Strichartz estimates and the Cauchy problem for the gravity water waves equations

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Abstract

This memoir is devoted to the proof of a well-posedness result for the gravity water waves equations, in arbitrary dimension and in fluid domains with general bottoms, when the initial velocity field is not necessarily Lipschitz. Moreover, for two-dimensional waves, we can consider solutions such that the curvature of the initial free surface does not belong to L^2.

The proof is entirely based on the Eulerian formulation of the water waves equations, using microlocal analysis to obtain sharp Sobolev and Hölder estimates. We first prove tame estimates in Sobolev spaces depending linearly on Hölder norms and then we use the dispersive properties of the water-waves system, namely Strichartz estimates, to control these Hölder norms.

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Chapter 1

Introduction

In this paper we consider the free boundary problem describing the motion of water waves over an incompressible, irrotational fluid flow. We are interested in the study of the possible emergence of singularities and would like to understand which quantities govern the boundedness of the solutions.

We shall work in the Eulerian coordinate system where the unknowns are the velocity field v and the free surface elevation η. Namely, consider a simply connected domain, Ω, located between a fixed bottom Γ and a free unknown surface Σ, given as a graph

$$\Sigma = \{(x,y) \in \mathbf{R}^d \times \mathbf{R}\,;\, y = \eta(t,x)\}.$$

In this framework, water waves are described by a system of coupled equations: the incompressible Euler equation in the interior of the domain and a kinematic equation describing the deformations of the domain. Moreover, the velocity will be assumed to be irrotational. Thus we are interested in the following system

$$\begin{cases} \partial_t v + v\cdot\nabla_{x,y} v + \nabla_{x,y}(P+gy) = 0 \text{ in } \Omega,\\ \operatorname{div}_{x,y} v = 0,\ \operatorname{curl}_{x,y} v = 0 \text{ in } \Omega,\\ v\cdot\nu = 0 \text{ on } \Gamma,\\ \partial_t\eta = (1+|\nabla_x\eta|^2)^{1/2}\, v\cdot n \text{ on } \Sigma,\\ P = 0 \text{ on } \Sigma, \end{cases} \tag{1.1}$$

augmented with initial data (η_0, v_0) at time $t=0$. Here n (resp. ν) is the outward unit normal to the free surface (resp. the bottom), P is the pressure and $g>0$ is the acceleration of gravity.

The first equation is the usual Euler equation in presence of a gravity force directed along the y coordinate, the third one is the solid wall boundary condition at the bottom, the fourth equation describes the movement of the interface under the action of the fluid and ensures that the fluid particles initially at the interface remain at the interface, while the last one expresses the continuity of the pressure through the interface (no tension surface).

In [3] we proved that the Cauchy problem for the system (1.1) is well-posed under the minimal assumptions that insure that at time $t = 0$, in terms of Sobolev embeddings, the initial velocity v_0 is Lipschitz up to the boundary (see also the improvement to velocities whose derivatives are in BMO by Hunter-Ifrim-Tataru [37]). This Lipschitz regularity threshold for the velocity appears to be very natural. However, it has been known for some time (see the work by Bahouri-Chemin [11] and Tataru [59]) that taking benefit of dispersive effects, it is possible to go beyond this threshold on some quasilinear wave-type systems. The goal of this article is to show that such an improvement is also possible on the water-waves system.

To describe our main result, we need to introduce some notations. For (η, v) as above, denote by $V = v_x \mid_\Sigma, B = v_y \mid_\Sigma$ the horizontal and vertical components of the velocity field at the interface. Since v is irrotational, and incompressible, we can write $v = \nabla_{x,y}\phi$, with $\Delta_{x,y}\phi = 0$ in the domain. We set $\psi = \phi \mid_\Sigma$. We shall prove the following result (see Section 1.4 for a more complete statement).

Theorem. *Let*

$$s > 1 + \frac{d}{2} - \mu, \qquad \textit{with} \begin{cases} \mu = \frac{1}{24} & \textit{if } d = 1, \\ \mu = \frac{1}{12} & \textit{if } d \geq 2, \end{cases}$$

and set $\mathcal{H}^s = H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times (H^s)^d \times H^s$ *where* $H^\sigma = H^\sigma(\mathbf{R}^d)$.

Then for any initial data (η_0, v_0) *such that* $(\eta_0, \psi_0, V_0, B_0) \in \mathcal{H}^s$ *and satisfying the Taylor sign condition, there exist* $T > 0$ *and a solution* (η, v) *of the water-waves system* (1.1) *(unique in a suitable space) such that* $(\eta, \psi, V, B) \in C^0((-T, T); \mathcal{H}^s)$.

Remark 1.1.

- The Taylor sign condition expresses the fact that the pressure increases going from the air into the fluid domain ($\partial_y P|_\Sigma \leq c < 0$). It is always satisfied when there is no bottom (see Wu [65]) or for small perturbations of flat bottoms (see Lannes [43]). Notice that the water-waves system is ill-posed when this condition is not satisfied (see Ebin [33]).
- The curvature κ_0 of the initial free surface involves two derivatives of η_0. Hence, we have $\kappa_0 \in H^{s-\frac{3}{2}}$ which, according to our assumption on s, can be negative in dimension 1. This shows that, when $d = 1$, no control on the $L^2(\mathbf{R})$-norm of the curvature of the initial free surface is to be assumed.
- In any dimension, in view of Sobolev embeddings, our assumptions require that V_0, B_0 belong to the Hölder space $W^{1-\mu,\infty}$. Consequently, our result applies to initial data for which the initial velocity field is not Lipschitz.
- In this article we consider only gravity water waves. We refer to the works [31, 32] by de Poyferré and Nguyen for the case with surface tension.
- At the end of this introduction, we explain the strategy of the proof. The main restriction on μ comes from the fact that we use a Strichartz estimate with loss of derivative (this means a loss compared to the estimate which holds for the linearized equation). This in turn comes from the fact that we prove a

dispersive estimate which holds only on a small time interval whose length is taylored to the size of the frequency.

To prove this result, we follow several steps. The first one is (roughly speaking) to reduce the water waves equations to a quasilinear wave type equation of the form

$$(\partial_t + T_V \cdot \nabla_x + iT_c)u = f, \tag{1.2}$$

where T_V is the paramultiplication by V, T_c is a paradifferential operator of order $\frac{1}{2}$ and almost self-adjoint (such that $T_c^* - T_c$ is of order 0) and f is a remainder term in the paradifferential reduction. Actually, this reduction is not new. It was already performed in our previous work [3] and was based on two facts: the Craig-Sulem-Zakharov reduction to a system on the boundary, introducing the Dirichlet-Neumann operator and (following Lannes [43] and [1, 8]) the use of paradifferential analysis to study the Dirichlet-Neumann operator in non smooth domains. The second major step in the proof consists in proving that the solutions of the water waves system enjoy dispersive estimates (Strichartz-type inequalities). For the equations with surface tension in the special case of dimension 1, Strichartz estimates were proved by Christianson, Hur and Staffilani in [22] for smooth enough data and in [2] for the low regularity solutions constructed in [1]. In the present context, the main difficulty will consist in proving these dispersive estimates for gravity waves at a lower level of regularity than the threshold where we proved the existence of the solutions in [3]. This will be done by constructing parametrices on small time intervals tailored to the size of the frequencies considered (in the spirit of the works by Lebeau [46], Bahouri-Chemin [11], Tataru [59], Staffilani-Tataru [58], and Burq-Gérard-Tzvetkov [17]).

The important new points in the present article with respect to our previous analysis are the following.

- To go beyond the analysis previously developed in [3] (under the assumption $s > 1 + \frac{d}{2}$), we need to develop much more technically involved approaches, in order to work with very rough functions and domains (most parts in our analysis extend to $s > \frac{1}{2} + \frac{d}{2}$). We believe that these results on the Dirichlet-Neumann operator in very rough domains can be of independent interest (see also the work by Dahlberg-Kenig [29] and Craig-Schanz-Sulem [27]).

- The *a priori* estimates we prove involve $L^\infty_t(\mathcal{H}^s_x)$ norms (energy estimates) and $L^2_t(C^\sigma_x)$ norms (dispersive estimates). For this, we need to estimate the non linear (and non local) remainder terms given by the paradifferential calculus using these norms. The loss of integrability in time (L^2_t) for the Hölder norms forces us to track down the precise dependence of the constants in our analysis and prove *tame* estimates depending linearly on Hölder norms.

- A simpler model operator describing our system is $(\partial_t + V\cdot\nabla_x) + i|D_x|^{\frac{1}{2}}$ (while in presence of surface tension, $|D_x|^{\frac{1}{2}}$ is replaced by $|D_x|^{\frac{3}{2}}$). Consequently, the dispersive properties exhibited on the water-waves system without surface tension are generated by the *lower order term* in the equation (the principal part, being a simple transport equation, induces no dispersive effects). To our knowledge, this is the only known example of such phenomenon.

- The main point in the proof of dispersive estimates is the construction of a parametrix for solutions to (1.2). One of the difficulties in the approach will be to get rid of this low regularity transport term by means of a parachange of variables (see the work by Alinhac [9]). Once this reduction is performed, we finally have to handle the low regularity in the parametrix construction.

In the rest of this section we shall describe more precisely the problem and outline the steps of the proof on the main result.

1.1 Equations and assumptions on the fluid domain

We consider the incompressible Euler equation in a time dependent fluid domain Ω contained in a fixed domain $\mathcal{O}$, located between a free surface and a fixed bottom. We consider the general case where the bottom is arbitrary which means that the only assumption we shall make on the bottom is that it is separated from the free surface by a strip of fixed length.

Namely, we assume that,

$$\Omega = \{(t,x,y) \in I \times \mathcal{O} \,:\, y < \eta(t,x)\},$$

where $I \subset \mathbf{R}_t$ and $\mathcal{O} \subset \mathbf{R}^d \times \mathbf{R}$ is a given open connected set. The spatial coordinates are $x \in \mathbf{R}^d$ (horizontal) and $y \in \mathbf{R}$ (vertical) with $d \geq 1$. We assume that the free surface

$$\Sigma = \Big\{(t,x,y) \in I \times \mathbf{R}^d \times \mathbf{R} \,:\, y = \eta(t,x)\Big\},$$

is separated from the bottom $\Gamma = \partial\Omega \setminus \Sigma$ by a curved strip. This means that we study the case where there exists $h > 0$ such that, for any t in I,

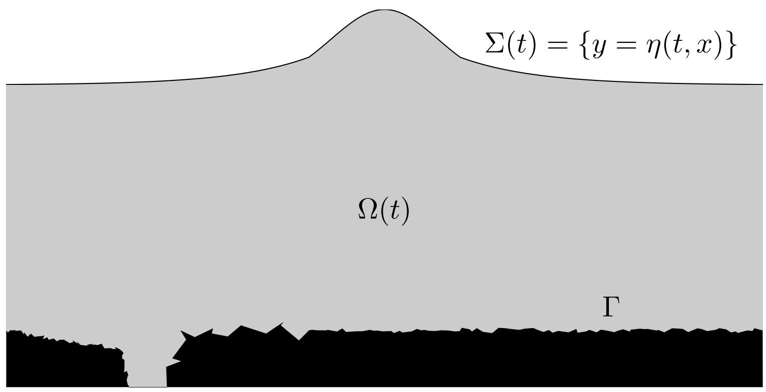
$$\Big\{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, \eta(t,x) - h < y < \eta(t,x)\Big\} \subset \mathcal{O}. \tag{1.3}$$

Examples.

