (2.6)) and the fact that $H^{s-1}(\mathbf{R}^d)$ is an algebra since $s>1+\frac{d}{2}$.

The proof of (5.32) is based on a classical argument : we commute $\langle D_x\rangle^{s-3/2}$ to (5.22)–(5.23) and perform an L^2 estimate. Then the key points are that (see point (iii) in Theorem 2.6)

$$(5.34)\qquad \begin{aligned} \big\|(T_{V_j}\cdot\nabla)^* + T_{V_j}\cdot\nabla\big\|_{L^2\to L^2} &\lesssim \|V_j\|_{W^{1,\infty}}\,,\\ \|T_\ell-(T_\ell)^*\|_{L^2\to L^2} &\le \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{W^{3/2,\infty}}\big)\end{aligned}$$

and that the commutators $[T_{V_j}\cdot\nabla,\langle D_x\rangle^{s-3/2}]$ are, according to (2.5), of order $s-\frac{3}{2}$.

Notice that since $(\varphi,\vartheta)\in C^1([0,T_0];H^{s-\frac{3}{2}})$, we do not need to regularize the equations. □

Finally, let us notice that an elementary argument allows to control lower norms of (V,B) (and hence also of $V+\zeta_1 B$):

$$(5.35)\qquad \|(V,B)\|_{L^\infty(I;H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)\big(N(0)+TN(T)\big).$$

Indeed, (the proof of) Theorem 5.2 implies that (with $a=a_1-a_2$)

$$(5.36)\qquad \|a\|_{L^\infty(I;H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).$$

Since $\partial_t B + V_1\cdot\nabla B = a - V\cdot\nabla B_2$, we have

(5.37)

$$\begin{aligned}\|B\|_{L^\infty(I;H^{s-2})} &\le \|B(0)\|_{H^{s-2}} + \int_0^T \Big(\|V_1\cdot\nabla B\|_{H^{s-2}} + \|a\|_{H^{s-2}} + \|V\cdot\nabla B_2\|_{H^{s-2}}\Big)dt'\\ &\le \|B(0)\|_{H^{s-2}} + T\mathcal{K}(M_1,M_2)N(T).\end{aligned}$$

Similarly, we have

$$(5.38)\qquad \|V\|_{L^\infty(I;H^{s-2})} \le \|V(0)\|_{H^{s-2}} + T\mathcal{K}(M_1,M_2)N(T).$$

Now we have

$$V+\zeta_1 B = T_{\sqrt{\lambda_1}^{-1}}\varphi + (\mathrm{Id} - T_{\sqrt{\lambda_1}^{-1}}T_{\sqrt{\lambda_1}})(V+\zeta_1 B),$$

where according to (2.5), the operator $\mathrm{Id} - T_{\sqrt{\lambda_1}^{-1}}T_{\sqrt{\lambda_1}}$ is of order$-1/2$. Hence, we deduce from (5.33), (5.31), (5.35), (5.38) and a bootstrap argument

$$(5.39)\qquad \|V+\zeta_1 B\|_{L^\infty(I;H^{s-1})} \le \mathcal{K}(M_1,M_2)\{N(0)+TN(T)\}.$$

5.4. Back to the original unknowns. Recall that $I=[0,T]$ (resp. $J=(-1,0)$) is an interval in the t variable (resp. in the z variable).

Lemma 5.8. *There holds*

$$(5.40)\qquad \|\eta\|_{L^\infty(I;H^{s-\frac{1}{2}})} \le \mathcal{K}(M_1,M_2)\{N(0)+TN(T)\}.$$

Proof. From the equation $\partial_t\eta_j = G(\eta_j)\psi_j$ we have,

$$\eta(t) = \eta(0) + \int_0^t G(\eta_1)\psi(t')dt' + \int_0^t \big(G(\eta_1)-G(\eta_2)\big)\psi_2(t')dt',$$

from which we deduce according to Theorem 5.2,

$$(5.41)\qquad \|\eta\|_{L^\infty(I;H^{s-2})} \le \|\eta(0)\|_{H^{s-2}} + T\mathcal{K}(M_1,M_2)\|\eta\|_{L^\infty(I;H^{s-\frac{1}{2}})}.$$

