Proceedings of the International Congress of Mathematicians
Hyderabad, India, 2010

Random data Cauchy theory for dispersive partial differential equations

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Abstract. In a series of papers in 1930-32, Paley and Zygmund proved that random series on the torus enjoy better L^p bounds than the bounds predicted by the deterministic approach (and Sobolev embeddings). The subject of random series was later largely studied and developed in the context of harmonic analysis. Curiously, this phenomenon was until recently not exploited in the context of partial differential equations. The purpose of this talk is precisely to present some recent results showing that in some sense, the solutions of dispersive equations such as Schrödinger or wave equations are better behaved when one consider initial data randomly chosen (in some sense) than what would be predicted by the deterministic theory. A large part of the material presented here is a collaboration with N. Tzvetkov.

Mathematics Subject Classification (2000). Primary 35LXX; Secondary 35Q55.

Keywords. Random series, Wave equations, Schrödinger equations

1. Introduction

In a series of papers in 1930-32, Paley and Zygmund [42] proved that for any square summable sequence $(c_n) \in \ell^2$, if one consider the trigonometric series

$$u(x) = \sum_{n=0}^{+\infty} c_n e^{inx},$$

then, changing the signs of the coefficients c_n randomly and independently ensures that almost surely, the sum of the series is in every space $L^p(\mathbb{T})$, $2 \leq p < +\infty$. In modern language, this result reads

Theorem 1.1. *Consider a sequence* $(c_n) \in \ell^2$ *and a family of* independent, mean zero *Bernouilli random variables,* (b_n^ω) *on a probability space* $(\Omega, \mathcal{P})$*:*

$$\mathcal{P}(b_n = \pm 1) = \frac{1}{2},$$

and the corresponding series on the torus,

$$u^\omega(x) = \sum_{n=0}^{+\infty} b_n^\omega c_n e^{inx}.$$

Then almost surely

$$\forall 2 \leq p < +\infty, u^\omega \in L^p(\mathbb{T}).$$

Actually, in 1930, the most difficult part in this result was precisely to define what is a "*family of independent, mean zero Bernouilli random variables*", and Paley-Zygmund proof relied on an explicit realization (see Rademacher [43] and Kolmogorov [27]). With modern technology, it is not difficult to give a quantitative version of this result and one can prove (see section 2)

$$\forall 2 \leq p < +\infty, \exists C > 0; \forall \lambda > 0, \mathcal{P}(\|u^\omega\|_{L^p(\mathbb{T})} > \lambda) \leq Ce^{-\lambda^2/C}.$$

This much celebrated result has been followed by many works on random series of functions (see in particular the books by Kahane [24] and Marcus-Pisier [36]) where the studies focused mostly on the question of giving criteria for the uniform convergence of the series. It is quite remarkable that this very active fields of research for the point of view of harmonic analysis were not, until recently investigated from the point of view of partial differential equations. To my knowledge, the first step toward this direction is due to Bourgain [7], where these properties of random series on the torus $\mathbb{T}^2$ were exploited in the context of the (renormalized by Wick ordering) two dimensional non linear cubic Schrödinger equations. The purpose of this talk is in fact to show that these properties of random series can be exploited in a number of situations including wave equations on manifolds and non linear harmonic oscillators. The examples we have in mind are the semilinear wave equation on a compact manifold

$$(\frac{\partial^2}{\partial t^2} - \mathbf{\Delta})u = -|u|^{p-1}u, \qquad u\mid_{t=0} = u_0, \quad \frac{\partial}{\partial t}u\mid_{t=0} = u_1, \tag{1}$$

and the semilinear Schrödinger equation on the line

$$(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2)u = -\kappa|u|^{p-1}u, \qquad u\mid_{t=0} = u_0, \qquad \kappa = 0; \pm 1. \tag{2}$$

As far as Cauchy theory is concerned, the (deterministic) behaviour of these equations has been investigated for a long time and the picture is by now fairly complete. Notice that up to now, the ideas presented in this talk do not apply to the case of nonlinear Schrödinger equations on compact manifolds (see Tzvetkov [47, Appendix] where some partial results are obtained in this case). Notice also that in this setting of deterministic theory of semi-linear Schrödinger equations on manifolds, the situation is much less well known, see Gérard [19] for a review of this question). Many of the questions which remain open on $\mathbb{R}^d$ are essentialy about the critical problems and the long time behaviour (or possibly explosion, see the works by Merle-Raphael [37, 38, 40, 39]) of the solutions. In particular, for both the wave equation on a compact manifold, and the Schrödinger equation, the Cauchy problem is known to be well posed above the scaling index

$$s_c = \frac{d}{2} - \frac{2}{p-1}.$$

(see Kapitanskii [25], Oh [41] and Carles [15], and the contributions by Bourgain [6], Colliander-Keel-Staffilani-Takaoka-Tao [17], and Kenig-Merle [29, 30] for the critical problems), while it is known to be ill posed below the scaling index. Indeed, the following result is known (see the works by Lebeau [32, 33], Christ-Colliander-Tao [16], Burq-Gérard-Tzvetkov [9], Alazard-Carles [1] and Thomann [48]).

Theorem 1.2. *Assume that* $0 < s < s_c$*. Then there exists a sequence of initial data* $(u_{0,n}, u_{1,n}) \to 0$ *in* $H^s(M) \times H^{s-1}(M)$ *as* $n \to 0$*, and there exists times* $t_n \to 0$ *such that the solutions of* (1) *exist (and are unique) in suitable spaces for* $|t| \leq t_n$*, but*

$$\lim_{n \to +\infty} \|u(t_n)\|_{H^s(M)} = +\infty.$$

There exists also a sequence of initial data $(u_{0,n}) \to 0$ *in* $H^s(\mathbb{R}))$ *as* $n \to 0$*, and there exists times* $t_n \to 0$ *such that the solutions of* (2) *exist (and are unique) in suitable spaces for* $|t| \leq t_n$*, but*

$$\lim_{n \to +\infty} \|u(t_n)\|_{H^s(\mathbb{R})} = +\infty.$$

In other word, the equations (1) and (2) admit no flow continuous at $t = 0$, $(u_0, u_1) = 0$ (resp $u_0 = 0$) for the H^s topology. Having this negative result in mind, a natural question to ask is whether one can still find initial states with super-critial regularity (i.e. $(u_0, u_1) \in H^s(M) \times H^{s-1}(M)$, (resp. $u_0 \in H^s(M)$), $s < s_c$), for which the Cauchy problem (1) (resp. (2)) is locally (or even better, globally) well posed. The purpose of this talk is precisely to present such examples.