1. $\mathcal{O} = \mathbf{R}^d \times \mathbf{R}$ corresponds to the infinite depth case ($\Gamma = \emptyset$);

2. The finite depth case corresponds to $\mathcal{O} = \{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, y > b(x)\}$ for some continuous function b such that $\eta(t,x) - h > b(x)$ for any time t (then $\Gamma = \{y = b(x)\}$). Notice that no regularity assumption is required on b.

3. See the picture below.



We consider a potential flow such that the velocity v is given by $v = \nabla_{x,y}\phi$ for some function $\phi\colon \Omega \to \mathbf{R}$, such that $\Delta_{x,y}\phi = 0$. The system (1.1) reads

$$\begin{cases} \partial_t\phi + \dfrac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0 \quad \text{ in } \Omega, \\ \partial_t\eta = \partial_y\phi - \nabla\eta\cdot\nabla\phi \quad \text{ on } \Sigma, \\ P = 0 \quad \text{ on } \Sigma, \\ \partial_\nu\phi = 0 \quad \text{ on } \Gamma, \end{cases} \tag{1.4}$$

where as above $g > 0$ is acceleration due to gravity, P is the pressure and ν denotes the normal vector to Γ (whenever it exists; for general domains, one solves the boundary value problem by a variational argument, see [1, 3]).

1.2 Regularity thresholds for the water waves

A well-known property of smooth solutions is that their energy is conserved

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\Omega(t)} \left|\nabla_{x,y}\phi(t,x,y)\right|^2\, dxdy + \frac{g}{2}\int_{\mathbf{R}^d}\eta(t,x)^2\, dx\right\} = 0.$$

However, we do not know if weak solutions exist at this level of regularity (even the meaning of the equations is not clear). This is the only known coercive quantity (see [14]).

Another regularity threshold is given by the scaling invariance which holds in the infinite depth case (that is when $\mathcal{O} = \mathbf{R}^d \times \mathbf{R}$). If ϕ and η are solutions of the gravity water waves equations, then ϕ_λ and η_λ defined by

$$\phi_\lambda(t,x,y) = \lambda^{-3/2}\phi(\sqrt{\lambda}t, \lambda x, \lambda y), \quad \eta_\lambda(t,x) = \lambda^{-1}\eta(\sqrt{\lambda}t, \lambda x),$$

solve the same system of equations. The (homogeneous) Hölder spaces invariant by this scaling (the scaling critical spaces) correspond to η_0 Lipschitz and ϕ_0 in $\dot{W}^{3/2,\infty}$ (one can replace the Hölder spaces by other spaces having the same invariance by scalings).

According to the scaling argument, one could expect that the problem exhibits some kind of "ill-posedness" for initial data such that the free surface is not Lipschitz. See e.g. [19, 20] for such ill-posedness results for semi-linear equations. However, the water waves equations are not semi-linear and it is not clear whether the scaling argument is the only relevant regularity threshold to determine the optimal regularity in the analysis of the Cauchy problem (we refer the reader to the discussion in Section 1.1.2 of the recent result by Klainerman-Rodnianski-Szeftel [41]). In particular, it remains an open problem to prove an ill-posedness result for the gravity water waves equations. We refer to the recent paper by Chen, Marzuola, Spirn and Wright [21] for a related result in the presence of surface tension.

Several additional criterions have appeared in the mathematical analysis of the water waves equations. The first results on the water waves equations required very smooth initial data. The literature on the subject is now well established, starting with the pioneering work of Nalimov [52] (see also Yosihara [67] and Craig [26]) who showed the unique solvability in Sobolev spaces under a smallness assumption. Wu proved that the Cauchy problem is well posed without smallness assumption ([65, 64]). Several extensions of this result were obtained by various methods and many authors. We shall quote only some recent results on the local Cauchy problem: [3, 18, 25, 43, 47, 48, 56], see also [7, 34, 37, 39, 66] for global existence results.

To ensure that the particles flow is well defined, it seems natural to assume that the gradient of the velocity is bounded (or at least in BMO, see the work by Hunter, Ifrim and Tataru in [37]). We refer to blow-up criteria by Christodoulou and Lindblad [23] or Wang and Zhang [63]. Below we shall construct solutions such that the velocity is still in $L^2((-T,T);W^{1,\infty})$ even though it is initially only in $W^{1-\mu,\infty}$.

Finally, notice that though the above continuation criterions are most naturally stated in Hölder spaces, the use of L^2-based Sobolev spaces seems unavoidable (recall from the appendix of [5] that the Cauchy problem for the linearized equations is ill-posed on Hölder spaces, as it exhibits a loss of $d/4$ derivatives). So let us rewrite the previous discussion in this framework. Firstly, the critical space for η_0 (resp. the trace $\underline{v}_0$ of the velocity at the free surface) is $\dot{H}^{1+\frac{d}{2}}(\mathbf{R}^d)$ (resp. $\dot{H}^{\frac{1}{2}+\frac{d}{2}}(\mathbf{R}^d)$). We proved in [3] that the Cauchy problem is well-posed for initial data $(\eta_0, \underline{v}_0)$ in $H^{\frac{3}{2}+\frac{d}{2}+\varepsilon}(\mathbf{R}^d) \times H^{1+\frac{d}{2}+\varepsilon}(\mathbf{R}^d)$ with $\varepsilon > 0$. This corresponds to the requirement that the initial velocity field should be Lipschitz. In this paper we shall prove that the Cauchy problem is well posed for initial data $(\eta_0, \underline{v}_0)$ belonging to $H^{\frac{3}{2}+\frac{d}{2}-\delta}(\mathbf{R}^d) \times H^{1+\frac{d}{2}-\delta}(\mathbf{R}^d)$ for $0 < \delta < \mu$. One important conclusion is that, in dimension $d = 1$, one can consider initial free surface whose curvature does not belong to L^2.

1.3 Reformulation of the equations

Following Zakharov ([68]) and Craig and Sulem ([28]) we reduce the water waves equations to a system on the free surface. To do so, notice that since the velocity potential ϕ is harmonic, it is fully determined by the knowledge of η and the knowledge of its trace at the free surface, denoted by ψ. Then one uses the Dirichlet-Neumann operator which maps a function defined on the free surface to the normal derivative of its harmonic extension. Namely, if $\psi = \psi(t,x) \in \mathbf{R}$ is defined by

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

and if the Dirichlet-Neumann operator is defined by

$$\begin{aligned}(G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta(t,x)}\\ &= (\partial_y\phi)(t,x,\eta(t,x)) - \nabla\eta(t,x)\cdot(\nabla\phi)(t,x,\eta(t,x)),\end{aligned}$$

then one obtains the following system for two unknowns (η,ψ) of the variables (t,x),

$$\begin{cases}\partial_t\eta - G(\eta)\psi = 0,\\ \partial_t\psi + g\eta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{\big(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\big)^2}{1+|\nabla\eta|^2} = 0.\end{cases} \tag{1.5}$$

We refer to [1, 3] for a precise construction of $G(\eta)$ in a domain with a general bottom. We also mention that, for general domains, we proved in [4] that if a solution (η,ψ) of System (1.5) belongs to $C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))$ for some $T>0$ and $s>1/2+d/2$, then one can define a velocity potential ϕ and a pressure P satisfying (1.4). Below we shall always consider solutions such that (η,ψ) belongs to $C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s+\frac{1}{2}}(\mathbf{R}^d))$ for some $s>1/2+d/2$ (which is the scaling index). It is thus sufficient to solve the Craig–Sulem–Zakharov formulation (1.5) of the water waves equations.

1.4 Main result

We shall work with the horizontal and vertical traces of the velocity on the free boundary, namely

$$B = (\partial_y\phi)|_{y=\eta}, \quad V = (\nabla_x\phi)|_{y=\eta}.$$

They are given in terms of η and ψ by means of the formula

$$B = \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \qquad V = \nabla\psi - B\nabla\eta. \tag{1.6}$$

Also, recall that the Taylor coefficient $\mathfrak{a}$ defined by

$$\mathfrak{a} = -\partial_y P|_{y=\eta} \tag{1.7}$$

can be defined in terms of η,ψ only (see [4] or Definition 1.5 in [3]).

For $\rho = k+\sigma$ with $k\in\mathbf{N}$ and $\sigma\in(0,1)$, recall that one denotes by $W^{\rho,\infty}(\mathbf{R}^d)$ the space of functions whose derivatives up to order k are bounded and uniformly Hölder continuous with exponent σ. Hereafter, we always consider indexes $\rho\notin\mathbf{N}$.

In our previous paper [3], we proved that the Cauchy problem is well-posed in Sobolev spaces for initial data such that, for some $s>1+d/2$, (η_0,ψ_0,V_0,B_0)

belongs to $H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times (H^s)^d \times H^s$. Our main result in this paper is a well-posedness result which holds for some $s < 1 + d/2$. In addition, we shall prove Strichartz estimates.

Theorem 1.2. *Let $d \geq 1$ and consider two real numbers s and r satisfying*

$$s > 1 + \frac{d}{2} - \mu, \quad 1 < r < s + \mu - \frac{d}{2} \quad \textit{where } \mu = \begin{cases} \frac{1}{24} \textit{ if } d = 1, \\ \frac{1}{12} \textit{ if } d \geq 2. \end{cases}$$

Consider an initial data (η_0, ψ_0) such that

(H1) $\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad \psi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad V_0 \in H^s(\mathbf{R}^d), \quad B_0 \in H^s(\mathbf{R}^d),$

(H2) there exists $h > 0$ such that condition (1.3) holds initially for $t = 0$,

(H3) (Taylor sign condition) there exists $c > 0$ such that, for all $x \in \mathbf{R}^d$, $\mathfrak{a}_0(x) \geq c$.

Then there exists $T > 0$ such that the Cauchy problem for (1.5) with initial data (η_0, ψ_0) has a unique solution such that

1. *η and ψ belong to $C^0\big([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}^d)\big) \cap L^p\big([0,T]; W^{r+\frac{1}{2},\infty}(\mathbf{R}^d)\big)$ where $p = 4$ if $d = 1$ and $p = 2$ for $d \geq 2$,*
2. *V and B belong to $C^0\big([0,T]; H^s(\mathbf{R}^d)\big) \cap L^p\big([0,T]; W^{r,\infty}(\mathbf{R}^d)\big)$ with p as above,*
3. *the condition (1.3) holds for $0 \leq t \leq T$, with h replaced with $h/2$,*
4. *for all $0 \leq t \leq T$ and for all $x \in \mathbf{R}^d$, $\mathfrak{a}(t,x) \geq c/2$.*

Remark 1.3.
- Notice that the last two assumptions in (H1) do not imply that $\psi_0 \in H^{s+1}(\mathbf{R}^d)$ since $\nabla \eta_0$ does not belong to $H^s(\mathbf{R}^d)$. Notice also that the assumption (H1) holds, for instance, when $\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ and $\psi_0 = 0$.
- The velocity is the gradient of the velocity potential and hence it could be interesting to make an assumption on $\nabla \psi_0$ instead of ψ_0 (indeed, the assumption that ψ_0 belongs to L^2 implies a restriction on the moment of the velocity in dimension 1). We refer to [64, 65, 43] for such well-posedness results in infinite depth or in the case with a smooth bottom. However, because we consider here general bottoms, it was convenient to make an assumption on ψ_0 instead of $\nabla \psi_0$ and we did not try to improve the result by weakening the low frequency assumption on ψ_0.