Let $R = Id - T_{\frac{1}{\sqrt{a_2}}}T_{\sqrt{a_2}}$, which, according to (2.5) and (4.18) is an operator of order $-\frac{1}{2}$ (with norm estimated by $\mathcal{K}(M_2)$). We have

$$\nabla\eta = R\nabla\eta + T_{\frac{1}{\sqrt{a_2}}}\vartheta.$$

Therefore we deduce from (5.41), (5.31) and a bootstrap argument,

$$\|\nabla \eta\|_{L^\infty(I;H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1, M_2)\big(N(0) + TN(T)\big).$$

Combining with (5.41) gives Lemma 5.8. □

We are now ready to estimate (V, B).

Proposition 5.9. *There holds*

$$\|(V, B)\|_{L^\infty(I;H^{s-1})} \le \mathcal{K}(M_1, M_2)\big\{N(0) + TN(T)\big\}. \tag{5.42}$$

The proof will require several preliminary Lemmas. We begin by noticing that it is enough to estimate B. Indeed, if

$$\|B\|_{L^\infty(I;H^{s-1})} \le \mathcal{K}(M_1, M_2)\big\{N(0) + TN(T)\big\}.$$

then, by using the triangle inequality, the estimate (5.39) for $V + \zeta_1 B$ implies that V satisfies the desired estimate.

Let $v = \widetilde{\phi}_1 - \widetilde{\phi}_2$, where $\widetilde{\phi}_j$ is the harmonic extension in $\tilde{\Omega}$ of the function ψ_j and set

$$b_2 := \frac{\partial_z \widetilde{\phi}_2}{\partial_z \rho_2}, \quad w = v - T_{b_2}\rho.$$

We notice that

$$w|_{z=0} = \psi - T_{B_2}\eta. \tag{5.43}$$

We first state the following result.

Lemma 5.10. *We have*

$$\|\psi - T_{B_2}\eta\|_{L^\infty(I;H^s)} \le \mathcal{K}(M_1, M_2)\big\{N(0) + TN(T)\big\}. \tag{5.44}$$

Proof. Indeed, the low frequencies are estimated by (5.35), while for the high frequencies, we write

$$\begin{aligned}
\nabla(\psi - T_{B_2}\eta) &= \nabla\psi - T_{B_2}\nabla\eta - T_{\nabla B_2}\eta \\
&= \nabla\psi_1 - \nabla\psi_2 - T_{B_2}\nabla\eta - T_{\nabla B_2}\eta \\
&= V_1 + B_1\nabla\eta_1 - V_2 - B_2\nabla\eta_2 - T_{B_2}\nabla\eta - T_{\nabla B_2}\eta \\
&= V + (B_1 - B_2)\nabla\eta_1 + B_2(\nabla\eta_1 - \nabla\eta_2) - T_{B_2}\nabla\eta - T_{\nabla B_2}\eta \\
&= V + \zeta_1 B + (B_2 - T_{B_2})\nabla\eta - T_{\nabla B_2}\eta,
\end{aligned}$$

where we used that, by definition, $\nabla\psi_j = V_j + B_j\nabla\eta_j$ and $\zeta_1 = \nabla\eta_1$.

The main term $V + \zeta_1 B$ is estimated using (5.39), while the two other terms are estimated using (5.40), the *a priori* estimate on B_2 and the product rules (2.9) and (2.11). □

We next relate w, ρ and B.

