STRICHARTZ ESTIMATES WITHOUT LOSS ON MANIFOLDS WITH HYPERBOLIC TRAPPED GEODESICS

Nicolas Burq, Colin Guillarmou and Andrew Hassell

Abstract. In [Do], Doi proved that the $L_t^2 H_x^{1/2}$ local smoothing effect for Schrödinger equations on a Riemannian manifold does not hold if the geodesic flow has one trapped trajectory. We show in contrast that Strichartz estimates and $L^1 \to L^\infty$ dispersive estimates still hold without loss for $e^{it\Delta}$ in various situations where the trapped set is hyperbolic and of sufficiently small fractal dimension.

The influence of the geometry on the behaviour of solutions of linear or non-linear partial differential equations has been widely studied recently, and especially in the context of wave or Schrödinger equations. In particular, the understanding of the smoothing effect for the Schrödinger flow and Strichartz type estimates has been related to the global behaviour of the geodesic flow on the manifold (see for example the works by Doi [Do] and Burq [Bu]). Let us recall that, for the Laplacian Δ on a d-dimensional non-compact Riemannian manifold (M, g), the *local smoothing effect* for bounded time $t \in [0, T]$ and Schrödinger waves $u = e^{it\Delta}u_0 : M \times \mathbb{R} \to \mathbb{C}$ is the estimate

$$\|\chi e^{it\Delta}u_0\|_{L^2((0,T);H^{1/2}(M))} \leq C_T \|u_0\|_{L^2(M)}, \quad \forall u_0 \in L^2(M),$$

where $C_T > 0$ is a constant depending a priori on T, and χ is a compactly supported smooth function (the assumption on χ can of course be weakened in many cases, e.g. for $M = \mathbb{R}^d$) [CoS]. In other words, although the solution is only L^2 in space uniformly in time, it is actually half a derivative better (locally) in an L^2-in-time sense. For its description in geometric settings, the picture now is fairly complete: the so called "nontrapping condition", stating roughly that every geodesic maximally extended goes to infinity, is known to be essentially necessary and sufficient (modulo reasonable conditions near infinity) [Bu].

Another tool for analyzing non-linear Schrödinger equations is the family of so-called *Strichartz estimates* introduced by [Str]: for Schrödinger waves on Euclidean space $\mathbb{R}^d$ with initial data u_0,

$$\|e^{it\Delta}u_0\|_{L^p((0,T);L^q(\mathbb{R}^d))} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)} \quad \text{if } p, q \geq 2, \ \tfrac{2}{p} + \tfrac{d}{q} = \tfrac{d}{2}, \quad (p,q) \neq (2,\infty). \tag{0.1}$$

If $\sup_{T \in (0,\infty)} C_T < \infty$, we will say that a global-in-time Strichartz estimate holds. Such a global-in-time estimate has been proved by Strichartz for the flat Laplacian on $\mathbb{R}^d$ while the local-in-time estimate is known in several geometric situations where

---

the manifold is non-trapping (asymptotically Euclidean, conic or hyperbolic); see
[BoT], [Bo3], [HaTW], [StT]. On the other hand, it is clear that such a global-
in-time estimate cannot hold on compact manifolds, for it suffices to consider the
function $u_0 = 1$. The situation is similar for the non-compact case in the presence
of elliptic (stable) non-degenerate periodic orbits of the geodesic flow: as remarked
by M. Zworski, the quasi-modes constructed by Babič [B] and Pyškina [Py] (see also
the work by Ralston [R]) show that, for Schrödinger solutions, some loss must occur
as far as Strichartz (or smoothing) estimates are concerned; and moreover, that
no Strichartz estimates can be true globally in time in the presence of such orbits.
On the other hand, Burq–Gérard–Tzvetkov [BuGT1] proved that (0.1) holds on
compact manifolds for finite time if one replaces $\|u_0\|_{L^2(M)}$ by $\|u_0\|_{H^{1/p}(M)}$, meaning
that a Strichartz estimate is satisfied if one accepts some loss of derivatives. It is
however certainly not optimal in general since Bourgain [Bou] proved that, for the
flat torus $(\mathbb{R}/2\pi\mathbb{Z})^2$, the Strichartz estimate for $p = q = 4$ holds with ϵ loss of
derivatives for any $\epsilon > 0$. Another striking example has been given by Takaoka
and Tzvetkov [TT] by adapting the ideas of Bourgain, namely the case of the two-
dimensional infinite flat cylinder $S^1 \times \mathbb{R}$ where (0.1) holds (with no loss of derivatives)
if $p = q = 4$; note that this manifold is trapping. An example with a repulsive
potential $V(x_1, x_2) = x_1^2 - x_2^2$ has also been studied by Carles [Ca], who proved that
global-in-time Strichartz estimates with no loss hold in this case. To summarize,
it is not really understood when (i.e. under what geometric conditions) a loss in
Strichartz estimates must occur, and if it does, how large that loss must be.

The purpose of this article is precisely to give some examples of Riemannian
manifolds where trapping does occur (and consequently loss is unavoidable for the
smoothing effect), but nevertheless, since the dynamics are *hyperbolic* near the
trapped set, we are able to prove (local-in-time) Strichartz estimates without loss
for Schrödinger solutions.

The first example, which we treat in section 1, is the case of a convex co-compact
hyperbolic manifold of dimension $d = n + 1$, with a limit set of Hausdorff dimension
$\delta < n/2$. The simplest example of such a manifold is the two-dimensional infinite
hyperbolic cylinder with one single trapped geodesic. In this case, the calculations
are quite explicit, representing the Schrödinger kernel as an average over the group of
the Schrödinger kernel on the hyperbolic space $\mathbb{H}^{n+1}$, and we are able to prove that
not only Strichartz estimates but also the stronger $L^1 \to L^\infty$ dispersive estimates
hold for the Schrödinger group.

Theorem 0.1. *Let X be an $(n + 1)$-dimensional convex co-compact hyperbolic
manifold such that its limit set has Hausdorff dimension $\delta < n/2$. Then the following
dispersive and Strichartz estimates without loss hold:*

$$\|e^{it\Delta_X}\|_{L^1(X)\to L^\infty(X)} \leq \begin{cases} C|t|^{-(n+1)/2}, & \text{for } |t| \leq 1, \\ C|t|^{-3/2}, & \text{for } |t| > 1, \end{cases}$$

$$\|e^{it\Delta_X}u_0\|_{L^p(\mathbb{R};L^q(X))} \leq C\|u_0\|_{L^2(X)},$$

for all (p,q) such that $(1/p, 1/q) \in T_n$ where

$$T_n := \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \left(0, \frac{1}{2} \right] \times \left(0, \frac{1}{2} \right) ; \ \frac{2}{p} \geq \frac{n+1}{2} - \frac{n+1}{q} \right\} \cup \left\{ \left(0, \frac{1}{2} \right) \right\}. \quad (0.2)$$

These manifolds are non-compact, infinite volume, with finitely many ends of funnel type, they have constant curvature -1 and possess *infinitely* many closed geodesics; it is remarkable that despite this last fact, a sharp dispersive estimate holds for all time. We also remark that Strichartz estimates for the same range of (p,q) have been recently shown by Anker–Pierfelice [AnP] for the model non-trapping case $\mathbb{H}^{n+1}$ (see also [Ba], [IS] for the estimate (0.1) in that setting). The triangle of admissibility for the Strichartz exponent (p,q) is a consequence of the exponential decay of the integral kernel of the Schrödinger operator at infinity. Notice that in the asymptotically hyperbolic setting, J.-M. Bouclet [Bo3] proved Strichartz estimates without loss of derivatives for bounded times and with admissibility exponents satisfying (0.1) for *non-trapping* such manifolds (in this case the sectional curvature is not assumed constant, but rather tending to -1 at infinity).

The second example is on the manifold Z given by the connected sum of two copies of Euclidean $\mathbb{R}^2$. This we provide with a Riemannian metric g by gluing two copies of the Euclidean metric on $\mathbb{R}^2$ with the metric on the 2-dimensional hyperbolic cylinder. Essentially because Z is formed from pieces all of which satisfy Strichartz estimates without loss, the same is true for (Z,g). Actually we need to use local smoothing estimates to control error terms in the transition region, but since this region is disjoint from the single trapped orbit, there are no losses in such local smoothing estimates. This example is given in section 2; the main result is Theorem 2.1.

Our last family of examples, in section 3, is a generalization of that in section 2 to higher dimensions and more complicated trapped sets. It is similar to the class of manifolds studied recently by Nonnenmacher and Zworski [NZ]: we consider asymptotically Euclidean (or more generally asymptotically conic) manifolds, the curvature of which is assumed to be negative in a geodesically convex compact part that includes the (projection of) the trapped set, and such that the trapped set is small enough in the sense that the topological pressure $P(1/2)$ of the trapped set evaluated at $1/2$ is negative (see the definition in section 3.1). This is a dynamical condition which generalizes the condition $\delta < n/2$ above to this more general setting and roughly speaking means that the trapped set is filamentary with small fractal dimension. For instance, for surfaces (dimension $d = 2$) this means that the trapped set (as a subset of the cosphere bundle S^*M) has Hausdorff dimension less than 2. More precisely, our result (which includes the example in section 2 as a special case) is

Theorem 0.2. *Let (M, g) satisfy assumptions (A1)–(A4) defined in section 3. Then Strichartz estimates without loss hold for M: there exists $C > 0$ such that*

$$\| e^{it\Delta} u_0 \|_{L^p((0,1), L^q(M))} \leq C \| u_0 \|_{L^2(M)}, \quad (0.3)$$

for all $u_0 \in L^2(M)$ and (p,q) satisfying (0.1) and $p > 2$.

Note that Christianson [Chr] and Datchev [D] showed that Strichartz estimates hold with an ϵ loss of derivatives for all $\epsilon > 0$ in that setting. Our method is based on the use of the local smoothing effect with log loss, which follows from the resolvent estimate of [NZ] (see also [D])

$$\big\|\chi(\Delta - \lambda + i0)^{-1}\chi\big\|_{L^2 \to L^2} \leq C \frac{\log(\lambda)}{\lambda^{1/2}}, \quad \text{for } \chi \in C_0^\infty(M),$$

together with a sharp dispersive estimate on the logarithmically *extended* time interval $t \in (0, h\log(h))$ for the frequency localized operator $e^{it\Delta}\varphi(h^2\Delta)$ where $\varphi \in C_0^\infty((0, \infty))$ and $h \in (0, h_0)$ is small. Roughly speaking this logarithmic extension of the time interval of validity of the dispersive estimate allows one to recover the log loss in the local smoothing estimate. This dispersive estimate is inspired by the works of Anantharaman [A], Anantharaman–Nonnenmacher [AN] and Nonnenmacher–Zworski [NZ]. In particular, the technique for proving the dispersive estimate for logarithmically extended time originates in [A, Th. 1.3.3], while the idea of combining the exponential decay provided by this theorem with the topological pressure assumption (see (A4) in section 3) is due to [NZ].

1   Hyperbolic Manifolds

A *convex co-compact subgroup* $\Gamma \subset \mathrm{SO}(n+1, 1)$ is a discrete group of orientation preserving isometries of hyperbolic space $\mathbb{H}^{n+1}$, consisting of hyperbolic isometries and such that the quotient $X := \Gamma \backslash \mathbb{H}^{n+1}$ has finite geometry and infinite volume. If one considers the ball model B^{n+1} of $\mathbb{H}^{n+1}$, a hyperbolic isometry is an isometry of $\mathbb{H}^{n+1}$ which fixes exactly two points on $\overline{B^{n+1}}$, and these points are on the boundary $S^n = \partial\overline{\mathbb{H}^{n+1}}$. The manifold $X := \Gamma \backslash \mathbb{H}^{n+1}$ is said to be *convex co-compact hyperbolic*; it is a smooth complete hyperbolic manifold which admits a natural conformal compactification $\bar{X}$ and the hyperbolic metric g on X is of the form $g = \bar{g}/x^2$ where x is a smooth boundary defining function of $\bar{X}$ and $\bar{g}$ a smooth metric on $\bar{X}$. The set of closed geodesics is in correspondence with the classes of conjugacy of the group Γ. The limit set of Γ is the set of accumulation points on the sphere $S^n = \partial\overline{\mathbb{H}^{n+1}}$ of the orbit $\Gamma.m$ where $m \in \mathbb{H}^{n+1}$ is any point. It has a Hausdorff dimension given by $\delta \in [0, n)$, and the trapped set of the geodesic flow on the unit tangent bundle SX has Hausdorff dimension $2\delta + 1$; see [Su], [Z].

The simplest example is $\Gamma = \mathbb{Z}$ acting by powers of a fixed dilation D on the upper half space model of $\mathbb{H}^{n+1}$. Then the limit set consists of two points $\{0, \infty\}$, $\delta = 0$, and $\mathbb{H}^{n+1}/\Gamma$ is the $(n+1)$-dimensional hyperbolic cylinder.

We prove the following.

Theorem 1.1. *Let X be an $(n+1)$-dimensional convex co-compact hyperbolic manifold such that its limit set has Hausdorff dimension $\delta < n/2$. Then $e^{it\Delta_X}$ has a smooth Schwartz kernel for $t \neq 0$, and there is a constant C such that the following dispersive estimate holds for all $t \neq 0$:*

$$\|e^{it\Delta_X}\|_{L^1 \to L^\infty} \leq \begin{cases} C|t|^{-(n+1)/2}, & \text{for } |t| \leq 1, \\ C|t|^{-3/2}, & \text{for } |t| > 1. \end{cases} \tag{1.1}$$

Moreover, the following global-in-time Strichartz estimates hold:

$$\|e^{it\Delta_X} u_0\|_{L^p(\mathbb{R};L^q(X))} \leq C\|u_0\|_{L^2(X)}, \tag{1.2}$$

for all (p,q) such that $(1/p, 1/q) \in T_n$ where T_n is given by (0.2).