1.5 Paradifferential reduction

The proof relies on a paradifferential reduction of the water-waves equations, as in [1, 3, 8]. This is the property that the equations can be reduced to a very simple form

$$\big(\partial_t + T_V \cdot \nabla + iT_\gamma\big)u = f \tag{1.8}$$

where T_V is a paraproduct and T_γ is a paradifferential operator of order $\frac{1}{2}$ with symbol

$$\gamma = \sqrt{\mathfrak{a}\lambda}$$

where

$$\lambda = \sqrt{(1+|\nabla\eta|^2)|\xi|^2 - (\xi\cdot\nabla\eta)^2}.$$

Here $\mathfrak{a}$ is the Taylor coefficient (see (1.7)) and λ is the principal symbol of the Dirichlet-Neumann operator (see Appendix A for the definition of paradifferential operators). When $d=1$, λ simplifies to $|\xi|$ so $T_\gamma u = T_{\sqrt{a}} |D_x|^{\frac{1}{2}} u$.

To prove Theorem 1.2 we need tame estimates which complement the estimates already proved in [3]. We shall study the Dirichlet-Neumann operator.

In the case without bottom ($\Gamma = \emptyset$), when η is a smooth function, it is known that, modulo a smoothing operator, $G(\eta)$ is a pseudo-differential operator whose principal symbol is given by λ.

Notice that λ is well-defined for any C^1 function η. In [3] we proved several results which allow to compare $G(\eta)$ to the paradifferential operator T_λ when η has limited regularity. In particular we proved that, for any $s > 1 + d/2$,

$$\left\| G(\eta)f - T_\lambda f \right\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \big) \|f\|_{H^s(\mathbf{R}^d)}. \tag{1.9}$$

When η is a smooth function, one expects that $G(\eta) - T_\lambda$ is of order 0 which means that $G(\eta)f - T_\lambda f$ has the same regularity as f (this holds true whenever η is much smoother than f). On the other hand, (1.9) gives only that this difference "is of order" $1/2$ (it maps H^s to $H^{s-1/2}$). This is because we allow η to be only $1/2$-derivative more regular than f. This is tailored to the analysis of gravity waves since, for scaling reasons, it is natural to assume that η is $1/2$-derivative more regular than the trace of the velocity on the free surface.

We shall improve (1.9) by proving that $G(\eta) - T_\lambda$ is of order $1/2$ assuming only that $s > 3/4 + d/2$ together with sharp Hölder regularity assumptions on both η and f (these Hölder assumptions are the ones that hold by Sobolev injections for $s > 1 + d/2$). Since Hölder norms are controlled only in some L^p spaces in time (by Strichartz estimates), we need to precise the dependence of the constants. We shall prove in Appendix B the following result that we believe is of independent interest.

Theorem 1.4. *Let $d \ge 1$ and consider real numbers s, r such that*

$$s > \frac{3}{4} + \frac{d}{2}, \qquad r > 1.$$

Consider $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \cap W^{r+\frac{1}{2},\infty}(\mathbf{R}^d)$ and $f \in H^s(\mathbf{R}^d) \cap W^{r,\infty}(\mathbf{R}^d)$, then $G(\eta)f$ belongs to $H^{s-\frac{1}{2}}(\mathbf{R}^d)$ and

$$\left\| G(\eta)f - T_\lambda f \right\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}} + \|f\|_{H^s} \big) \Big\{ 1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{W^{r,\infty}} \Big\}, \tag{1.10}$$

for some non-decreasing function $\mathcal{F} \colon \mathbf{R}^+ \to \mathbf{R}^+$ depending only on s and r.

Remark 1.5. This estimate is tame in the following sense. In the context we will be mostly interested in ($s < 1 + d/2$), for oscillating functions, we have ($r > 1$)

$$\Big\| u\Big(\frac{x}{\varepsilon}\Big)\Big\|_{W^{r+\frac{1}{2},\infty}} \sim \Big(\frac{1}{\varepsilon}\Big)^{r+\frac{1}{2}} \gg \Big(\frac{1}{\varepsilon}\Big)^{s+\frac{1}{2}-\frac{d}{2}} \sim \Big\| u\Big(\frac{x}{\varepsilon}\Big)\Big\|_{H^{s+\frac{1}{2}}},$$

and consequently, the estimate (1.10) is *linear* with respect to the highest norm.

1.6 Strichartz estimates

Using the previous paradifferential reduction, the key point is to obtain estimates in Hölder spaces coming from Strichartz ones. Most of the analysis is devoted to the proof of the following result. Set for $I = [0, T]$,

$$\begin{aligned} &\|V\|_{E_0} = \|V\|_{L^\infty(I, L^\infty(\mathbf{R}^d))} + \|V\|_{L^p(I, W^{1,\infty}(\mathbf{R}^d))},\\ &\mathcal{N}_k(\gamma) = \sum_{|\alpha|\le k} \sup_{\xi\in\mathcal{C}} \|D_\xi^\alpha \gamma\|_{L^\infty(I, L^\infty(\mathbf{R}^d))} + \sum_{|\alpha|\le k} \sup_{\xi\in\mathcal{C}} \|D_\xi^\alpha \gamma\|_{L^p(I, W^{\frac{1}{2},\infty}(\mathbf{R}^d))}, \end{aligned}$$

where $\mathcal{C} = \{\xi \in \mathbf{R}^d : \frac{1}{10} \le |\xi| \le 10.\}$.

Theorem 1.6. *Let $I = [0, T]$, $d \ge 1$, μ such that $\mu < \frac{1}{24}$ if $d = 1$, $\mu < \frac{1}{12}$ if $d \ge 2$ and $p = 4$ if $d = 1$, $p = 2$ if $d \ge 2$.*

There exists $k = k(d)$ such that, for all $s \in \mathbf{R}$, one can find a non decreasing function $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that the following property holds. If $u \in C^0(I; H^s(\mathbf{R}^d))$ and $f \in L^p(I; H^s(\mathbf{R}^d))$ solve (1.8), then

$$\|u\|_{L^p(I; C_*^{s-\frac{d}{2}+\mu}(\mathbf{R}^d))} \le \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_k(\gamma)\big)\Big\{\|f\|_{L^p(I;H^s(\mathbf{R}^d))} + \|u\|_{C^0(I;H^s(\mathbf{R}^d))}\Big\},$$

where C_*^r is the Zygmund space of order $r \in \mathbf{R}$ (see Definition A.2),

This last theorem is proved in Chapter 2. We conclude this introduction by explaining the strategy of its proof.

Linearized equation

To explain our strategy, let us first consider as a simple model the linearized equation at ($\eta = 0, \psi = 0$) when $d = 2, g = 1$, in the case without bottom. Then $G(0) = |D_x|$ and the linearized system reads

$$\partial_t \eta - |D_x|\psi = 0, \qquad \partial_t \psi + g\eta = 0,$$

which, with $u = \eta + i|D_x|^{1/2}\psi$ can be written under the form

$$\partial_t u + i\,|D_x|^{1/2}\,u = 0.$$

Since the operators $e^{-it|D_x|^{1/2}}$ are unitary on Sobolev spaces, the Sobolev embedding $H^{1+\varepsilon}(\mathbf{R}^2) \subset L^\infty(\mathbf{R}^2)$ $(\varepsilon > 0)$ implies that

$$\left\| e^{-it|D_x|^{1/2}} u_0 \right\|_{L_t^\infty(]0,1[;L^\infty(\mathbf{R}_x^2))} \leq C \left\| u_0 \right\|_{H^{1+\varepsilon}(\mathbf{R}_x^2)} .$$

We shall recall the proof of the following Strichartz estimate

$$\exists C > 0, \forall 2 < p \leq +\infty, \quad : \quad \left\| e^{-it|D_x|^{1/2}} u_0 \right\|_{L_t^p(]0,1[;L^{\frac{2p}{p-2}}(\mathbf{R}_x^2))} \leq C \left\| u_0 \right\|_{H^{\frac{3}{2p}}(\mathbf{R}^2)} ,$$

which (taking p close to 2) allows to gain almost 1/4 derivative with respect to the Sobolev embedding.

The strategy of the proof is classical (see Ginibre-Velo [35] and Keel-Tao [40]). Firstly, by using the Littlewood-Paley decomposition, one can reduce the analysis to the case where the spectrum of u_0 is in a dyadic shell. Namely, it is sufficient to prove that

$$\left\| e^{-it|D_x|^{1/2}} \chi(h|D_x|) u_0 \right\|_{L_t^p(]0,1[;L^{\frac{2p}{p-2}})} \leq C h^{-\frac{3}{2p}} \left\| u_0 \right\|_{L^2(\mathbf{R}^2)} ,$$

where C is uniform with respect to $h \in]0,1[$ and $\chi \in C_0^\infty(\mathbf{R} \setminus 0)$ equals 1 on $[1,2]$.

To prove that $T = e^{-it|D_x|^{1/2}} \chi(h|D_x|)$ is bounded from L_x^2 to $L_t^p(L_x^q)$ $(q = \frac{2p}{p-2})$ with norm bounded by $A := Ch^{-\frac{3}{2p}}$, it suffices to prove that the operator TT^* is bounded from $L_t^{p'}(L_x^{q'})$ to $L_t^p(L_x^q)$ with norm bounded by $A^2 = C^2 h^{-\frac{3}{p}}$. Now, write

$$TT^* f = \chi(h|D_x|) e^{-it|D_x|^{1/2}} \int_0^1 e^{is|D_x|^{1/2}} \chi(h|D_x|) f(s,\cdot) ds.$$

Using the Hardy-Littlewood-Sobolev inequality, the desired estimate for TT^* will be a consequence of the following dispersive estimate :

$$\| \chi(h|D_x|) e^{-i(t-s)|D_x|^{1/2}} \chi(h|D_x|) \|_{L_x^1 \to L_x^\infty} \leq \frac{C}{h^{\frac{3}{2}} |t-s|} .$$

The proof of this estimate is classical: we have

$$\chi(h|D_x|) e^{-it|D_x|^{1/2}} \chi(h|D_x|) u = \frac{1}{(2\pi)^2} \int e^{-it|\xi|^{1/2} + i(x-y)\cdot\xi)} \chi^2(h|\xi|) u(y) dy d\xi,$$

and the estimate follows after changing variables $(\eta = h\xi)$ from the stationary phase inequality.

The nonlinear system

We now consider the nonlinear equation (1.8), which reads

$$(\partial_t + T_V \cdot \nabla + iT_\gamma) u = f.$$

To apply the strategy recalled in the previous paragraph, the main difficulties are the following:

- this is a paradifferential equation with non constant coefficients,
- the coefficients are not smooth. Indeed, V is in $L^\infty_t(C^1_x)$ and the symbol $\gamma = \gamma(t,x,\xi)$, of order $1/2$ in ξ, is only $L^\infty_t(C^{1/2}_x)$ in the space-time variables,
- the dispersion is due to the operator T_γ of order $1/2$, while the equation contains the term $T_V \cdot \nabla$ of order 1.