Lemma 5.11. *We have*

$$B = \Big[\frac{1}{\partial_z\rho_1}\Big(\partial_z w - (b_2 - T_{b_2})\partial_z\rho + T_{\partial_z b_2}\rho\Big)\Big]\Big|_{z=0}.$$

Proof. Write

$$\begin{aligned} B_1 - B_2 &= \frac{\partial_z \widetilde{\phi}_1}{\partial_z \rho_1} - \frac{\partial_z \widetilde{\phi}_2}{\partial_z \rho_2} \Big|_{z=0} \\ &= \frac{1}{\partial_z \rho_1} \big(\partial_z \widetilde{\phi}_1 - \partial_z \widetilde{\phi}_2\big) + \Big(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2}\Big) \partial_z \widetilde{\phi}_2 \Big|_{z=0} \\ &= \frac{1}{\partial_z \rho_1} \partial_z v - \frac{1}{\partial_z \rho_1} \frac{\partial_z \widetilde{\phi}_2}{\partial_z \rho_2} \partial_z \rho \Big|_{z=0} \end{aligned}$$

and replace v by $w + T_{b_2}\rho$ in the last expression. □

Lemma 5.12. *Recall that $b_2 := \frac{\partial_z \widetilde{\phi}_2}{\partial_z \rho_2}$. For $k = 0, 1, 2$, we have*

$$\left\| \partial_z^k b_2 \right\|_{C^0([-1,0], L^\infty(I, H^{s-\frac{1}{2}-k}))} \leq C \left\| \psi_2 \right\|_{H^{s+\frac{1}{2}}},$$

for some constant C depending only on $\left\| \eta_2 \right\|_{H^{s+\frac{1}{2}}}$.

Proof. We estimate $\nabla_{x,z} \widetilde{\phi}_2$ in $C^0([-1,0], L^\infty(I, H^{s-\frac{1}{2}}))$ by using the elliptic regularity (see Proposition 3.21 and Remark 3.20). Now, using the equation satisfied by $\widetilde{\phi}_2$ and the product rule in Sobolev spaces, we successively estimate $\partial_z^2 \widetilde{\phi}_2$ and $\partial_z^3 \widetilde{\phi}_2$. This proves the lemma since the derivatives of ρ_2 are estimated directly from the definition of ρ_2. □

Notice that η and hence ρ are estimated in $L^\infty(I; H^{s-\frac{1}{2}})$ (see (5.40)). Now, use Lemma 5.12 and Proposition 2.10 (applied with $s > 1 + d/2$, $\gamma = s - 1$, $r = s - 1/2$, $\mu = s - 3/2$) to obtain

$$\| (b_2 - T_{b_2}) \partial_z \rho \|_{H^{s-1}} \lesssim \| b_2 \|_{H^{s-\frac{1}{2}}} \| \eta \|_{H^{s-1}}.$$

Now, (2.12) implies that

$$\| T_{\partial_z b_2} \rho \|_{H^{s-1}} \lesssim \| b_2 \|_{H^{s-1/2}} \| \eta \|_{H^{s-1}},$$

and hence, to complete the proof of the Proposition 5.9, it remains only to estimate $\partial_z w|_{z=0}$ in $L^\infty(I, H^{s-1})$. This is the purpose of the following result.

Lemma 5.13. *For $t \in [0, T]$ we have*

$$\| \nabla_{x,z} w \|_{C^0([-1,0], H^{s-1})} \leq \mathcal{K}(M_1, M_2) \big\{ N(0) + T N(T) \big\}. \tag{5.45}$$

Proof. To prove this estimate, we are going to show that w satisfies an elliptic equation in the variables (x, z) to which we may apply the results of Proposition 3.21. We have

$$\partial_z^2 v + \alpha_1 \Delta v + \beta_1 \cdot \nabla \partial_z v - \gamma_1 \partial_z v = (\gamma_1 - \gamma_2) \partial_z \widetilde{\phi}_2 + F_1,$$

where (see (5.10))

$$F_1 = (\alpha_2 - \alpha_1) \Delta \widetilde{\phi}_2 + (\beta_2 - \beta_1) \cdot \nabla \partial_z \widetilde{\phi}_2.$$