The paper is organized as follows: In Section 2, I will present a short proof of Paley-Zygmund's result which, using Hörmander-Sogge's Laplace eigenfunctions estimates [21, 45], or Hermite eigenfunctions estimates [31] extends readily to the more general setting of random series on manifolds (or on $\mathbb{R}^d$). In section 3, I will show how these estimates, combined with the usual Strichartz estimates [46, 20, 28] allow to obtain a nice "probabilistic" Cauchy theory for the wave equation on compact manifolds and I will give a particular example where this local theory combined with Bourgain's [7, 6] Gibbs measure arguments gives a global (in time) result. In Section 4, I will follow the same program for the semi-linear Schrödinger equation on the line $\mathbb{R}$, with or without harmonic oscillator. Finally, in a last section, I will focus on some different randomizations in connexion with Sobolev embeddings.

2. Random series

2.1. Random series on the torus. In this section I will give a simple proof of Paley-Zygmund's theorem, to show the versatility of the result.

Theorem 2.1 (see [2, 13, 14]). *Assume that the random variable* b_n^ω *are*

1. *independent,*

2. *have mean equal to* 0*,*
3. *have super-exponential decay*

$$\exists C, \delta > 0; \forall \alpha \in \mathbb{R}, \mathbb{E}(e^{\alpha |b_n^\omega|}) \leq Ce^{\delta \alpha^2}. \tag{3}$$

Notice that this latter assumption is satisfied for Bernouilli, or more generally for families of random variables having a (fixed) compact support, or for standard Gaussian random variables.

Then, almost surely, $u_n^\omega \in L^p(\mathbb{T}), \forall q < +\infty$*. More precisely, the following large deviation estimate holds*

$$\forall q < +\infty, \exists C; \qquad \mathcal{P}(\|u^\omega\|_{L^q(\mathbb{T})} > \lambda) \leq Ce^{-\Lambda^2/C}.$$

The remaining of this section is devoted to the proof of Theorem 2.1.

2.2. Proof of Theorem 2.1.

The proof relies on

Proposition 2.2. *[Large deviation estimate] Assume that the random variables satisfy the assumptions of Theorem 2.1. Then there exists* $\delta > 0$ *such that for any* $\Lambda > 0$*, and any sequence* $(v_n) \in \ell^2$*,*

$$\mathcal{P}(|\sum_n v_n b_n^\omega| > \Lambda) \leq e^{-\delta \frac{\Lambda^2}{\sum_n |v_n|^2}}.$$

2.2.1. Proof of Theorem 2.1 assuming Proposition 2.2.

Fix $r \geq q$. Remark that the norm of an integral is always smaller that the integral of the norm. As a consequence,

$$\begin{aligned} \|\|u^\omega(x)\|_{L^q_x}\|_{L^r_\omega} &= \Big(\|\int_x |u^\omega(x)|^q dx\|_{L^{r/q}_\omega}\Big)^{1/q}, \\ &\leq \Big(\int_x \||u^\omega(x)|^q\|_{L^{r/q}_\omega} dx\Big)^{1/q}, \\ &= \|\|u^\omega(x)\|_{L^r_\omega}\|_{L^q_x}. \end{aligned} \tag{4}$$

Notice (x is a fixed parameter) that

$$\|u^\omega(x)\|_{L^r_\omega} = \int_0^{+\infty} r\lambda^{r-1}\mathcal{P}(|u^\omega(x)| > \lambda) d\lambda,$$

and according to Proposition 2.2 applied to $v_n = u_n e^{inx}$, with x a fixed parameter (and the change of variables $\mu = \frac{\sqrt{2}\delta^{1/2}}{(\sum_n |u_n e^{inx}|^2)^{1/2}}$),

$$\begin{aligned} \|u^\omega(x)\|^r_{L^r_\omega} &\leq C\int_0^{+\infty} r\lambda^{r-1} e^{-\delta \frac{\lambda^2}{\sum_n |u_n e^{inx}|^2}} d\lambda \\ &\leq (C\sum_n |u_n|^2)^{r/2} r\int_0^{+\infty} \mu^{r-1} e^{-\frac{\mu^2}{2}} d\mu, \\ &\leq (C\sum_n |u_n|^2)^{r/2} \times r \times r-2 \times \cdots \times 1 \leq (C'r\sum_n |u_n|^2)^{r/2}. \end{aligned} \tag{5}$$

Notice now that the norm with respect to the x parameter is harmless (as the bound does not depend on x). For future use, it should be noticed that we use here that the L^q norm of the functions e^{inx} are uniformly bounded. As a conclusion, we just proved

$$\| \| u^\omega(x) \|_{L^q_x} \|_{L^r_\omega} \leq (C' r \sum_n |u_n|^2)^{1/2}.$$

To conclude, let us recall Tchebytchev inequality:

$$\forall \lambda, \qquad \lambda \mathcal{P}(f^\omega > \Lambda) \leq \mathbb{E}(f).$$

Apply this inequality to the random variable $f^\omega = \|u^\omega(x)\|^r_{L^q_x}$ and $\lambda = \Lambda^r$. We get

$$\begin{aligned} \mathcal{P}(\|u^\omega(x)\|_{L^q_x} > \Lambda) &= \mathcal{P}(\|u^\omega(x)\|^r_{L^q_x} > \Lambda^r = \lambda) \\ &\leq \frac{1}{\Lambda^r} \mathbb{E}(\|u^\omega(x)\|^r_{L^q_x}) = \frac{1}{\Lambda^r} \| \|u^\omega(x)\|_{L^q_x} \|^r_{L^r_\omega} \\ &\leq \Big(\frac{(C' r \sum_n |u_n|^2)}{\Lambda^2} \Big)^{r/2}. \end{aligned} \tag{6}$$