The first step of the proof is classical in the context of quasi-linear wave equations (see the works by Lebeau [44], Smith [57], Bahouri-Chemin [11], Tataru [59] and Blair [15]). It consists, after a dyadic decomposition at frequency h^{-1}, in regularizing the coefficients at scale $h^{-\delta}$, where $\delta \in (0,1)$ is to be chosen properly. Using the Littlewood-Paley decomposition $u = \sum_{j\geq -1} \Delta_j u$, we can write

$$\Big(\partial_t + S_{j-2}(V)\cdot\nabla + iT_\gamma\Big)\Delta_j u = f_j,$$

where $S_{j-2}u = \sum_{k=-1}^{j-3} \Delta_k u$ and f_j is easily estimated. Then, for some $\delta \in]0,1[$ (here $\delta = \frac{2}{3}$), one considers instead the equation with smoothed coefficients:

$$\Big(\partial_t + S_{j\delta}(V)\cdot\nabla + iT_{\gamma_\delta}\Big)\Delta_j u = f_j + g_{j\delta}, \qquad \gamma_\delta = \chi_0(2^{-j\delta}D_x)\gamma$$

where

$$g_{j\delta} = \big(S_{j\delta}(V) - S_j(V)\big)\cdot\nabla + i(T_\gamma - T_{\gamma_\delta})\Big)\Delta_j u.$$

Since the dispersion is due to the sub-principal term, we chose to straighten the vector field $\partial_t + S_{j\delta}(V)\cdot\nabla$ by means of a parachange of variables (following Alinhac [9]). To do so, we solve the system $\dot{X}(t) = S_{j\delta}(V)(t, X(t))$ with $X(0) = x$ to obtain a mapping $x \mapsto X(t,x)$ which is a small perturbation of the identity in small time, satisfying

$$\|\frac{\partial X}{\partial x}(t,\cdot) - Id\|_{L^\infty(\mathbf{R}^d)} \leq C(\|V\|_{C^1})|t|^{\frac{1}{2}}.$$

However, as the vector field V is only Lipschitz, we have only the following estimates for the higher order derivatives:

$$\|(\partial_x^\alpha X)(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \leq C_\alpha(\|V\|_{E_0})h^{-\delta(|\alpha|-1)}|t|^{\frac{1}{2}}, \quad |\alpha|\geq 2, \quad h = 2^{-j}.$$

So one controls $\partial_x^\alpha X$ only on small time intervals whose sizes depend of $h = 2^{-j}$ and α. This is one reason why we will prove a dispersive estimate only in short time intervals whose size is tailored to the frequency.

Then, one makes the change of variables

$$v_h(t,y) = (\Delta_j u)(t, X(t,y)), \quad h = 2^{-j},$$

to obtain an equation of the form

$$\partial_t v_h + iA_h(t,y,D_y)v_h = g_h,$$

for an explicit operator A of order $1/2$. For convenience we reduce the equation to a semi classical form by changing variables

$$z = h^{-\frac{1}{2}}y, \quad \tilde{h} = h^{\frac{1}{2}}, \quad w_{\tilde{h}}(t,z) = v_h(t,\tilde{h}y)$$

and multiplying the equation by $\tilde{h}$. We get an equation of the form

$$(\tilde{h}\partial_t + iP(t,\tilde{h}z,\tilde{h}D_z,\tilde{h}))w_{\tilde{h}} = \tilde{h}F_{\tilde{h}}.$$

Finally, we are able to write a parametrix for this reduced system, which allows to prove Strichartz estimates using the classical strategy outlined above, on a small time interval $|t| \leq \tilde{h}^\delta = h^{\frac{\delta}{2}}$. The key step here is to prove that, on such time intervals one has a parametrix of the form

$$\mathcal{K}v(t,z) = (2\pi\tilde{h})^{-d}\iint e^{\frac{i}{\tilde{h}}(\phi(t,z,\xi,\tilde{h})-z'\cdot\xi)}b(t,z,\xi,\tilde{h})v(z')dz'd\xi$$

where b is a symbol and ϕ a real-valued phase function, such that

$$\phi|_{t=0} = z\cdot\xi, \quad b|_{t=0} = \chi(\xi), \quad \mathrm{supp}\chi \subset \{\xi : \frac{1}{3} \leq |\xi| \leq 3\}.$$

Using the parametrix, the stationary phase estimate and coming back to the original variable $z \to y = h^{\frac{1}{2}}z \to x = X(t,y)$ we obtain a dispersive estimate (see Theorem 2.35). This gives a Strichartz estimate on a time interval of size $h^{\delta/2}$. Finally, splitting the time interval $[0,T]$ into $Th^{-\delta/2}$ time intervals of size $h^{\delta/2}$, and gluing together all these estimates, we obtain a Strichartz estimate with loss on the time interval $[0,T]$.

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Chapter 2

Strichartz estimates

In this chapter we shall prove Strichartz estimates for rough solutions of the gravity water waves equations. This is the main new point in this paper. Namely, we shall prove Theorem 1.6 stated in the introduction.

The first section of this chapter is based on a paradifferential analysis of the water-wave equations — we refer the reader to Appendix A for the definition of paradifferential operators. This paradifferential analysis relies on the approach given in [1, 3, 8] combined with some tame estimates which are proved in the appendix.

2.1 Symmetrization of the equations

We begin by recalling from [3] that the water-waves equations can be reduced to a very simple form

$$\big(\partial_t + T_V \cdot \nabla + iT_\gamma\big)u = f \tag{2.1}$$

where T_V is a paraproduct and T_γ is a para-differential operator of $1/2$. To do so, we begin by recalling a formulation of the water waves system which involves the unknowns

$$\zeta = \nabla\eta, \quad B = \partial_y\phi|_{y=\eta}, \quad V = \nabla_x\phi|_{y=\eta}, \quad \mathfrak{a} = -\partial_y P|_{y=\eta}, \tag{2.2}$$

where recall that ϕ is the velocity potential, $P = P(t,x,y)$ is the pressure given by

$$-P = \partial_t\phi + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + gy, \tag{2.3}$$

and $\mathfrak{a}$ is the Taylor coefficient.

We consider smooth solutions (η,ψ) of (1.5) defined on the time interval $[0,T_0]$ and satisfying the following assumptions on that time interval.

Assumption 2.1. *We consider smooth solutions of the water waves equations such that* *i)* (η, ψ) *belongs to* $C^1([0, T_0]; H^{s_0}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d))$ *for some* s_0 *large enough and* $0 < T_0$;

ii) *there exists* $h > 0$ *such that* (1.3) *holds for any* t *in* $[0, T_0]$ *(this is the assumption that there exists a curved strip of width* h *separating the free surface from the bottom)*;

iii) *there exists* $c > 0$ *such that the Taylor coefficient* $\mathfrak{a}(t, x) = -\partial_y P|_{y=\eta(t,x)}$ *is bounded from below by* c *for any* (t, x) *in* $[0, T_0] \times \mathbf{R}^d$.

We begin by recalling two results from [3].

Proposition 2.2 (from [3]). *For any* $s > \frac{1}{2} + \frac{d}{2}$ *one has*

$$G(\eta)B = -\operatorname{div} V + \widetilde{\gamma} \tag{2.4}$$

where

$$\|\widetilde{\gamma}\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}(\|(\eta, V, B)\|_{H^{s+\frac{1}{2}} \times H^{\frac{1}{2}} \times H^{\frac{1}{2}}}). \tag{2.5}$$

Proposition 2.3 (from [3]). *We have*

$$(\partial_t + V \cdot \nabla)B = \mathfrak{a} - g, \tag{2.6}$$

$$(\partial_t + V \cdot \nabla)V + \mathfrak{a}\zeta = 0, \tag{2.7}$$

$$(\partial_t + V \cdot \nabla)\zeta = G(\eta)V + \zeta G(\eta)B + \gamma, \tag{2.8}$$

where the remainder term $\gamma = \gamma(\eta, \psi, V)$ *satisfies the following estimate : if* $s > \frac{1}{2} + \frac{d}{2}$ *then*

$$\|\gamma\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}(\|(\eta, \psi, V)\|_{H^{s+\frac{1}{2}} \times H^s \times H^s}). \tag{2.9}$$

Remark 2.4. With γ and $\widetilde{\gamma}$ as above, there holds $\gamma = -\zeta\widetilde{\gamma}$. In particular, it follows from (2.4) and (2.8) that

$$(\partial_t + V \cdot \nabla)\zeta = G(\eta)V - (\operatorname{div} V)\zeta. \tag{2.10}$$

Moreover, in the case without bottom ($\Gamma = \emptyset$), one can see that $\widetilde{\gamma} = 0$ and hence $\gamma = 0$.

The analysis then uses in an essential way the introduction of a new unknown (following Alinhac, see [8, 1, 9, 10]) which allows us to circumvent the classical issue that there is a loss of 1/2 derivative when one works with the Craig-Sulem-Zakharov system. By working with the unknowns (η, V, B), the introduction of this good unknown amounts to work with $U = V + T_\zeta B$ where recall that $\zeta = \nabla\eta$ (the ith component ($i = 1, \dots, d$) of this vector valued unknown is $U_i = V_i + T_{\partial_i \eta} B$).

To prove Sobolev estimates, it is convenient to work with

$$\begin{aligned} U_s &:= \langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B, \\ \zeta_s &:= \langle D_x \rangle^s \zeta. \end{aligned} \tag{2.11}$$

Now we can state the following result which complements [3, Prop. 4.8].

Proposition 2.5. *Let $d \geq 1$ and consider real numbers s, r such that*

$$s > \frac{3}{4} + \frac{d}{2}, \qquad r > 1.$$

There holds

$$(\partial_t + T_V \cdot \nabla)U_s + T_{\mathfrak{a}}\zeta_s = f_1, \tag{2.12}$$

$$(\partial_t + T_V \cdot \nabla)\zeta_s = T_\lambda U_s + f_2, \tag{2.13}$$

where recall that λ is the symbol

$$\lambda(t; x, \xi) := \sqrt{(1 + |\nabla\eta(t,x)|^2)\,|\xi|^2 - (\nabla\eta(t,x)\cdot\xi)^2},$$

and where, for each time $t \in [0, T]$,

$$\begin{aligned}(2.14)\quad & \|(f_1(t), f_2(t))\|_{L^2\times H^{-\frac{1}{2}}} \\ & \leq \mathcal{F}\big(\|(\eta,\psi)(t)\|_{H^{s+\frac{1}{2}}}, \|(V,B)(t)\|_{H^s}\big)\left\{1 + \|\eta(t)\|_{W^{r+\frac{1}{2},\infty}} + \|(V,B)(t)\|_{W^{r,\infty}}\right\}.\end{aligned}$$

By using the tame estimates for the paralinearization of the Dirichlet-Neumann operator proved in Appendix B, there is nothing new in the proof of Proposition 2.5 compared to the proof of Prop. 4.8 in [3]. Indeed, the proof of Prop. 4.8 in [3] applies verbatim (up to replacing $\mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s}\big)$ by the right-hand side of (2.14)).

The next step is a symmetrization of the non-diagonal part of the equations. We have the following result, whose proof follows directly from the proof of Proposition 4.10 in [3] and the estimates on the Taylor coefficient proved in the appendix (see Proposition C.1 in Appendix C).

Proposition 2.6. *Let $d \geq 1$, $s > \frac{3}{4} + \frac{d}{2}$ and $r > 1$. Introduce the symbols*

$$\gamma = \sqrt{\mathfrak{a}\lambda}, \quad q = \sqrt{\frac{\mathfrak{a}}{\lambda}},$$

and set $\theta_s = T_q\zeta_s$. Then

$$\partial_t U_s + T_V \cdot \nabla U_s + T_\gamma\theta_s = F_1, \tag{2.15}$$

$$\partial_t\theta_s + T_V \cdot \nabla\theta_s - T_\gamma U_s = F_2, \tag{2.16}$$

for some source terms F_1, F_2 satisfying

$$\begin{aligned}& \|(F_1(t), F_2(t))\|_{L^2\times L^2} \\ & \leq \mathcal{F}\big(\|(\eta,\psi)(t)\|_{H^{s+\frac{1}{2}}}, \|(V,B)(t)\|_{H^s}\big)\left\{1 + \|\eta(t)\|_{W^{r+\frac{1}{2},\infty}} + \|(V,B)(t)\|_{W^{r,\infty}}\right\}.\end{aligned}$$

We are now in position to obtain the following corollary, which will be the starting point of the proof of the Strichartz estimate.

