SEMI-CLASSICAL ESTIMATES FOR THE RESOLVENT IN NON TRAPPING GEOMETRIES

by

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Abstract. — The purpose of this note is to present a very natural and simple method to prove (for non trapping geometries) the existence of resonance-free regions and to give estimates for the resolvent in these regions. The method is valid in any dimensions of space and does not rely on Lax and Phillips theory.

Résumé. — On présente dans cette note une méthode simple et naturelle permettant d'obtenir (dans le cas d'une géométrie non captante) l'existence de régions ne contenant pas de résonance et de démontrer des estimations sur la résolvante dans ces régions. Cette approche est indépendante de la dimension d'espace et n'utilise pas la théorie de Lax et Phillips.

1. Introduction

The purpose of this article is the study of the localization of resonances for the semi-classical Schrödinger equation near positive energy levels. The shape resonances are the metastable states of a system whose evolution is described by a Hamiltonian, H, depending upon the Planck constant h. An important feature of the resonances is their lifetime (the reverse of their width) which describes the time needed by a wave packet associated to the resonance and initially localized in a compact region of the space, to develop a non zero probability density in the outside region. In this paper we shall study the semi-classical approximation when the Planck constant tends to 0. In the case of non trapping geometries (see definition below), it is known that there exist some pole free regions. The proofs of these results rely on the "propagation of singularities" theorem of R. Melrose and J. Sjöstrand [**MS82**] on the one hand and on the other hand on Lax and Phillips [**LP89**] theory in odd space dimensions and on some resolvent computations in even (or odd) space dimensions (see Vainberg [**Vai88**]). The purpose of this note is to present a very natural and simple method relying on P. Gérard, E. Leichtnam [**GL93**] and P.L. Lions, T. Paul [**LP93**] Wigner measures (or semi-classical measures), to prove these results (see also the survey by P. Gérard, P. Markowich, N. Mauser and F. Poupaud [**GMMP97**] and the work by L. Miller [**Mil97, Mil00**]). The method does also provide a direct proof for the estimates on the outgoing resolvent in these regions. For the sake of simplicity we restrict the study to the case of compactly supported perturbations of the semi-classical Laplace operator.

1.1. Asumptions. — Let $\Theta \subset \mathbb{R}^d$ be a (possibly empty) smooth compact obstacle whose complementary, $\Omega = \Theta^c$, is connected. Suppose that Ω has no infinite order contact with its boundary (see in section 4 the precise definition). Consider a semi-classical *self adjoint* operator defined on $L^2(\Omega)$:

$$P = \sum_{i,j=1}^{d} a_{i,j}(x) hD_{x_i} hD_{x_j} + \sum_{i=1}^{d} (b_i(x)) hD_{x_i} + V(x) \tag{1.1}$$

with domain

$$\mathcal{D}(P) = \{u \in H^2(\Omega); u \mid_{\partial\Omega} = 0\} \tag{1.2}$$

Observe that $\partial\Omega$ can be empty.

Suppose that the functions $a_{i,j}, b_j$ and V are *smooth*, that P is elliptic ($\exists c > 0; (a_{i,j}(x)) \geq c\mathrm{Id}$) and that the operator P is for $|x|$ large enough equal to $P_0 = -h^2\Delta$. Let $p(x,\xi)$ be the semi-classical principal symbol of the operator P:

$$p(x,\xi) = \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j + \sum_{i=1}^{d} b_i(x)\xi_i + V(x) \tag{1.3}$$

Consider the resolvent of the operator P: $R(z,h) = (P-z)^{-1}$ which is (since P is self adjoint) defined (and holomorphic) for $\mathrm{Im}z \neq 0$. It is well known that this resolvent admits a meromorphic continuation (as a bounded operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$ across the continuous spectrum $\mathbb{R}^+$ to the logarithmic plane if d is even and to the z^2-plane if d is odd. We will denote these spaces by $\tilde{\mathbb{C}}$ in the sequel. The resonances are the poles of this meromorphic continuation. In this paper we'll restrict our attention to the resonances close to the real axis and obtained by the continuous meromorphic extension from $\{\mathrm{Im}z > 0\}$. Remark that the framework above contains naturally the two main models we have in mind: the semi classical Schrödinger operator $P = -h^2\Delta + V(x)$ and the high frequency Helmoltz operator $P = -\Delta - \lambda^2$ with $|\lambda| = h^{-1} >> 1$. In this article we'll be interested only in the non trapping case (see [**Bur98, Bur00**] for general results in the trapping case).

Definition 1.1. — *The energy $e \in \mathbb{R}^+$ is non trapping if every integral curve of the hamiltonian vector field associated to the symbol $p(x,\xi)$, drawn on the surface $S_e = \{(x,\xi); p(x,\xi) = e\}$ and reflecting on the boundary according to the law of the geometric optics (the so called generalized bicharacteristics) does leave any compact set in finite time (remark that the equation $p(x,\xi) = e$ implies a uniform bound on $|\xi|$ hence "leaving any compact set" means "going to the infinity in the x variable"). We refer to* [**MS82**] *for a precise definition of generalized bicharacteristics)*

If d is odd, using Lax and Phillips semi-group theory [**LP89**] (see also [**TZ00, Vai88**] for a result independent of the parity of the dimension of space), it is possible to show that the following result is a consequence of the "propagation of singularities theorem" (see [**MS82**])

Theorem 1. — *Consider K a compact subset of $\mathbb{R}^+$. Suppose that every energy level in K is non trapping. Then for any $N \in \mathbb{N}$ there exists $h_N > 0$ such that for any $0 < h < h_N$ the operator $P(h)$ has no resonance in the set*

$$\{z \in \tilde{C}; Rez \in K, |Imz| \leq Nh\log 1/h\} \tag{1.4}$$

In this text we are going to give a very natural and simple (at least when the boundary is empty) proof of the following (weaker)