Proof. The integral kernel of the Schrödinger operator $e^{it\Delta_{\mathbb{H}^{n+1}}}$ on hyperbolic space was computed by V. Banica [Ba]. It is a function of the hyperbolic distance

$$K\big(t; \rho(z,z')\big) = c|t|^{-1/2} e^{-itn^2/4} \big(\sinh(\rho)^{-1}\partial_\rho\big)^{n/2} e^{i\rho^2/4t}, \quad n \text{ even},$$

$$K\big(t; \rho(z,z')\big) = c|t|^{-3/2} e^{-itn^2/4} \big(\sinh(\rho)^{-1}\partial_\rho\big)^{\frac{n-1}{2}} \int_\rho^\infty \frac{e^{is^2/4t} s}{\sqrt{\cosh s - \cosh \rho}} ds, \quad n \text{ odd},$$
$$\tag{1.3}$$

where $\rho = \rho(z,z') := d_{\mathbb{H}^{n+1}}(z,z')$. In both cases we remark, like for the heat kernel, that the kernel $K(t; \cdot, \cdot)$ is smooth on $\mathbb{H}^{n+1} \times \mathbb{H}^{n+1}$ for $t \neq 0$; this is clear when $n+1$ is odd, and needs a bit more analysis when $n+1$ is even. From this expression we obtain an upper bound for $|K(t; z, z')|$ (see [Ba, Prop. 4.1 & §4.2]) for $t \neq 0$ of the form

$$\begin{cases} C|t|^{-(n+1)/2} \left(\frac{\rho}{\sinh\rho}\right)^{n/2}, & \text{for } |t| \leq 1, \\ C|t|^{-3/2} \left(\frac{\rho}{\sinh\rho}\right)^{n/2}, & \text{for } |t| > 1, \end{cases} \tag{1.4}$$

for some constant $C > 0$. Using the inequality $(\rho/\sinh\rho) \leq (1+\rho)e^{-\rho}$, and since

$$i\partial_t K(t; z, z') = -\Delta_z K(t; z, z') = -\Delta_{z'} K(t; z, z'),$$

one can deduce that for $t \neq 0$ bounded

$$\big|\Delta_z^j K(t; z, z')\big| + \big|\Delta_{z'}^j K(t; z, z')\big| \leq \begin{cases} C'|t|^{-(n+1)/2-2j}(1+\rho)^{\frac{n}{2}+2j} e^{-\frac{n}{2}\rho}, & \text{for } |t| \leq 1, \\ C'|t|^{-3/2}(1+\rho)^{\frac{n}{2}+2j} e^{-\frac{n}{2}\rho}, & \text{for } |t| > 1, \end{cases} \tag{1.5}$$

and in particular $K(t, z, z')$ is smooth in z, z'.

To proceed we use the celebrated result of Patterson and Sullivan [Pa], [Su] that the dimension of the limit set δ is the exponent of convergence of the Poincaré series

$$P_s(z, z') := \sum_{\gamma \in \Gamma} e^{-s\rho(z, \gamma.z')}, \quad z, z' \in \mathbb{H}^{n+1}.$$

LEMMA 1.2. *Let $\mathcal{F} \in \mathbb{H}^{n+1}$ be a fundamental domain of the convex co-compact group Γ, and let x be a boundary defining function of the compactification $\bar{X}$ of*

$X = \Gamma\backslash\mathbb{H}^{d+1}$, *which we also view as a function on* $\mathcal{F}$. *For each* $\gamma \in \Gamma$, *define by* ℓ_γ *the translation length of* γ. *Then there exists* $R > 0$ *such that, for all* $\epsilon > 0$, *there is* $C_\epsilon > 0$ *such that, for* $s > \delta + \epsilon$ *and all* $z, z' \in \mathcal{F}$,

$$\sum_{\gamma \in \Gamma, \ell_\gamma > R} e^{-s\rho(z,\gamma z')} \leq C_\epsilon \big(x(z)x(z')\big)^s. \tag{1.6}$$

Proof. In [GuMP, Lem. 5.2], it is shown that there exist constants $C_1, C_2 > 0$ such that for all $\gamma \in \Gamma$ such that $\ell_\gamma > C_1$

$$e^{-\rho(z,\gamma z')} \leq C_2 e^{-\ell_\gamma} x(z)x(z').$$

Now it suffices to sum after raising to the power s and to use the fact that $\sum_{\gamma \in \Gamma} e^{-s\ell_\gamma} < C_\epsilon$ for some C_ϵ if $s > \delta + \epsilon$. $\square$

Combining (1.6) and (1.4), we deduce that for $z, z' \in \mathcal{F}$, the series

$$K_X(t; z, z') := \sum_{\gamma \in \Gamma} K(t; z, \gamma.z')$$

converges uniformly and for all $s < n/2$ there exists $C_s > 0$ such that for all $z, z' \in \mathcal{F}$

$$\big|K_X(t; z, z')\big| \leq \begin{cases} C_s|t|^{-(n+1)/2}\big(\sum_{\gamma \in \Gamma, \ell_\gamma \leq R} e^{-s\rho(z,\gamma z')} + x(z)^s x(z')^s\big), & \text{for } |t| \leq 1, \\ C_s|t|^{-3/2}\big(\sum_{\gamma \in \Gamma, \ell_\gamma \leq R} e^{-s\rho(z,\gamma z')} + x(z)^s x(z')^s\big), & \text{for } |t| > 1, \end{cases} \tag{1.7}$$

where R is the constant in (1.6). This leads directly to the dispersive estimate

$$\sup_{z,z' \in \mathcal{F}} \big|K_X(t; z, z')\big| \leq \begin{cases} C'|t|^{-(n+1)/2}, & \text{for } |t| \leq 1, \\ C'|t|^{-3/2}, & \text{for } |t| > 1. \end{cases} \tag{1.8}$$

for some constants C'. Moreover, using (1.5), the same argument shows that the series $K_X(t; z, z')$ is smooth in z, z' for $t \neq 0$. Let $\mathcal{F}$ be a fundamental domain of Γ. For any $u_0 \in C_0^\infty(X)$ the function $u(t) := \int_{\mathcal{F}} K_X(t; z, z')u_0(z')dz'$ is smooth on $\mathbb{H}^{n+1}$ and satisfies $u(t, \gamma z) = u(t, z)$ for any $\gamma \in \Gamma$, thus $u(t)$ is smooth on X. Moreover, it solves the Schrödinger equation on X with initial data $u(0) = u_0$, so $u(t) = e^{it\Delta_X}u_0$. This implies that K_X is the Schwartz kernel of $e^{it\Delta_X}$ on X.

We next prove the global-in-time Strichartz estimates (1.2) (notice that these estimates for a *finite* time interval follow immediately from the small-time dispersive estimate (1.1) and from Keel–Tao [KT], following the method of Anker–Pierfelice [AnP]). Let $\gamma \in \Gamma$ be such that $\ell_\gamma \leq R$ where R is the constant in (1.6), then define $K_\gamma^1(t)$ to be the operator acting on $\mathcal{F}$ with L^∞ kernel $\mathbb{1}_{\mathcal{F}}(z)K(t; z, \gamma z')\mathbb{1}_{\mathcal{F}}(z')$. Since γ is an isometry of $\mathbb{H}^{n+1}$, this operator can also be written as $f \to \mathbb{1}_{\mathcal{F}} e^{it\Delta_{\mathbb{H}^{n+1}}}\gamma_*(\mathbb{1}_{\mathcal{F}} f)$. Then from Theorem 3.4 of Anker–Pierfelice [AnP] and the fact that push-forward γ_* is an isometry on any $L^{r'}(\mathbb{H}^{n+1})$, we get the estimate

$$\big\|K_\gamma^1(t)u_0\big\|_{L^q(\mathbb{H}^{n+1})} \leq C\|\mathbb{1}_{\mathcal{F}} u_0\|_{L^{r'}(\mathbb{H}^{n+1})} \times \begin{cases} |t|^{-(n+1)\max(\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{r})}, & \text{if } |t| \leq 1, \\ |t|^{-\frac{3}{2}}, & \text{if } |t| > 1, \end{cases} \tag{1.9}$$

for all $2 < q, r \leq \infty$ and $1/r' + 1/r = 1$, so the same estimate holds for $K^1(t) := \sum_{\gamma \in \Gamma, \ell_\gamma \leq R} K_\gamma^1(t)$. Now consider the operator $K^2(t)$ acting on $\mathcal{F}$ whose L^∞ kernel is

$K^2(t; z, z') := K_X(t; z, z') - K^1(t; z, z')$. From (1.6) and (1.4), this kernel is bounded (for all $s < n/2$) by

$$\left|K^2(t; z, z')\right| \leq \begin{cases} C_s|t|^{-(n+1)/2}x(z)^s x(z')^s, & \text{for } |t| \leq 1, \\ C_s|t|^{-3/2}x(z)^s x(z')^s, & \text{for } |t| > 1. \end{cases}$$

Since the hyperbolic metric on $\mathcal{F}$ induces a measure of the form $x^{-n-1}\mu$ for some bounded measure μ on $\mathcal{F}$, we see that the function x^s is in $L^\alpha(\mathcal{F}, dv_{\mathbb{H}^{n+1}})$ for all $\alpha > n/s$, and hence deduce directly that $K^2(t)$ satisfies

$$\|K^2(t)u_0\|_{L^q(\mathcal{F}, dv_{\mathbb{H}^{n+1}})} \leq C\|u_0\|_{L^{r'}(\mathcal{F}, dv_{\mathbb{H}^{n+1}})} \times \begin{cases} |t|^{-\frac{n+1}{2}}, & \text{if } |t| \leq 1, \\ |t|^{-\frac{3}{2}}, & \text{if } |t| > 1, \end{cases}$$

for all $q, r \in (2, \infty]$ and r' the conjugate exponent of r, so the same estimate holds for $K_X(t)$ when combining with (1.9). Then it suffices to conclude using the standard TT^* argument exactly as in the proof of Theorem 3.6 of Anker–Pierfelice [AnP], and we obtain the claimed Strichartz estimate. $\qquad\square$

REMARK 1.3. For hyperbolic quotients with dimension of limit set $\delta > n/2$, the positive number $\delta(n - \delta)$ is an L^2 eigenvalue with multiplicity one and smooth eigenvector ψ_δ. It follows easily from section 2 of [Pe] (or the general result of Mazzeo–Melrose [MM] about the structure of the resolvent) that $\psi_\delta \in L^p(X)$ for all $p > n/\delta$, thus in particular for all $p \geq 2$. This implies that for $q \geq 2$, $2 < p < \infty$ and all $\chi \in L^\infty(X)$

$$\|\chi e^{it\Delta_X}\psi_\delta\|_{L^q(X)} = \|\chi\psi_\delta\|_{L^q(X)} \notin L^p_t((0, \infty)),$$

so global-in-time Strichartz estimates cannot hold when $\delta > n/2$, even with a space cut-off.

2 Connected Sum of Two Copies of $\mathbb{R}^2$

In this section, we give an example of a Riemannian manifold (Z, g) which is topologically the connected sum of two copies of $\mathbb{R}^2$ and is geometrically Euclidean near infinity and hyperbolic near the 'waist' (and hence with a single trapped ray), for which Strichartz estimates without loss are valid. The idea is simple; since Strichartz estimates without loss are valid on flat $\mathbb{R}^2$, and on the hyperbolic cylinder (thanks to Theorem 1.1), then they should also be valid on a space obtained by gluing pieces of these manifolds together, provided that no additional trapping is created by the gluing procedure.

Let us consider an asymptotically Euclidean manifold (Z, g) which is the connected sum of two copies of $\mathbb{R}^2$, joined by a neck which has a neighbourhood U isometric to a neighbourhood U' of the short closed geodesic, or 'waist', on the hyperbolic two-cylinder C^2. We denote this short closed geodesic by γ, whether on Z or on C^2. We can write down an explicit metric g for such a manifold, on $\mathbb{R} \times S^1$, in the form $dr^2 + f(r)^2 d\theta^2$, where $d\theta^2$ is the metric on S^1 of length 2π, and where $f(r) = \cosh r$ for small r, say $r \leq 3\eta$ for some small $\eta > 0$, and is equal to $|r| + a$

for large $|r|$, say $|r| \geq R$ (where a is a constant). We also choose f so that $f'(r)$ has the same sign as r; it is easy to see that this is compatible with the condition that $f(r) = \cosh r$ for small $|r|$ and $|r| + a$ for large $|r|$. The equations of motion for geodesic flow then give $\ddot{r} = 2f'(r)f(r)\dot{\theta}^2$, which has the same sign as r, and it is straightforward to deduce from this that there can be no trapped geodesic other than the waist γ at $r = 0$. For any such manifold (Z, g) we have
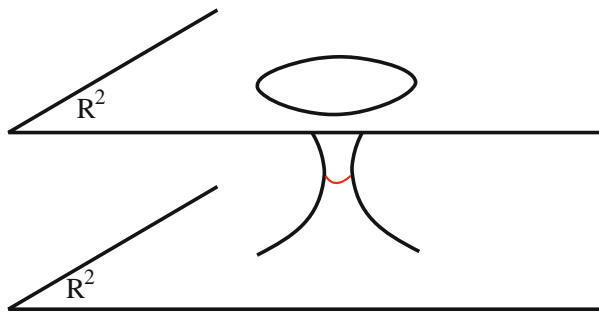

Figure 1: The manifold (Z, g)

Theorem 2.1. *For any finite T there is a constant C_T such that*

$$\|e^{it\Delta_Z}u_0\|_{L^p([0,T];L^q(Z))} \leq C_T\|u_0\|_{L^2(Z)} \tag{2.1}$$

for all (p, q) satisfying (0.1) with $d = 2$ and all $u_0 \in L^2(Z)$.

Before giving the proof we introduce some further notation and definitions. We will compare Z to the hyperbolic cylinder C^2 and to the auxiliary Riemannian manifold $(\tilde{Z} = \mathbb{R}^2, \tilde{g})$, given in standard polar coordinates (r, θ) on $\mathbb{R}^2$ by $\tilde{g} = dr^2 + \tilde{f}(r)^2 d\theta$, where $\tilde{f}(r) = f(r)$ for $r \geq \eta$, $\tilde{f}'(r) > 0$, and is equal to r for small r. Reasoning as above, we see that the metric $\tilde{g}$ is nontrapping. We will take a Schrödinger wave u on Z and decompose it so that one piece lives on C^2 and the other lives on $\tilde{Z}$, and we will deduce Strichartz without loss on Z from the fact that Strichartz without loss holds for both C^2 and $\tilde{Z}$.