Corollary 2.7. *With the above notations, set*

$$u = \langle D_x \rangle^{-s} (U_s - i\theta_s) = \langle D_x \rangle^{-s} (U_s - iT_{\sqrt{a/\lambda}} \nabla \eta_s). \tag{2.17}$$

Then u satisfies the complex-valued equation

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = f, \tag{2.18}$$

where f satisfies for each time $t \in [0,T]$,

$$\begin{aligned} &\left\| f(t) \right\|_{H^s} \\ &\le \mathcal{F}\big(\left\| (\eta, \psi)(t) \right\|_{H^{s+\frac{1}{2}}}, \left\| (V,B)(t) \right\|_{H^s} \big) \left\{ 1 + \left\| \eta(t) \right\|_{W^{r+\frac{1}{2},\infty}} + \left\| (V,B)(t) \right\|_{W^{r,\infty}} \right\}. \end{aligned} \tag{2.19}$$

Proof. It immediately follows from (D.2)–(D.3) that $U_s - i\theta_s$ satisfies

$$(\partial_t + T_V \cdot \nabla + iT_\gamma)(U_s - i\theta_s) = h, \tag{2.20}$$

where, for each time $t \in [0,T]$, the $L^2(\mathbf{R}^d)$-norm of $h(t)$ is bounded by the right-hand side of (2.19). We now have to commute this equation with $\langle D_x \rangle^{-s}$. If follows from (A.5) that

$$\left\| \left[\langle D_x \rangle^{-s}, T_V \cdot \nabla \right] u \right\|_{H^s} \lesssim \left\| V \right\|_{W^{1,\infty}} \left\| u \right\|_{H^s}. \tag{2.21}$$

Now we claim that

$$\left\| u \right\|_{H^s} \le \mathcal{F}\big(\left\| (\eta, \psi) \right\|_{H^{s+\frac{1}{2}}}, \left\| (V,B) \right\|_{H^s} \big). \tag{2.22}$$

To see this, write, by definition of u,

$$\begin{aligned} \left\| u \right\|_{H^s} &\le \left\| U_s \right\|_{L^2} + \left\| \theta_s \right\|_{L^2} \\ &\le \left\| \langle D_x \rangle^s V \right\|_{L^2} + \left\| T_\zeta \langle D_x \rangle^s B \right\|_{L^2} + \left\| T_q \zeta_s \right\|_{L^2} \\ &\le \left\| V \right\|_{H^s} + \left\| \zeta \right\|_{L^\infty} \left\| B \right\|_{H^s} + \mathcal{M}_0^{-\frac{1}{2}}(q) \left\| \zeta \right\|_{H^{s-\frac{1}{2}}} \end{aligned}$$

where we used that $\left\| T_p v \right\|_{H^{\mu-m}} \lesssim \mathcal{M}_0^m(p) \left\| v \right\|_{H^\mu}$ for any paradifferential operator with symbol p in Γ_0^m (cf (A.4) and (A.3) for the definition of $M_0^m(p)$). Now we have, using the Sobolev embedding,

$$\left\| \zeta \right\|_{L^\infty} + \mathcal{M}_0^{-\frac{1}{2}}(q) \le \mathcal{F}\big(\left\| \eta \right\|_{W^{1,\infty}}, \left\| a \right\|_{L^\infty} \big) \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac{1}{2}}}, \left\| a \right\|_{H^{s-\frac{1}{2}}} \big)$$

so the claim (2.22) follows from the Sobolev estimates for a (see Proposition C.1 in the appendix). As a result, it follows from (2.22) and (2.21) that the H^s-norm of $[\langle D_x \rangle^{-s}, T_V \cdot \nabla] u$ is bounded by the right-hand side of (2.19).

To complete the proof of the Corollary it remains to prove a similar bound for $\left\| [\langle D_x \rangle^{-s}, T_\gamma] u \right\|_{H^s}$. To do so, we use again (A.5) to infer that

$$\left\| \left[\langle D_x \rangle^{-s}, T_\gamma \right] u \right\|_{H^s} \lesssim M_{1/2}^{1/2}(\gamma) \left\| u \right\|_{H^s}$$

(see (A.3) for the definition of $M_{1/2}^{1/2}(\gamma)$). Thus, in view of (2.22), it remains only to estimate $M_{1/2}^{1/2}(\gamma)$, which is done below in Lemma 2.8. □

2.2 Smoothing the paradifferential symbol

In this section we smooth out the paradifferential symbol γ and we give some estimates on it.

From now on we fix $r > 1$ and we assume $s > 1 + \frac{d}{2} - \sigma_d$ where $\sigma_1 = \frac{1}{24}$, $\sigma_d = \frac{1}{12}$ if $d \geq 2$ and we set $s_0 := \frac{1}{2} - \sigma_d > 0$. Then $s - \frac{1}{2} > \frac{d}{2} + s_0$.

Now, with $I = [0, T]$, we introduce the spaces

$$\begin{cases} E := C^0(I; H^{s-\frac{1}{2}}(\mathbf{R}^d)) \cap L^p(I; W^{\frac{1}{2},\infty}(\mathbf{R}^d)), \\ F := C^0(I; H^s(\mathbf{R}^d)) \cap L^p(I; W^{1,\infty}(\mathbf{R}^d)), \\ G := C^0(I; W^{s_0,\infty}(\mathbf{R}^d)) \cap L^p(I; W^{\frac{1}{2},\infty}(\mathbf{R}^d)) \end{cases} \tag{2.23}$$

where $p = 4$ if $d = 1$, $p = 2$ if $d \geq 2$, endowed with their natural norms.

We shall assume that

$$\begin{aligned} &(i) \quad \mathfrak{a} \in E, \quad \nabla\eta \in E, \quad V \in F, \\ &(ii) \quad \exists c > 0 : \mathfrak{a}(t,x) \geq c, \quad \forall (t,x) \in I \times \mathbf{R}^d. \end{aligned} \tag{2.24}$$

Let us recall that

$$\begin{aligned} &\gamma(t,x,\xi) = \big(\mathfrak{a}^2 \mathcal{U}(t,x,\xi)\big)^{\frac{1}{4}} \\ &\mathcal{U}(t,x,\xi) := (1 + |\nabla\eta|^2(t,x))|\xi|^2 - (\xi \cdot \nabla\eta(t,x))^2. \end{aligned} \tag{2.25}$$

Now we have, for $\xi \in \mathcal{C}_0 := \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ considered as a parameter,

$$\mathfrak{a}^2 \mathcal{U} \in G$$

uniformly in ξ.

Lemma 2.8. *There exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\|\gamma\|_G \leq \mathcal{F}(\|\nabla\eta\|_E + \|(V,B)\|_{F\times F})$ for all $\xi \in \mathcal{C}_0$.*

Proof. By the Cauchy-Schwartz inequality we have $\mathcal{U}(t,x,\xi) \geq |\xi|^2$ from which we deduce that

$$\gamma(t,x,\xi) \geq c_0 > 0 \quad \forall (t,x,\xi) \in I \times \mathbf{R}^d \times \mathcal{C}_0. \tag{2.26}$$

Moreover $\gamma \in C^0(I; L^\infty(\mathbf{R}^d \times \mathcal{C}_0))$. On the other hand, since

$$\gamma^4(t,x,\xi) - \gamma^4(t,y,\xi) = (\gamma(t,x,\xi) - \gamma(t,y,\xi)) \sum_{j=0}^{3} (\gamma(t,x,\xi))^{3-j} (\gamma(t,y,\xi))^j$$

we have, using (2.26),

$$\frac{|\gamma(t,x,\xi) - \gamma(t,y,\xi)|}{|x-y|^\sigma} \leq \frac{1}{4c_0^3} \frac{|(\mathfrak{a}^2\mathcal{U})(t,x,\xi) - (\mathfrak{a}^2\mathcal{U})(t,y,\xi)|}{|x-y|^\sigma}.$$

Taking $\sigma = s_0$, $\sigma = 1/2$ and using (C.1) in Appendix C we deduce the lemma. □

Guided by works by Lebeau [44], Smith [57], Bahouri-Chemin [11], Tataru [59] and Blair [15], we smooth out the symbol of the operator.

Definition 2.9. *Let $\psi \in C_0^\infty(\mathbf{R}^d)$, even, $\psi(\xi) = 1$ if $|\xi| \le \frac{1}{2}$, $\psi(\xi) = 0$ if $|\xi| \ge 1$. With $h = 2^{-j}$ and $\delta > 0$, which will be chosen later on, we set*

$$\gamma_\delta(t,x,\xi) = \psi(h^\delta D_x)\gamma(t,x,\xi). \tag{2.27}$$

Lemma 2.10. (i) $\quad \forall \alpha, \beta \in \mathbf{N}^d \quad \exists C_{\alpha,\beta} > 0 : \forall t \in I, \forall \xi \in \mathbf{R}^d$

$$|D_x^\alpha D_\xi^\beta \gamma_\delta(t,x,\xi)| \le C_{\alpha,\beta} h^{-\delta|\alpha|} \|D_\xi^\beta \gamma(t,\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)}.$$

(ii) $\quad \forall \alpha, \beta \in \mathbf{N}^d, |\alpha| \ge 1 \quad \exists C_{\alpha,\beta} > 0 : \forall t \in I, \forall \xi \in \mathbf{R}^d$

$$|D_x^\alpha D_\xi^\beta \gamma_\delta(t,x,\xi)| \le C_{\alpha,\beta} h^{-\delta(|\alpha| - \frac{1}{2})} \|D_\xi^\beta \gamma(t,\cdot,\xi)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)}.$$

Proof. (i) follows from the fact that

$$\gamma_\delta(t,x,\xi) = (2\pi)^{-d} h^{-\delta d} \widehat{\psi}\Big(\frac{\cdot}{h^\delta}\Big) * \gamma(t,\cdot,\xi).$$

(ii) We write

$$D_x^\alpha D_\xi^\beta \gamma_\delta(t,x,\xi) = \sum_{k=-1}^{+\infty} \Delta_k D_x^\alpha \psi(h^\delta D_x) D_\xi^\beta \gamma(t,x,\xi) := \sum_{k=-1}^{+\infty} v_k$$

where Δ_k denotes the usual Littlewood-Paley frequency localization.
If $\frac{1}{2} 2^k \ge h^{-\delta} = 2^{j\delta}$ we have $\Delta_k \psi(h^\delta D_x) = 0$. Therefore

$$D_x^\alpha D_\xi^\beta \gamma_\delta(t,x,\xi) = \sum_{k=-1}^{2+[j\delta]} v_k.$$

Now

$$v_k = 2^{k|\alpha|} \varphi_1(2^{-k} D_x) \psi(h^\delta D_x) \Delta_k D_\xi^\beta \gamma(t,x,\xi)$$

where $\varphi_1(\xi)$ is supported in $\{\frac{1}{3} \le |\xi| \le 3\}$. Therefore,

$$\|v_k\|_{L^\infty(\mathbf{R}^d)} \le 2^{k|\alpha|} \|\Delta_k D_\xi^\beta \gamma(t,\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \le C 2^{k|\alpha|} 2^{-\frac{k}{2}} \|D_\xi^\beta \gamma(t,\cdot,\xi)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)}.$$

It follows that

$$\|D_x^\alpha D_\xi^\beta \gamma_\delta(t,\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \le C \sum_{k=-1}^{2+[j\delta]} 2^{k(|\alpha| - \frac{1}{2})} \|D_\xi^\beta \gamma(t,\cdot,\xi)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)}.$$

Since $|\alpha| - \frac{1}{2} > 0$ we deduce that

$$\|D_x^\alpha D_\xi^\beta \gamma_\delta(t,\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \le C 2^{j\delta(|\alpha| - \frac{1}{2})} \|D_\xi^\beta \gamma(t,\cdot,\xi)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)}.$$

This completes the proof. □

We introduce now the Hessian matrix of γ_δ,

$$\mathrm{Hess}_\xi(\gamma_\delta)(t,x,\xi) = \Big(\frac{\partial^2\gamma_\delta}{\partial\xi_j\partial\xi_k}(t,x,\xi)\Big). \tag{2.28}$$

Our purpose is to prove the following result.