Theorem 2. — *Consider K a compact subset of $\mathbb{R}^{+,*}$. Suppose that every energy level in K is non trapping. Then for any $N \in \mathbb{N}$ there exists $h_N > 0$ such that for any $0 < h < h_N$ the operator $P(h)$ has no resonance in the set*

$$\Lambda_N(h) = \{z \in \tilde{C}; Rez \in K, |Imz| \leq Nh\} \tag{1.5}$$

Furthermore for any $\chi \in C_0^\infty(\mathbb{R}^d)$ and any $N \in \mathbb{N}$ there exists $h_N > 0$ and $C > 0$ such that for any $0 < h < h_N$ and any $z \in \Lambda_N(h)$

$$\|\chi R(z,h)\chi\|_{\mathcal{L}(L^2(\Omega))} \leq \frac{C}{h} \tag{1.6}$$

Remark 1.2. — *Remark that the pole free region in theorem 2 is smaller than the one in theorem 1. However, the estimate* (1.6) *which is the natural one (it is satisfied by the free resolvent $\chi(P_0 - z)^{-1}\chi$, seeproposition 2.1) is exactly what is needed to prove exponential energy decay or asymptotic expansion of solutions to wave equations (see* [**Mor61, Mor75, MRS77**]*). One could also remark that very few regularity on the coefficients is needed for the proof of theorem 2 (C^2 coefficients or even less would be enough, see* [**Bur97a**] *for results in this direction). We'll not develop this aspect of the problem in this paper.*

Remark 1.3. — *It is well known that if χ is equal to 1 on the set where the operator P is not the standard Laplacean, the resonances are the poles of the meromorphic family of operators $\chi R(z,h)\chi$. Hence estimate* (1.6) *implies the first part of theorem 2.*

Remark 1.4. — *It is not difficult to prove that if an energy level e_0 is non trapping then every energy levels sufficient close to e_0 are non trapping. This implies that the conclusions in the theorems above hold for K a neighborhood of e_0.*

The note is organized as follows: In section 2 we give some basic results about the behaviour of the free resolvent. In section 3 we give the proof of theorem 2 in the simpler case when $\Omega = \mathbb{R}^d$. In the general case the idea is the same but the proof is technically more complicated. In section 4 we give the modifications required for the general case and the necessary references.

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2. The free resolvent

In this section we have gathered the estimates on the free resolvent $(-\Delta - \lambda^2)^{-1}$, that we shall use in the sequel. These estimates are well known, at least for λ real (especially in odd dimensions of space, see [**Zwo89**] and for a simpler argument valid also in even dimensions see [**Vod94a**] and [**Vod94b**]). The exposition in this part follows a method of M. Zworski [**Zwo00**] which has the advantage of reducing the use of explicit calculus.

2.1. Estimates on the free cut-off resolvent. — Let $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$, $\Delta = \sum_{j=1}^d \partial_{x_j}^2$, be the free resolvent in $\mathbb{R}^d$, normalized so that $\mathrm{Im}\lambda < 0$ is the resolvent half-plane.

Proposition 2.1. — *Consider $\chi \in C_0^\infty(\mathbb{R}^d)$. There exists $C > 0$ such that for any $|\lambda| > 1$, $Re\lambda \geq 0$, we have*

$$\|\chi R_0(\lambda)\chi\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \frac{\exp(C(Im\lambda)_+)}{|\lambda|} \tag{2.1}$$

In odd dimensions of space, the estimate 2.1 follows easily from the sharp Huygens principle and the relation

$$\lambda R_0(\lambda) = \int_0^{+\infty} e^{-i\lambda t} \cos\sqrt{-\Delta} dt. \tag{2.2}$$

For $\mathrm{Im}\lambda < -C < 0$ this follows from the resolvent estimate,

$$\|R_0(\lambda)\|_{L^2 \to L^2} \leq C/|\mathrm{Im}\lambda^2| \tag{2.3}$$

To prove (2.1) in all dimensions, we use the formula relating the spectral measure of $-\Delta$ to the difference of resolvents (see for exemple [**Mel94**, section 1.5]):

$$\begin{gathered} \chi(R_0(\lambda) - R_0(-\lambda))\chi = \lambda^{d-2}\, {}^t\mathbb{E}^\chi(\lambda)\mathbb{E}^\chi(-\lambda)\,, \\ \mathbb{E}^\chi(\lambda) : L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})\,, \quad \mathbb{E}^\chi(\theta, x) = \exp(i\lambda\langle x, \theta\rangle)\chi(x)\,. \end{gathered} \tag{2.4}$$

Hence, the estimate (2.1) for $\mathrm{Im}\lambda > C > 0$ will follow from the estimate for $\mathrm{Im}\lambda < -C < 0$ and another estimate:

$$\|\mathbb{E}^\chi(\lambda)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})} \leq C|\lambda|^{-\frac{d-1}{2}} e^{C|\mathrm{Im}\lambda|}\,. \tag{2.5}$$

This is the same as proving the bound for $\mathbb{E}^\chi(\lambda) \circ \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform. By Schur's lemma this in turn reduces to the estimate

$$\int_{\mathbb{S}^{d-1}} |\hat{\chi}(\xi - \lambda\omega)| d\omega \leq C|\lambda|^{1-d} e^{C|\mathrm{Im}\lambda|}\,, \quad \xi \in \mathbb{R}^d\,,$$

which follows from the Paley-Wiener theorem. In fact, we write $\xi = t(\omega)\omega + \xi^\perp(\omega)$ where $\langle \xi^\perp(\omega), \omega\rangle = 0$. We then need to estimate

$$C_M e^{C|\mathrm{Im}\lambda|} \int_{\mathbb{S}^{d-1}} (1 + |t(\omega) - \mathrm{Re}\lambda| + |\xi^\perp(\omega)|)^{-M} d\omega\,.$$

Thus we can assume that $|t(\omega) - \mathrm{Re}\lambda| < \mathrm{Re}\lambda/3$ and change the variable of integration to $y \in \mathbb{R}^{d-1}$: $\omega = (\xi - y)/|\xi - y|$, $\mathrm{Re}\lambda/3 < |\xi - y| < 2\mathrm{Re}\lambda$. Since $|D\omega/Dy| \sim (\mathrm{Re}\lambda)^{1-d}$, the estimate follows.

We now have (2.1) for $|\mathrm{Im}\lambda| > C$, $\mathrm{Re}\lambda > 0$, $|\lambda| > 1$ (the last restriction only in even dimensions to avoid the discussion of the logarithmic plane). The general case follows from the Phragmén-Lindelöf principle.

The next result is a microlocal traduction of the outgoing behaviour of the resolvent.