We claim that for $t \in [0, T]$

$$\| F_1(t, \cdot) \|_{L^2(J, H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2) \{ N(0) + T N(T) \}. \tag{5.46}$$

The two terms in F_1 are estimated by the same way. We will only consider the first one. Using the product rule (2.13) with $s_0 = s - \frac{3}{2}, s_1 = s - 1, s_2 = s - \frac{3}{2}$ we can write for fixed t

$$\| (\alpha_2 - \alpha_1) \Delta \widetilde{\phi}_2 \|_{L^2(J, H^{s-\frac{3}{2}})} \leq C \| \alpha_2 - \alpha_1 \|_{L^2(J, H^{s-1})} \| \Delta \widetilde{\phi}_2 \|_{L^\infty(J, H^{s-\frac{3}{2}})}.$$

Then we use (5.11), Proposition 3.21 with $\sigma = s - \frac{1}{2}$ and Lemma 5.8 to conclude that the term above is estimated by the right hand side of (5.46).

Now we introduce the operators

$$P_j := \partial_z^2 + \alpha_j \Delta + \beta_j \cdot \nabla \partial_z, \qquad L_j = P_j - \gamma_j \partial_z, \qquad (j = 1, 2).$$

With these notations we have $\gamma_j = \frac{1}{\partial_z \rho_j} P_j \rho_j$ and

$$L_1 v = (\gamma_1 - \gamma_2) \partial_z \widetilde{\phi}_2 + F_1. \tag{5.47}$$

Moreover

$$\begin{aligned}
\gamma_1 - \gamma_2 &= \frac{1}{\partial_z \rho_1} P_1 \rho_1 - \frac{1}{\partial_z \rho_2} P_2 \rho_2 = \frac{1}{\partial_z \rho_2} P_1 \rho_1 + \Big(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \Big) P_1 \rho_1 - \frac{1}{\partial_z \rho_2} P_2 \rho_2 \\
&= \frac{1}{\partial_z \rho_2} P_1 \rho + \frac{1}{\partial_z \rho_2} (P_1 - P_2) \rho_2 + \Big(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \Big) P_1 \rho_1 \\
&= \frac{1}{\partial_z \rho_2} P_1 \rho + \Big(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \Big) P_1 \rho_1 + F_2,
\end{aligned}$$

where

$$F_2 = \frac{1}{\partial_z \rho_2} \Big((\alpha_1 - \alpha_2) \Delta \rho_2 + (\beta_1 - \beta_2) \cdot \nabla \partial_z \rho_2 \Big). \tag{5.48}$$

Now we observe that

$$\Big(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \Big) P_1 \rho_1 = - \frac{\partial_z \rho}{\partial_z \rho_2} \frac{P_1 \rho_1}{\partial_z \rho_1} = - \frac{\partial_z \rho}{\partial_z \rho_2} \gamma_1,$$

which implies

$$\gamma_1 - \gamma_2 = \frac{1}{\partial_z \rho_2} P_1 \rho - \frac{\partial_z \rho}{\partial_z \rho_2} \gamma_1 + F_2 = \frac{1}{\partial_z \rho_2} L_1 \rho + F_2.$$

Plugging this into (5.47) yields

$$L_1 v - b_2 (L_1 \rho) = F_1 + (\partial_z \widetilde{\phi}_2) F_2. \tag{5.49}$$

We claim that for fixed t we have

$$\left\| (\partial_z \widetilde{\phi}_2) F_2(t, \cdot) \right\|_{L^2(J, H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1, M_2) \{ N(0) + T N(T) \}. \tag{5.50}$$

Indeed we first use the product rule (2.13) to write

$$\left\| (\partial_z \widetilde{\phi}_2) F_2(t, \cdot) \right\|_{L^2(J, H^{s-\frac{3}{2}})} \le \| (\partial_z \widetilde{\phi}_2)(t, \cdot) \|_{L^2(J, H^{s-\frac{1}{2}})} \| F_2(t, \cdot) \|_{L^2(J, H^{s-\frac{3}{2}})}.$$