Now we optimize this inequality by choosing r so that

$$\frac{(C' r \sum_n |u_n|^2)}{\Lambda^2} = \frac{1}{2}$$

(notice that the assumption $r \geq p$ requires that λ is large enough, but for bounded λ's, the large deviation estimate in Theorem 2.1 is straightforward). This gives

$$\mathcal{P}(\|u^\omega(x)\|_{L^q_x} > \Lambda) \leq \Big(\frac{1}{2}\Big)^{r/2} = e^{-\delta \frac{\Lambda^2}{\sum_n |u_n|^2}},$$

which ends the proof of Theorem 2.1. □

2.2.2. Proof of Proposition 2.2. The proof we give is very classical. In the special case where the random variables g_n are gaussian random variables of variance 1, the result is straightforward. Indeed, $\sum_n v_n g_n$ is a Gaussian random variable of variance $\sum_n |v_n|^2$ and the result follows. In the general case, it is enough to prove

$$\mathcal{P}(\sum_n v_n b_n^\omega > \lambda) \leq e^{-\delta \frac{\lambda^2}{\sum_n |v_n|^2}}.$$

Indeed, the estimate for the other part, $\mathcal{P}(\sum_n v_n b_n^\omega < -\lambda)$ is obtained by changing v_n to $-v_n$. Let us fix $t > 0$ and compute (using the fact that the random variables are independent)

$$\begin{aligned} \mathbb{E}(e^{t \sum_n v_n b_n^\omega}) &= \mathbb{E}(\prod_n e^{t v_n b_n^\omega}) = \prod_n \mathbb{E}(e^{t v_n b_n^\omega}) \\ &\leq \prod_n e^{\delta t^2 |v_n|^2} \leq e^{t^2 \sum_n |v_n|^2}, \end{aligned} \tag{7}$$

where in the last but one inequality, we used the super-exponential decay assumption(3). Now, using Tchebytchev inequality,

$$
\begin{aligned}
\mathcal{P}(\sum_n v_n b_n^\omega > \lambda) &= \mathcal{P}(e^{t\sum_n v_n b_n^\omega} > e^{t\lambda}),\\
&\leq e^{-t\lambda}\mathbb{E}(e^{t\sum_n v_n b_n^\omega}),\\
&\leq e^{\delta t^2 \sum_n |v_n|^2 - t\lambda}.
\end{aligned}
\tag{8}
$$

Optimize by choosing $\delta t^2 \sum_n |v_n|^2 = t\lambda/2$, i.e. $t = \lambda/(2\delta \sum_n |v_n|^2)$, which gives

$$\mathcal{P}(\sum_n v_n b_n^\omega > \lambda) \leq e^{-\alpha \frac{\lambda^2}{\sum_n |v_n|^2}},$$

which ends the proof of Proposition 2.2 and consequently the proof of Theorem 2.1. □

2.3. Random series on manifolds and on the line. Consider M a riemanian manifold and H a non-negative self adjoint operator on $L^2(M)$ with compact resolvent (the examples we have in mind are M a compact riemanian manifold with $H = -\boldsymbol{\Delta}$ and $M = \mathbb{R}$ with $H = -\frac{d^2}{dx^2} + x^2$ the harmonic oscillator). It is well known that the eigenfunctions of H, e_n, associated to eigenvalues $-\lambda_n^2$ provide a Hilbert base of $L^2(M)$

$$u \in L^2(M) \Leftrightarrow u = \sum_{n\in\mathbb{N}} u_n e_n(x), \|u\|^2_{L^2(M)} = \sum_{n\in\mathbb{N}} |u_n|^2 < +\infty$$

Definition 1. For any $s \in \mathbb{R}$, let $\mathcal{H}^s(M)$ be the space of distributions such that $(\mathrm{Id} + H)^s u \in L^2(M)$, and let $\mathcal{W}^{s,p}(M)$ be the space of distributions u such that $(\mathrm{Id} + H)^{s/2} u \in L^p(M)$ endowed with their natural norm. In particular, we have

$$u \in \mathcal{H}^s(M) \Leftrightarrow u = \sum_{n\in\mathbb{N}} u_n e_n(x), \sum_{n\in\mathbb{N}} (1 + \lambda_n^2)^s)|u_n|^2 = \|u_n\|^2_{\mathcal{H}^s(M)} < +\infty$$

and notice that if M is a compact manifold, $\mathcal{H}^s(M)$ coincides with the usual Sobolev space $H^s(M)$ while if M is the real line and H the harmonic oscillator, and $s \geq 0$

$$\mathcal{W}^{s,p}(M) = \{u \in \mathcal{D}'(\mathbb{R}); \langle x\rangle^s u, \langle |D_x|\rangle^s u \in L^p(\mathbb{R})$$

(endowed with its natural norm [18]).

The starting point of the analysis is Hörmander-Sogge's estimates for the growth of the L^p norm of eigenfunctions on (compact) manifolds

Theorem 2.3. *Consider M a compact riemanian manifold and $(e_n)_{n\in\mathbb{N}}$, the L^2-normalized eigenfunctions of the Laplace operator on M, associated to the eigenvalues $-\lambda_n^2$. Then, there exists $C > 0$ such that for any $n \in \mathbb{N}$, and any $2 \leq p \leq +\infty$*

$$\|e_n\|_{L^p(M)} \leq C\lambda_n^{\delta(p)} \tag{9}$$

where

$$\delta(p) = \begin{cases} \frac{d-1}{2} - \frac{d}{p} \text{ if } p \geq \frac{2(d+1)}{d-1}, \\ \frac{d-1}{2}(\frac{1}{2} - \frac{1}{p}) \text{ if } p \leq \frac{2(d+1)}{d-1}. \end{cases} \tag{10}$$

The end point $p = \infty$ is due to Hörmander [21] while the point $p = \frac{2(d+1)}{d-1}$ is due to Sogge [45] (notice that the last extremal point $p = 2$ is trivial).

Consider now the (L^2 normalized) eigenfunctions of the Harmonic oscillator, $h_n(x)$,

$$(-\frac{d^2}{dx^2} + x^2)h_n = \lambda_n^2 h_n, \lambda_n = \sqrt{2n+1}$$

Then the analog of Sogge's result is the following (see Yajima-Zhang [49] and Koch-Tataru [31]

Theorem 2.4. *For any $2 \leq p \leq +\infty$, there exists $C > 0$ such that for any $n \in \mathbb{N}$,*

$$\|h_n\|_{L^p(\mathbb{R})} \leq C\lambda_n^{\sigma(p)} \tag{11}$$

with

$$\sigma(p) = \begin{cases} -\big(\frac{1}{6} + \frac{1}{3p}\big) \text{ if } 4 < p \leq +\infty \\ = -\big(\frac{1}{2} - \frac{1}{p}\big) \text{ if } 2 \leq p < 4 \end{cases} \tag{12}$$

and

$$\|h_n\|_{L^4(\mathbb{R})} \leq C\lambda_n^{-\frac{1}{4}} \log(\lambda_n)^{1/4} \tag{13}$$