Proposition 2.2. — *Consider $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^d)$, with disjoint supports, and K a compact subset of $\mathbb{S}^{d-1}$. Suppose that*

$$\forall x \in supp(\chi_1), \forall y \in \; supp(\chi_2), \frac{x - y}{|x - y|} \notin K \tag{2.6}$$

then

$$(2.7)\qquad \begin{gathered}\forall N, \exists C_N, \forall u \in L^2(\mathbb{R}^d), \forall \xi \in \mathbb{R}^+ \times K, \forall \lambda; Re\lambda > 0, |\lambda| > 1,\\ |\mathcal{F}_{x\mapsto\xi}(\chi_1 R_0(\lambda)\chi_2 u)(\xi)| \leq \frac{C_N}{(|\lambda| + |\xi|)^N} e^{CIm(\lambda)_+}\end{gathered}$$

The proof for $\mathrm{Im}\lambda \leq -C < 0$ is a straightforward consequence of elliptic regularity (and the asumption (2.6) is not used here). According to (2.4), the proof for $\mathrm{Im}\lambda \geq C > 0$ follows from the similar estimate for ${}^t\mathbb{E}^{\chi_1}(\lambda)\mathbb{E}^{\chi_2}(-\lambda)u$. We have

$$(2.8)\qquad \mathcal{F}_{x\mapsto\xi}({}^t\mathbb{E}^{\chi_1}(\lambda)\mathbb{E}^{\chi_2}(-\lambda)u)(\xi) = \int_{\substack{x,y\in\mathbb{R}^d\\ \omega\in\mathbb{S}^{d-1}}} e^{i\lambda(\langle x-y,\rangle\omega - \langle x,\xi\rangle)}\chi_1(x)\chi_2(y)u(y)dxdyd\omega$$

Consider the following operator (with $(x-y)^\perp(\omega) = (x-y) - \langle(x-y),\omega\rangle\omega)$

$$(2.9)\qquad L = \frac{i\left(\langle\xi - \lambda\omega\rangle\cdot\nabla_x - \lambda\langle(x-y)^\perp(\omega),\omega\rangle\cdot\nabla_\omega\right)}{(\xi-\lambda\omega)^2 + \lambda^2((x-y)^\perp(\omega))^2},$$

which is tangential to $\mathbb{R}^d_x \times \mathbb{S}^{d-1}_\omega$. Integrating by parts N times in (2.8) with the operator L (and using (2.6)) we obtain the announced decay for $\mathrm{Im}\lambda > C$, $\mathrm{Re}\lambda > 0$, $|\lambda| > 1$. Now the general case follows as above from the Phragmén-Lindelöf principle.

3. The case $\Omega = \mathbb{R}^d$

3.1. Preliminaries. — The proof of theorem 2 is a contradiction argument (the idea of arguying by contradiction to obtain estimates and to use defect measures dates to [**Leb96**]): take $\chi \in C_0^\infty(\mathbb{R}^d)$ equal to 1 near the set where $P \neq P_0$ and fix $N \in \mathbb{N}$. Suppose that (1.6) is false. Then there exist sequences $(h_n) \to 0$, $(z_n) \in \Lambda_N(h_n)$ and $(u_n) \in L^2(\mathbb{R}^d)$ such that

$$(3.1)\qquad \|\chi R(z_n,h_n)\chi u_n\|_{L^2(\Omega)} > \frac{n}{h_n}\|u_n\|_{L^2}.$$

By dividing u_n by a suitable constant we can suppose that

$$(3.2)\qquad \|\chi R(z_n,h_n)\chi u_n\|_{L^2(\Omega)} = 1.$$

Denote by $f_n = R(z_n,h_n)\chi u_n$ the solution to

$$(3.3)\qquad (P(h_n) - z_n)f_n = \chi u_n, \qquad f_n \in \mathcal{D}(P)_{\mathrm{loc}},$$

with (from (3.1) (3.2))

$$(3.4)\qquad \|\chi u_n\|_{L^2} = o(h_n), \qquad \|\chi f_n\|_{L^2} = 1.$$

Lemma 3.1. — *The sequence (f_n) is bounded in $L^2_{loc}(\mathbb{R}^d)$.*

Take a function $\Psi \in C^\infty(\mathbb{R}^d)$ equal to 0 where P is not equal to P_0, equal to 1 for $|x|$ large and such that χ is equal to 1 on the support of $\nabla\Psi$. We have

$$(3.5)\qquad (P(h_n) - z_n)\Psi f_n = (P_0(h_n) - z_n)\Psi f_n = \Psi\chi u_n + [P(h_n),\Psi]f_n$$

Since χf_n and $(P(h_n)-z_n)f_n$ are bounded in L^2 we deduce that near the support of $\nabla\Psi$, $h_n\nabla f_n$ is also bounded in L^2. This implies that the right hand side in (3.5) is bounded in L^2 by Ch_n. Furthermore, every term in (3.5) is supported in the set where $P = P_0$. Hence

$$(3.6)\qquad \Psi f_n = R_0(z_n,h_n)(\Psi\chi u_n + [P(h_n),\Psi]f_n)$$

is, according to proposition 2.1 with $\lambda_n = \sqrt{z_n}/h_n$ and other function χ, bounded in $L^2_{\text{loc}}(\mathbb{R}^d)$.

3.2. Semi-classical measures. — Consider for $a(x,\xi) \in C_0^\infty(\mathbb{R}^{2d})$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ equal to 1 near the x-projection of the support of a the operator $\text{Op}_\varphi(a)(x, hD_x)$ defined by

$$\text{Op}_\varphi(a)(x, hD_x)f = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} a(x, h\xi)\widehat{\varphi f}(\xi) d\xi \tag{3.7}$$

The operator $\text{Op}_\varphi(a)(x, hD_x)$ is (uniformly with respect to $0 < h < 1$) bounded on L^2 and we have the following weak form of Gårding inequality

Proposition 3.2. — *For any $a \in C_0^\infty(\mathbb{R}^{2d})$*

$$a(x,\xi) \geq 0 \Rightarrow \liminf_{n\to+\infty} Re\,(\,Op(a)_\varphi(x, h_n D_x)f_n, f_n)_{L^2(\mathbb{R}^d)} \geq 0 \tag{3.8}$$

To prove this result consider for $\varepsilon > 0$, $\psi \in C_0^\infty(\mathbb{R}^{2d})$ equal to 1 near the support of a and $b = \psi\sqrt{\varepsilon + a} \in C_0^\infty(\mathbb{R}^{2d})$. Then the symbolic calculus shows

$$0 \leq \big(\,\text{Op}(b)^*_\varphi\, \text{Op}(b)_\varphi f_n, f_n\big) = (\,\text{Op}(a)_\varphi(x, hD_x)f_n, f_n)_{L^2} + \varepsilon\left(\psi^2 f_n, f_n\right)_{L^2} + \mathcal{O}(h_n) \tag{3.9}$$

hence

$$\liminf_{n\to+\infty} \text{Re}\,(\,\text{Op}(a)_\varphi(x, hD_x)f_n, f_n)_{L^2(\mathbb{R}^d)} + \varepsilon \liminf_{n\to+\infty} \|\psi f_n\|^2 \geq 0 \tag{3.10}$$

When $\varepsilon > 0$ tends to 0 we get proposition 3.2.

