Random data for Partial Differential Equations

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Hyderabad, aug. 20th, 2010

International congress of mathematicians

based on joint works with

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A result by Paley and Zygmund (1932)

Consider a sequence $(\alpha_k)_{k\in\mathbb{N}} \in \ell^2$

$$\sum_k |\alpha_k|^2 < +\infty.$$

Let u be the trigonometric series on $\mathbb{T}$

$$u = \sum_k \alpha_k e^{ik\theta}$$

This series is convergent in $L^2(\mathbb{T})$ but "in general" (generically for the ℓ^2 topology), the function u is in

no $L^p(\mathbb{T}), p > 2$ space.

If one changes the signs in front of the coefficients α_k randomly and independently, i.e. if one considers

$$\sum_k g_k(\omega)\alpha_k e^{ik\theta} = u^\omega(\theta)$$

where Ω, P is a probabilistic space and $g_k(\omega)$ are Bernouilli *independent* random variables on Ω,

$$P(g_k(\omega) = \pm 1) = \frac{1}{2}$$

then

Theorem (Paley- Zygmund 1930-32)

For any $p < +\infty$, almost surely, the series $u^\omega = \sum_k g_k(\omega)\alpha_k e^{ik\theta}$ is convergent in $L^p(\mathbb{T})$. Furthermore, large deviation estimate:

$$\mathcal{P}(\{\|u^\omega\|_{L^p(\mathbb{T})} > \lambda\}) \leq Ce^{-c\lambda^2}$$

Why study PDEs with low regularity initial data?

- Global existence of smooth solutions: local (in time) classical strategy for initial data in a space X such that the norm in X is essentially preserved by the flow. If it is possible to solve the equation between $t = 0$ and $t = T(\|u_0\|_X)$ and if $\|u(T)\|_X \leq \|u_0\|_X$ then it is possible to solve between $t = T$ and $t = 2T$, etc...
- Large time behaviour of solutions with *smooth* initial data (scattering), or large time behaviour of the norms of these solutions (exponential/polynomial increase rate, etc...)
- Informations about the behaviour of the blowing up solutions in some cases.
- ...

Super/sub critical PDEs (in Sobolev spaces)

While solving non linear PDEs, very often a critical threshold of regularities appears, s_c, for which

- If the initial data are smooth enough, $u_0 \in H^s, s > s_c$ then local existence holds (with a time existence depending only on the norm of u_0 in H^s)
- If the initial data are not smooth enough i.e $u_0 \in H^s, s < s_c$ (and not better) then the PDE is unstable, or even ill posed

For example for the Navier-Stokes equation, the critical index is

- $s_c = 0$ in space dimension 2
- $s_c = 1/2$ in space dimension 3

Some unstabilities

- The solution ceases to exist after a finite time: finite time blow up.
- No continuous flow (on any ball in H^s) ill posedness.
- The flow defined by the PDE (if it exists) is not uniformly continuous on the balls of H^s.

N.B. This type of instabilities say very little about the smooth solutions.

The 3-dimensional wave equation: a model dispersive PDE

Let (M,g) be a 3-dimensional riemannian manifold (without boundary) and

$$\mathbf{\Delta} = \sum_{i,j=1,\cdots 3} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} g^{i,j}(x) \sqrt{\det g} \frac{\partial}{\partial x_j},$$

be the Laplace operator on functions, and

$$(\text{NLW}) \quad \begin{cases} (\frac{\partial^2}{\partial t^2} - \mathbf{\Delta})u + u^3 = 0, \\ (u, \partial_t u)_{t=0} = (u_0(x), u_1(x)) \in H^s(M) \times H^{s-1}(M) \end{cases}$$

the cubic defocusing non linear wave equation.

Critical index: $s_c = 1/2$

Theorem (Strichartz, Ginibre-Velo, Kapitanskii, ...)

Let $s \geq 1/2$*. For any initial data* $(u_0, u_1) \in \mathcal{H}^s(M) = H^s(M) \times H^{s-1}(M)$*, there exists* $T > 0$ *and a unique solution to the system (NLW) in the space*

$$C^0([0,T]; H^s(M)) \cap C^1([0,T]; H^{s-1}(M)) \cap L^4((0,T) \times M)$$

Furthermore, if $s > \frac{1}{2}$*,* $T = T(\|(u_0, u_1)\|_{\mathcal{H}^s(M)})$*.*

The super-critical wave equation is *ill posed*

Theorem (Lebeau 01, Christ-Colliander-Tao 04, Burq-Tzvetkov 07)