By the elliptic regularity the first term in the right hand side is bounded by $\mathcal{K}(M_2)$. It is therefore sufficient to bound the second one. We have, for fixed t

$$\left\| \frac{1}{\partial_z \rho_2} (\alpha_1 - \alpha_2) \Delta \rho_2 \right\|_{L^2(J, H^{s-\frac{3}{2}})} \le \mathcal{K}(M_2) \| \alpha_1 - \alpha_2 \|_{L^2(J, H^{s-1})} \| \Delta \rho_2 \|_{L^\infty(J, H^{s-\frac{3}{2}})}.$$

Using (5.11) and (3.11) we see that the right hand side is bounded by the right hand side of (5.50). The second term in F_2 is estimated by the same way.

To estimate $v - T_{b_2} \rho$ we paralinearize in writing

$$b_2 (L_1 \rho) = T_{b_2} L_1 \rho + T_{L_1 \rho} b_2 + F_3. \tag{5.51}$$

We claim that for $t \in [0, T]$

$$\| F_3(t, \cdot) \|_{L^2(J, H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1, M_2) \{ N(0) + T N(T) \}. \tag{5.52}$$

To prove it we shall use (2.9) with $\alpha = s - \frac{1}{2}, \beta = s - 2$. Then $\alpha + \beta - \frac{d}{2} > s - \frac{3}{2}$. It follows that, for fixed z and t we have

$$\|F_3(t,\cdot,z)\|_{H^{s-\frac{3}{2}}} \leq C\|b_2\|_{H^{s-\frac{1}{2}}}\|L_1\rho\|_{H^{s-2}}.$$

Therefore

$$\|F_3(t,\cdot)\|_{L^2(J,H^{s-\frac{3}{2}})} \leq C\|b_2\|_{L^\infty(J,H^{s-\frac{1}{2}})}\|L_1\rho\|_{L^2(J,H^{s-2})}.$$

Now as we have seen before we have $\|b_2(t\cdot)\|_{L^\infty(J,H^{s-\frac{1}{2}})} \leq \mathcal{K}(M_2)$ and due to the smoothing of the Poisson kernel $\|L_1\rho\|_{L^2(J,H^{s-2})} \leq \mathcal{K}(M_1)\|\eta\|_{H^{s-\frac{1}{2}}}$. The estimate (5.52) thus follows from (5.40).

Setting $F_4 = T_{L_1\rho}b_2$ we claim that for fixed t we have

$$\|F_4(t,\cdot)\|_{L^2(J,H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1,M_2)\{N(0) + TN(T)\}. \tag{5.53}$$

To see this we use (2.12) with $s_0 = s - \frac{3}{2}, s_1 = s - 2, s_2 = s - \frac{1}{2}$. We get

$$\|F_4(t,\cdot)\|_{L^2(J,H^{s-\frac{3}{2}})} \leq \|L_1\rho(t,\cdot)\|_{L^2(J,H^{s-2})}\|b_2(t,\cdot)\|_{L^\infty(J,H^{s-\frac{1}{2}})}$$

and (5.53) follows from estimates used above.

Now according to (5.49), (5.51) we have

$$L_1 v - T_{b_2}L_1\rho = F_1 + (\partial_z\widetilde{\phi}_2)F_2 + F_3 + F_4.$$

We claim that we have

$$L_1 T_{b_2}\rho = T_{b_2}L_1\rho - F_5$$

with

$$\|F_5(t,\cdot)\|_{L^2(J,H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1,M_2)\{N(0) + TN(T)\}.$$

To see this we use (5.12) and (2.12). It follows then that we have

$$L_1 w = L_1(v - T_{b_2}\rho) = F_1 + (\partial_z\widetilde{\phi}_2)F_2 + F_3 + F_4 + F_5 := F$$

where $\|F(t,\cdot)\|_{L^2(J,H^{s-\frac{3}{2}})}$ is bounded by the right hand side of (5.53).