Remark that by the symbolic calculus the operator $\text{Op}_\varphi(a)(x, hD_x)$ is, modulo operators bounded in L^2 by $\mathcal{O}(h^\infty)$, independent of the choice of the function φ. For conciseness, we'll dropp in the sequel the index φ. As in [**GL93**] (see also [**Bur97b**]) we can prove the following

Proposition 3.3. — *There exist a subsequence (n_k) and a positive Radon measure on $\mathbb{R}^{2d}$, μ, such that for any $a \in C_0^\infty(\mathbb{R}^{2d})$*

$$\lim_{k\to+\infty} (Op(a)(x, h_{n_k} D_x)f_{n_k}, f_{n_k})_{L^2} = \langle \mu, a(x,\xi)\rangle \tag{3.11}$$

The idea for extracting such a sequence is to fix a and consider the bounded sequence $(L(a)_n) = (\text{Op}(a)(x, h_n D_x)f_n, f_n)_{L^2}$. By compactness we can extract a subsequence which converges. Iterating this process for a sequence (a_j) dense in C_0^∞, we obtain, by diagonal extraction, a sequence (f_{n_k}) such that the limit exists for any a_j. By (3.8) the limit defines a positive functional on a dense subset of C_0^∞ (hence this limit is continuous for the C^0_{comp} topology). It is consequently a Radon measure and the limit (3.11) exists for any $a \in C_0^\infty$.

We can extract subsequences further so that

$$\begin{aligned} \text{Re} z_{n_k} &\to e \in K \\ h_{n_k}^{-1} \text{Im} z_{n_k} &\to \alpha \in [-N, 0] \end{aligned} \tag{3.12}$$

For the sake of conciseness we shall note again $(f_n), (u_n), (z_n)$ the extracted subsequences.

The measure μ represents at points (x_0, ξ_0) the oscillations of the sequence f_{n_k} at point x_0 and scale ξ_0/h_{n_k}. The oscillations at frequencies smaller than $h_{n_k}^{-1}$ are concentrated in $\{\xi_0 = 0\}$ whereas the oscillations at higher frequencies are lost. The following lemma shows that nothing is lost and that the measure μ contains the relevant information about the sequence (f_{n_k}):

Lemma 3.4. — *The sequence (f_n) is h_n-oscillating:*

$$\forall \psi \in C_0^\infty(\mathbb{R}^d), \lim_{R\to+\infty} \limsup_{n\to+\infty} \int_{|\xi|\geq R/h_n} |\widehat{\psi f_n}(\xi)|^2 d\xi = 0 \tag{3.13}$$

Indeed, this result is a simple consequence of the equation

$$(P(h_n) - z_n) f_n = \chi u_n \text{ bounded in } L^2 \tag{3.14}$$

and of the fact that the operator $(P(h_n) - z_n)$ is uniformly elliptic in the semi-classical meaning for $|\xi| \geq R$ with R large enough.

From lemma 3.4 we deduce easily that (according to (3.4))

$$\langle \mu, \chi(x) \times 1_\xi \rangle = \lim_{n\to+\infty} \int |\chi f_n|^2 = 1 \neq 0 \tag{3.15}$$

We'll get the announced contradiction if we show that $\mu \equiv 0$.

3.3. Invariance of the defect measure. — Take $a \in C_0^\infty$ and consider first

$$\begin{aligned} 0 &= ((a(x, h_n D_x)(P(h_n) - z_n)) f_n, f_n)_{L^2} \\ &= ((a(x, h_n D_x)(P(h_n) - e)) f_n, f_n)_{L^2} + o(1) \end{aligned} \tag{3.16}$$

Taking into account that the operator $\mathrm{Op}(a)(x, hD_x)(P - e)$ is equal to $\mathrm{Op}(a \times (p - e))(x, hD_x)$ modulo $\mathcal{O}(h)$ in $\mathcal{L}(L^2_{\mathrm{loc}}; L^2_{\mathrm{comp}})$ and passing to the limit in (3.16) we obtain

$$\langle \mu, a(x, \xi)(p - e)(x, \xi) \rangle = 0 \tag{3.17}$$

from which we deduce that the measure μ is supported in the characteristic set of the operator:

$$\{(x, \xi); p(x, \xi) = e\} \tag{3.18}$$

Consider now

$$h_n^{-1} ([a(x, D_x), P(h_n) - z_n] f_n, f_n)_{L^2} \tag{3.19}$$

taking into account that the adjoint of $P(h) - z$ is $P(h) - \overline{z}$ we obtain from (3.4) (3.12)

$$\begin{aligned} &h_n^{-1} ([a(x, D_x), P(h_n) - z_n] f_n, f_n)_{L^2} \\ &\quad = h_n^{-1} ((P(h_n) - z_n) a(x, h_n D_x) f_n, f_n)_{L^2} + o(1) \\ &\quad = h_n^{-1} ((\overline{z_n} - z_n) a(x, h_n D_x) f_n, f_n)_{L^2} + o(1) \\ &\quad = (-2i(\alpha + o(1)) a(x, h_n D_x) f_n, f_n)_{L^2} + o(1) \end{aligned} \tag{3.20}$$

Passing to the limit in (3.20) and taking into account that the operator

$$h_n^{-1} [a(x, D_x), P(h_n) - z_n] \tag{3.21}$$

is equal to

$$\frac{1}{i} \{a, p\}(x, hD_x) + O(h)_{\mathcal{L}(L^2_{\mathrm{loc}}; L^2_{\mathrm{comp}})} \tag{3.22}$$

where the Poisson bracket of a and p, $\{a, p\}$, is defined by

$$\{a, p\} = \nabla_\xi a \cdot \nabla_x p - \nabla_x a \nabla_\xi p, \tag{3.23}$$

we obtain the following:

$$\langle \mu, \{a, p\} \rangle = \langle \mu, 2\alpha a \rangle \tag{3.24}$$

or equivalently (with H_p the Hamiltonian vector field of p)

$$H_p(\mu) - 2\alpha\mu = 0 \tag{3.25}$$

The equation (3.25) is a transport equation for the measure μ, with a damping term. This transport equation (and (3.18)) implies that the measure μ is (modulo the damping coefficient) invariant along the integral curves of the vector field H_p drawn on the surface $\{p - e = 0\}$.