- *There exists sequences* $(u_{0,n}, u_{1,n}) \in C_0^\infty(M), (t_n) \in \mathbb{R}$ *such that the solution of (NLW) with initial data* (u_0, u_1) *exists on* $[0, t_n]$ *but*

$$\lim_{n\to+\infty} t_n = 0, \lim_{n\to+\infty} \|(u_{0,n}, u_{1,n})\|_{\mathcal{H}^s(M)} = 0,$$
$$\lim_{n\to+\infty} \|u_n(t_n)\|_{H^s(M)} = +\infty$$

- *There exists an initial data* $(u_0, u_1) \in \mathcal{H}^s(M)$ *such any weak solution of (NLW) associated to this initial data, satisfying the finite speed of propagation principle ceases instantaneously to belong to* $\mathcal{H}^s(M)$

Is instability a generic situation?

- Lebeau's initial data are very particular:

$$(u_{0,n}, u_{1,n}) = n^{\frac{3}{2}-s}(\phi(nx), n^{-1}\psi(nx)), \quad \phi, \psi \in C_0^\infty(\mathbb{R}^3).$$

- Question: are the initial data exhibiting the pathological behaviour described by Lebeau's result rares or on the contrary generic?

A first answer is that in some sense the situation is much better behaved than what Lebeau's theorem might let think: the phenomenon described by lebeau's theorem appears to be rare (in some sense). We show that for random initial data, the situation is much better behaved.

Random initial data: the case of $\mathbb{T}^3$.

$(e_n = e^{in\cdot x}\sqrt{|n|^2}, n \in \mathbb{Z}^3)$ be the Hilbert basis of $L^2(\mathbb{T}^3)$. Any function $u \in H^s(\mathbb{T}^3)$ writes

$$u = \sum_n \alpha_n e_n(x), \qquad \sum_n (1+|n|^2)^s |\alpha_n|^2 = \|u\|^2_{H^s(P^{1/2})} < +\infty.$$

Let $\Omega, \mathcal{A}, p$ be a probabilistic space and (g_n) a sequence of *independent* random random variables *with mean equal to* 0 and super exponential decay at infinity:

$$\exists C, \delta > 0; \forall \alpha > 0, \sup_n \mathbb{E}(e^{\alpha |g_n|}) < Ce^{\delta\alpha^2}$$

a random function in $H^s(\mathbb{T}^3)$ takes the form

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^3} g_n^\omega \alpha_n e_n(x), \qquad \sum_n (1+|n|^2)^s |\alpha_n|^2 < +\infty$$

Almost sure local well posedness for random initial data in $\mathcal{H}^s = H^s \times H^{s-1}, \forall s \geq 0$

Theorem (Tzvetkov-B. 2007)

Consider $s \geq 0$ *and a random initial data*

$$(u_0, u_1) = \Big(\sum_{n \in \mathbb{Z}^3} g_n^\omega \alpha_n e_n(x), \sum_{n \in \mathbb{Z}^3} \widetilde{g}_n^\omega \beta_n e_n(x) \Big)$$

Notice that a.s. $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$. *Then a.s. there exists* $T > 0$ *and a unique solution* $u^\omega(t, x)$ *of (NLW) in a space*

$$X_T \subset C([0, T]; H^s(M)) \cap C^1([0, T]; H^{s-1}(M)).$$

Furthermore

$$\mathcal{P}(T_{max}(\omega) < T) \leq C e^{-c/T^\delta}.$$

From local to global existence

The result above shows that we have a good Cauchy theory at the probabilistic level in $\mathcal{H}^s(\mathbb{T}^3), s \geq 0$. and we can almost surely solve the non linear wave equation on a maximal time interval $(0, T)$.

Natural question: $T = +\infty$? (global existence).

Theorem (Tzvetkov-B.2010)

For any $0 \leq s$, the solution of (NLW) constructed above exists almost surely globally in time and satisfies:

$$\|(u^\omega(t, \cdot), \partial_t u^\omega(t, \cdot))\|^2_{\mathcal{H}^s(\mathbb{T}^3)} \leq C((M^\omega + t))^{\frac{(1-s)}{s}+0}$$

with

$$\mathbb{P}(M^\omega > \Lambda) \leq Ce^{-c\Lambda}$$

Rk 1. Almost surely, the initial data $(u_0^\omega, u_1^\omega) \in \mathcal{H}^s(\mathbb{T}^3)$, $s > 0$, but as soon as

$$\sum_{n \in \mathbb{Z}^3} (1+|n|^2)^{s'} |\alpha_n|^2 + (1+|n|^2)^{s'-1} |\beta_n|^2 = +\infty$$

and the random variables $g_n, \widetilde{g}_n$ do not accumulate at 0 (say they are i.i.d. non trivial), then almost surely

$$(u_0^\omega, u_1^\omega) \notin \mathcal{H}^{s'}(\mathbb{T}^3)$$

and the result provides many initial data for which the classical Cauchy theory does not apply (even locally in time)