Proof. Let $U \subset Z$ be a neighbourhood of $\{r = 0\}$, say $\{|r| < 2\eta\}$, thus containing the projection of the trapped set of Z. By construction, the metric is exactly hyperbolic in a neighbourhood of $\overline{U}$. We decompose $u = u_i + u_e$, where $u_i = \chi u$ is supported in U and $u_e = (1 - \chi)u$ is supported where the metric g is identical to $\tilde{g}$. (Thus $\nabla\chi$ is supported where $\eta \leq |r| \leq 2\eta$.) We prove the estimate (2.1) separately for u_i and u_e. As stated above, the idea is to regard u_e as solving a PDE on $\tilde{Z}$ and to regard u_i as solving a PDE on C^2.

We first prove a local smoothing result for Z, $\tilde{Z}$ and C^2. This is essentially standard, but we give the details for the reader's convenience (and in keeping with the expository character of this section). For applications in the following section, we give a result in any dimension.

LEMMA 2.2. (i) *Suppose that X is a d-dimensional manifold with Euclidean ends and with trapped set $K \subset T^*X$. Suppose that u solves the Schrödinger equation on*

(X, g) with initial condition u_0, and suppose that $\phi \in C_c^\infty(M)$ is supported away from the projection of the trapped set $\pi(K)$. Then

$$\|\phi u\|_{L_t^2([0,T];H^{1/2}(X))} \leq C\|u_0\|_{L^2(X)} \,.$$

(ii) *Suppose that v solves the Schrödinger equation on (C^2, g_{hyp}) with initial condition v_0, and suppose that $\phi \in C_c^\infty(C^2)$ is supported away from the closed geodesic γ. Then*

$$\|\phi v\|_{L_t^2([0,T];H^{1/2}(C^2))} \leq C\|\tilde{v}_0\|_{L^2(C^2)} \,.$$

Proof. This result can be deduced from the resolvent estimate of Cardoso–Vodev [CV], but for completeness we give a proof via a positive commutator argument. We construct a zeroth-order pseudodifferential operator A on X such that $i[\Delta, A]$ has a nonnegative principal symbol which is elliptic on the support of ϕ. Then we use the identity

$$\big\langle Au(\,\cdot\,, T), u(\,\cdot\,, T)\big\rangle - \big\langle Au(\,\cdot\,, 0), u(\,\cdot\,, 0)\big\rangle = \int_0^T \big\langle i[\Delta, A]u, u\big\rangle dt \qquad (2.2)$$

valid for any Schrödinger wave u. Since $i[\Delta, A]$ is order one and elliptic on the support of ϕ, the right-hand side is equal to $c\|\phi u\|_{L^2([0,T];H^{1/2})}^2$ plus terms which are essentially positive, while the left-hand side is bounded by $C\|u_0\|_{L^2}^2$, giving the estimate.

Let us set $A = A_1 + A_2$, where A_1 is supported in the region where X is Euclidean and A_2 is properly supported. We take $R > 0$ large enough so that each end of X has a neighbourhood isometric to $\mathbb{R}^d \setminus \overline{B(0, R)}$ and use Euclidean coordinates $\mathbf{x} = (x_1, x_2, \ldots x_d)$ on this neighbourhood with dual cotangent coordinates ξ. We write $r = |\mathbf{x}|$ and take A_1 to have principal symbol

$$a_1 = \zeta^2(r)\langle\xi\rangle^{-1}r^{-1}\mathbf{x} \cdot \xi(1 - r^{-\epsilon}) \,. \qquad (2.3)$$

Here $\zeta(t)$ is chosen to be 0 for $t < R$ and 1 for $t \geq 2R$ and to be nondecreasing, where R is sufficiently large that $1 - R^{-\epsilon} > 1/2$, say. We understand this to mean that a_1 is defined as above on *each* end of X. Explicitly, we could take $A_1 = \zeta(r)r^{-1}(1 + \Delta)^{-1/2}\frac{1}{2}(\mathbf{x} \cdot D_{\mathbf{x}} + D_{\mathbf{x}} \cdot \mathbf{x})\zeta(r)(1 - r^{-\epsilon})$, where here Δ denotes the flat Laplacian on $\mathbb{R}^d$; notice that $(1 + \Delta)^{-1/2}$ makes sense since it is both pre- and post-multiplied by $\zeta(r)$ which is supported where the metric is Euclidean.

Then the derivative a_1 along the Hamilton vector field of $\sigma(\Delta_X)$, namely the geodesic flow $2\xi \cdot \partial_{\mathbf{x}}$, is

$$2\zeta^2(r)\langle\xi\rangle^{-1}r^{-3}(1 - r^{-\epsilon})\big(r^2|\xi|^2 - (\mathbf{x} \cdot \xi)^2\big)$$
$$+ 4\zeta(r)\zeta'(r)\langle\xi\rangle^{-1}r^{-2}(1 - r^{-\epsilon})(\mathbf{x} \cdot \xi)^2$$
$$+ 2\zeta^2(r)\langle\xi\rangle^{-1}r^{-3}\epsilon r^{-\epsilon}(\mathbf{x} \cdot \xi)^2.$$

We see that this is nonnegative everywhere, and bounded below by $C\langle\xi\rangle r^{-1-\epsilon}$ for $r \geq 2R$.

Now we define a symbol a_2 which will be supported in the region $r \leq 4R$. First we introduce some notation: for $\tilde{R} \geq R$, let $E_{\tilde{R}}$ denote the union of ends $\mathbb{R}^d \setminus B(0, \tilde{R})$,

and let $U_{\tilde{R}} \subset T^*X$ denote $\pi^{-1}(X \setminus E_{\tilde{R}})$. We choose conic neighbourhoods $U_<$ and U of K such that $\overline{U_<} \subset U$ and $\pi(U)$ is disjoint from $\operatorname{supp}\phi$. For any $p \in \pi^{-1}(\operatorname{supp}\phi)$ we let $\beta = \beta(p)$ denote the maximally extended geodesic through p, and we denote by β_+, resp. β_-, the forward, resp. backward geodesic ray starting at p. Standard topological arguments show that one can choose $U_<$ so that for every $p \in \pi^{-1}(\operatorname{supp}\phi)$, at least one of β_+ or β_- does not meet $U_<$, which we will now assume.

Now choose an arbitrary $p \in \pi^{-1}(\operatorname{supp}\phi)$ and consider the geodesic $\beta(p)$. Because of the way we chose $U_<$, either β_- or β_+ does not intersect $\overline{U_<}$. Suppose, for the sake of definiteness that β_+ does not meet $\overline{U_<}$ (the argument for $\beta_- \cap \overline{U_<} = \emptyset$ is similar). Let V be a conic neighbourhood of β_+. We may construct a symbol of order 0 that is

- supported on V;
- non-decreasing with respect to geodesic flow on U_{2R};
- strictly increasing with respect to geodesic flow on $V_< \cap (U_{2R} \setminus U) \cap \{|\xi| \geq 1/2\}$, where $V_<$ is a conic neighbourhood of β_+ such that $\overline{V_<} \subset V$; and
- vanishing outside U_{4R}.

To do this, we let t be an arc-length parameter along β with $\beta(0) = p$ and $\beta_+ = \{\beta(t) \mid t \geq 0\}$, and let $t_1 = \sup\{t \mid \beta(t) \in \overline{U_<}\}$, which is negative by assumption. Also let $t_2 = \sup\{t \mid \beta(t) \in U_{2R}\}$, and $t_4 = \sup\{t \mid \beta(t) \in U_{4R}\}$, both of which are positive by assumption. We choose a function along β that is 0 for $t \leq t_1$, strictly increasing for $t_1 < t < t_2$ and zero for $t \geq t_4$. This can be extended to a symbol of order 0 supported in V.

Using compactness, we can select a finite number of conic neighbourhoods $V_<$ as above, covering $\pi^{-1}\operatorname{supp}\phi \setminus \{0\}$. Summing the corresponding symbols defined above, we obtain a symbol a_2 supported in U_{4R} such that the Hamilton vector field of Δ_X applied to a_2 is positive and elliptic on $\pi^{-1}\operatorname{supp}\phi$. Let A_2 be a properly supported pseudodifferential operator with symbol a_2. Let A be the sum of A_2 and a sufficiently large multiple of A_1. Then $i[\Delta, A]$ has nonnegative symbol, and (if the symbol a_2 is specified appropriately, i.e. so that $\{\sigma(\Delta), a_2\}$ is a sum of squares of symbols, which is always possible) may be expressed in the form $\sum_i B_i^* B_i + B_0$, where the B_i are order $1/2$ and $\sum_i B_i$ is elliptic on $\pi^{-1}\operatorname{supp}\phi$ and B_0 is order 0. Then substituting $i[\Delta, A] = \sum_i B_i^* B_i + B_0$, and using the sharp Gårding inequality in the form $C \sum_i \|B_i u\|_2^2 \geq \|\phi u\|_{H^{1/2}(X)}^2 - C'\|u\|_{L^2(X)}^2$ which is valid for sufficiently large C, we deduce that

$$\int_0^T \|\phi u\|_{H^{\frac{1}{2}}(M)}^2 dt \leq C\|u_0\|_{L^2(M)}^2,$$

proving (i).

The proof of (ii) is very similar in spirit. Again we construct a pseudodifferential operator A with the property that $i[\Delta, A]$ has a nonnegative principal symbol which is elliptic on the support of ϕ. We construct A as $A_1 + A_2$, where A_2 is constructed exactly as above, but A_1 is modified to reflect the hyperbolic rather than Euclidean structure at infinity. We shall take A_1 to be a zeroth-order pseudodifferential operator in the 0-calculus of Mazzeo–Melrose [MM]. Recall that the 0-calculus of

pseudodifferential operators on a manifold with boundary is the natural class of pseudodifferential operators associated to differential operators generated by vector fields that *vanish* at the boundary. This calculus is appropriate here since, if we compactify C^2 by adding circles at $r = \pm\infty$ with boundary defining functions $e^{\mp r}$, then the Laplacian Δ_{C^2} on C^2 is an elliptic combination of such vector fields.

Using coordinates (ρ, ω) dual to (r, θ), we define A_1 to be a zeroth-order 0-pseudodifferential operator with symbol $\zeta^2(r)(1 - e^{-\epsilon r})\rho \cdot (1 + \sigma(\Delta_{C^2}))^{-1/2}$. In these coordinates, the symbol of Δ_{C^2} is

$$\sigma(\Delta_{C^2}) = \rho^2 + (\operatorname{sech} r)^2 \omega^2$$

and the Hamilton vector field is

$$2\left(\rho\frac{\partial}{\partial r} + \tanh r(\operatorname{sech} r)^2\omega^2\frac{\partial}{\partial \rho} + \operatorname{sech}^2 r\,\omega\frac{\partial}{\partial \theta}\right).$$

Applying this to the symbol of A_1 gives the positive term

$$2\zeta(r)\big(1 + \sigma(\Delta_{C^2})\big)^{-1/2}\big(2\zeta'(r)(1 - e^{-\epsilon r})\rho^2 + \epsilon\zeta(r)e^{-\epsilon r}\rho^2$$
$$+ \tanh r(\operatorname{sech} r)^2\zeta(r)(1 - e^{-\epsilon r})\omega^2\big)$$

which is nonnegative everywhere and bounded below by a multiple of $\sigma(\Delta_{C^2})^{1/2}$ on the support of ϕ. The rest of the proof is the same as in part (i), using the fact that zeroth-order operators in the 0-calculus are bounded on $L^2(C^2)$. $\qquad\square$

REMARK 2.3. Exactly the same result holds if X is replaced by an asymptotically conic manifold, with the same proof. We only have to replace $r^{-1}\langle\xi\rangle^{-1}\mathbf{x}\cdot\xi$ in (2.3) by the cotangent variable dual to dr. We shall use this remark in the next section.

REMARK 2.4. We can rephrase this result as follows: the operators $T_X = \phi e^{-it\Delta_X}$ and $T_{C^2} = \phi e^{-it\Delta_{C^2}}$, for $\phi \in C_c^\infty(M)$ supported away from the trapped set, and $T_{\tilde{X}} = \tilde{\phi}e^{-it\Delta_{\tilde{x}}}$, for $\tilde{\phi} \in C_c^\infty(\tilde{X})$, are bounded from $L^2(X)$ to $L^2([0, T]); H^{1/2}(X)$, resp. $L^2(C^2)$ to $L^2([0, T]); H^{1/2}(C^2)$, resp. $L^2(\tilde{X})$ to $L^2([0, T]); H^{1/2}(\tilde{X})$.

We return to the proof of Theorem 2.1. Consider the function $u_e = (1 - \chi)u$. We can regard it as a function on $\tilde{Z}$, and as such it satisfies on the time interval $[0, T]$

$$(i\partial_t - \Delta_{\tilde{Z}})u_e = w \equiv -2\nabla\chi \cdot \nabla u + (\Delta_{\tilde{Z}}\chi)u; \quad u_e\big|_{t=0} = (1 - \chi)u_0 \in L^2(\tilde{Z}). \quad (2.4)$$

By Lemma 2.2, $w \in L^2([0, T]; H^{-1/2}(\tilde{Z}))$. Let us write $u_e = u_e' + u_e''$, where u_e' solves the PDE above with zero initial condition, and u_e'' solves the homogeneous equation $(i\partial_t - \Delta_{\tilde{Z}})u_e'' = 0$ with initial condition $(1 - \chi)u_0$. By [StT], the Strichartz estimate (2.1) holds for u_e''. The function u_e' is given by Duhamel's formula

$$u_e'(\,\cdot\,, t) = \int_0^t e^{-i(t-s)\Delta_{\tilde{Z}}}w(\,\cdot\,, s)ds\,.$$

We want to show that this is in $L_t^p L_x^q$ for Strichartz pairs (p, q). Since we are in dimension $d = 2$, we have $p > 2$, and hence we can apply the Christ–Kiselev lemma [ChK], which tells us that it is sufficient to show boundedness of the operator

$$w \mapsto \int_0^1 e^{-i(t-s)\Delta_{\tilde{Z}}}w(\,\cdot\,, s)ds$$

from $L^2([0,T]; H^{-1/2}(\tilde{Z}))$ to $L_t^p L_x^q$. But, defining $T_{\tilde{Z}}$ as above, this is $e^{it\Delta_{\tilde{z}}} T_{\tilde{Z}}^* w$ (for any $\tilde{\phi}$ equal to 1 on the support of $\nabla\chi$). By Lemma 2.2 and duality, $T_{\tilde{Z}}^*$ maps $L^2([0,T]; H^{-1/2}(\tilde{Z}))$ to $L^2(\tilde{Z})$, while by [StT], $e^{it\Delta_{\tilde{z}}}$ maps L^2 to $L_t^p L_x^q$. This shows that (2.1) holds for the function u_e.