According to the non trapping condition, these integral curves (parametrized by the variable $s \in \mathbb{R}$) do leave any compact set in x both for $s << 0$ and $s >> 0$.

The following result is a microlocal interpretation of the "outgoing" terminology: roughly speaking, any outgoing solution to Helmoltz equation is microlocally concentrated near "outgoing" directions. Hence the measure is equal to 0 near incoming directions.

Proposition 3.5. — *Take R large enough such that $supp(\chi) \subset B(0, R-1)$. Then the measure μ is null close to the incoming points:*

$$Inc = \{(x,\xi); |x| \geq R, x \cdot \xi \leq 0\} \tag{3.26}$$

Indeed, this result is a direct consequence of proposition 2.2 and (3.6) (with $\lambda_n = \sqrt{z_n}/h_n$). The contradiction is now straightforward: consider a ray starting at a point (x_0, ξ_0). Suppose that $|x_0| < R$. When the parameter along the ray tends to $-\infty$, the ray tends to the infinity (in x variables); hence it leaves the ball $B_x(0,R)$ at a point (x_1, ξ_1). Since for this point $\dot{x} = 2\xi$, the corresponding point is an incoming point where the measure μ is null. Since the measure μ is invariant along this ray; it is identically null which contradicts (3.15).

Remark 3.6. — *In this proof, the nature of free problem ($-h^2\Delta - e$ in our particular case) intervenes only for 2 purposes: first to obtain the L^2_{loc} boundedness of (f_n) from the boundedness of (χf_n), the only fact used is that the free outgoing resolvent satisfies the estimates we want to prove for the perturbed problem; and the second point where the nature of the free problem intervenes is to prove that the measure is null near incoming directions.*

4. The general case

In this section we are going to outline the modifications needed to perform the same proof as above in the case of a boundary value problem. The proof is again a contradiction argument. Suppose that we have two sequences, $(f_n), (u_n)$ satisfying (3.2), (3.3) and (3.4) and (f_n) satisfies the boundary condition $f_n \mid_{\partial\Omega} = 0$. We have chosen to deal with Dirichlet boundary condition. The general case of systems of boundary problems satisfying strong Lopatinski conditions is treated (in a different framework) in [**BL99**].

4.1. Geometry. — Denote by ${}^bT\overline{\Omega}$ the bundle of rank d whose sections are the vector fields tangent to $\partial\Omega$, ${}^bT^*\overline{\Omega}$ the dual bundle (Melrose's compressed cotangent bundle) and $j : T^*\overline{\Omega} \to {}^bT^*\overline{\Omega}$ the canonical map. In any coordinate system where $\Omega = \{x = (x_n > 0, x')\})$, the bundle ${}^bT\overline{\Omega}$ is generated by the fields $\frac{\partial}{\partial x'}$, $x_n \frac{\partial}{\partial x_n}$ and j is defined by

$$j(x_n, x', \xi_n, \xi') = (x_n, x', v = x_n\xi_n, \xi'). \tag{4.1}$$

Denote by Car$P$ the characteristic manifold of $P - e$ and Z its projection

$$\mathrm{Car}P = \left\{(x,\xi) \in T^*\mathbb{R}^d \mid_{\overline{\Omega}}; p(x,\xi) = e\right\}, \qquad Z = j(\mathrm{Car}P) \tag{4.2}$$

The set Z is a locally compact metric space.

Consider, near a point $z_0 \in \partial\Omega$ a geodesic system of coordinates for which (near z_0) $\Omega = \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^{d-1}\}$ and the operator P has the form

$$P = -h^2\partial_{x_n}^2 + \alpha(x_n,x')h\partial_{x_n} + h\partial_{x_n}\alpha(x_n,x') + R(x_n,x',hD_{x'}) \tag{4.3}$$

with R a second order tangential operator ans (since P is self adjoint) $i\alpha(x_n,x') \in \mathbb{R}$. Conjugating the operator P by $e^{\int_{x_n}\alpha(x_n,x')/h}$ (which at the L^2 level is an isometry), and modifying R, we can suppress the $\alpha(x_n,x')h\partial_{x_n} + h\partial_{x_n}\alpha(x_n,x')$ terms. We recall now the usual decomposition of $T^*\partial\Omega$ (in this coordinate system). Denote by $r(x_n,x',\xi')$ the semi-classical principal symbol of R and $r_0 = r\mid_{x=0}$. $T^*\partial\Omega$ is the disjoint union of $\mathcal{E}\cup\mathcal{G}\cup\mathcal{H}$ with

$$\mathcal{E} = \{r_0 < e\}, \mathcal{G} = \{r_0 = e\}, \mathcal{H} = \{r_0 > e\} \tag{4.4}$$

In $\mathcal{G}$ we distinguish between the *diffractive* points $\mathcal{G}^{2,+} = \{r_0 = 0, \partial_x r\mid_{x=0} < 0\}$ and the *gliding* points $\mathcal{G}^- = \{r_0 = 0, \partial_x r\mid_{x=0} \geq 0\}$.

4.2. Semi-classical measures. — Consider functions $a = a_i + a_\partial$ with $a_i \in C_0^\infty(\Omega\times\mathbb{R}^d)$ and $a_\partial(x_n,x',\xi') \in C_0^\infty(\mathbb{R}^{2d-1}$. Such symbols are quantized in the following way: Take $\varphi_i \in C_0^\infty(\Omega)$ (resp $\varphi_\partial \in C_0^\infty(\mathbb{R}^d)$) equal to 1 near the x-projection of $\mathrm{supp}(a_i)$ (resp the x-projection of $\mathrm{supp}(a_\partial)$) and define

$$\begin{aligned}\mathrm{Op}_{\varphi_i,\varphi_\partial}(a)(x,hD_x)f = {}& \frac{1}{(2\pi h)^d}\int e^{i(x-y)\cdot\xi/h}a_i(x,\xi)\varphi_i(y)f(y)dyd\xi \\ &+ \frac{1}{(2\pi h)^{d-1}}\int e^{i(x'-y')\cdot\xi'/h}a_i(x_n,x',\xi)\varphi_i(y_n,y')f(y_n,y')dy'd\xi'\end{aligned} \tag{4.5}$$

Remark that according to the symbolic semi-classical calculus, the operator $\mathrm{Op}_{\varphi_i,\varphi_\partial}(a)$ does not depend on the choice of functions $\varphi_i,\varphi_\partial$, modulo operators of norms $O(h^\infty)$ on $\mathcal{L}(L^2_{\mathrm{loc}};L^2_{\mathrm{comp}})$.