Remark. *Notice that in the case of the harmonic oscillator, the situation is much more favorable as the L^p norms of the Hermite functions h_n tend to be small as n tend to infinity. This is of course natural, as, by elliptic regularity, the functions h_n are essentially concentrated in the set $\{|x| \leq \lambda_n\}$, whose measure is growing.*

Remark. *Following the $X^{s,b}$ approach by Bourgain [4, 3, 5], multilinear versions of estimates* (9) *proved to be crucial in the analysis of the (deterministic) well posedness of non linear Schrödinger equations on general compact manifolds and spheres (see the works by Burq-Gérard-Tzvetkov [8, 10, 19]), while the bilinear version of* (13) *was the starting point of our work on the non linear harmonic oscillator (see [12] and Section 4.1).*

Now the analog of Paley-Zygmund's theorem is (see Burq-Tzvetkov [13])

Theorem 2.5. *Consider a compact riemanian manifold, M. Fix $2 \leq p < +\infty$ and consider*

$$u = \sum_n u_n e_n(x) \in \mathcal{H}^s(M),$$

and random variables (b_n) satisfying the assumptions in Theorem 2.1. Assume that $s > \delta(p)$. Then almost surely the random series

$$u^\omega = \sum_{n \in \mathbb{N}} b_n^\omega u_n e_n(x)$$

belongs to $L^p(M)$ *and more precisely*

$$\exists C > 0; \mathcal{P}(\|u^\omega\|_{L^p(M)} > \lambda) \leq Ce^{-\lambda^2/C}. \tag{14}$$

Furthermore, for any $s' > s$*, if* $u \notin \mathcal{H}^{s'}(M)$*, then*

$$\mathcal{P}(\|u^\omega\|_{\mathcal{H}^{s'}(M)} < +\infty) = 0. \tag{15}$$

In the case of the harmonic oscillator, the analog of Paley-Zygmund's theorem is (see Burq-Thomann-Tzvetkov [12])

Theorem 2.6. *Fix* $2 \leq p < +\infty$ *and consider*

$$u = \sum_n u_n h_n(x) \in \mathcal{H}^s(\mathbb{R}),$$

and random variables (b_n) *satisfying the assumptions in Theorem 2.1. Assume that* $s > \sigma(p)$*. Then almost surely the random series*

$$u^\omega = \sum_{n\in\mathbb{N}} b_n^\omega u_n h_n(x)$$

belongs to $L^p(\mathbb{R})$ *and more precisely*

$$\exists C > 0; \mathcal{P}(\|u^\omega\|_{L^p(\mathbb{R})} > \lambda) \leq Ce^{-\lambda^2/C}. \tag{16}$$

Furthermore, for any $s' > s$*, if* $u \notin \mathcal{H}^{s'}(\mathbb{R})$*, then*

$$\mathcal{P}(\|u^\omega\|_{\mathcal{H}^{s'}(\mathbb{R})} < +\infty) = 0. \tag{17}$$

Remark. *Notice that these results exhibit gains of derivatives with respect to the Sobolev embeddings. Indeed, it is of course clear for the harmonic oscillator case as the* L^p *norms are better behaved almost surely than the* L^2 *norms, while in the case of a compact manifold, Sobolev embeddings read*

$$\|u\|_{L^p(M)} \leq C\|u\|_{\mathcal{H}^s(M)}, \qquad s = d\big(\frac{1}{2} - \frac{1}{p}\big), \quad 2 \leq p < +\infty.$$

3. Wave equations and random series

3.1. Local theory. In this section, for simplicity, I shall consider the simplest model on semi-linear wave equation, which is obtained for cubic non linearities on three dimensional manifolds.

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad (u, \partial_t u)|_{t=0} = (u_1, u_2) \in H^s(M) \times H^{s-1}(M). \tag{18}$$

Notice that for this equation, the critical index is $s_c = \frac{1}{2}$. The following result (Burq-Tzvetkov [13]) shows that neverthless, the Cauchy problem is locally well posed for a large number of supercritical initial data

Theorem 3.1. *Consider a compact riemanian manifold, M. Let us fix $s > \frac{1}{4}$ and*

$$(u_1, u_2) = \sum_n (u_{n,1} e_n(x), u_{n,2} e_n(x)) \in H^s(M) \times H^{s-1}(M).$$

Let (g_n) and (h_n) be two families or independent random variables satisfying the assumptions in Theorem 2.1. Consider

$$(u_0^\omega, u_1^\omega) = \sum_n (u_{n,1} g_n^\omega e_n(x), h_n^\omega u_{n,2} e_n(x))$$

the associated random function. Then for almost every initial data (u_0^ω, u_1^ω), there exists $T > 0$ such that there exists a unique solution u of (18) in a space continuously embedded in $C([-T, T]; H^s(M))$, and furthermore, there exist $C > 0, \delta > 0$ such that

$$p(T \geq T_0) \geq 1 - Ce^{-c/T_0^\delta}, \quad c,\, \delta > 0. \tag{19}$$

Remark. *Notice that if the initial data (u_0, u_1) are in $\mathcal{H}^s(M) \times \mathcal{H}^{s-1}(M)$ but* not *in $\mathcal{H}^\sigma(M) \times \mathcal{H}^{\sigma-1}(M)$, then almost surely (u_0^ω, u_1^ω) are* not *in $\mathcal{H}^\sigma(M) \times \mathcal{H}^{\sigma-1}(M)$. Consequently, this theorem provides us with a large number of initial data of supercritical regularity, for which local existence of a strong solution holds.*

Sketch of proof. Let us recall first how, using purely deterministic arguments, one can prove that (18) is locally well posed for initial data in $\mathcal{H}^s(M) \times \mathcal{H}^{s-1}(M)$ when $s \geq 1/2$. We shall use the following Strichartz estimate due to Kapitanskii [26]

Theorem 3.2. *The solution of the linear wave equation*

$$(\partial_t^2 - \Delta)u = f, \quad (u, \partial_t u)|_{t=0} = (u_1, u_2) \tag{20}$$

satisfies

$$\|u\|_{L^4((0,T\times\Omega))} \leq C\Big(\|(u_1, u_2)\|_{H^{1/2}(M)\times H^{-1/2}(M)} + \|f\|_{L^{4/3}((0,T)\times M)}\Big).$$