It remains to consider $u_i = \chi u$. We regard u_i as a function on the hyperbolic cylinder C^2 since it is supported in the region where the metric is hyperbolic, and as such it satisfies on the time interval $[0,T]$

$$(i\partial_t - \Delta_{C^2})u_i = w \equiv -2\nabla\chi \cdot \nabla u + (\Delta_{C^2}\chi)u; \quad u_i\big|_{t=0} = \chi u_0 \in L^2(C^2). \quad (2.5)$$

As we have seen, $w \in L^2([0,T]; H^{-1/2}(C^2))$. Let us write $u_i = u_i' + u_i''$, where u_i' solves the PDE above with zero initial condition, and u_i'' solves the homogeneous equation $(i\partial_t - \Delta_{C^2})u_i'' = 0$ with initial condition χu_0. By Theorem 1.1, the Strichartz estimate (2.1) holds for u_i''. The function u_i' is given by Duhamel's formula

$$u_e'(\,\cdot\,,t) = \int_0^t e^{-i(t-s)\Delta_{C^2}} w(\,\cdot\,,s) ds \,.$$

We want to show that this is in $L_t^p L_x^q$ for Strichartz pairs (p,q). We use the Christ–Kiselev trick again and show that the operator

$$w \mapsto \int_0^1 e^{-i(t-s)\Delta_{C^2}} w(\,\cdot\,,s) ds$$

is bounded from $L^2([0,T]; H^{-1/2}(\tilde{Z}))$ to $L_t^p L_x^q$. But, defining T_{C^2} as above, this is $e^{it\Delta_{C^2}} T_{C^2}^* w$ (for any $\tilde{\phi}$ equal to 1 on the support of $\nabla\chi$). By (ii) of Lemma 2.2, $T_{C^2}^*$ by duality maps $L^2([0,T]; H^{-1/2}(C^2))$ to $L^2(C^2)$, while by Theorem 1.1, $e^{it\Delta_{\tilde{z}}}$ maps L^2 to $L_t^p L_x^q$. This shows that (2.1) holds for the function u_i, and completes the proof of Theorem 2.1. $\qquad\square$

3   Asymptotically Euclidean (or Conic) Manifolds with Filamentary Hyperbolic Trapped Set

In the previous section, we used the dispersive estimate from section 1 for constant negative curvature manifolds to prove Strichartz estimates without loss. It is natural to ask if this result can be generalized to a variable negative curvature setting. In this section, we shall show that a more general class of manifolds with hyperbolic trapped set has this 'Strichartz without loss' property. The class of manifolds we will consider are asymptotically Euclidean (and more generally asymptotically conic) but the projection of their trapped set is contained in an open set where the metric has (variable) negative curvature, so that the flow is hyperbolic there, and we will assume, as in [NZ], that the topological pressure $P(s)$ of the unstable Jacobian on the trapped set satisfies $P(1/2) < 0$; this last condition roughly means that the trapped set is thin enough, also called *filamentary*, although it may contain an infinite number of closed geodesics.

An asymptotically conic manifold (or scattering manifold in the sense of [Me]) is a complete non-compact Riemannian manifold (M,g) which is the interior of a

smooth compact manifold with boundary $\overline{M}$ and such that a collar neighbourhood of the boundary is isometric to

$$\left([0,\epsilon)_x \times \partial\overline{M}, \frac{dx^2}{x^4} + \frac{h(x)}{x^2}\right)$$

where $h(x)$ is a one-parameter family of metrics on $\partial\overline{M}$ depending smoothly on $x \in [0,\epsilon)$. Here $\partial\overline{M}$ has really to be considered as the 'points at infinity' of (M,g). The function x can be extended to a nonnegative smooth function on $\overline{M}$ and the function $r = 1/x$ is analogous to the radial function on Euclidean space.

The geodesic flow Φ^t, $t \in \mathbb{R}$, is the flow of the Hamiltonian vector field V_H associated to $H \in C^\infty(T^*M)$ defined by $H(m,\xi) := |\xi|_g^2$. The trapped set K is defined by $K := \Gamma^+ \cap \Gamma^-$ where

$$\Gamma^\pm := \left\{(m,\xi) \in T^*M \mid \Phi^t(m,\xi) \not\to \infty\,,\ t \to \mp\infty\right\} \subset T^*M\,. \qquad (3.1)$$

Let us denote by $\pi : T^*M \to M$ the projection on the base, and let $d = \dim M$.

The geodesic flow is said to be hyperbolic on $U \subset S^*M$ if, for all $m \in U$, the tangent space at m splits into flow unstable and stable subspaces such that

(i) $T_m S^* U = \mathbb{R}V_H(m) \oplus E_m^+ \oplus E_m^-$, $\quad \dim E_m^\pm = d - 1$,

(ii) $d\Phi_m^t(E_m^\pm) = E_m^\pm$, $\quad \forall t \in \mathbb{R}$, $\qquad\qquad\qquad\qquad (3.2)$

(iii) $\exists \lambda > 0$, $\quad \left\|d\Phi_m^t(v)\right\| \le Ce^{-\lambda|t|}\|v\|$, $\quad \forall v \in E_m^\mp$, $\pm t \ge 0$.

for some uniform $\lambda > 0$; here the norm can be taken with respect to the Sasaki metric on the cotangent bundle (see [P, Def. 1.17]). This is true in particular for $U = S^*M$ if M is a complete manifold with negative sectional curvatures contained in an interval $[-k_1, -k_0]$ for some $k_i > 0$, see for instance [Kl, Th. 3.9.1].

We define the unstable Jacobian $J_t^u(m)$ and the weak unstable Jacobian $J_t^{wu}(m)$ for the flow Φ^t at the point m to be

$$\begin{aligned}
J_t^u(m) &= \det\left(d\Phi^{-t}(\Phi^t(m))|_{E_{\Phi^t(m)}^+}\right), \\
J_t^{wu}(m) &= \det\left(d\Phi^{-t}(\Phi^t(m))|_{E_{\Phi^t(m)}^+ \oplus \mathbb{R}V_H(m)}\right),
\end{aligned} \qquad (3.3)$$

where the volume form on d-dimensional subspaces of $T(T^*M)$ is induced by the Sasaki metric. It follows from (iii) of (3.2) that $J_t^u(m), J_t^{wu}(m) \le e^{-\lambda t}$ for $t > 0$.

If the geodesic flow on the trapped set K is hyperbolic, and $s : K \to \mathbb{R}$ is a continuous function, then the *topological pressure* of the unstable Jacobian at s is a real number $P(s)$, whose definition is given by (3.7). The topological pressure of the unstable Jacobian can be viewed as a real function P of s. The quantity $P(0)$ is known as the topological entropy of the flow. For positive s, $P(s)$ in a sense measures two competing effects of the flow: the density of K (the denser K, the longer points near K stay close by under the flow) and the instability of the flow (the more unstable, the more quickly points near K move away from K under the flow). In our analysis we encounter products of square roots of the unstable Jacobian, which in view of (3.6) and (3.7) make it natural to consider the topological pressure at $s = 1/2$; if $P(1/2) < 0$ then the instability dominates, which is crucial in our main estimate (Lemma 3.13) of this section. The first use, to our knowledge, of topological

pressure in analytical estimates was by Nonnenmacher–Zworski [NZ] following work
of Gaspard–Rice [GR] in the physics literature.

Our assumptions on (M, g) and on the trapped set K in this section are

(A1) (M, g) is asymptotically conic.

(A2) There is an open set $X_- \subset X$ containing $\pi(K)$ which can be extended to
a complete manifold $\tilde{M}$ with sectional curvatures bounded above by a nega-
tive constant (in particular, M_- has sectional curvatures bounded above by a
negative constant).

(A3) M_- is geodesically convex in M, i.e. any geodesic entering $\pi^{-1}(M \setminus M_-)$ from
$\pi^{-1}M_-$ remains in this region thereafter.

(A4) The topological pressure $P(s)$ of K evaluated at $s = 1/2$ is negative:

$$P\left(\tfrac{1}{2}\right) < 0. \tag{3.4}$$

For examples, see section 3.2.

REMARK 3.1. With these assumptions, the geodesic flow is hyperbolic on $S^*\tilde{M}$ and
K is the trapped set on both $S^*\tilde{M}$ and S^*M. On S^*M, the splitting satisfying (3.2)
only makes sense at points of K, but we can still consider the splitting $TS^*M_- =
\mathbb{R}V_H(m) \oplus E_m^+ \oplus E_m^-$ coming from the inclusion $S^*M_- \subset S^*\widetilde{M}$: in particular, for all
$m \in S^*M_-$ and all t such that $\Phi^t(m) \in \pi^{-1}(M_-)$ we have

$$d\Phi_m^t(E_m^\pm) = E_m^\pm,$$

$$\exists \lambda > 0, \quad \left\| d\Phi_m^t(v) \right\| \le Ce^{-\lambda|t|}\|v\|, \quad \forall v \in E_m^\mp, \text{ if } \pm t \ge 0.$$

REMARK 3.2. It seems likely that (A2) actually follows from (A1) and (A3); that
is, that if M_- is negatively curved and geodesically convex, then we expect that it
can always be extended to a complete manifold with negative sectional curvature.
We do not pursue this question further here as it is a purely differential-geometric
question.

Our main result in this section is

Theorem 3.3. *Let (M, g) satisfy assumptions* (A1)–(A4) *above. Then local-in-
time Strichartz estimates without loss hold for M: there exists $C > 0$ such that*

$$\|e^{it\Delta}u_0\|_{L^p((0,1), L^q(M))} \le C\|u_0\|_{L^2(M)} \tag{3.5}$$

for all $u_0 \in L^2(M)$ and (p, q) satisfying (0.1).

REMARK 3.4. In dimension $d = 2$, the condition $P(1/2) < 0$ is equivalent to
the Hausdorff dimension of the trapped set satisfying $d_H(K) < 3$, or equivalently
$d_H(K \cap S^*M) < 2$. Note that this is the natural generalization of the condition
$\delta < n/2$ in our hyperbolic quotients examples above, since $d_H(K \cap S^*M) = 2\delta + 1$
in that case (recall $d = n + 1$).

3.1   Topological pressure.   We now define the topological pressure of the flow
$P(s)$ on the trapped set, following [NZ] (which follows from Definition 20.2.1 of
[HK]): a set $E \subset K \cap S^*M$ is said to be (ϵ, T) separated if, given $(x_1, v_1) \ne (x_2, v_2)$

in E, there exists $t \in [0, T]$ for which the distance between $\Phi^t(x_1, v_1)$ and $\Phi^t(x_2, v_2)$ is at least $\epsilon > 0$. For any $s \in \mathbb{R}$ we define

$$Z_T(\epsilon, s) := \sup_E \sum_{m \in E} \left(J_T^{wu}(m) \right)^s, \tag{3.6}$$

where the sup is taken over all sets E which are (ϵ, T)-separated. The pressure of s is

$$P(s) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log Z_T(\epsilon, s) . \tag{3.7}$$

REMARK. For later convenience we define the topological pressure using the weak unstable Jacobian rather than the unstable Jacobian which is more standard. However, making this change only changes $\log Z_T(\epsilon, s)$ by $O(1)$, uniformly in ϵ, and thus leads to the same value of $P(s)$.