Denote by $\mathcal{A}$ the space of the operators which are a finite sum of operators obtained as above in suitable coordinates systems near the boundary. Denote, for $A\in\mathcal{A}$, by $a = \sigma(A)$ the semiclassical symbol of the operator A. For such functions a we can define $\kappa(a) \in C^0(Z)$ by

$$\kappa(a)(\rho) = a(j^{-1}(\rho)). \tag{4.6}$$

(the value is independant of the choice of $j^{-1}(\varrho)$). The set

$$\{\kappa(a), a = \sigma(A), A \in \mathcal{A}\} \tag{4.7}$$

is a locally dense subset of $C^0(Z)$.

The same argument as in section 3 shows that if $a_i \in C_0^\infty(\Omega\times\mathbb{R}^d)$ is equal to 0 near $\mathrm{Car}P$ then

$$\lim_{n\to+\infty}(\mathrm{Op}(a_i)(x,hD_x)f_n,f_n)_{L^2} = 0 \tag{4.8}$$

and the analysis of the boundary value problem shows easily that if $a_\partial \in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^{d-1})$ is equal to 0 near Z (i.e. a_∂ is supported in the elliptic region) then

$$\lim_{k\to+\infty}\big(\mathrm{Op}(a_\partial)(x_n,x',hD_{x'})f_k,f_k\big)_{L^2} = 0 \tag{4.9}$$

The analog of proposition 3.3 is the following:

Proposition 4.1. — *There exists a subsequence (k_p) and a Radon positive measure mesure μ on Z such that*

$$\forall Q \in \mathcal{A} \quad \lim_{p\to\infty} (Qf_{k_p}, f_{k_p})_{L^2} = \langle \mu, \kappa(\sigma(Q))\rangle. \tag{4.10}$$

The proof of this result in the interior of Ω is the same as in section 2 and near a boundary point, it relies on Gårding inequality for tangential operators (see G. Lebeau [**Leb96**] for a proof in the classical context and [**GL93, Bur97b**] for the semi-classical construction). As before, we note again (f_k) the extracted sequence.

Proposition 4.2 **(First properties of the measure μ).** —

$$\mu(\mathcal{H}) = 0 \tag{4.11}$$

$$\limsup_{k\to+\infty} \left| \left(Op(a)\frac{1}{i}\partial_x f_k, f_k\right)_{L^2} \right| \leq \sup_{\varrho\in\ supp(a)} |r-e|^{1/2}|a| \tag{4.12}$$

The relation (4.11) is a simple consequence of the micro-local analysis of the boundary problem near a point $\varrho_0 \in \mathcal{H}$. To prove (4.12) compute

$$\begin{aligned}
\Big|\Big(\operatorname{Op}(a)\frac{1}{i}h_k\partial_x f_k, f_k\Big)_{L^2}\Big| &\leq \| \operatorname{Op}(a)\frac{1}{i}h_k\partial_x f_k\|_{L^2}\|f_k\|_{L^2} \\
&\leq \Big(\frac{-1}{i}h_k\partial_x \operatorname{Op}(a)^* \operatorname{Op}(a)\frac{1}{i}h_k\partial_x f_k, f_k\Big)_{L^2}^{1/2}\|f_k\|_{L^2} \\
&\leq \big(\operatorname{Op}(a)^* \operatorname{Op}(a) h_k^2\partial_x^2 f_k, f_k\big)_{L^2}^{1/2} \|f_k\|_{L^2} + o(1) \\
&\leq \big(\operatorname{Op}(a)^* \operatorname{Op}(a)(P-e-R+e) f_k, f_k\big)_{L^2}^{1/2} \|f_k\|_{L^2} + o(1) \\
&\leq \big(\operatorname{Op}(a)^* \operatorname{Op}(a)(-R+e) f_k, f_k\big)_{L^2}^{1/2} \|f_k\|_{L^2} + o(1)
\end{aligned} \tag{4.13}$$

and we obtain

$$\limsup_{k\to+\infty}\Big|\Big(\operatorname{Op}(a)\frac{1}{i}\partial_x f_k, f_k\Big)_{L^2}\Big| \leq |\langle \mu, -a^2(r-e)\rangle|^{1/2} \leq C \sup_{\varrho\in\ \operatorname{supp}(a)} |a||r-e|^{1/2} \tag{4.14}$$

The following result is the analog of lemma 3.1:

Lemma 4.3. — *The sequence (f_k) is h_k-oscillating:*

$$\begin{aligned}
&\forall \psi \in C_0^\infty(\Omega) \lim_{R\to+\infty} \limsup_{k\to+\infty} \int_{|\zeta|\geq R/h_k} |\widehat{\psi f}(\zeta)|^2 d\xi = 0 \\
&\forall \psi \in C_0^\infty(\overline{\Omega}) \lim_{R\to+\infty} \limsup_{k\to+\infty} \int_{|\eta|\geq R/h_k, x>0} |\widehat{\psi f}(x,\eta)|^2 dx d\eta = 0
\end{aligned} \tag{4.15}$$

The proof of this result is again a straightforward consequence of the equation.

4.3. Invariance of the defect measure. —

Proposition 4.4. — *Consider $q \in C^\infty(Z)$. Then the measure μ satisfies*

$$\langle \mu, \{p,q\} \; - 2\alpha q\rangle = 0 \tag{4.16}$$

Remark 4.5. — *For $q \in C^\infty(Z)$, in general $\{p,q\} = -2\xi_n \partial_{x_n} + \{r,q\}$ is not a function defined on Z (because of the ξ_n dependance). To obtain a (μ-integrable) function on Z, we have to take the convention*

$$\{p,q\} \stackrel{def}{=} -2\xi_n \partial_{x_n} q 1_{\varrho \notin \mathcal{H}} + \{r,q\}. \tag{4.17}$$

The proof of proposition 4.4 is simply integration by parts (and some carefull study of the terms arising). We give it below: Since in Ω, the equation (4.16) is simply (3.24), we restrict the study to the case where q is supported near a point $\varrho_0 \in T^*\partial\Omega$. From Malgrange preparation theorem, there exist functions $q_0(x_n, x', \xi'), q_1(x_n, x', \xi') \in C^\infty$ such that

$$q \mid_{\mathrm{Car} P} = q_0 \mid_{\mathrm{Car\ p}} + \xi_n q_1 \mid_{\mathrm{Car\ p}} . \tag{4.18}$$