Proposition 2.11. *There exist $c_0>0, h_0>0$ such that*

$$|\det \mathrm{Hess}_\xi(\gamma_\delta)(t,x,\xi)| \geq c_0$$

for all $t\in I, x\in \mathbf{R}^d, \xi\in \mathcal{C}_0, 0<h\leq h_0$.

With the notation in (2.25) we can write

$$\mathcal{U}(t,x,\xi) = \langle A(t,x)\xi,\xi\rangle,$$

where $A(t,x)$ is a symmetric matrix. Since we have

$$|\xi|^2 \leq \mathcal{U}(t,x,\xi) \leq C(1+\|\nabla\eta\|^2_{L^\infty(I\times\mathbf{R}^d)})|\xi|^2 \tag{2.29}$$

we see that the eigenvalues of A are greater than one; therefore we have

$$\det A(t,x) \geq 1 \quad \forall (t,x)\in I\times\mathbf{R}^d. \tag{2.30}$$

We shall need the following lemma.

Lemma 2.12. *With $\alpha=\frac{1}{4}$ we have*

$$|\det \mathrm{Hess}_\xi(\gamma)(t,x,\xi)| = \mathfrak{a}^{\frac{d}{2}}(2\alpha)^d|2\alpha-1|\det A(t,x)\big(\mathcal{U}(t,x,\xi)\big)^{(\alpha-1)d}.$$

Proof. Here t and x are fixed parameters. The matrix A being symmetric one can find an orthogonal matrix B such that $B^{-1}AB = D = \mathrm{diag}(\mu_j)$ where the $\mu_j's$ are the eigenvalues of A. Setting $C=\mathrm{diag}(\sqrt{\mu_j})$ and $M=CB^{-1}$ we see that $\mathcal{U}(t,x,\xi)=|M\xi|^2$ which implies that

$$\gamma(t,x,\xi) = \mathfrak{a}^{\frac{1}{2}}g(M\xi) \quad \text{where} \quad g(\zeta)=|\zeta|^{2\alpha}, \tag{2.31}$$

so that $\mathrm{Hess}_\xi(\gamma)(t,x,\xi) = \mathfrak{a}^{\frac{1}{2}}\, {}^tM\big(\mathrm{Hess}_\zeta(g)(M\xi)\big)M$. Since $|\det M|^2 = |\det C|^2 = \det A$ we obtain

$$|\det \mathrm{Hess}_\xi(\gamma)(t,x,\xi)| = \mathfrak{a}^{\frac{d}{2}}\det A(t,x)\,|\det \mathrm{Hess}_\zeta(g)(M(t,x)\xi)|. \tag{2.32}$$

Now we have,

$$\frac{\partial^2 g}{\partial\zeta_j\partial\zeta_k}(\zeta) = 2\alpha|\zeta|^{2\alpha-2}\big(\delta_{jk}+2(\alpha-1)\omega_j\omega_k\big), \quad \omega_j=\frac{\zeta_j}{|\zeta|}. \tag{2.33}$$

Let us consider the function $F:\mathbf{R}\to\mathbf{R}$ defined by

$$F(\lambda) = \det\big(\delta_{jk}+\lambda\omega_j\omega_k\big).$$

It is a polynomial in λ and we have

$$F(0) = 1. \tag{2.34}$$

Let us denote by $C_j(\lambda)$ the j^{th} column of this determinant. Then

$$F'(\lambda) = \sum_{k=1}^{d} \det\big(C_1(\lambda), \ldots, C'_k(\lambda), \ldots C_d(\lambda)\big).$$

We see easily that

$$\det\big(C_1(0), \ldots, C'_k(0), \ldots C_d(0)\big) = \omega_j^2$$

which ensures that

$$F'(0) = 1. \tag{2.35}$$

Now since $C_j(\lambda)$ is linear with respect to λ we have $C''_j(\lambda) = 0$ therefore

$$F''(\lambda) = \sum_{j=1}^{d} \sum_{k=1, k\neq j}^{d} \det\big(C_1(\lambda), \ldots, C'_j(\lambda), \ldots, C'_k(\lambda), \ldots, C_d(\lambda)\big).$$

Now $C'_j(\lambda) = \omega_j(\omega_1, \ldots, \omega_d)$ and $C'_k(\lambda) = \omega_k(\omega_1, \ldots, \omega_d)$. It follows that $F''(\lambda) = 0$ for all $\lambda \in \mathbf{R}$. We deduce from (2.34), (2.35) that $F(\lambda) = 1 + \lambda$ and from (2.33) that

$$\det \text{Hess}_\zeta(g)(\zeta) = \big(2\alpha|\zeta|^{2\alpha-2}\big)^d (2\alpha - 1).$$

The lemma follows then from (2.32) since $\mathcal{U}(t, x, \xi) = |M(t, x)\xi|^2$. $\square$

Corollary 2.13. *One can find $c_0 > 0$ such that*

$$|\det \text{Hess}_\xi(\gamma)(t, x, \xi)| \geq c_0,$$

for all $t \in I, x \in \mathbf{R}^d, \xi \in \mathcal{C}_0$.

Proof. This follows from the previous lemma and from (2.29), (2.30). $\square$

Proof of Proposition 2.11. Recall that we have for all $\alpha \in \mathbf{N}^d$

$$\sup_{t\in I} \sup_{|\xi|\leq 2} \|D_\xi^\alpha \gamma(t, \cdot, \xi)\|_{W^{s_0,\infty}(\mathbf{R}^d)} < +\infty.$$

For fixed $j, k \in \{1, \ldots, d\}$ we write

$$\frac{\partial^2 \gamma_\delta}{\partial\xi_j \partial\xi_k}(t, x, \xi) = \frac{\partial^2 \gamma}{\partial\xi_j \partial\xi_k}(t, x, \xi) - (I - \psi(h^\delta D_x))\frac{\partial^2 \gamma}{\partial\xi_j \partial\xi_k}(t, x, \xi). \tag{2.36}$$

Setting $\gamma_{jk} = \frac{\partial^2 \gamma}{\partial\xi_j \partial\xi_k}$ we have, since $\psi(0) = 1$,

$$(I - \psi(h^\delta D_x))\gamma_{jk}(t, x, \xi) = h^{-\delta d} \int_{\mathbf{R}^d} \overline{\mathcal{F}}\psi\Big(\frac{y}{h^\delta}\Big)\big[\gamma_{jk}(t, x, \xi) - \gamma_{jk}(t, x - y, \xi)\big]dy,$$

where $\overline{\mathcal{F}}$ denotes the inverse Fourier transform with respect to x. Then

$$\begin{aligned}
&\|(I-\psi(h^\delta D_x))\gamma_{jk}(t,\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)}\\
&\leq h^{-\delta d}\Big(\int_{\mathbf{R}^d}|\overline{\mathcal{F}}\psi(\frac{y}{h^\delta})||y|^{s_0}dy\Big)\|\gamma_{jk}(t,\cdot,\xi)\|_{W^{s_0,\infty}(\mathbf{R}^d)}\\
&\leq Ch^{\delta s_0}\sup_{t\in I}\sup_{|\xi|\leq 2}\|\gamma_{jk}(t,\cdot,\xi)\|_{W^{s_0,\infty}(\mathbf{R}^d)}.
\end{aligned}$$

Then Proposition 2.11 follows from Corollary 2.13 and (2.36) if h_0 is small enough. □

2.3 The pseudo-differential symbol

In this paragraph, we study the pseudo-differential symbol of T_{γ_δ}

Let $\chi\in C^\infty(\mathbf{R}^d\setminus\{0\}\times\mathbf{R}^d\setminus\{0\})$ be such that

$$(2.37)\qquad \begin{aligned}
&(i) && \chi(-\zeta,\xi)=\chi(\zeta,\xi),\\
&(ii) && \chi \text{ is homogeneous of order zero },\\
&(iii) && \chi(\zeta,\xi)=1 \text{ if } |\zeta|\leq\varepsilon_1|\xi|, \quad \chi(\zeta,\xi)=0 \text{ if } |\zeta|\geq\varepsilon_2|\xi|
\end{aligned}$$

where $0<\varepsilon_1<\varepsilon_2$ are small constants. Let us set

$$(2.38)\qquad \sigma_{\gamma_\delta}(t,x,\xi)=\Big(\int_{\mathbf{R}^d}e^{ix\cdot\zeta}\chi(\zeta,\xi)\widehat{\gamma}_\delta(t,\zeta,\xi)\,d\zeta\Big)\psi_0(\xi)$$

where $\widehat{\gamma}_\delta$ denotes the Fourier transform of γ_δ with respect to the variable x and ψ_0 is a cut-off function such that $\psi_0(\xi)=0$ for $|\xi|\leq\frac{1}{4}$, $\psi_0(\xi)=1$ for $|\xi|\geq\frac{1}{3}$. Then modulo an operator of order zero we have

$$(2.39)\qquad \begin{aligned}
T_{\gamma_\delta}u(x)&=(2\pi)^{-d}\iint e^{i(x-y)\cdot\xi}\,\sigma_{\gamma_\delta}(t,x,\xi)\,u(y)\,dy\,d\xi,\\
(T_{\gamma_\delta})^*u(x)&=(2\pi)^{-d}\iint e^{i(x-y)\cdot\xi}\,\overline{\sigma}_{\gamma_\delta}(t,y,\xi)\,u(y)\,dy\,d\xi.
\end{aligned}$$

In the sequel we shall set

$$(2.40)\qquad \mathfrak{T}_{\gamma_\delta}=\frac{1}{2}\big(T_{\gamma_\delta}+(T_{\gamma_\delta})^*\big).$$

The following result will be useful.

Lemma 2.14. *For all $N\in\mathbf{N}$, $M>0$ and all $\alpha,\beta\in\mathbf{N}^d$ there exists $C>0$ such that*

$$\sup_{|\eta|\leq M}|D_\mu^\alpha D_\eta^\beta\widehat{\chi}(\mu,\eta)|\leq\frac{C}{\langle\mu\rangle^N}$$

where the Fourier transform is taken with respect to the first variable of χ.

Proof. Recall that we have $\operatorname{supp}\chi \subset \{(\zeta,\eta) : |\zeta| \le \varepsilon_2 |\eta|\}$. Then for $|\eta| \le M$ we see easily that for any $k \in \mathbf{N}$ we can write

$$|\mu|^{2k} D_\mu^\alpha D_\eta^\beta \widehat{\chi}(\mu,\eta) = \int_{\mathbf{R}^d} e^{-i\mu\cdot\zeta} (-\Delta_\zeta)^k \big[(-\zeta)^\alpha D_\eta^\beta \chi(\zeta,\eta)\big]\, d\zeta.$$

The result follows then from the fact that when $|\eta| \le M$ the domain of integration is contained in the set $\{\zeta : |\zeta| \le \varepsilon_2 M\}$. □

2.4 Several reductions

Let us recall the Littlewood-Paley decomposition. There exists an even function $\psi \in C_0^\infty(\mathbf{R}^d)$ such that $\psi(\xi) = 1$ for $|\xi| \le 1/2$ and $\psi(\xi) = 0$ for $|\xi| \ge 1$, and an even function $\varphi \in C_0^\infty(\mathbf{R}^d)$ whose support is contained in the shell $\mathcal{C}_0 := \{\xi : \frac{1}{2} \le |\xi| \le 2\}$ such that

$$\psi(\xi) + \sum_{k=0}^{N-1} \varphi(2^{-k}\xi) = \psi(2^{-N}\xi).$$

We set

$$\Delta_{-1}u = \psi(D)u, \quad \Delta_j u = \varphi(2^{-j}D)u, \quad S_j u = \sum_{k=-1}^{j-1} \Delta_k u = \psi(2^{-j}D)u.$$

We introduce te following definition.