Now, to solve (18), we simply look for a fixed point of the operator

$$K : u \mapsto \cos(t\sqrt{-\mathbf{\Delta}})u_1 + \frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}} u_2 + \int_0^t \frac{\sin((t-s)\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}} u^3(s) ds$$

in the space $C^0((0, T); H^s(M)) \cap L^4((0, T) \times M)$, and using Theorem 3.2, it is not difficult to see the existence of such a fixed point (notice that $u \in L^4 \Rightarrow u^3 \in L^{4/3}$). The idea of the proof of Theorem 3.1 is now very simple. Instead of performing, the previous fixed point in the Strichartz type space, we perform a first iteration and search for a solution under the form

$$u = \cos(t\sqrt{-\mathbf{\Delta}})u_1 + \frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}} u_2 + v = u_{\text{free}} + v.$$

The function v is solution of

$$(\partial_t^2 - \mathbf{\Delta})v + (u_{\text{free}} + v)^3 = 0, \quad (v, \partial_t v)|_{t=0} = (0,0) \tag{21}$$

and we can rewrite this equation as a fixed point

$$v = \widetilde{K}(v) = \int_0^t \frac{\sin((t-s)\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}}(u_{\text{free}} + v)^3(s)ds.$$

Now, according to Theorem 2.5, almost surely, there exists $T > 0$ such that

$$u_{\text{free}} \in L^4((0,T) \times M)$$

(notice that the additional time dependence plays no role and the proof of Theorem 2.1 applies). As a consequence, the same proof as in the deterministic case for the operator K applies and shows the existence of a fixed point for $\widetilde{K}$.

3.2. A global existence result. Having the previous local result in mind, a natural question is whether one can exhibit cases where it is possible to prove global (in time) existence for the solutions. It turns out that it is the case for a very particular model problem: Consider the case of the wave equation in the unit ball of $\mathbb{R}^3$, $\mathbf{B}$, with Dirichlet boundary conditions

$$(\partial_t^2 - \Delta)u + |u|^{p-1}u = 0, \quad u\,|_{\partial\mathbf{B}} = 0, \qquad (u, \partial_t u)|_{t=0} = (u_1, u_2) \tag{22}$$

In this case, the critical index is

$$s_c = \frac{3}{2} - \frac{2}{p-1},$$

and for $k > 3$, $s_c > \frac{1}{2}$. Consider now $(e_n)_{n\in\mathbb{N}}$ the sequence of *radial* eigenfunctions of the Laplace operator with Dirichlet boundary conditions in $\mathbf{B}$. The following result [14] shows that the Cauchy problem is, in this particular case globally well posed for a large number of super-critical initial data.

Theorem 3.3. *Suppose that $k < 4$. Fix a real number p such that $\max(4, 2\alpha) < p < 6$. Let $((h_n(\omega), l_n(\omega))_{n=1}^{\infty}$ be a sequence of independent standard real Gaussian random variables on a probability space $(\Omega, \mathcal{A}, p)$. Consider (22) with initial data*

$$f_1^\omega(r) = \sum_{n=1}^{\infty} \frac{h_n(\omega)}{n\pi} e_n(r), \quad f_2^\omega(r) = \sum_{n=1}^{\infty} l_n(\omega) e_n(r)\,, \tag{23}$$

where $(e_n(r))_{n=1}^{\infty}$ is the orthonormal basis consisting in radial eigenfunctions of the Laplace operator with Dirichlet boundary conditions, associated to eigenvalues $-(\pi n)^2$. Then for every $s < 1/2$, almost surely in $\omega \in \Omega$, the problem (22) has a unique global solution

$$u^\omega \in C(\mathbb{R}, H^s(\mathbf{B})) \cap L^p_{loc}(\mathbb{R}_t; L^p(\mathbf{B}))\,.$$

Furthermore, the solution is a perturbation of the linear *solution*

$$u^\omega(t) = U(t)(f_1^\omega, f_2^\omega) + v^\omega(t) = \cos(t\sqrt{-\mathbf{\Delta}})f_1^\omega + \frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}} f_2^\omega + v^\omega(t),$$

where $v^\omega \in C(\mathbb{R}, H^\sigma(\mathbf{B}))$ *for some* $\sigma > 1/2$*. Moreover there exists* $C > 0$*, and almost surely* D^ω *such that*

$$\|u^\omega(t)\|_{H^s(\mathbf{B})} \leq C\log(D^\omega + |t|)^{\frac{1}{2}}, \mathcal{P}(D^\omega > \Lambda) \leq Ce^{-\lambda^2/C}.$$

Notice that as soon as $s < 1/2$, the initial data given by (23) are almost surely in $H^s(\mathbf{B}) \times H^{s-1}(\mathbf{B})$ and as soon as $s \geq \frac{1}{2}$ are almost surely not in $H^s(\mathbf{B}) \times H^{s-1}(\mathbf{B})$, and consequently in the range of non linearities $3 < k < 4$, the initial data we consider are super-critical. The proof of this result combines a local Cauchy at the probabilistic level with the Gibbs measure strategy performed by Bourgain [7], following the trend iniciated by Lebowitz-Rose-Speer [34].

4. Non linear harmonic oscillators

In this section, I will present some results on the non linear harmonic oscillator

$$\begin{cases} i\partial_t u + \partial_x^2 u - x^2 u = \kappa_0 |u|^{k-1} u, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = f(x), \end{cases} \tag{24}$$

where $k \geq 3$ is an odd integer and where either $\kappa_0 = 1$ or $\kappa_0 = -1$. Our main result [12] shows once again that the Cauchy problem is globally well posed for a large number of initial data.