Rk 2. In the deterministic setting, global well posedness below H^1 iniciated by Bourgain using high/low decomposition. Then global well posedness obtained for $s > \frac{3}{4}$ by Kenig-Ponce-Vega (see also Gallagher-Planchon), and then for $s = \frac{3}{4}$ by Bahouri-Chemin

Deterministic theory: Strichartz estimates

$$S(t)(u_0, u_1) = \cos(\sqrt{-\Delta})u_0 + \frac{\sin(\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1$$

the solution to the linear wave equation with data (u_0, u_1).

$$(\partial_t^2 - \Delta)u = 0, \qquad u \mid_{t=0} = u_0, \partial_t u \mid_{t=0} = u_1.$$

Theorem (Strichartz)

$$\|S(t)(u_0, u_1)\|_{L^4((0,T)\times M)} \leq C\|(u_0, u_1)\|_{\mathcal{H}^{1/2}(M)}$$

Rk.:

- Similar estimates for norms $L_t^p; L_x^q$, $\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - s, 2 < p \leq +\infty$
- Sobolev inequalities imply

$$\|u\|_{L^4(M)} \leq C\|u\|_{H^{3/4}(M)}$$

Deterministic theory: local Cauchy theory in H^1

Theorem

Assume that

$$\|u_0\|_{H^1} + \|u_1\|_{L^2} \leq \Lambda.$$

There exists a unique solution of (NLW)

$$u \in L^\infty([0, C^{-1}\Lambda^{-3}], H^1 \times L^2(M))$$

Moreover the solution satisfies

$$\|(u, \partial_t u)\|_{L^\infty([0,C\Lambda^{-3}],H^1\times L^2)} \leq C\Lambda$$

and $(u, \partial_t u)$ *is unique in the class*

$$L^\infty([0, C\Lambda^{-3}], H^1 \times L^2)$$

Proof: Fixed point in the ball centered on $S(t)(u_0, u_1)$ in

$$L^\infty((0,T); H^1_x)$$

Use that u satisfies $(\partial_t^2 - \Delta)u = -u^3$, and hence Duhamel formula gives

$$u = S(t)(u_0, u_1) - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} u^3(s)ds$$
$$= S(t)(u_0, u_1) + K(t)(u)$$

where $K(t)$ satisfies (using Sobolev embeddings $H^1_x \hookrightarrow L^6_x$)

$$\|K(t)(u)\|_{L^\infty(0,T);H^1(M)} \leq \|u^3\|_{L^1(0,T);L^2(M)}$$
$$\leq T\|u\|^3_{L^\infty(0,T);L^6(M)} \leq CT\|u\|^3_{L^\infty(0,T);H^1(M)}$$

Deterministic theory in H^1: a remark

Theorem

Assume that

$$\|u_0\|_{H^1} + \|u_1\|_{L^2} + \|f\|_{L^\infty(\mathbb{R};L^6(M))} \leq \Lambda.$$

There exists a unique solution in $L^\infty([0, C\Lambda^{-3}], H^1 \times L^2)$ *of*

$$(\partial_t^2 - \Delta)u + (f + u)^3 = 0, (u, \partial_t u) \mid_{t=0} = (u_0, u_1)$$

Moreover the solution satisfies

$$\|(u, \partial_t u)\|_{L^\infty([0,\tau],H^1\times L^2)} \leq C\Lambda.$$

(same proof as before)

Paley-Zygmund Theorem, quantitative version

Theorem

Assume $\mathcal{H}^0$ random initial data.

$$\mathcal{P}(\{\|S(t)(u_0^\omega, u_1^\omega)\|_{L^p((0,T)\times\mathbb{T}^3)} > \lambda\}) \leq Ce^{-c\frac{\lambda^2}{(u_0,u_1)\|^2_{\mathcal{H}^0(M)}}}$$

Assume $\mathcal{H}^s$ random initial data.