Definition 2.15. *A right hand side function F_j is said to be controlled if it satisfies the following estimate*

$$\begin{aligned} \|F_j\|_{L^p(I,H^s(\mathbf{R}^d))} &\le \|f\|_{L^p(I,H^s(\mathbf{R}^d))} \\ &\quad + C\big(\|V\|_{L^p(I,W^{1,\infty}(\mathbf{R}^d))} + \|\gamma\|_E\big)\|u\|_{L^\infty(I,H^s(\mathbf{R}^d))} \end{aligned} \tag{2.41}$$

with C independent of j.

We proceed now to several reductions.

Let $u \in L^\infty(I; H^s(\mathbf{R}^d))$ be a solution of the paradifferential equation (2.1) namely

$$Lu := \partial_t u + T_V \cdot \nabla u + iT_\gamma u = f \in L^2(I; H^s(\mathbf{R}^d)). \tag{2.42}$$

Point 1. We have

$$L\Delta_j u = f_j^1,$$

where $f_{1,j}$ is controlled.

Indeed we can write

$$f_{1,j} = \Delta_j f + [\Delta_j, T_V] \cdot \nabla u + i[\Delta_j, T_\gamma]u.$$

Moreover $[\Delta_j, T_V]\cdot\nabla u = [\Delta_j, T_V]\cdot\nabla\widetilde{\Delta}_j u$, $[\Delta_j, T_\gamma]u = [\Delta_j, T_\gamma]\widetilde{\Delta}_j u$ and the estimate follows from the symbolic calculus.

Point 2. The solution u satisfies

$$\big(\partial_t + S_j(V)\cdot\nabla + iT_\gamma\big)\Delta_j u = f_{2,j} \tag{2.43}$$

where $f_{2,j}$ is controlled.

Indeed we have $f_{2,j} = f_{1,j} - \big(T_V\cdot\nabla\Delta_j u - S_j(V)\cdot\nabla\Delta_j u\big)$. Since $\Delta_k\Delta_j = 0$ if $|k-j|\geq 2$ we can write

$$\begin{aligned} T_V\cdot\Delta_j\nabla u &= \sum_{|j-k|\leq 1} S_{k-3}(V)\cdot\Delta_k\Delta_j\nabla u = S_j(V)\cdot\sum_{|j-k|\leq 1}\Delta_k\Delta_j\nabla u + R_j u\\ &= S_j(V)\Delta_j\nabla u + R_j u, \end{aligned}$$

where $R_j u = -\sum_{|j-k|\leq 1}\big(S_j(V) - S_{k-3}(V)\big)\cdot\Delta_k\Delta_j\nabla u$. We have three terms in the sum defining $R_j u$. Each of them is a finite sum of terms of the form $A_j = \Delta_{j+\mu}(V)\cdot\Delta_{j+\nu}\Delta_j\nabla u$. Since the spectrum of A_j is contained in a ball of radius $C2^j$ we can write for fixed t

$$\begin{aligned} \|A_j\|_{H^s} &\leq C_1 2^{js}\|A_j\|_{L^2} \leq C_1 2^{js}\|\Delta_{j+\mu}V\|_{L^\infty}\|\Delta_{j+\nu}\Delta_j\nabla u\|_{L^2}\\ &\leq C_2 2^{js}2^{-j}\|V\|_{W^{1,\infty}}2^{-js}2^j\|\Delta_j u\|_{H^s} \leq C_2\|V\|_{W^{1,\infty}}\|\Delta_j u\|_{H^s}. \end{aligned}$$

Thus

$$\|T_V\cdot\nabla\Delta_j u - S_j(V)\cdot\nabla\Delta_j u\|_{L^p(I,H^s(\mathbf{R}^d))} \leq C\|V\|_{L^\infty(I,W^{1,\infty}(\mathbf{R}^d))}\|\Delta_j u\|_{L^\infty(I,H^s(\mathbf{R}^d))}.$$

which proves our claim since $f_{1,j}$ is controlled.

Point 3. As already mentioned, we need to smooth out the symbols. To do so, we replace $S_j(V)$ and γ by

$$S_{j\delta}(V) = \psi(2^{-j\delta}D)V, \quad \gamma_\delta = \psi(2^{-j\delta}D)\gamma$$

where $\delta = \frac{2}{3}$. Then the solution u satisfies

$$\big(\partial_t + S_{j\delta}(V)\cdot\nabla + iT_{\gamma_\delta}\big)\Delta_j u = f_{3,j}$$

where

$$f_{3,j} = f_{2,j} + (S_{j\delta}(V)\cdot\nabla - S_j(V)\cdot\nabla)\Delta_j u + i(T_{\gamma_\delta} - T_\gamma)\Delta_j u. \tag{2.44}$$

Finally for later use we write the equation under a symmetric form.

Point 4. The solution u satisfies

$$\mathcal{L}_\delta\Delta_j u := \big(\partial_t + \frac{1}{2}(S_{j\delta}(V)\cdot\nabla + \nabla\cdot S_{j\delta}(V)) + \frac{i}{2}(T_{\gamma_\delta} + T^*_{\gamma_\delta})\big)\Delta_j u = f_{4,j} \tag{2.45}$$

where

$$\begin{aligned} f_{4,j} &= f_{3,j} + \frac{1}{2}\big\{S_{j\delta}(V)\cdot\nabla - \nabla\cdot S_{j\delta}(V) + i(T_{\gamma_\delta} - T^*_{\gamma_\delta})\big\}\Delta_j u,\\ &= f_{3,j} + \big\{\frac{1}{2}S_{j\delta}(\operatorname{div} V) + i(T_{\gamma_\delta} - T^*_{\gamma_\delta})\big\}\Delta_j u. \end{aligned} \tag{2.46}$$

If we set

$$(2.47) \qquad \mathfrak{T}_{\gamma_\delta} = \frac{1}{2}(T_{\gamma_\delta} + T^*_{\gamma_\delta})$$

then we can write

$$(2.48) \qquad \mathcal{L}_\delta = \partial_t + S_{j\delta}(V)\cdot\nabla + \frac{1}{2}S_{j\delta}(\mathrm{div}V) + i\mathfrak{T}_{\gamma_\delta}.$$

Notice that the operator

$$\frac{1}{2}(S_{j\delta}(V)\cdot\nabla + \nabla\cdot S_{j\delta}(V)) + i\mathfrak{T}_{\gamma_\delta}$$

is anti-symmetric.

Our goal now is to prove a Strichartz estimate for $\mathcal{L}_\delta$.

2.5 Straightening the vector field

We want to straighten the vector field $\partial_t + S_{j\delta}(V)\cdot\nabla$. Consider the system of differential equations

$$(2.49) \qquad \begin{cases} \dot{X}_k(s) = S_{j\delta}(V_k)(s, X(s)), \quad 1\leq k\leq d, \quad X = (X_1,\dots,X_d) \\ X_k(0) = x_k. \end{cases}$$

For $k = 1,\dots,d$ we have $S_{j\delta}(V_k)\in L^\infty(I; H^\infty(\mathbf{R}^d))$ and

$$|S_{j\delta}(V_k)(s,x)| \leq C\|V_k\|_{L^\infty(I\times\mathbf{R}^d)} \quad \forall(s,x)\in I\times\mathbf{R}^d.$$

Therefore System (2.49) has a unique solution defined on I which will be denoted $X(s;x,h)(h = 2^{-j})$ or sometimes simply $X(s)$.

We shall set

$$(2.50) \qquad E_0 = L^p(I; W^{1,\infty}(\mathbf{R}^d))^d \cap L^\infty(I; L^\infty(\mathbf{R}^d))^d$$

where $p = 4$ if $d = 1$, $p = 2$ if $d\geq 2$, endowed with its natural norm.

Proposition 2.16. *For fixed (s,h) the map $x\mapsto X(s;x,h)$ belongs to $C^\infty(\mathbf{R}^d,\mathbf{R}^d)$. Moreover there exist functions $\mathcal{F},\mathcal{F}_\alpha : \mathbf{R}^+\to\mathbf{R}^+$ such that*

$$(i) \qquad \Big\|\frac{\partial X}{\partial x}(s;\cdot,h) - Id\Big\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}(\|V\|_{E_0})|s|^{\frac{1}{2}},$$

$$(ii) \qquad \|(\partial_x^\alpha X)(s;\cdot,h)\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}_\alpha(\|V\|_{E_0})h^{-\delta(|\alpha|-1)}|s|^{\frac{1}{2}}, \quad |\alpha|\geq 2$$

for all $(s,h)\in I\times(0,h_0]$.

Proof. To prove (i) we differentiate the system with respect to x_l and we obtain

$$\begin{cases} \dfrac{\partial \dot{X}_k}{\partial x_l}(s) = \displaystyle\sum_{q=1}^{d} S_{j\delta}\Big(\frac{\partial V_k}{\partial x_q}\Big)(s, X(s))\frac{\partial X_q}{\partial x_l}(s) \\ \dfrac{\partial X_k}{\partial x_l}(0) = \delta_{kl} \end{cases}$$

from which we deduce

$$\frac{\partial X_k}{\partial x_l}(s) = \delta_{kl} + \int_0^s \sum_{q=1}^{d} S_{j\delta}\Big(\frac{\partial V_k}{\partial x_q}\Big)(\sigma, X(\sigma))\frac{\partial X_q}{\partial x_l}(\sigma)\, d\sigma. \tag{2.51}$$

Setting $|\nabla X| = \sum_{k,l=1}^{d} |\frac{\partial X_k}{\partial x_l}|$ we obtain from (2.51)

$$|\nabla X(s)| \leq C_d + \int_0^s |\nabla V(\sigma, X(\sigma))|\, |\nabla X(\sigma)|\, d\sigma.$$

The Gronwall inequality implies that

$$|\nabla X(s)| \leq \mathcal{F}(\|V\|_{E_0}) \quad \forall s \in I. \tag{2.52}$$

Coming back to (2.51) and using (2.52) we can write

$$\Big|\frac{\partial X}{\partial x}(s) - Id\Big| \leq \mathcal{F}(\|V\|_{E_0}) \int_0^s \|\nabla V(\sigma, \cdot)\|_{L^\infty(\mathbf{R}^d)}\, d\sigma \leq \mathcal{F}_1(\|V\|_{E_0})|s|^{\frac{1}{2}}.$$

Notice that in dimension one we have used the inequality $|s|^{\frac{3}{4}} \leq C|s|^{\frac{1}{2}}$ when $s \in I$.
To prove (ii) we shall show by induction on $|\alpha|$ that the estimate

$$\|(\partial_x^\alpha X)(s; \cdot, h)\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}_\alpha(\|V\|_{E_0}) h^{-\delta(|\alpha|-1)}$$

for $1 \leq |\alpha| \leq k$ implies (ii) for $|\alpha| = k+1$. The above estimate is true for $|\alpha| = 1$ by (i). Let us differentiate $|\alpha|$ times the system (2.49). We obtain

$$\frac{d}{ds}\big(\partial_x^\alpha X\big)(s) = S_{j\delta}(\nabla V)(s, X(s))\partial_x^\alpha X + (1) \tag{2.53}$$

where the term (1) is a finite linear combination of terms of the form

$$A_\beta(s,x) = \partial_x^\beta\big(S_{j\delta}(V)\big)(s, X(s)) \prod_{i=1}^{q} \big(\partial_x^{L_i} X(s)\big)^{K_i}$$

where

$$2 \leq |\beta| \leq |\alpha|, \quad 1 \leq q \leq |\alpha|, \quad \sum_{i=1}^{q} |K_i| L_i = \alpha, \quad \sum_{i=1}^{q} K_i = \beta.$$