For instance, if the metric has constant curvature, one has

LEMMA 3.5. *Let (M, g) be a convex cocompact hyperbolic manifold of dimension $n+1$ with limit set of dimension δ. Then the topological pressure at $s = 1/2$ is given by $P(1/2) = \delta - n/2$.*

Proof. For a constant curvature -1 manifold, $J_t^u(m) = e^{-tn}$. It follows from (3.6), (3.7) and the remark above that $P(s) = P(0) - ns$. But $P(0)$, which is the topological entropy, is equal to δ by a result of Sullivan [Su]. $\square$

Finally, we recall the alternate definition of topological pressure given in [NZ, §5.2], which turns out to be easier to use. If $\mathcal{V} = (V_b)_{b \in B}$ is an open finite cover of $K \cap S^*M$, let $\mathcal{V}^T$ ($T \in \mathbb{N}$) be the refined cover made of T-fold intersections

$$V_\beta := \bigcap_{k=0}^{T-1} \Phi^{-k}(V_{b_k}), \quad \beta := b_0 b_1 \ldots b_{T-1} \in B^T,$$

and consider the set $\mathcal{B}'_T \subset B^T$ of β such that $V_\beta \cap K \neq \emptyset$. For any $W \subset S^*M$ with $W \cap K \neq \emptyset$, define the coarse-grained unstable Jacobian

$$S_T^K(W) := \sup_{m \in W \cap K} \log J_T^u(m) = - \inf_{m \in W \cap K} \log \det \left(d\Phi^T(m)|_{E_m^+ \oplus \mathbb{R}V_H(m)} \right). \tag{3.8}$$

The topological pressure is defined by

$$P(s) := \lim_{\text{diam} \mathcal{V} \to 0} \lim_{T \to \infty} \frac{1}{T} \log \inf \left\{ \sum_{\beta \in \mathcal{B}_T} \exp\left(s S_T^K(V_\beta) \right); \mathcal{B}_T \subset \mathcal{B}'_T, K \cap S^*M \subset \bigcup_{\beta \in \mathcal{B}_T} V_\beta \right\}.$$

In particular, for all $\epsilon_0 > 0$ small, there exists $\epsilon_1 > 0$ such that, for all $\epsilon < \epsilon_1$ and all covers $\mathcal{V}$ of $K \cap S^*M$ as above with diameter smaller than ϵ, there is a $T_0 \in \mathbb{N}$, a set $\mathcal{B}_{T_0} \subset \mathcal{B}'_{T_0}$ such that $\{V_\beta, \beta \in \mathcal{B}_{T_0}\}$ is an open cover of $K \cap S^*M$ and

$$\sum_{\beta \in \mathcal{B}_{T_0}} \exp\left(s S_{T_0}^K(V_\beta) \right) \leq \exp\left(T_0(P(s) + \epsilon_0/2) \right) . \tag{3.9}$$

Moreover, V_β are all included in $\pi^{-1}(M_-)$ since they are ϵ close to K. Since by the chain rule one has

$$J_{T_0}^u(m) = \prod_{j=1}^{T_0} J_1^u\left(\Phi^{j-1}(m) \right),$$

and since the unstable foliation is γ-Hölder [SS], we deduce that for any V_β with $\beta \in \mathcal{B}_{T_0}$

$$\left| S^K_{T_0}(V_\beta) - S_{T_0}(V_\beta) \right| \le \exp(CT_0\epsilon^\gamma), \quad \text{where } S_{T_0}(V_\beta) := \sup_{m \in V_\beta} \log J^u_{T_0}(m)$$

for some constant C depending only on ϵ_0 (notice that E^+_m makes sense since $\pi(V_\beta)$ is in the negative curved part M_-). Therefore, renaming the family $(V_\beta)_{\beta \in \mathcal{B}_{T_0}}$ by $(W_a)_{a \in A_1}$, we get, by taking for instance $C\epsilon^\gamma \le \epsilon_0/2$

$$\sum_{a \in A_1} \exp\left(sS_{T_0}(W_a)\right) \le \exp\left(T_0(P(s) + \epsilon_0)\right). \tag{3.10}$$

3.2 Some examples. We first give examples of Riemannian manifolds satisfying assumptions (A1)–(A4). Consider any convex co-compact hyperbolic manifold (M, g). Near infinity, it is conformally compact. That is, it admits a compactification to a compact manifold $\overline{M}$ with a boundary defining function x, such that $x^2 g$ is a smooth nondegenerate metric up to the boundary of $\overline{M}$. In other words, near infinity g takes the form $g = h/x^2$ for some smooth metric h on $\overline{M}$. We shall now modify the metric g near infinity to a metric that is asymptotically conic, in such a way that the trapped set is left unchanged. (We recall that both asymptotically conic and conformally compact metrics are nontrapping near infinity.) This is straightforward: we have near infinity, in suitable coordinates (x, y) where x is a boundary defining function for $\overline{M}$ and y is a local coordinate on $Y = \partial \overline{M}$

$$g = \frac{dx^2 + h(x)}{x^2}.$$

Here $h(x)$ is a smooth family of metrics on Y, i.e. is smooth in all its arguments. Changing variables to the 'geometric' coordinate $r = \log(1/x)$, this reads

$$g = dr^2 + e^{2r}h(e^{-r}).$$

Assume that this is valid for $r \ge R$. Then, for some $R' \ge R$ we choose a function $f(r)$ such that $f(r) = e^r$ for $r \le 2R'$ and $f(r) = cr$ for $r \ge 4R'$, and such that $f'(r) > f(r)/2r$ for all r. This is possible for all $R' \ge R$ and some c depending on R'. Define the metric

$$g_{ac} = dr^2 + f(r)^2 h(e^{-r}), \quad r \ge 2R; \quad g_{ac} \equiv g \text{ for } r \le 2R. \tag{3.11}$$

This is an asymptotically conic metric on M and satisfies assumption (A2). The symbol of the Laplacian with respect to this metric is $\rho^2 + f(r)^{-2}(h^{-1}(e^{-r}))_{ij}\eta_i\eta_j$ and along geodesics we have

$$\ddot{r} = 2\left(\frac{f'(r)}{f(r)^3} |\eta|^2_{h^{-1}(e^{-r})} + \frac{e^{-r}}{f(r)^2} |\eta|^2_{\dot{h}^{-1}(e^{-r})} \right).$$

Here $\dot{h}^{-1}(e^{-r})$ means $d/ds(h^{-1}(s)) \mid s = e^{-r}$. The metric $d/ds(h^{-1}(s))$ is bounded above by a constant times $h^{-1}(s)$ uniformly for $s \in [0, \log R^{-1}]$. Also, using $f'(r) > f(r)/2r$, we see that $f'(r)/f(r)^3 \gg e^{-r}/f(r)^2$ for large r. It follows that for R' sufficiently large, and $r \ge R'$, we have $\ddot{r} \ge 0$ and hence there is no trapped set in

$r \geq R'$ for the metric g_{ac}; moreover, the set $\{r \leq R'\}$ is geodesically convex. Hence the metric satisfies condition (A3).

Finally to verify assumption (A4) it suffices to use Lemma 3.5.

3.3 Strategy of the proof.   The proof of Theorem 3.3 is much more involved than that of Theorem 2.1. We will need to localize both in frequency and in time. To explain the idea, we first show how the Strichartz estimates on an asymptotically conic *nontrapping* manifold M may be deduced via frequency and time localization. Here we focus on estimating the solution on a compact set contained in M. Thus, we consider χu, where $\chi \in C_0^\infty(M)$ vanishes for small x.

We introduce the semiclassical parameter h, where h^{-1} will be (up to a constant) the frequency of our frequency-localized wave u. Let $\psi \in C_0^\infty(1/2, 2)$. Then the semiclassical dispersion estimate from [BuGT1] says that for sufficiently small c,

$$\left\| \psi(h^2\Delta_M)e^{-it\Delta_M} \right\|_{L^1(M)\to L^\infty(M)} \leq Ct^{-n/2} \quad \text{for } t \in [0, ch]. \qquad (3.12)$$

Let us assume that u_0 is localized near frequencies $\approx h^{-1}$ in the sense that $\psi(h^2\Delta_M)u_0 = u_0$ (this will then be true for all times t). This assumption is harmless as a Littlewood–Paley argument (see [BuGT1, §2.3.2], using the Littlewood–Paley estimate from [Bo1]) shows that if Strichartz is true for frequency-localized u, then it holds for all u. It follows from (3.12) and Keel–Tao that Strichartz holds for u on a time interval of length ch:

$$\|e^{-it\Delta_M}u_0\|_{L^p[0,ch];L^q(M)} \leq C\|u_0\|_{L^2(M)} \qquad (3.13)$$

for (p, q) satisfying (0.1). Here, c depends on the injectivity radius of M and the support of ψ; for simplicity, below we assume that $c = 1$.

To extend this to a fixed-length time interval, we use time cutoffs and local smoothing estimates. To define the time cutoffs, let $\varphi(s) \in C_0^\infty[-1, 1]$ satisfy $\varphi(0) = 1$ and $\sum_{j\in\mathbb{Z}} \varphi(s - j) = 1$. Then we can write, for any Schrödinger wave $u(\,\cdot\,, t) = e^{-it\Delta_M}u_0$,

$$\chi u = \sum_{j\in\mathbb{Z}} \varphi(t/h - j)\chi u \equiv \sum_j u_j \,,$$

where each u_j is supported on a time interval of length $2h$. We work on the time interval $[0, 1]$ and assume that $h^{-1} = N \in \mathbb{N}$; thus, we need to consider u_j for $j = 0, 1, \ldots, N$. The functions u_0 and u_N are dealt with from the semiclassical Strichartz estimate (3.13). So consider u_j for $1 \leq j \leq N-1$. These functions satisfy the equation

$$(i\partial_t - \Delta_M)u_j = h^{-1}\varphi'(t/h - j)\chi u + 2\varphi(t/h - j)\big(\nabla\chi \cdot \nabla u - \Delta_M\chi u\big) \equiv w_j \,. \quad (3.14)$$

Since M is nontrapping, we have from Lemma 2.2 and Remark 2.3 the local smoothing estimate

$$\|\tilde{\chi}u\|_{L^2[0,1];H^{1/2}(M)} \sim h^{-1/2}\|\tilde{\chi}u\|_{L^2[0,1];L^2(M)} \leq C\|u_0\|_{L^2(M)} \qquad (3.15)$$

for any $\tilde{\chi} \in C_0^\infty(M)$. Choose $\tilde{\chi}$ to be 1 on the support of χ. It follows that

$$\sum_{j=1}^{N} \|w_j\|_{L_t^2;L^2(M)}^2 \leq Ch^{-1}\|u_0\|_{L^2(M)}^2 \, . \tag{3.16}$$

We can express u_j in terms of w_j using Duhamel's formula,

$$u_j(t) = \tilde{\chi} \int_{(j-1)h}^{t} e^{-i(t-s)\Delta_M} \tilde{\chi} w_j(s) ds \, . \tag{3.17}$$

By the Christ–Kiselev lemma, if $p > 2$, in order to estimate the L_t^p norm of u_j in terms of the L_t^2 norm of w_j it is sufficient to estimate the L^p norm of $\tilde{u}_j$ defined by

$$\tilde{u}_j(t) = \int_{(j-1)h}^{(j+1)h} e^{-i(t-s)\Delta_M} w_j(s) \, ds = e^{-it\Delta_M} \int_{(j-1)h}^{(j+1)h} e^{is\Delta_M} w_j(s) ds \, . \tag{3.18}$$

Now we can use the semiclassical Strichartz estimate since the time interval is $O(h)$. The dual estimate to (3.15) gives

$$\left\| \int_{(j-1)h}^{(j+1)h} e^{is\Delta_M} w_j(s) ds \right\|_{L^2(M)} \leq Ch^{1/2} \|w_j\|_{L_t^2;L^2(M)} \, . \tag{3.19}$$

Then (3.13) applied to this L^2 function shows that

$$\|\tilde{u}_j\|_{L_t^p;L^q(M)} \leq Ch^{1/2}\|w_j\|_{L_t^2;L^2(M)} \, , \tag{3.20}$$

and the same estimate holds for u_j by Christ–Kiselev. Squaring this inequality, summing over j and using (3.16) shows that

$$\sum_{j=1}^{N-1} \|u_j\|_{L_t^p;L^q(M)}^2 \leq C\|u_0\|_{L^2(M)}^2 \, . \tag{3.21}$$

Using the continuous embedding from $l^2(\mathbb{N})$ to $l^p(\mathbb{N})$ if $p \geq 2$, we obtain

$$\left(\sum_{j=1}^{N-1} \|u_j\|_{L_t^p;L^q(M)}^p \right)^{2/p} \leq C\|u_0\|_{L^2(M)}^2 \, , \tag{3.22}$$

and this gives

$$\|u\|_{L^p[0,1];L^q(M)} \leq C\|u_0\|_{L^2(M)} \tag{3.23}$$

with C independent of h.

Now suppose that M is trapping, but obeys assumptions (A1)–(A4) above. In that case, the local smoothing estimate definitely fails [Bu], [Do], and then the argument only gives Strichartz estimate with a loss (i.e. with additional negative powers of h on the right-hand side) arising from the loss in the local smoothing estimate. If the trapped set has negative topological pressure, the local smoothing loss is $|\log h|^{1/2}$ as follows from work of Nonnenmacher–Zworski [NZ] and Datchev [D], Theorem 3.7 below (see also [Bu], [BuZ]). In combination with the argument above, this gives Strichartz with logarithmic loss, as shown for example in the case of the exterior of several convex obstacles by Burq [Bu].

However, when the topological pressure is negative then more is true: essentially from the work of Anantharaman [A] and Nonnenmacher–Zworski [NZ], it follows that the semiclassical Strichartz estimate can be *improved* by a logarithm: it is valid not just on a time interval of length $O(h)$, but actually on an interval of length $h|\log h|$; see Theorem 3.8 below. Then it turns out that this logarithmic improvement exactly compensates for the logarithmic loss in the local smoothing estimate and we recover the Strichartz estimate on a fixed finite time interval without loss. This is achieved by localizing in time on intervals of length $h|\log h|$ rather than h for the part of u localized near the trapped set.

More precisely, we proceed as above but with $\varphi(t/h - j)$ replaced by $\varphi(t/h|\log h| - j)$; that is, we localize to time intervals of length $h|\log h|$ which is the maximum for which we can apply the semiclassical Strichartz estimate. We then write $w_j = w_j' + w_j''$, where

$$w_j' = \frac{1}{h|\log h|}\varphi'\big(t/h|\log h|-j\big)u_j\,,\quad w_j''=2\varphi\big(t/h|\log h|-j\big)(\nabla\chi\cdot\nabla u-\Delta_M\chi u)\,. \quad (3.24)$$

Then w_j'' is of size $\sim h^{-1}$ (since a derivative of u costs h^{-1}). On the other hand, it is supported in the nontrapping region and may be dealt with as above, as the local smoothing estimate is valid without loss in the nontrapping region. The other term, w_j', is supported in the trapped region but is of size $O((h|\log h|)^{-1})$. On this term we apply the local smoothing estimate, losing $|\log h|^{1/2}$ as compared to the argument above. When we apply the dual estimate at the step (3.19) we lose a further $|\log h|^{1/2}$, and then applying semiclassical Strichartz completes the argument with no overall loss. The details are given in section 3.7.

In summary, the key ingredients of the proof will be the following three results:

Theorem 3.6 (Strichartz in the nontrapping region). *Let M be an asymptotically conic manifold, and suppose that $\chi \in C^\infty(M) \cap L^\infty(M)$ vanishes in a neighbourhood of $\pi(K)$ where $K \subset T^*M$ is the trapped set. Then we have Strichartz estimates without loss for $\chi e^{-it\Delta_M}u_0$:*

$$\|\chi e^{-it\Delta_M}u_0\|_{L^p[0,1];L^q(M)} \le C\|u_0\|_{L^2(M)} \quad (3.25)$$

for all (p,q) satisfying (0.1), $p > 2$.