Put $Q = \mathrm{Op}(q_0) + \mathrm{Op}(q_1)\frac{1}{i} h \partial_{x_n}$ and compute

$$h_k^{-1} \left((P - z_k)^* Q - Q(P - z_k) f_k, f_k\right)_{L^2} . \tag{4.19}$$

Two integrations by part, (4.3) and the boundary condition $f_k \mid_{x_n=0} = 0$ show that

$$\begin{aligned} &h_k^{-1} \left((P - z_k)^* Q - Q(P - z_k) f_k, f_k\right)_{L^2} \\ &\qquad = h_k^{-1} \left(P - z_k)^* Q f_k, f_k\right)_{L^2} + o(1) \\ &\qquad\qquad = -\left(Q_1 \mid_{x_n=0} h_k \partial_{x_n} f_k \mid_{x_n=0}, h_k \partial_{x_n} f_k \mid_{x_n=0}\right)_{L^2(\mathbb{R}^{d-1}_{x'})} \end{aligned} \tag{4.20}$$

On the other hand

$$\begin{aligned} h_k^{-1} \left((P - z_k)^* Q - Q(P - z_k) f_k, f_k\right)_{L^2} &= h_k^{-1} \left((P - z_k)^* Q f_k, f_k\right)_{L^2} + o(1) \\ &= h_k^{-1} \left(([(P - z_k), Q] + 2ih_k \alpha Q + o(h_k))\, f_k, f_k\right)_{L^2} \\ &= \left(-i(A_0 + A_1 \frac{1}{i}\partial_{x_n} + A_2(P - e) + o(1)) f_k, f_k\right)_{L^2} \end{aligned} \tag{4.21}$$

Where A_0, A_1 and A_2 are tangential operators, and on Car P we have

$$a_0 + a_1 \xi_n = \{p,q\} - 2\alpha q. \tag{4.22}$$

From (4.11) we deduce that μ-almost everywhere

$$a_0 + a_1 \xi_n 1_{x_n > 0} = \{p,q\} - 2\alpha q. \tag{4.23}$$

Passing to the limit in (4.21) we obtain

$$\begin{aligned} &\lim_{k \to +\infty} \left((A_0 + A_1 \frac{1}{i}\partial_{x_n} + A_2(P - e) + o(1)) f_k, f_k\right)_{L^2} \\ &\qquad\qquad = \langle \mu, a_0 \rangle + \lim_{k \to +\infty} \left(A_1 \frac{1}{i}\partial_{x_n} f_k, f_k\right)_{L^2} \end{aligned} \tag{4.24}$$

Take $\varepsilon > 0$ and $\varphi \in C_0^\infty(]-1,1[)$ equal to 1 near 0. Note

$$\begin{aligned} A_1 = (1 - \varphi(\frac{x_n}{\varepsilon}))A_1 &+ \mathrm{Op}\left(\varphi(\frac{x_n}{\varepsilon})\varphi(\frac{r(x_n, x', \xi') - e}{\varepsilon})\right) A_1 \\ &+ \mathrm{Op}\left(\varphi(\frac{x_n}{\varepsilon})(1 - \varphi(\frac{r(x_n, x', \xi') - e}{\varepsilon}))\right) A_1 \end{aligned} \tag{4.25}$$

The first term in the right hand side of (4.25) is supported in the interior of Ω; its contribution to the limit in (4.21) is equal to

$$\langle \mu, (1-\varphi(\frac{x_n}{\varepsilon}))a_1\xi_n\rangle \tag{4.26}$$

The contribution of the second term is, according to (4.12), smaller than

$$\sup_{\varrho\in \text{supp}(\varphi(\frac{x_n}{\varepsilon})(\varphi(\frac{r(x_n,x',\xi')-e}{\varepsilon})))} |r-e|^{1/2}|a_1| \leq C\varepsilon^{1/2} \tag{4.27}$$

and the contribution of the last term smaller than

$$\begin{aligned} \|h_k\partial_{x_n}f_k\| \Big\| A_1^*\text{Op}\left(\varphi(\frac{x_n}{\varepsilon})(1-\varphi(\frac{r(x_n,x',\xi')-e}{\varepsilon}))\right)^* f_k\Big\| \\ \leq C\langle \mu, a_1^2\varphi^2(\frac{x_n}{\varepsilon})(1-\varphi(\frac{r(x_n,x',\xi')-e}{\varepsilon}))^2\rangle^{1/2}+o(1) \end{aligned} \tag{4.28}$$

Passing to the limit $\varepsilon\to 0$ we obtain that the contribution of the first term is equal to

$$\langle \mu, a_1\xi_n 1_{x_n>0}\rangle = \langle \mu, a_1\xi_n 1_{\varrho\notin\mathcal{H}}\rangle, \tag{4.29}$$

the contribution of the second term is (according to (4.27)) equal to 0 and the contribution of the last term is smaller than

$$\langle \mu, a_1^2 1_{x_n=0}1_{r\neq e}\rangle = \langle \mu, a_1^2 1_{\varrho\in\mathcal{H}}\rangle = 0. \tag{4.30}$$

Finally we have proved

$$\lim_{n\to+\infty} -(Q_1\mid_{x_n=0} h_k\partial_{x_n}f_k\mid_{x_n=0}, h_k\partial_{x_n}f_k\mid_{x_n=0})_{L^2(\mathbb{R}^{d-1}_{x'})} = -i\langle \mu, \{p,q\}-2\alpha q\rangle \tag{4.31}$$

But, if $q\in C^0(Z)$, the function $q\mid_{x_n=0}$ is independant of ξ_n Hence $q_1\mid_{x_n=0}=0$ and we get proposition 4.4.