Using (5.43) and Lemma 5.10 we may then apply to w Proposition 3.21 with $\sigma = s-1$ to conclude the proof of Lemma 5.13 and thus that of Proposition 5.9. □

6. Well-posedness of the Cauchy problem

Here we conclude the proof of Theorem 1.2 about the Cauchy theory for the system

$$\begin{cases} \partial_t\eta + G(\eta)\psi = 0, \\ \partial_t\psi + g\eta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = 0. \end{cases} \tag{6.1}$$

We previously proved the uniqueness of solutions (see Theorem 5.1). To complete the proof of Theorem 1.2, it remains to prove the existence. We obtain solutions to the water waves system as limits of smooth solutions to approximate systems. This approach has been detailed in [1], where we considered the problem with surface tension. The analysis is actually easier without surface tension. One reason is that with surface tension, we needed in [1] to use some mollifiers with various properties (since we need good estimates for commutators with the principal part of the operator). Here it is possible to use a simpler regularization of the equations since the reduced paradifferential system involves only operator of order less than or equal to 1.

To explain the scheme of the proof, we first consider the case without bottom ($\Gamma = \emptyset$). Then we know, from previous results (see Wu [**51, 52**], Lannes [**37**], Lindblad [**39**]), that the Cauchy problem is well-posed for smooth initial data. Then, one can obtain the existence of smooth approximate solutions in a straightforward way : by smoothing the initial data. Namely, denote by J_ε the usual Friedrichs mollifiers, defined by $J_\varepsilon = \jmath(\varepsilon D_x)$ where $\jmath \in C_0^\infty(\mathbf{R}^d)$, $0 \le \jmath \le 1$, is such that

$$\jmath(\xi) = 1 \quad \text{for } |\xi| \le 1, \qquad \jmath(\xi) = 0 \quad \text{for } |\xi| \ge 2.$$

Set $\psi_0^\varepsilon = J_\varepsilon \psi_0$ and $\eta_0^\varepsilon = J_\varepsilon \eta_0$. Then $(\psi_0^\varepsilon, \eta_0^\varepsilon) \in H^\infty(\mathbf{R}^d)^2$ and the Cauchy problem for (6.1) has a unique smooth solution $(\psi^\varepsilon, \eta^\varepsilon)$ defined on some time interval $[0, T_\varepsilon^*)$. It follows from Proposition 4.1 that there exists a function $\mathcal{F}$ such that, for all $\varepsilon \in (0,1]$ and all $T < T_\varepsilon$, we have

$$M_s^\varepsilon(T) \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s^\varepsilon(T)\big)\big), \tag{6.2}$$

with obvious notations. Then by standard arguments, we infer that the lifespan of $(\eta_\varepsilon, \psi_\varepsilon)$ is bounded from below by a positive time T_0 independent of ε and that we have uniform estimates on $[0, T_0]$. The fact that one can pass to the limit in the equations follows from the previous contraction estimates (see (5.1)), which allows us to prove that $(\eta_\varepsilon, \psi_\varepsilon, B_\varepsilon, V_\varepsilon)$ is a Cauchy sequence (this argument has been explained in [**1**]). Notice that these estimates were proved under the assumption $a(t) > a_0/2$. This actually follows from the *a priori* bound (6.2), (4.18), (4.19) and a bootstrap method. Then, it remains to prove that the limit solution has the desired regularity properties. Again, this follows from the analysis in [**1**].

In the case with a general bottom, to apply the strategy explained above, the only remaining point is to prove that, for smooth enough initial data, the Cauchy problem has a smooth solution. This can be proved using a parabolic regularization of the equations. For the sake of conciseness, we omit the details.

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