Theorem 3.7 (Local smoothing with logarithmic loss). *Suppose that M satisfies assumptions (A1)–(A4). Then for any $\chi \in C_0^\infty(M)$ and $\psi \in C_c^\infty(1/2, 2)$, we have*

$$\|\chi e^{-it\Delta_M}\psi(h^2\Delta)u_0\|_{L^2[0,1];L^2(M)} \le C\big(h|\log h|\big)^{1/2}\|u_0\|_{L^2(M)}\,. \quad (3.26)$$

Moreover, if χ is supported outside the trapping region, then the estimate holds without the logarithmic loss in h on the right-hand side.

Theorem 3.8 (Semiclassical Strichartz on a logarithmic interval). *Suppose that M satisfies assumptions (A1)–(A4). Then for any χ supported in M_-, we have on a time interval of length $h|\log h|$*

$$\big\|\chi e^{-it\Delta_M}\psi(h^2\Delta)u_0\big\|_{L^p[0,h|\log h|];L^q(M)} \le C\|u_0\|_{L^2(M)}$$

for all (p,q) satisfying (0.1).

3.4 Strichartz in the nontrapping region. In this section we sketch how to prove Theorem 3.6. The argument is quite related to similar ideas in [StT]. Here, we follow fairly straightforward modifications of the argument in [HaTW] for nontrapping metrics. Notice that none of the results about 'local Schrödinger integral operators' in section 3 of [HaTW] use the nontrapping property, which only enters when the local smoothing estimate is used. To adapt the results of [HaTW] to prove (3.25), we modify the definition of the Banach space X in (4.6) of that paper to include a cutoff function χ_+, supported in the nontrapping region and equal to 1 on the support of χ in (3.25) in the $\|f\|_{H^{1/2,-1/2-\rho/2}(M)}$ term. This cutoff function then needs to be included in Lemma 4.4 and Lemma 5.5 of [HaTW]. The proof in section 6 then goes through provided that $\psi_\alpha^{(0)}$ is supported in the nontrapping region.

3.5 Local smoothing effect. Theorem 3.7 follows fairly directly from the resolvent estimate from [D] (which generalizes [NZ, Th. 5] to scattering manifolds):

$$\left\|\chi(h^2\Delta - (1 \pm i\epsilon))^{-1}\chi\right\|_{L^2 \to L^2} \le C\frac{|\log h|}{h}, \quad 0 < h < h_0 \ll 1, \tag{3.27}$$

with C independent of ϵ (and h). From this, we deduce (3.26), following [BuGT2]. Indeed, denote by T the operator

$$T = \chi e^{it\Delta}\psi(h^2\Delta).$$

The boundedness of T from L^2 to $L^2((-\infty, +\infty); L^2(M))$ is equivalent to the boundedness of its adjoint T^* from $L^2((-\infty, +\infty); L^2(M))$ to L^2 (with same norm), which in turn is equivalent to the boundedness of TT^* from $L^2((-\infty, +\infty); L^2(M))$ to itself (with same norm squared). But

$$TT^*f(t) = \int_{-\infty}^{+\infty} \chi e^{i(t-s)\Delta}\psi^2(h^2\Delta)\chi f(s)ds = \chi\int_{-\infty}^{t} +\chi\int_{t}^{+\infty} \equiv \chi A_1 f(t) + \chi A_2 f(t),$$

and it is enough to estimate for example $\chi A_1 f$. For this we can assume that f has compact support, and consider $u_\epsilon = e^{-\epsilon t}A_1 f(t)$ and $f_\epsilon = e^{-\epsilon t}f$ (notice that u_ϵ is supported in the set $\{t \ge C\}$) which satisfy

$$(i\partial_t + \Delta + i\epsilon)u_\epsilon = \psi^2(h^2\Delta)\chi f_\epsilon.$$

Taking Fourier transform with respect to the variable t, we get

$$\chi\widehat{u_\epsilon}(\tau) = \chi(\Delta - (\tau - i\epsilon))^{-1}\psi^2(h^2\Delta)\chi\widehat{f_\epsilon},$$

and according to the Plancherel formula (recall that the Plancherel formula is true for functions taking values in any (separable) Hilbert space), and using (3.27) we obtain

$$\|\chi u_\epsilon\|_{L^2((-\infty,+\infty);L^2(M))} \le Ch\log(1/h)\|\chi f_\epsilon\|_{L^2((-\infty,+\infty);L^2(M))}.$$

Letting $\epsilon > 0$ tend to 0 we obtain that the contribution of χA_1 to TT^* satisfies the required estimate. The other contribution χA_2 is dealt with similarly.

The improved estimate when the support of χ does not meet the trapped set is a consequence of Lemma 2.2 and Remark 2.3.

3.6 Semiclassical Strichartz on a logarithmic interval. Using the work of Nonnenmacher–Zworski [NZ] (which follows techniques of Anantharaman [A]), we shall obtain a sharp dispersive estimate for the propagator $e^{-it\Delta_M}$ for time $t \in (0, h|\log h|)$ and in frequency localization in windows of size h. We want to prove the following:

PROPOSITION 3.9. *There exists $\delta > 0$ and $C > 0$ such that for all $\psi \in C_0^\infty((1-\delta/2, 1+\delta/2))$, all $t \in (0, h|\log h|)$ with $h \in (0, h_0)$ small, we have for every $\chi \in C_0^\infty(M)$ supported in M_-*

$$\left\|\chi e^{it\Delta}\psi(h^2\Delta)\chi\right\|_{L^1 \to L^\infty} \leq Ct^{-n/2}.$$

Then Theorem 3.8 follows immediately from this by applying the main result of Keel–Tao [KT].

The proof of Proposition 3.9 decomposes into several parts. Let us first introduce the objects taken from [NZ] that we need to use for the proof.

Without loss of generality and to simplify notation, we assume as in [A] that the injectivity radius of M is larger than 1. For $t \in (0, h)$, the result is essentially contained in [BuGT1], [BoT] since we are localized in a compact set of M. Now take $s_0 \in [0, 1]$ and an integer L, with $1 \leq L \leq \log(1/h)$. We want to obtain a dispersive estimate for $U(L + s_0)$, where

$$U(t) := e^{ith\Delta}, \quad h \in (0, h_0),$$

by following [A], [NZ]. We consider, as in section 6.3 of [NZ], a microlocal partition of unity $(\Pi_a)_{a \in A}$ of the energy layer $\mathcal{E}^\delta := \{(m, \xi) \in T^*M, |\xi| \in (1-\delta, 1+\delta)\}$ for some $\delta > 0$ small. Let us recall how the partition (Π_a) is defined. The operators Π_a are associated to an open covering $(W_a)_{a \in A}$ of $\mathcal{E}^\delta$ in the sense that the semiclassical wavefront set $\text{WF}_h(\Pi_a) \subset W_a$ and $\sum_{a \in A} \Pi_a = I$ microlocally near $\mathcal{E}^{\delta/2}$, i.e. $\text{WF}_h(\Pi_\infty) \cap \mathcal{E}^{\delta/2} = \emptyset$ if Π_∞ is defined by $\Pi_\infty := I - \sum_{a \in A} \Pi_a$. Following section 5.2 and 5.3 in [NZ], the set A is decomposed into 3 parts, $A = A_1 \sqcup A_2 \sqcup \{0\}$. The open set W_0 is defined by

$$W_0 := \mathcal{E}^\delta \cap \pi^{-1}(M \setminus M_-).$$

$(W_a)_{a \in A_1}$ is chosen so that $W_a \subset M_-$ if $a \in A_1$ and, as a ranges over A_1, these sets cover $K \cap \mathcal{E}^\delta$ in such a way that for any $\epsilon_0 > 0$ fixed small, there exist $\delta, \epsilon > 0$, $T_0 \in \mathbb{N}$ with $W_a \subset \{m \in T^*M, d(W_a, K) \leq \epsilon\}$ and

$$\sum_{a \in A_1} \exp\big(sS_{T_0}(W_a)\big) \leq \exp\big(T_0(P(s) + \epsilon_0)\big), \tag{3.28}$$

where $S_{T_0}(W_a)$ is defined by (3.8). This is possible as explained in section 3.1 and using the homogeneity of the Hamiltonian on T^*M to deal with $\mathcal{E}^\delta$ instead of S^*M. Finally, the W_a for $a \in A_2$ are defined so that there exists $d_1 > 0$ such that

$$d(W_a, \Gamma^+ \cap \mathcal{E}^\delta) + d(W_a, \Gamma^- \cap \mathcal{E}^\delta) > d_1$$

where $\Gamma^\pm$ are the forward/backward trapped sets defined in (3.1). By [NZ, Lem. 5.1], there exists $L_0 \in \mathbb{N}$ such that for all $a \in A_2$

$$\Phi^t(W_a) \subset W_0 \quad \text{for } t \geq L_0 \text{ or } t \leq -L_0. \tag{3.29}$$

For notational simplicity we replace both T_0 and L_0 by $\max(T_0, L_0)$; hence we have

$$\Phi^t(W_a) \subset W_0 \quad \text{for } t \geq T_0 \text{ or } t \leq -T_0. \tag{3.30}$$

To summarize, the energy layer $\mathcal{E}^\delta$ is decomposed into the part W_0 covering the (spatial) infinity of $\mathcal{E}^\delta$, the part $\cup_{a \in A_1} W_a$ covering the trapped set $K \cap \mathcal{E}^\delta$, and finally the part covering the complementary, whose flowout by Φ^t lies in W_0 after some large (positive or negative) time.

We write $U(t) = e^{ith\Delta}$ and we shall prove

$$\left\| \chi\psi(h^2\Delta)U(T)u \right\|_{L^\infty} \leq C(Th)^{-n/2}\|u\|_{L^1} \tag{3.31}$$

for $T \in (2T_0, |\log h|)$. First we need a technical lemma.

LEMMA 3.10.   (i) Let (M, g) be a scattering manifold and Δ_M its Laplacian. Then for any $\psi \in C_0^\infty(\mathbb{R})$, $\psi(h^2\Delta_M)$ is a semiclassical scattering operator of order $(-\infty, 0, 0)$ in the sense of Wunsch–Zworski [WZ].

(ii) Let ψ be as above, then for each fixed t, the operator $\psi(h^2\Delta_M)e^{-ith\Delta_M}$ is a semiclassical Fourier Integral operator associated to the canonical relation

$$\left\{ ((z, \zeta), (z', \zeta')) \in T^*M \times T^*M \mid (z, t\zeta) = \exp(z', t\zeta') \right\}.$$

Proof. (i) This follows from the argument in [HaV] for scattering pseudodifferential operators. (We also remark that a similar, weaker result proved under more general assumptions about the nature of the ends of the manifold by Bouclet [Bo1] would also suffice for our purposes.)

(ii) It is shown in [NZ] that $e^{ith\Delta_h}\phi(h^2\Delta)$ is a semiclassical FIO for each fixed t and all $\phi \in C_0^\infty(\mathbb{R})$. Thus, using the result of (i), $\psi(h^2\Delta)e^{ith\Delta}\phi(h^2\Delta)$ is a semiclassical FIO. If $\phi = 1$ on the support of ψ then this is precisely $\psi(h^2\Delta)e^{ith\Delta_h}$ by functional calculus, proving the result. $\square$

We decompose $T > 2T_0$ into the form $T = L - 1 + s_0 = (2 + N)T_0 + t_0$, where $L, N \in \mathbb{N}$ and $s \in (0, 1]$, $t_0 \in (0, T_0]$. Choosing $\psi_+ \in C_c^\infty(\mathbb{R})$ to be 1 on the support of ψ, we have

$$\psi(h^2\Delta)U(T) = \psi(h^2\Delta)\psi_+^L(h^2\Delta)U(T) = \psi(h^2\Delta)U(s_0)\left(\psi_+(h^2\Delta)U(1)\right)^{L-1}\psi_+(h^2\Delta),$$

and we can decompose

$$\psi_+(h^2\Delta)U(1) = \sum_{a \in A \cup \infty} U_a, \quad U_a := \psi_+(h^2\Delta)U(1)\Pi_a. \tag{3.32}$$

Hence we may write

$$\psi(h^2\Delta)U(T) = \sum_{\alpha \in A^L} \psi(h^2\Delta)U(s_0)\Pi_{\alpha_L}U_{\alpha_{L-1}}\dots U_{\alpha_1}\psi_+(h^2\Delta) + R_T \tag{3.33}$$

where the R_T term is the sum over all sequences α containing at least one index $\alpha_j = \infty$. The first estimate we obtain corresponds to Lemma 6.5 in [NZ].

LEMMA 3.11.   If $\psi, \psi_+(x, hD)$ are chosen as above, we have

$$\|\chi R_T \chi\|_{L^1 \to L^\infty} = O(h^\infty)$$

for $1 \leq T \leq |\log h|$, with implied constants independent of T.

Proof. Since both Π_∞ and $\psi_+(h^2\Delta)$ are order zero pseudodifferential operators and they have disjoint operator wavefront set, the composition $\Pi_\infty\psi_+(h^2\Delta)$ is $O(h^\infty)$ as an operator from L^2 to L^2. The other factors in (3.33) are all bounded from L^2 to L^2, and there are at most Ce^{CT} terms in the sum. As $T \leq |\log h|$ this contributes at most a factor of a fixed power of h. Hence $\|R_T\|_{L^2 \to L^2} = O(h^\infty)$.