Theorem 4.1. *Consider the* L^2 *Wiener measure on* $\mathcal{D}'(\mathbb{R})$*,* μ*, constructed on the harmonic oscillator eigenbasis, i.e.* μ *is the distribution of the random variable*

$$\sum_{n=0}^{\infty} \sqrt{\frac{2}{2n+1}} g_n(\omega) h_n(x),$$

where $(h_n)_{n=0}^\infty$ *are the Hermite functions and* $(g_n)_{n=0}^\infty$ *is a system of standard independent complex Gaussian random variables. Then in the defocusing case, for any order of nonlinearity* $k < +\infty$*, and in the focusing case for the cubic non linearity, the Cauchy problem* (24) *is globally well posed for* μ*-almost every initial data. Furthermore, in both cases, there exists a Gibbs measure, absolutely continuous with respect to* μ*, which is invariant by this flow.*

An interesting by-product of our analysis is the following result for the L^2 critical and super-critical equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = |u|^{k-1} u, & k \geq 5, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x) \end{cases} \tag{25}$$

Theorem 4.2. *[[12]] For any $0 < s < 1/2$, the equation (25) has for μ-almost every initial data a unique global solution satisfying*

$$u(t,\cdot) - e^{-it\mathbf{\Delta}}u_0 \in C\big(\mathbb{R}; \mathcal{H}^s(\mathbb{R})\big)$$

(the uniqueness holds in a space continuously embedded in $C\big(\mathbb{R}; \mathcal{H}^s(\mathbb{R})\big)$). Moreover, the solution scatters in the following sense. There exists μ-almost surely states $g_\pm \in \mathcal{H}^s(\mathbb{R})$ so that

$$\|u(t,\cdot) - e^{it\mathbf{\Delta}}(f + g_\pm)\|_{\mathcal{H}^s(\mathbb{R})} \longrightarrow 0, \quad \textit{when} \quad t \longrightarrow \pm\infty.$$

Remark. *Notice that Theorem 4.2 gives global existence whilst no invariant measure is involved in the proof (see Colliander-Oh [23, 22] for results in this direction).*

The proof of Theorem 4.2 uses the pseudo-conformal transform (see [15] for a use of this transform in the context of L^2 scattering problems).

Proposition 4.3. *Suppose that $v(s,y)$ is a solution of the problem*

$$i\partial_s v + \partial_y^2 v = |v|^{k-1}v, \quad s \in \mathbb{R}, \quad y \in \mathbb{R}. \tag{26}$$

We define $u(t,x) = \mathcal{L}(v)(t,x)$ for $|t| < \frac{\pi}{4}$, $x \in \mathbb{R}$ by

$$u(t,x) = \frac{1}{\cos^{\frac{1}{2}}(2t)} v\big(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\big) e^{-\frac{ix^2 \mathrm{tg}(2t)}{2}}. \tag{27}$$

Then u solves the problem

$$i\partial_t u - (\partial_x^2 - x^2)u = \cos^{\frac{k-5}{2}}(2t)|u|^{k-1}u, \quad |t| < \frac{\pi}{4}, \, x \in \mathbb{R}. \tag{28}$$

As a consequence, in the case $k = 5$, (28) reduces to (25), and Theorem 4.2 follows rather directly from Theorem 4.1 In the case $k \geq 7$, the proof is more involved and relies on an analog of Theorem 4.1 for (28) (notice that this latter equation is non autonomous).

Sketch of proof of Theorem 4.1. For low order non linearities ($p \leq 7$), the proof follows the same lines as in the case of wave equations, and relies on Theorem 2.6 (or more precisely on similar estimates for the solution of the linear harmonic Schrödinger equation $u = e^{itH}u_0$). However, as soon as $p \geq 9$, these estimates are not sufficient, because they allow only for a gain of at most 1/4 space derivatives, and the exponent for which $s_c = \frac{1}{4}$ is precisely $k = 9$. As a consequence, our analysis requires a full bi-linear analysis at the probabilistic level. The bilinear nature of our probabilistic analysis can be seen though the following statement which shows that by considering nonlinear quantities, a gain of (almost) 1/2 space derivatives can be achieved.

$$\forall\, \theta < 1/2, \ \forall\, t \in \mathbb{R}, \quad \|(e^{-itH}u^\omega)^2\|_{\mathcal{H}^\theta} < +\infty, \quad \mu \text{ almost surely}. \tag{29}$$

4.1. Bilinear estimates. In this section we give a proof of (29) which was pointed to us by P. Gérard. Observe that (29), applied with $t=0$ implies that $(u_0^\omega)^2(x)$ is a.s. in $\mathcal{H}^\theta$ for every $\theta<1/2$ which is a remarkable smoothing property satisfied by the random series $(u_0^\omega)(x)$. The key point in the proof of (29) is the following bilinear estimate for Hermite functions.

Lemma 4.4. *There exists $C>0$ so that for all $0\leq\theta\leq 1$ and $n,m\in\mathbb{N}$*

$$\|h_n\, h_m\|_{\mathcal{H}^\theta(\mathbb{R})}\leq C\max{(n,m)}^{-\frac14+\frac\theta2}\big(\log\big(\min{(n,m)}+1\big)\big)^{\frac12}. \tag{30}$$

Proof. We give an argument we learned from Patrick Gérard. It suffices to prove (30) for $\theta=0$ and $\theta=1$ (the general case then follows by interpolation). The case $\theta=1$ can actually be directly reduced to the case $\theta=0$ by taking space derivatives. We are going to use the generating series:

$$\begin{aligned}E(x,y,\alpha)&=\sum_{n\geq0}\alpha^n\, h_n(x)\, h_n(y)\\ &=\frac{1}{\sqrt{\pi(1-\alpha^2)}}\exp\big(-\frac{1-\alpha}{1+\alpha}\frac{(x+y)^2}{4}-\frac{1+\alpha}{1-\alpha}\frac{(x-y)^2}{4}\big).\end{aligned} \tag{31}$$

Therefore, if we set

$$I(\alpha,\beta)\equiv\int_{\mathbb{R}}E(x,x,\alpha)E(x,x,\beta)\mathrm{d}x,$$

then we get

$$\begin{aligned}I(\alpha,\beta)&=\frac1\pi(1-\alpha^2)^{-\frac12}(1-\beta^2)^{-\frac12}\int_{\mathbb{R}}e^{-\big(\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}\big)x^2}\mathrm{d}x\\ &=\frac{1}{\sqrt{2\pi}}(1-\alpha)^{-\frac12}(1-\beta)^{-\frac12}(1-\alpha\beta)^{-\frac12}.\end{aligned} \tag{32}$$

On the other hand, coming back to the definition

$$I(\alpha,\beta)=\sum_{n,m\geq0}\alpha^n\beta^m\int_{\mathbb{R}}h_n^2(x)\, h_m^2(x)\mathrm{d}x.$$

Hence to get a useful expression for the L^2 norm of the product of two Hermite functions, it suffices to expand (32) in entire series in α and β. Write

$$(1-x)^{-\frac12}=\sum_{p\geq0}c_px^p,\quad c_0=1,\qquad c_p=\frac{(2p-1)\,!}{2^{2p-1}\,p\,!\,(p-1)\,!},\quad p\geq1.$$