***Proposition 4.6* (see [BG97]and [Mil97]).** — *We have*

$$\mu(\mathcal{G}^{2,+})=0 \tag{4.32}$$

Consider a point $\varrho_0\in\mathcal{G}^{2,+}$. Apply (4.31) to a family of functions $q=\xi_n\times q_\varepsilon$ with

$$q_\varepsilon = \varphi(\frac{x_n}{\varepsilon^{1/3}})\varphi(\frac{(r(x_n,x',\xi')-e)}{\varepsilon})a(x_n,x',\xi'). \tag{4.33}$$

Then we get

$$\begin{aligned} \lim\nolimits_{n\to+\infty}\left(\varphi(\tfrac{(r(0,x',h_kD_{x'})-e)}{\varepsilon})a(0,x',h_kD_{x'})h_k\partial_{x_n}f_k\mid_{x_n=0}, h_k\partial_{x_n}f_k\mid_{x_n=0}\right)_{L^2(\mathbb{R}^{d-1}_{x'})} \\ = \langle \mu, -2\xi_n^2\partial_{x_n}(\varphi(\frac{x_n}{\varepsilon^{1/3}})\varphi(\frac{(r(x_n,x',\xi')-e)}{\varepsilon})a(x_n,x',\xi'))\rangle \\ -\langle \mu, \partial_{x_n}r\varphi(x_n/\varepsilon^{1/3})\varphi((r(x_n,x',\xi')-e)/\varepsilon)a(x_n,x',\xi')+\xi_n\{r,q_\varepsilon\}'+2\xi_n\alpha q_\varepsilon\rangle \end{aligned} \tag{4.34}$$

where $\{r,q\}'$ is the Poisson bracket with respect to the x',ξ' variables. Remark that on the support of the measure μ, $\xi_n^2=e-r(x_n,x',\xi')$. Hence we can apply the dominated convergence theorem and obtain that the right hand side in (4.34) tends to

$$\langle \mu, -\partial_{x_n}r(0,x',\xi')a(x_n,x',\xi')1_{x_n=0}1_{r=e}\rangle = \langle \mu, -\partial_{x_n}r(0,x',\xi')a(x_n,x',\xi')1_{\rho\in\mathcal{G}}\rangle \tag{4.35}$$

Remark that $\partial_{x_n} r < 0$ at the point ϱ_0. If $\partial_{x_n} r < 0$ on the support of a then the right hand side in (4.35) is non positive. On the other hand by Gårding inequality the limit on the left hand side is non negative. Both sides are then equal to 0. this implies proposition 4.2.

It is now possible to prove as in [**BL99**, theorème 1], by measure theory methods, that the invariance of the measure μ along the bicharacteristic flow is equivalent to propositions 4.4 and 4.6 (in fact the proof of this result is presented in [**BL99**, section 3.3] for classical measures, in the more general context of systems, but the proof for semi-classical measures is the same word by word). The proof of this equivalence is in two steps. First one shows that if a measure is invariant then it satisfies propositions 4.4 and 4.6. The proof of proposition 4.4 is straightforward whereas the proof of proposition 4.6 uses simply that $\mathcal{G}^{2,+}$ is transversal to the flow. The second step is the proof of the fact that propositions 4.4 and 4.6 imply the invariance of the measure. This last result is the measure analog of R. Melrose and J. Sjöstrand's theorem of propagation of singularities. It is much simpler in this framework but remains a litle bit technical. Let's give an outline of this proof: There is a hierarchy in the points:

$$Z = \mathcal{E} \cup \bigcup_{j=0}^{\infty} Z^j \tag{4.36}$$

with

$$\begin{aligned}
Z^0 &= Z \cap (x_n > 0)\\
Z^1 &= \mathcal{H} = \big\{(x', \xi'); r_0(x', \xi') > e\big\}\\
Z^2 &= \big\{(x', \xi'); r_0(x', \xi') = 0, r_1(x', \xi') \neq 0\big\}\\
&\dots\\
Z^{k+3} &= \big\{(x', \xi'); r_0(x', \xi') = 0, H^j_{r_0}(r_1) = 0, \forall j \leq k; H^{k+1}_{r_0}(r_1) \neq 0\big\}\\
&\dots\\
Z^{\infty} &= \big\{(x', \xi'); r_0(x', \xi') = 0, H^j_{r_0}(r_1) = 0, \forall j\big\}.
\end{aligned}$$

$$Z^2 = \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}, \quad \mathcal{G}^{2,\pm} = \big\{(x', \xi') \in Z^2, \pm r_1(x', \xi') > 0\big\}. \tag{4.37}$$

The asumption that Ω has no infinite order contact with its boundary is simply the fact that $Z^{\infty} = \emptyset$. The measure μ is null near any point ϱ_0 in $\mathcal{E}$, near a point in Z_0 the invariance theorem has been proved in section 3. Near a point in Z^1 (a hyperbolic point) it can be proved by constructing the parametrix (or some symetry arguments). Of course the most difficult cases are the glancing case. Now suppose that $\varrho_0 \in Z^j$ to prove the invariance result near ϱ_0 the idea is to proceed by induction on the index of the layer Z^j. At each step, since the invariance result is assumed to be true near points in Z^k, $k < j$, one considers the part of the measure μ for which the result is already known (i.e. the measure μ restricted to points whose flow do not encounter $Z^l, l \geq j$ near the point ϱ_0). Denote by μ_j this part of the measure μ. Since μ_j is by the induction asumption, invariant, it does satisfy (according to the first step) the same relations as μ (4.16), (4.32). Hence $\mu - \mu_j$ does also satisfy these relations. We are going to argue on $\mu - \mu_j$ which is supported along generalized bicharacteristics which encounter $Z^l; l \geq j$ (and because there is no point in Z^l, $l > j$ near a point in Z^j, these bicharacteristics encounter Z^j). A transversality argument shows that any such bicharacteristic encounter (locally) at most only one point in Z^j. Then the measure is (by the induction asumption) invariant before and after that point and the jump formula (to show that there is no jump) in conjunction with (4.16) and (4.32) near $\mathcal{G}^{2,+}$ implies the invariance of the measure $\mu - \mu_j$. One can remark that the case $\varrho_0 \in \mathcal{G}^{2,+}$ is inspired from the study by V. Ivrii of the propagation of singularities near diffractive points (see [**Ivr98**], [**Hör85**, Chap. 24]).

The rest of the proof of the estimate (1.6) in the case of a boundary value problem is now the same as in section 3.

Remark 4.7. — *As pointed out in remark 3.6, the nature of the free problem intervenes only for two purposes: First we use that the free problem satisfies the estimate we want to prove for the perturbed problem. About this asumption remark that (at least on the real axis) it is a consequence of the limiting absorbtion principle and Mourre estimates (see* [**GM88, TZ00**]*). The second result on the free resolvent we use is that the outgoing solutions have some decay in the incoming region (which implies that the measure is equal to* 0 *in the incoming region). This result is also a consequence of Mourre estimates and it follows from a slight modification of the argument in* [**TZ00**]. *These two facts show that our strategy applies to long range perturbation of the Laplace operator and gives an estimate as in* (1.6) *(for real z of course since resonances can non longer be defined).*

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