To get an $L^1 \to L^\infty$ estimate from this, we compose on the left with $\psi(h^2\Delta_M)$ and observe that we still obtain an $O(h^\infty)$ estimate if we pre- and post-multiply by $(1+\Delta_M)^m$ for any m, since this has the effect of increasing the operator norm by at most Ch^{-4m}. This is equivalent to an $O(h^\infty)$ estimate from Sobolev spaces $H^{-m}(M)$ to $H^m(M)$, from which we obtain $L^1 \to L^\infty$ by Sobolev embedding for m larger than half the dimension of M. $\qquad\square$

The second estimate needed is similar to Lemma 6.6 of [NZ], but it is even better since we cut on the left on a compact set. Let $\mathcal{A}_L \subset (A \setminus \{0\})^L$ defined by

$$\alpha \in \mathcal{A}_L \iff \begin{cases} \Phi^1(W_{\alpha_j}) \cap W_{\alpha_{j+1}} \neq \emptyset, & j = 1, \ldots, L-1, \\ \text{and } \alpha_j \in A_1 \text{ for all } j = T_0, \ldots, L - T_0. \end{cases} \tag{3.34}$$

LEMMA 3.12. *If $\alpha \in A^L \setminus \mathcal{A}_L$ and $\chi, \psi, \psi_+(x, hD)$ are chosen as above, then*

$$\big\|\chi\psi(h^2\Delta)U(s_0)\Pi_{\alpha_L}U_{\alpha_{L-1}}\ldots U_{\alpha_1}\psi(x,hD)\chi\big\|_{L^1 \to L^\infty} = O(h^\infty) \tag{3.35}$$

for $2T_0 < T \leq |\log h|$, with $T = L - 1 + s_0$ and $s_0 \in [0,1]$.

Proof. This is proved in the same way as the previous lemma. We only need to show that each term in (3.33) corresponding to a multi-index $\alpha \in A^L \setminus \mathcal{A}_L$ has a factor which is $O(h^\infty)$ as a map from L^2 to L^2. This follows directly from Egorov's theorem if there is a j such that $\Phi^1(W_{\alpha_j}) \cap W_{\alpha_{j+1}} = \emptyset$. Indeed, referring back to Lemma 3.10 we can write

$$U_{\alpha_{j+1}} \circ U_{\alpha_j} = \psi_+(h^2\Delta)U(1)\Pi_{\alpha_{j+1}}\psi_+(h^2\Delta)e^{ih\Delta\psi_{++}(h^2\Delta)}\Pi_{\alpha_j}$$

where $\psi_{++} \in C_c^\infty(\mathbb{R})$ is 1 on the support of ψ_+. By Egorov, we have $e^{ih\Delta\psi_{++}(h^2\Delta)}\Pi_{\alpha_j} = Qe^{ih\Delta\psi_{++}(h^2\Delta)}$ for some pseudodifferential operator Q with wavefront set given by $\Phi^{-1}(WF'(\Pi_{\alpha_j}))$. Since this is disjoint from the operator wavefront set of $\Pi_{\alpha_{j+1}}$ by hypothesis, this factor is $O(h^\infty)$ as a map from L^2 to L^2.

If either $\alpha_1 = 0$ or $\alpha_L = 0$ then the $O(h^\infty)$ estimate is immediate because Π_0 is microsupported in $M \setminus M_-$ and χ is supported in M_-. If any of the other $\alpha_j = 0$ then the $O(h^\infty)$ estimate follows because of assumption (A3), which implies that either $a_1 = 0$ or $a_L = 0$, or else the condition $\Phi^1(W_{\alpha_j}) \cap W_{\alpha_{j+1}} = \emptyset$ has to hold for some intermediate j, showing that we are back in the situation considered above. Similarly, if $\alpha_j \in A_2$ for some $T_0 \leq j \leq L - T_0$ then (3.30) shows that we are again back in the situation considered above. $\qquad\square$

This lemma clearly implies the bound

$$\sum_{\alpha \notin \mathcal{A}_L} \big\|\chi\psi(h^2\Delta)U(s_0)\Pi_{\alpha_L}U_{\alpha_{L-1}}\ldots U_{\alpha_1}\chi\big\|_{L^1 \to L^\infty} = O(h^\infty), \tag{3.36}$$

since $|A|^L = O(h^{-\log|A|})$.

It remains to deal with the elements $\alpha \in \mathcal{A}_L$. We can obtain the following bounds, again essentially from the analysis of [A] (and in a comparable way to [NZ, Prop. 6.3]).

LEMMA 3.13. *Let χ be as above and let ϵ_0 be the small parameter in (3.28). Then for all small $\epsilon > 0$, there exists $C > 0$ such that*

$$\sum_{\alpha \in \mathcal{A}_L} \left\| \chi\psi(h^2\Delta)U(s_0)\Pi_{\alpha_L}U_{\alpha_{L-1}}\ldots U_{\alpha_1}\chi \right\|_{L^1 \to L^\infty} \leq Ch^{-d/2}e^{T(P(1/2)+\epsilon_0+\epsilon)} \quad (3.37)$$

for all $h \in (0, h_0)$ and $4T_0 \leq T \leq |\log h|$, where $T = L - 1 + s_0$ with $s_0 \in [0, 1]$.

Proof. We start by proceeding as in [A, §3]. If the cover is taken thin enough, we may use coordinates (z, ξ) in each W_a, $a \in A_1$, where $z \in \pi(W_a)$ and $\xi \in T_z^*M$ are cotangent variables. We can write, for $u \in L^1(M)$ and $z \in \pi(W_{\alpha_1})$,

$$\Pi_{\alpha_1}\chi u(z) = \int_{\pi(W_{\alpha_1})} \delta_y(z)u(y)dy + O(h^\infty), \quad \text{with}$$

$$\delta_y(z) := \frac{1}{(2\pi h)^d} \int_{(z,\xi)\in W_{\alpha_1}} e^{i\frac{(z-y)\xi}{h}} \sigma(z,\xi)d\xi.$$

where $\sigma(x, \xi)$ is the local symbol of $\Pi_{\alpha_1}\chi$ in W_{α_1}. An upper bound for the left-hand side of (3.37) is then the sum over all $\alpha \in \mathcal{A}_L$ of

$$\sup_{y,z} \left| ((\psi(h^2\Delta)U(t)\Pi_{\alpha_{J+1}})U_{\alpha_J}\ldots U_{\alpha_2}e^{ih\Delta_0}\delta_y)(z) \right|. \quad (3.38)$$

where we shall choose $t = s_0$ and $J = L - 1$. Thus we take $\Pi_{\alpha_1}\chi u$ and evolve it through $e^{ih\Delta_0}$ then microlocally cutoff in W_{α_2}, evolve again, microlocally cut off again, and so on. For Anosov flows, it is shown in [A, §3] that, for any $J, K \in \mathbb{N}$ fixed (independently of L), there exists a function $S_J(\cdot, t) \in C^\infty(\pi(W_{\alpha_J}))$ and $b_J(\cdot, h, t) \in C^\infty(\pi(W_{\alpha_J}))$ with b_J smooth in $h \in [0, h_0)$ such that for $t \in [0, 1]$

$$((\psi(h^2\Delta)U(t)\Pi_{\alpha_{J+1}})U_{\alpha_J}\ldots U_{\alpha_2}e^{ih\Delta_0}\delta_y)(z) = (2\pi h)^{-\frac{d}{2}}e^{\frac{iS_J(z,t)}{h}}b_{J,K}(z,h,t) + R_{J,K}(h,t),$$

$$b_{J,K}(z, h, t) = \sum_{k=0}^{K} h^k b_{J;k}(z, t), \quad (3.39)$$

with $\|R_{J,K}(h,t)\|_{L^2} = C_K Jh^K$ for some $C_K > 0$ uniform in t, J (this estimate is shown in [A, Lem. 3.2.2]). The function $S_J(z, t)$ generates a smooth Lagrangian submanifold $\mathcal{L}_{J+t} = \mathcal{L}_J(t) = \{(z, d_z S_J(z, t)) \in T^*M; z \in \pi(W_{\alpha_J})\}$ which is part of the graph of the canonical transformation Φ^{J+t}, namely that part with first coordinate lying in the Lagrangian $\{(y, \xi) \mid 1 - \epsilon < |\xi| < 1 + \epsilon\}$.

REMARK 3.14. The key to the proof of Lemma 3.13 which we owe to [A] is the following fact: as $J \to \infty$ and since we only consider $\alpha \in \mathcal{A}_L$, the geodesics generating $\mathcal{L}_J(t)$ lie entirely within $\pi^{-1}(M_-)$ which has sectional curvatures bounded above by a negative constant, these Lagrangians $\mathcal{L}_{J+t}$ converge uniformly (indeed, exponentially) to the weak unstable foliation as $J \to \infty$. This will allow us to compare the size of b_J with the weak unstable Jacobians $J_t^{wu}(m)$, as we will shortly show.

We now take $J = L - 1$, $t = s_0$ and K large. We obviously have $\|\chi R_{L,K}(h, s_0)\|_{L^2}$ $\leq C_K h^K |\log h|$ so using Sobolev embedding arguments as in the proof of Lemma 3.11, we can replace the L^2 norm by the L^∞ norm, up to a loss of $h^{-d/2-1}$, so

$$\left\| \chi R_{L,K}(h, s_0) \right\|_{L^\infty} \leq C h^{K-d/2-1} |\log h| \,.$$

Taking K large enough and summing over the $\alpha \in \mathcal{A}_L$, the number of which is bounded by $h^{-\log|A|}$, we conclude that these terms do not contribute.

It thus remains to study the L^∞ norm of elements of the form

$$(2\pi h)^{-d/2} \chi e^{\frac{i S_{L-1}(z, s_0)}{h}} b_{L-1,K}(z, h, s_0) \,;$$

that is, the L^∞ norm of $b_{L-1,K}$. We can essentially use the estimates in [A] but first we need to make some remarks on the different partitions of unity used here as compared to [A]. There, the quantum partition of unity is implemented by multiplication operators that cut off at the scale $h^{-\kappa}$, $0 < \kappa < 1/2$, while here we use semiclassical pseudodifferential operators with symbols smooth in h. There are two main differences: in [A], the multiplication operators are trivially bounded on $L^2(M)$ with operator norm 1, while in our case, the operator norm of our microlocal cutoffs is $1 + O(h)$ since the principal symbols are bounded by 1. This is inessential since it contributes at most a factor $(1 + Ch)^{|\log h|}$ to each term, which is bounded uniformly as $h \to 0$. Second, we need to replace the estimate on the derivatives of the microlocal cutoffs from $|D^m A_a| \leq C h^{-m\kappa}$ in [A] to $\|\operatorname{ad}_m(D, \Pi_a)\|_{L^2 \to L^2} \leq C$, where D indicates differentiation and ad_m indicates the mth iterated commutator (which is even better than in [A], as we do not get any negative powers of h in our case).

With these remarks made, we can follow the analysis of section 3.2 of [A]. Let us define $J^t_{\mathcal{L}_{L+s_0}}(z)$ to be the Jacobian of the map Φ^t, restricted to $\mathcal{L}_{L+s_0}$, and evaluated at $\overline{z} = (z, dS_L(s_0)(z)) \in \mathcal{L}_{L+s_0}$. Then the construction of [A] shows that $b_{L-1;k}(z, s_0)$ is only nonzero if $\Phi^{-j}(z, dS_{L-1}(z, 0)) \in \pi(W_{\alpha_{L-j}})$ for all $j = 1, \ldots, L$ and $k = 0, \ldots, K$, in which case

$$\left| b_{L-1;k}(z, s_0) \right| \leq C_k L^{3k} \big(J^{L-1+s_0}_{\mathcal{L}_{-(L-1+s_0)}}(\overline{z}) \big)^{1/2}. \tag{3.40}$$

Notice that this is the analogue of [A, Lem. 3.2.1]. Let us write $T = T_1 + N T_0 + T_0$, where $T_1 = T_0 + t_0 \in [T_0, 2T_0]$. Then we can decompose

$$J^{-T}_{\mathcal{L}_{L-1+s_0}}(\overline{z}) = J^{-T_1}_{\mathcal{L}_{L-1+s_0}}(\overline{z}) \times J^{-T_0}_{\mathcal{L}_{(N+1)T_0}}\big(\Phi^{-T_1}(\overline{z}) \big) \times J^{-T_0}_{\mathcal{L}_{N T_0}}\big(\Phi^{-T_1-T_0}(\overline{z}) \big)$$
$$\ldots \times J^{-T_0}_{\mathcal{L}_{T_0}}\big(\Phi^{-T+T_0}(\overline{z}) \big) \,. \tag{3.41}$$

The first and last Jacobian factors are uniformly bounded with respect to L; they only depend on T_0 since they can be written as a supremum of the Jacobian of the flow at some time bounded by $2T_0$ on some set independent of L. Now using Remark 3.14, by assuming that T_0 is large enough, the Lagrangians $\mathcal{L}_{jT_0}$, $j \geq 1$, are arbitrarily close to the weak unstable foliation. Thus we can replace the Jacobian of the flow by the weak unstable Jacobian, up to an $\epsilon > 0$ error which can be taken

as small as we like (possibly after increasing T_0 sufficiently). Thus

$$J_{\mathcal{L}_{jT_0}}^{-T_0}\big(\Phi^{-T+jT_0}(\bar{z})\big) \le J_{T_0}^{wu}\big(\Phi^{-T+jT_0}(\bar{z})\big)(1+\epsilon)$$

where $J_t^{wu}(m)$ is defined in (3.3). But the right-hand side is uniformly bounded by

$$\exp\big(S_{T_0}(W_{\alpha_{j'}})\big)(1+\epsilon), \quad \text{with } j' := T - jT_0.$$

Consequently, using (3.39), (3.40) and (3.41), we find that (3.38) is bounded uniformly by

$$C(1+\epsilon)^N \exp\bigg(\sum_{j=1}^{N} \frac{1}{2} S_{T_0}(W_{\alpha_{j'}})\bigg)$$

for some subsequence $(\alpha_{j'})_{j'} \in A_1^N$ and some $C > 0$ depending only on T_0. Now summing over all α in $\mathcal{A}_L$, we clearly obtain the bound

$$\sum_{\alpha \in \mathcal{A}_L} \big\|\chi U(s_0)\Pi_{\alpha_L} U_{\alpha_{L-1}} \ldots U_{\alpha_1}\chi\big\|_{L^1 \to L^\infty} \le Ch^{-d/2}(1+\epsilon)^N \sum_{\alpha \in \mathcal{A}_L} \prod_{j'=1}^{N} e^{\frac{1}{2}S_{T_0}(W_{\alpha'_j})}$$

$$\le Ch^{-d/2}(1+\epsilon)^N \bigg(\sum_{a \in A_1} e^{\frac{1}{2}S_{T_0}(W_a)}\bigg)^N$$

which from (3.28) proves the Lemma since NT_0 is comparable to T. $\square$

Completion of the proof of Proposition 3.9. We first note that for times $T \le 1$, the estimate

$$\big\|\chi\psi(h^2\Delta)e^{iTh\Delta}\chi\big\|_{L^1 \to L^\infty} \le C(Th)^{-n/2}$$

follows from the parametrix construction in [BuGT1, §2.2].