Therefore, by the Stirling formula, there exists $C>0$ so that $|c_p|\leq\frac{C}{\sqrt{p+1}}$ for all $p\geq0$. Now by (32) and the previous estimate

$$\begin{aligned}\int_{\mathbb{R}}h_n^2(x)\, h_m^2(x)\mathrm{d}x&=\frac{1}{\sqrt{2\pi}}\sum_{\substack{p,q,r\geq0\\ p+r=n,\ q+r=m}}c_p\, c_q\, c_r\\ &\leq C\sum_{0\leq r\leq\min(n,m)}(n-r+1)^{-\frac12}\,(m-r+1)^{-\frac12}\,(r+1)^{-\frac12}.\end{aligned}$$

Without restricting the generality we may suppose that $m \geq n$. If $m \leq 2n$ then we obtain the needed bound by considering separately the cases when the sum runs over $r < m/2$ and $r \geq m/2$. If $m > 2n$, then we can write $(m-r+1)^{-\frac{1}{2}} \leq c(1+m)^{-\frac{1}{2}}$ and the needed bound follows directly. Therefore we get (30) in the case $\theta = 0$. This completes the proof of Lemma 4.4. □

Denote by $u^\omega_{\mathrm{free}}(t,x)$ the free Schrödinger solution with initial condition $u^\omega_0\phi(\omega,x)$, i.e.

$$u^\omega_{\mathrm{free}}(t,x) = e^{-itH}u^\omega_0 = \sum_{n\geq 0} \frac{\sqrt{2}}{\lambda_n} e^{-it\lambda_n^2}\, g^\omega_n h_n(x).$$

Write the decomposition $u = \sum_N u_N$, where the summation is taken over the dyadic integers and for N a dyadic integer

$$u_N(\omega,t,x) = \sum_{N\leq n<2N} \alpha_n(t) h_n(x) g^\omega_n, \quad \alpha_n(t) = \sqrt{\frac{2}{2n+1}} e^{-i(2n+1)t}\, .$$

Let us fix $t \in \mathbb{R}$ and $0 \leq \theta < \frac{1}{2}$. It suffices to show that the expression

$$J(t,x,\omega) \equiv |\sum_M \sum_N H^{\theta/2}\big(u_N\, u_M\big)|$$

belongs to $L^2(\mathbb{R}\times\Omega)$ (here the summation is again taken over the dyadic values of M, N). Using the Cauchy-Schwarz inequality, a symmetry argument and summing geometric series, for all $\epsilon > 0$ we can write

$$J(t,x,\omega) \leq C\big(\sum_{N\leq M} M^\epsilon |H^{\theta/2}\big(u_N\, u_M\big)|^2\big)^{\frac{1}{2}}\, . \tag{33}$$

Coming back to the definition we can write

$$H^{\theta/2}\big(u_N\, u_M\big) = \sum_{\substack{N\leq n\leq 2N \\ M\leq m\leq 2M}} \alpha_n\, \alpha_m\, g_n\, g_m\, H^{\theta/2}\big(h_n\, h_m\big).$$

We now estimate $\mathbb{E}(|H^{\theta/2}\big(u_N\, u_M\big)|^2$. We make the expansion

$$|H^{\theta/2}\big(u_N\, u_M\big)|^2 = \sum_{\substack{N\leq n_1,n_2\leq 2N \\ M\leq m_1,m_2\leq 2M}} \alpha_{n_1}\, \overline{\alpha}_{n_2}\, \alpha_{m_1}\, \overline{\alpha_{m_2}}\, g_{n_1}\, \overline{g_{n_2}}\, g_{m_1}\, \overline{g_{m_2}}\, H^{\theta/2}\big(h_{n_1}\, h_{m_1}\big)\, \overline{H^{\theta/2}\big(h_{n_2}\, h_{m_2}\big)}.$$

The random variables g_n are centered and independent, and consequently, we have $\mathbb{E}\big[\, g_{n_1}\, \overline{g_{n_2}}\, g_{m_1}\, \overline{g_{m_2}}\, \big] = 0$, unless the indexes are pairwise equal (i.e. ($n_1 = n_2$ and $m_1 = m_2$), or ($n_1 = m_2$ and $n_2 = m_1$). This implies that

$$\mathbb{E}(|H^{\theta/2}\big(u_N\, u_M\big)|^2) \leq C \sum_{\substack{N\leq n\leq 2N \\ M\leq m\leq 2M}} |\alpha_n|^2 |\alpha_m|^2 |H^{\theta/2}\big(h_n\, h_m\big)|^2. \tag{34}$$

We integrate (34) in x and by (30) we deduce that for all $\epsilon > 0$

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} |H^{\theta/2}(u_N\, u_M)|^2 &\leq C \sum_{\substack{N\leq n\leq 2N\\ M\leq m\leq 2M}} |\alpha_n|^2|\alpha_m|^2 \int_{\mathbb{R}} |H^{\theta/2}\big(h_n\, h_m\big)|^2 \mathrm{d}x \\ &\leq C \sum_{\substack{N\leq n\leq 2N\\ M\leq m\leq 2M}} (\max(M,N))^{-\frac{1}{2}+\theta+\epsilon}|\alpha_n|^2|\alpha_m|^2.\end{aligned}$$

Therefore using that $|\alpha_n| \leq \langle n\rangle^{-\frac{1}{2}}$, we get

$$\begin{aligned}\mathbb{E}(J(t,x,\omega)^2) &\leq C \sum_{N\leq M} \sum_{\substack{N\leq n\leq 2N\\ M\leq m\leq 2M}} M^{-\frac{1}{2}+\theta+2\epsilon}|\alpha_n|^2|\alpha_m|^2 \\ &\leq C \sum_{N\leq M} \sum_{\substack{N\leq n\leq 2N\\ M\leq m\leq 2M}} M^{-\frac{1}{2}+\theta+2\epsilon}(MN)^{-1} < \infty,\end{aligned}$$

provided ϵ is small enough, namely ϵ such that $-\frac{1}{2}+\theta+2\epsilon < 0$. This completes the proof of (29).