$$\mathcal{P}(\{\|S(t)(u_0^\omega, u_1^\omega)\|_{L^\infty((0,T)\times\mathbb{T}^3)} > \lambda\}) \leq Ce^{-c\lambda^2}$$

$$\mathcal{P}(\{\|S(t)(Id - \Pi_N)(u_0^\omega, u_1^\omega)\|_{L^\infty((0,T)\times\mathbb{T}^3)} > N^{-s}\}) \leq Ce^{-cN^{2s-0}}$$

where

$$\Pi_N(\sum_n w_n e^{in\cdot x}) = \sum_{|n|\leq N} w_n e^{in\cdot x}$$

Proof: same as (modern) proof of Paley-Zygmund theorem

Local existence

We look for the solution u^ω under the form

$$u^\omega = S(t)(u_0^\omega, u_1^\omega) + v^\omega = u_f^\omega + v^\omega$$

v^ω is solution of an equation of the form

$$(\partial_t^2 - \Delta)v^\omega + (S(t)(u_0^\omega, u_1^\omega) + v)^3 = 0, \qquad (v, \partial_t v) \mid_{t=0} = (0, 0)$$

which is essentially a cubic non linear wave equation with a source term $(S(t)(u_0^\omega, u_1^\omega)^3$. According to Paley-Zygmund, a.s. this source term is admissible, and according to the deterministic H^1 theory, there exists a time $T^\omega > 0$ such that this equation is well posed in H^1: notice that

$$L^{p/3}(I; L^2(\mathbb{T}^3)) \subset L^1_t; L^2_x.$$

In some sense, this result shows that the seemingly super-critical problem is in fact sub-critical

Global existence: probabilistic version of Bourgain's high/low decomposition method

(See Colliander-Oh for a related idea on the well posedness of NLS below L^2 on $\mathbb{T}$).
Fix $T > 0$. Want to prove almost surely existence up to time T of a solution. Fix $N \gg 1$. Seek u^ω as

$$u = w_N^\omega + v_N^\omega = S(t)(Id - \Pi_N)(u_0^\omega, u_1^\omega) + v_N^\omega$$

$$(\partial_t^2 - \Delta)v_N^\omega + (S(t)(Id - \Pi_N)(u_0^\omega, u_1^\omega) + v^\omega)^3 = 0,$$
$$(v_N^\omega, \partial_t v^\omega)\mid_{t=0} = \Pi_N(u_0^\omega, u_1^\omega)$$

AIM: Prove that v_N^ω exists on $[0, T]$ with probability $p_N \to 1, N \to +\infty$.
H^1-norm controls local Cauchy theory, hence enough to prove that H^1 norm of v_N^ω remains bounded on $[0, T]$ with

$$p_N \geq 1 - Ce^{-cN^\epsilon}$$

A priori bound

$$E(v) = \int_{\mathbb{T}^3} \frac{1}{2}|\partial_t v|^2 + \frac{1}{2}|\nabla_x v|^2 + \frac{1}{4}|v|^4 dx$$

$$\frac{d}{dt}E(t) = \int_{\mathbb{T}^3} \partial_t u(v^3 - (w_N^\omega + v^\omega)^3) dx$$

$$= \int_{\mathbb{T}^3} \partial_t u(-3v_N^2 w_N - 3w_N^2 v - w_N^3) dx$$

$$\leq 3\|\partial_t v_N\|_{L_x^2} \|v_N\|_{L_x^4}^2 \|w_N\|_{L_x^\infty} + 3\|\partial_t v_N\|_{L_x^2} \|v_N\|_{L_x^4} \|w_N\|_{L_x^\infty}^2 + \|\partial_t v_N\|_{L_x^2} \|w_N\|_{L_x^\infty}^3$$

$$\leq C(E(t)\|w_N\|_{L_x^\infty} + E(t)^{3/4}\|w_N\|_{L_x^\infty}^2 + E(t)^{1/2}\|w_N\|_{L_x^\infty}^3)$$

ϵ Sobolev injection in time gives

$$\mathcal{P}(\|w_N\|_{L^\infty((0,T)\times\mathbb{T}^3)} > N^{-s+\epsilon}) \leq \mathcal{P}(\|w_N\|_{L_t^p;L_x^\infty} > N^{-s}) \leq Ce^{-cN^{2s}}$$

Conclude using Gronwall

Further developments

- Extends to other non-linearities (but require the use of Strichartz estimates for the proof)
- Allow correlations in the random variables (using some slack in the arguments)
- Relax the mean equal to 0 assumption on the random variables i.e. perform the randomization around a given solution (e.g. smooth, or given by the preceding procedure) instead of the trivial (vanishing) solution
- Extend the result to other manifolds. Possible (but not in the full range $s \geq 0$) using this Paley-Zygmund randomization, or use other randomization to get the range $s > 0$ (work with Lebeau)
- Extend to other dispersive equations such as non linear Schrödinger equations with or without harmonic potential (with L. Thomann)