For times $1 \le T \le 4T_0$, the estimate can be obtained in the following way: by replacing ψ with ψ^2 in the estimate above, we obtain by the TT^* trick the following estimates:

$$\big\|\psi(h^2\Delta)e^{ih\Delta/2}\chi\big\|_{L^1 \to L^2} \le Ch^{-n/4},$$
$$\big\|\chi\psi(h^2\Delta)e^{ih\Delta/2}\big\|_{L^2 \to L^\infty} \le Ch^{-n/4}.$$

If we compose these estimates with the trivial intermediate estimate

$$\big\|e^{ih(T-1)\Delta/2}\big\|_{L^2 \to L^2} \le 1,$$

then we obtain the estimate for $1 \le T \le 4T_0$.

For $T \ge 4T_0$ we can apply Lemmas 3.11, 3.12 with the estimate of Lemma 3.13 where $\epsilon_0 + \epsilon$ is chosen smaller than $-P(1/2)$; we obtain the estimate

$$\big\|\chi\psi(h^2\Delta)e^{iTh\Delta}\chi\big\|_{L^1 \to L^\infty} \le Ch^{-n/2}e^{-\beta T}$$

for some $\beta > 0$, and all $T \in (0, \log(1/h))$. It suffices to set $t = Th$ and we get the desired result since $e^{-\beta T}T^{n/2} \le C$. $\square$

3.7   Proof of Theorem 3.3. We shall be brief here since the proof was already outlined in section 3.3. We use the notation from that section. Thus, $u_j = \varphi(t/h|\log h| - j)\chi u$ satisfies

$$(i\partial_t - \Delta_M)u_j = w'_j + w''_j$$

where w'_j, w''_j are defined in (3.24). Choose $\chi_- \in C_0^\infty(M)$ supported in M_- and identically 1 on the support of χ, and $\chi_+ \in C^\infty(M)$ so that $1 - \chi_+ \in C_0^\infty(M)$ is identically 1 on $\pi(K)$ and is 0 on the support of $\nabla\chi$. Then $u_j = \chi_+ u_j$ and $w'_j = \chi_- w'_j$, $w''_j = \chi_+ w''_j$. We define u'_j by

$$u'_j(t) = \chi_- \int_{(j-1)h|\log h|}^t e^{-i(t-s)\Delta_M} \chi_+ w'_j(s)\, ds \qquad (3.42)$$

with u''_j defined analogously. Clearly $u'_j + u''_j = u_j$.

To treat w''_j, consider $\tilde{u}''_j$ defined by

$$\tilde{u}''_j(t) = \chi_- \int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{-i(t-s)\Delta_M} \chi_+ w''_j(s)\, ds$$

$$= \chi_- e^{-it\Delta_M} \int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_M} \chi_+ w''_j(s)\, ds\,. \qquad (3.43)$$

Using Lemma 2.2 and Remark 2.3 we see that

$$\left\| \int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_M} \chi_+ w_j(s)\, ds \right\|_{L^2(M)} \le C h^{1/2} \| w''_j \|_{L_t^2; L^2(M)}\,.$$

Then Theorem 3.8 applied to this L^2 function shows that

$$\| \tilde{u}''_j \|_{L_t^p; L^q(M)} \le C h^{1/2} \| w''_j \|_{L_t^2; L^2(M)}\,, \qquad (3.44)$$

and the same estimate holds for u''_j by Christ–Kiselev. To treat w'_j, consider $\tilde{u}'_j$ defined by

$$\tilde{u}'_j(t) = \chi_- \int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{-i(t-s)\Delta_M} \chi_+ w'_j(s)\, ds$$

$$= \chi_- e^{-it\Delta_M} \int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_M} \chi_+ w'_j(s)\, ds\,. \qquad (3.45)$$

The dual estimate to Theorem 3.7 implies

$$\left\| \int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_M} \chi_+ w'_j(s)\, ds \right\|_{L^2(M)} \le C\big(h|\log h|\big)^{\frac{1}{2}} \| w'_j \|_{L_t^2; L^2(M)} \le \frac{C\|\chi u\|_{L_t^2; L^2(M)}}{(h|\log h|)^{1/2}}\,,$$

using also $\omega'_j = i(h|\log h|)^{-1}\varphi'(t/h|\log h| - j)\chi u$. Then we can use Theorem 3.8 applied to this L^2 function shows that

$$\| \tilde{u}'_j \|_{L_t^p; L^q(M)} \le C\big(h|\log h|\big)^{-1/2} \| \chi u \|_{L_t^2; L^2(M)}\,, \qquad (3.46)$$

and the same estimate holds for u'_j by Christ–Kiselev.

Squaring and summing over j gives

$$\sum_{j=1}^{N-1} \| u_j \|_{L_t^p; L^q(M)}^2 \le C \sum_{j=1}^{N-1} \left(h \| w''_j \|_{L_t^2; L^2(M)}^2 + \frac{1}{h|\log h|} \| w'_j \|_{L_t^2; L^2(M)}^2 \right), \qquad (3.47)$$

and the right-hand side is no bigger than $C\|u_0\|^2_{L^2(M)}$ using Lemma 2.2 for w''_j and Theorem 3.7 for w'_j. Using the continuous embedding from $l^2(\mathbb{N})$ to $l^p(\mathbb{N})$ as in section 3.3 gives

$$\left\|\chi e^{-it\Delta_M}\psi(h^2\Delta)u_0\right\|_{L^p[0,1];L^q(M)} \leq C\|u_0\|_{L^2(M)}.$$

Together with Theorem 3.6 this gives the Strichartz estimate without the space cutoff χ:

$$\left\|e^{-it\Delta_M}\psi(h^2\Delta)u_0\right\|_{L^p[0,1];L^q(M)} \leq C\|u_0\|_{L^2(M)}.$$

Finally using Bouclet's Littlewood–Paley estimate (equation (1.4) of [Bo1]) and the argument in [BuGT1], we remove the frequency cutoff and obtain (3.5), which completes the proof.

REMARK 3.15. The restriction $p > 2$ in Theorem 0.2 is only required because we use the Christ–Kiselev lemma. It is likely that this condition could be eliminated (for $d > 2$) with a more careful analysis.

References

[A]   N. ANANTHARAMAN, Entropy and the localization of eigenfunctions, Annals of Math. 168:2 (2008), 435–475.

[AN]   N. ANANTHARAMAN, S. NONNENMACHER, Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold, Ann. Inst. Fourier (Grenoble) 57:7 (2007), 1465–2523

[AnP]   J-P. ANKER, V. PIERFELICE, Nonlinear Schrödinger equation on real hyperbolic spaces, Ann. Henri Poincaré Analyse Non-Linéaire 26:5 (2009), 1853–1869.

[B]   V.M. BABIČ, Eigenfunctions which are concentrated in the neighborhood of a closed geodesic (in Russian), Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 9 (1968), 15–63.

[Ba]   V. BANICA, The nonlinear Schrödinger equation on the hyperbolic space, Comm. PDE 32:10 (2007), 1643–1677.

[Bo1]   J.-M. BOUCLET, Semi-classical functional calculus on manifolds with ends and weighted L^p estimates, Arxiv:0711.358.

[Bo2]   J.-M. BOUCLET, Littlewood–Paley decompositions on manifolds with ends, Bull. Soc. Math. Fr. 138:1 (2010), 1–37.

[Bo3]   J.-M. BOUCLET, Strichartz estimates for asymptotically hyperbolic manifolds, Analysis and PDE, in press.

[BoT]   J.-M. BOUCLET, N. TZVETKOV, Strichartz estimates for long range perturbations, Amer. J. Math. 129:6 (2007), 1665–1609.

[Bou]   J. BOURGAIN, Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations I. Schrödinger equations, Geom. Funct. Anal. 3 (1993), 107–156.

[Bu]   N. BURQ, Smoothing effect for Schrödinger boundary value problems, Duke Math. J. 123:2 (2004), 403–427.

[BuGT1] N. BURQ, P. GERARD, N. TZVETKOV, Strichartz inequalities and the non-linear Schrödinger equation on compact manifolds, Amer. J. Math, 126 (2004), 569–605.

[BuGT2] N. BURQ, P. GERARD, N. TZVETKOV, On Nonlinear Schrödinger equations in exterior domains, Ann. I.H.P. Ana. non lin. 21 (2004), 295–318.

[BuZ]   N. Burq, M. Zworski, Geometric control in the presence of a black box, J. Amer. Math. Soc. 17:2 (2004), 443–471.

[CV]   F. Cardoso, G. Vodev, Uniform estimates of the Laplace–Beltrami operator on infinite volume Riemannian manifolds II, Ann. H. Poincaré 3 (2002), 673–691.

[Ca]   R. Carles, Global existence results for nonlinear Schrödinger equations with quadratic potentials, Discrete Contin. Dyn. Syst. 13:2 (2005), 385–398.

[ChK]   M. Christ, A. Kiselev, Maximal functions associated to filtrations, J. Funct. Anal. 179 (2001), 409–425.

[Chr]   H. Christianson, Cutoff resolvent estimates and the semilinear Schrödinger equation, Proc. Amer. Math. Soc. 136:10 (2008), 3513–3520.

[CoS]   P. Constantin, J.-C. Saut, Effets régularisants locaux pour des équations dispersives générales, C. R. AcadṠci. Paris Sér. I Math. 304:14 (1987), 407–410.

[D]   K. Datchev, Local smoothing for scattering manifolds with hyperbolic trapped sets, Comm. Math. Phys. 286:3, 837–850.

[Do]   S.-I. Doi, Smoothing effects for Schrödinger evolution equation and global behaviour of geodesic flow, Math. Ann. 318 (2000), 355–389.

[GR]   P. Gaspard, S.A. Rice, Semiclassical quantization of the scattering from a classically chaotic repellor, J. Chem. Phys. 90 (1989), 2242–2254.

[GuMP]   C. Guillarmou, S. Moroianu, J. Park, Eta invariant and Selberg Zeta function of odd type over convex co-compact hyperbolic manifolds., Advances in Math., to appear; Arxiv 0901.4082

[HK]   B. Hasselblatt, A. Katok, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1997.

[HaTW]   A. Hassell, T. Tao, J. Wunsch, Sharp Strichartz estimates on nontrapping asymptotically conic manifolds, Amer. J. Math. 128:4 (2006), 963–1024.

[HaV]   A. Hassell, A. Vasy, Symbolic functional calculus and N-body resolvent estimates, J. Funct. Anal. 173:2 (2000), 257–283.

[IS]   A. Ionescu, G. Staffilani, Semilinear Schrödinger flow on hyperbolic space: scattering in H^1, Math. Ann. 345:1 (2009), 133–158.

[KT]   M. Keel, T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 15 (1998), 955–980.

[Kl]   W. Klingenberg, Riemannian Geometry, Studies in Mathematics, de Gruyter. Berlin, New-York 1982.

[MM]   R. Mazzeo, R. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, J. Funct. Anal. 75 (1987), 260–310.

[Me]   R.B. Melrose, Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces, in "Spectral and Scattering Theory" (M. Ikawa, ed.), Marcel Dekker, 1994.

[NZ]   S. Nonnenmacher, M. Zworski, Quantum decay rates in chaotic scattering, Acta Math. 203:2 (2009), 149–233

[P]   G.P. Paternain, Geodesic Flows. Progress in Mathematics 180, Birkhäuser Boston Inc., Boston MA (1999).

[Pa]   S.J. Patterson, The limit set of a Fuchsian group, Acta Math. 136 (1976), 241–273.

[Pe]   P. Perry, Asymptotics of the length spectrum for hyperbolic manifolds of infinite volume, Geom. Funct. Anal. 11 (2001), 132–141.

[Py]   M.F. Pyškina, The asymptotic behavior of eigenfunctions of the Helmholtz equation that are concentrated near a closed geodesic (in Russian), Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 15 (1969), 154–160.

[R]   J.V. Ralston, On the construction of quasimodes associated with stable periodic orbits, Comm. Math. Phys. 51-3 (1976), 219–242.

[SS]   J. Schmeling, R. Siegmund-Schultze, Hölder continuity of the holonomy maps for hyperbolic basic sets. I., Ergodic Theory and Related Topics, III (Güstrow, 1990), Springer Lecture Notes in Math. 1514 (1992), 174–191.

[StT]   G. Staffilani, D. Tataru, Strichartz estimates for a Schrödinger operator with nonsmooth coefficients, Comm. Part. Diff. Eq. 27:7-8 (2002), 1337–1372.

[Str]   R. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. 44:3 (1977), 705–714.

[Su]   D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, Publ. IHES 50 (1979), 172-202.

[TT]   H. Takaoka, N. Tzvetkov, On 2D nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}$, J. Funct. Anal. 182:2 (2001), 427–442.

[WZ]   J. Wunsch, M. Zworski, Distribution of resonances for asymptotically Euclidean manifolds, J. Diff. Geom. 55 (2000), 43–82.

[Z]   M. Zworski, Dimension of the limit set and the density of resonances for convex co-compact hyperbolic surfaces, Invent. Math. 136:2 (1999), 353–409.

Nicolas Burq, Laboratoire de Mathématiques, Bât. 425, Université Paris-Sud 11 F-91405 Orsay Cedex, France   nicolas.burq@math.u-psud.fr

Colin Guillarmou, Département de Mathématiques et Applications, École Normale Supérieure, 45 rue d'Ulm, F-75230 Paris Cedex 05, France   Colin.Guillarmou@unice.fr

Andrew Hassell, Department of Mathematics, Australian National University, Canberra ACT 0200 Australia   Andrew.Hassell@anu.edu.au