5. Improved Sobolev embeddings

As shown in the previous section, our applications to partial differential equations of Paley-Zygmund's result rely on the simple observation that "typical" functions n $\mathcal{H}^s(M)$ enjoy better L^p properties than what the Sobolev embeddings would predict. Namely, the L^∞ norm is essentially bounded (modulo logarithmic loss) by the $\mathcal{H}^{\frac{d-1}{2}}$ norm (versus the $\mathcal{H}^{d/2}$ norm for classical Sobolev embeddings). Notice that this bound is improved, in the case of the tora $\mathbb{T}^d$ all the way down to $\mathcal{H}^0$. In this section, I will present some other randomizations obtained with G. Lebeau [11] on compact manifolds. Notice that other applications to linear and non linear problems are developped in [11], and we expect these constructions to be of interest in view of further applications to partial differential equations.

5.1. Construction of the measure. Let M be a compact riemanian manifold, let $I = [c,c'], 0 \leq c < c' < \infty$ be an interval and $E_{I,h}$ the subspace of $L^2(M)$ of dimension $N(I,h)$ defined by

$$E_{I,h} = \{u = \sum_{k\in I_h} z_k e_k(x),\ z_k \in \mathbb{C}\}, \quad I_h = \{k,\ h\omega_k \in I\}. \tag{35}$$

According to the precised Weyl formula (see [21, Theorem 1.1]), we have for $h \in]0,1]$

$$N(I,h) = (2\pi h)^{-d}\mathrm{Vol}(M)\mathrm{Vol}(S^{d-1}) \int_I \rho^{d-1}d\rho + O(h^{-d+1}). \tag{36}$$

Let us recall that Sobolev injections read

$$\|u\|_{L^\infty(M)} \leq Ch^{-d/2}\|u\|_{L^2(M)} \quad \forall u \in E_{I,h}. \tag{37}$$

Notice that these estimates are optimal as can easily bee seen by considering the sequence $h^{-d/2}\chi(x/h)$, where $\chi \in C_0^\infty$ a fixed function (in a local coordinate chart). Denote by $S_{I,h}$ the unit sphere of the euclidean space $E_{I,h} = \mathbb{C}^{N(I,h)}$, and $P_{I,h}$ the uniform probability on $S_{I,h}$. We can now define probability measures on $E_{I,h}$ by picking a probability measure in the radial variable $\rho(r)$, with sufficient fast decay near infinity (e.g. Gaussian), and defining

$$d\mu_{I,h} = dP_{I,h} \otimes d\rho.$$

A typical example (to which all other examples reduce eventually) is of course the simplest choice $\rho = \delta_{r=1}$ for which the measure $\mu_{I,h}$ is simply the uniform measure on the unit sphere of $E_{I,h}$, which in the sequel will still be denoted by $P_{I,h}$. Finally, taking any family of positive real numbers $(\alpha_n) > 0$, we can rescale (in the radial variable) the measure by defining

$$d\mu_{I,h,\alpha_h} = dP_{I,h} \otimes \alpha_h d\rho(\frac{r}{\alpha\, h}).$$

The choice $I = [1/2, 2[, h_k = 2^{-k}$, $k \in \mathbb{N}$ (with a suitable modification for $k = 0$) gives

$$L^2(M) = \{u = \sum_k u_k; u_k \in E_{I,h_k}; \sum_k \|u_k\|_{L^2}^2 < +\infty\}$$

and the Sobolev space $H^s(M)$ can also be expressed in terms of this decomposition

$$H^s(M) = \{u = \sum_k u_k; u_k \in E_{I,h_k}; \sum_k 2^{2ks}\|u_k\|_{L^2}^2 < +\infty\}.$$

As a consequence, the choice of $\alpha_{h_k} = 2^{-k}\beta_k$ with $\beta_k \in \ell^2$ ensures that the measure

$$d\mu_{s,(\beta_n)} = \otimes_k d\mu_{I,h_k,\alpha_{h_k}}$$

defines a measure on $\oplus_k E_{I,h_k}$ which is supported by $H^s(M)$.

5.2. Improved Sobolev embeddings.

The measures constructed in the previous section satisfy:

Theorem 5.1.

- *For any choice of sequence $(\beta_n) \in \ell^2$, the measure $d\mu_{s,(\beta_n)}$ is supported in $H^s(M)$.*
- *For any $s' > s$, if the sequence (β_n) satisfies*

$$\sum_n |\beta_n|^2(1 + 2^{2(s'-s)}) = +\infty,$$

then the space $H^{s'}(M)$ has $d\mu_{s,(\beta_n)}$-measure equal to 0.

- *For any $s > 0$, the measure $d\mu_{s,(\beta_n)}$ is supported in $L^\infty(M)$. In other words, "for any $\epsilon > 0$, almost surely, $H^\epsilon(M) \subset L^\infty(M)$".*

Remark. *A similar result was obtained by Shiffman-Zelditch in [44] in the different context of random sequences of holomorphic sections of high powers of a positive line bundle.*

The main step for the proof of Theorem 5.1 is the proof of the following semi-classical result

Theorem 5.2. *For any $c <$ Vol(M), there exists $C > 0$ such that for any $h \in (0,1]$, and any $\lambda > 0$,*

$$P_{I,h}(\{u \in E_{I,h}; \|u\|_{L^\infty} > \lambda\}) \leq C h^{-d(1+d/2)} e^{-c_2 \lambda^2}. \tag{38}$$

Indeed, taking $\lambda = h^{-\epsilon}$ in (38), we obtain

$$P_{I,h}(\{u \in E_{I,h}; \|u\|_{L^\infty} > h^{-\epsilon}\}) \leq C' e^{-c' h^{-2\epsilon}},$$

and Theorem 5.1 follows after suitable resummations. Now, in turn, Theorem 5.2 follows from the classical concentration of measure phenomenon (see Ledoux [35])

Theorem 5.3. *Consider a Lipshitz function F, on the N dimensional sphere $\mathbb{S}^N$, endowed with its natural geodesic metric, and with the uniform probability measure μ_N. Let us define the mediane, $\mathcal{M}(F)$, of the function F by the relation*

$$\mu_N(\{x \in \mathbb{S}^N; F(x) \geq \mathcal{M}(F)\}) \geq \frac{1}{2}, \qquad \mu(\{x \in \mathbb{S}^N; F(x) \leq \mathcal{M}(F)\}) \geq \frac{1}{2}.$$

Then for any $r > 0$,

$$\mu(\{x \in \mathbb{S}^N;\ F(x) - \mathcal{M}(F)| > r\}) \leq 2 e^{-(N-1)\frac{r^2}{\|f\|^2_{Lip}}}.$$

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