Then $1 \leq |L_i| < |\alpha|$ which allows us to use the induction. Therefore we can write

$$\begin{aligned}\|A_\beta(s,\cdot)\|_{L^\infty(\mathbf{R}^d)} &\leq \big\|\partial_x^\beta\big(S_{j\delta}(V)\big)(s,\cdot)\big\|_{L^\infty(\mathbf{R}^d)} \prod_{i=1}^q \Big\|\partial_x^{L_i} X(s,\cdot)\Big\|_{L^\infty(\mathbf{R}^d)}^{|K_i|} \\ &\leq Ch^{-\delta(|\beta|-1)}\|V(s,\cdot)\|_{W^{1,\infty}(\mathbf{R}^d)} h^{-\delta\sum_{i=1}^q |K_i|(|L_i|-1)}\mathcal{F}(\|V\|_{E_0}) \\ &\leq \mathcal{F}(\|V\|_{E_0})h^{-\delta(|\alpha|-1)}\|V(s,\cdot)\|_{W^{1,\infty}(\mathbf{R}^d)}.\end{aligned}$$

It follows then from (2.53) that

$$\begin{aligned}|\partial_x^\alpha X(s)| \leq \mathcal{F}(\|V\|_{E_0})h^{-\delta(|\alpha|-1)} &\int_0^s \|V(\sigma,\cdot)\|_{W^{1,\infty}(\mathbf{R}^d)}\, d\sigma \\ + C\int_0^s &\|V(\sigma,\cdot)\|_{W^{1,\infty}(\mathbf{R}^d)} |\partial_x^\alpha X(\sigma)|\, d\sigma.\end{aligned} \tag{2.54}$$

The Hölder and Gronwall inequalities imply immediately (ii). □

Corollary 2.17. *There exist $T_0 > 0, h_0 > 0$ such that for $t \in [0, T_0]$ and $0 < h \leq h_0$ the map $x \mapsto X(t;x,h)$ from $\mathbf{R}^d$ to $\mathbf{R}^d$ is a C^∞ diffeomorphism.*

Proof. This follows from a result by Hadamard (see [13]). Indeed if T_0 is small enough, Proposition 2.16 shows that the matrix $\big(\frac{\partial X_k}{\partial x_j}(t;x,h)\big)$ is invertible. On the other hand since

$$|X(t;x,h) - x| \leq \int_0^t |S_{j\delta}(V)(\sigma, X(\sigma))|\, d\sigma \leq T_0 \|V\|_{L^\infty([0,T_0]\times\mathbf{R}^d)}$$

we see that the map $x \mapsto X(t;x,h)$ is proper. □

2.6 Reduction to a semi-classical form

In the sequel we shall set

$$U = \Delta_j u.$$

According to (2.45) we see that the function U is a solution of the equation

$$\big(\partial_t + \frac{1}{2}\{S_{j\delta}(V)\cdot\nabla + \nabla\cdot S_{j\delta}(V)\} + i\mathfrak{T}_{\gamma_\delta}\varphi_1(hD_x)\big)U(t,x) = f_{4,j}(t,x), \quad h = 2^{-j}$$

where

$$\varphi_1 \in C^\infty(\mathbf{R}^d), \quad \operatorname{supp}\varphi_1 \subset \{\xi : \frac{1}{4} \leq |\xi| \leq 4\}, \quad \varphi_1 = 1 \text{ on } \{\xi : \frac{1}{3} \leq |\xi| \leq 3\} \tag{2.55}$$

and $f_{4,j}$ has been defined in (2.46). According to (2.39), (2.40) we have

$$\begin{aligned}\mathfrak{T}_{\gamma_\delta}\varphi_1(hD_x)U(t,x) &= (2\pi)^{-d}\iint e^{i(x-y)\cdot\xi}\, a(t,x,y,\xi,h)u(y)\, dy\, d\xi, \\ a(t,x,y,\xi,h) &= \frac{1}{2}\big(\sigma_{\gamma_\delta}(t,x,\xi) + \overline{\sigma}_{\gamma_\delta}(t,y,\xi)\big)\varphi_1(h\xi).\end{aligned} \tag{2.56}$$

We make now the change of variable $x = X(t; x', h)$. Let us set

$$v_h(t, x') = U(t, X(t; x', h)) \quad t \in [0, T_0]. \tag{2.57}$$

Then it follows from (2.49) that

$$\partial_t v_h(t, x') = -i\big(\mathfrak{T}_{\gamma_\delta}\varphi_1(hD_x)U\big)(t, X(t; x', h)) + F_j(t, X(t; x', h)). \tag{2.58}$$

Our next purpose is to give another expression to the quantity

$$A = \big(\mathfrak{T}_{\gamma_\delta}\varphi_1(hD_x)U\big)(t, X(t; x', h)). \tag{2.59}$$

In the computation below, $t \in [0, T_0]$ and h being fixed, we will omit them. We have

$$A(x') = (2\pi)^{-d} \iint e^{i(X(x')-y)\cdot\xi} a(X(x'), y, \xi) U(y)\, dy\, d\xi.$$

Notations: we set

$$\begin{aligned} H(x', y') &= \int_0^1 \frac{\partial X}{\partial x}(\lambda x' + (1-\lambda) y')\, d\lambda, \quad M(x', y') = \big({}^t H(x', y')\big)^{-1} \\ M_0(x') &= \Big({}^t\Big(\frac{\partial X}{\partial x}(x')\Big)\Big)^{-1}, \quad J(x', y') = \Big|\det\Big(\frac{\partial X}{\partial x}(y')\Big)\Big| |\det M(x', y')|. \end{aligned} \tag{2.60}$$

Let us remark that M, M_0 are well defined by Proposition 2.16. Moreover $M_0(x') = M(x', x')$ and $J(x', x') = 1$.

In the integral defining A, we make the change of variables $y = X(y')$. Then using the equality $X(x') - X(y') = H(x', y')(x' - y')$ and setting $\xi = M(x', y')\zeta$ we get,

$$A(x') = (2\pi)^{-d} \iint e^{i(x'-y')\cdot\zeta} a\big(X(x'), X(y'), M(x', y')\zeta\big) J(x', y') v_h(y')\, dy'\, d\zeta.$$

Now we set

$$z = h^{-\frac{1}{2}} x', \quad w_h(z) = v_h(h^{\frac{1}{2}} z), \quad \widetilde{h} = h^{\frac{1}{2}}. \tag{2.61}$$

Then

$$A(\widetilde{h}z) = (2\pi)^{-d} \iint e^{i(\widetilde{h}z - y')\cdot\zeta} a\big(X(\widetilde{h}z), X(y'), M\big(\widetilde{h}z, y'\big)\zeta\big) J\big(\widetilde{h}z, y'\big) v_h(y')\, dy' d\zeta.$$

Then setting $y' = \widetilde{h}z'$ and $\widetilde{h}\zeta = \zeta'$ we obtain

$$\begin{aligned} A(\widetilde{h}z) = (2\pi)^{-d} \iint e^{i(z-z')\cdot\zeta'} a\big(X(\widetilde{h}z), X(\widetilde{h}z'), &M\big(\widetilde{h}z, \widetilde{h}z'\big)\widetilde{h}^{-1}\zeta'\big) \\ &J\big(\widetilde{h}z, \widetilde{h}z'\big) w_h(z')\, dz'\, d\zeta'. \end{aligned} \tag{2.62}$$

Our aim is to reduce ourselves to a semi-classical form, after multiplying the equation by $\widetilde{h}$. However this not straightforward since the symbol a is not homogeneous in ξ although γ is homogeneous of order $\frac{1}{2}$. We proceed as follows.

First of all on the support of the function φ_1 (see (2.55)) the function ψ_0 appearing in the definition of σ_{γ_δ} (see (2.38)) is equal to one.

Therefore we can write for $X \in \mathbf{R}^d, \rho \in \mathbf{R}^d$, (skipping the variable t),

$$\begin{aligned}\sigma_{\gamma_\delta}(X, \widetilde{h}^{-1}\rho) &= \int e^{iX\cdot\zeta}\chi\big(\zeta, \widetilde{h}^{-1}\rho\big)\widehat{\gamma}_\delta(\zeta, \widetilde{h}^{-1}\rho)\, d\zeta \\ &= \iint e^{i(X-y)\cdot\zeta}\chi\big(\zeta, \widetilde{h}^{-1}\rho\big)\gamma_\delta(y, \widetilde{h}^{-1}\rho)\, dy\, d\zeta \\ &= \int \widehat{\chi}\big(\mu, \widetilde{h}^{-1}\rho\big)\gamma_\delta\big(X-\mu, \widetilde{h}^{-1}\rho\big)\, d\mu.\end{aligned}$$

Now since χ is homogeneous of degree zero we have,

$$\widehat{\chi}(\mu, \lambda\eta) = \lambda^d \widehat{\chi}(\lambda\mu, \eta),$$

which follows from the fact that $\chi(\zeta, \lambda\eta) = \chi(\lambda\lambda^{-1}\zeta, \lambda\eta) = \chi(\lambda^{-1}\zeta, \eta)$.
Applying this equality with $\lambda = \widetilde{h}^{-2}$ and $\eta = \widetilde{h}\rho$ we obtain,

$$\begin{aligned}\sigma_{\gamma_\delta}(X, \widetilde{h}^{-1}\rho) &= \widetilde{h}^{-2d}\int \widehat{\chi}(\widetilde{h}^{-2}\mu, \widetilde{h}\rho)\gamma_\delta(X-\mu, \widetilde{h}^{-1}\rho)\, d\mu \\ &= \int \widehat{\chi}(\mu', \widetilde{h}\rho)\gamma_\delta(X-\widetilde{h}^2\mu', \widetilde{h}^{-1}\rho)\, d\mu'.\end{aligned}$$

Using the fact that γ and γ_δ are homogeneous of order $\frac{1}{2}$ in ξ we obtain

$$\widetilde{h}\sigma_{\gamma_\delta}(X, \widetilde{h}^{-1}\rho) = \int \widehat{\chi}(\mu, \widetilde{h}\rho)\gamma_\delta(X-\widetilde{h}^2\mu, \widetilde{h}\rho)\, d\mu$$

Now γ_δ is real and by (2.37) (i) $\widehat{\chi}$ is also real, therefore since $\widetilde{h}^{-1}h = \widetilde{h}$ using (2.56) we obtain

$$\widetilde{h}a(X, Y, \widetilde{h}^{-1}\rho) = \frac{1}{2}\int \widehat{\chi}(\mu, \widetilde{h}\rho)\big[\gamma_\delta(X-\widetilde{h}^2\mu, \widetilde{h}\rho) + \gamma_\delta(Y-\widetilde{h}^2\mu, \widetilde{h}\rho)\big]\, d\mu)\varphi_1(\widetilde{h}\rho).$$

It follows then from (2.58),(2.59),(2.62) that the function w_h defined in (2.61) is solution of the equation

$$(\widetilde{h}\partial_t + \widetilde{h}c + iP)w_h(t, z) = \widetilde{h}f_{4,j}(t, X(t, \widetilde{h}z, h)) \tag{2.63}$$

where $c(t, z, \widetilde{h}) = \frac{1}{2}S_{j\delta}(\mathrm{div}V)(t, X(t, \widetilde{h}z))$ and

$$Pw(t, z) = (2\pi\widetilde{h})^{-d}\iint e^{i\widetilde{h}^{-1}(z-z')\cdot\zeta}\widetilde{p}(t, z, z', \zeta, \widetilde{h})w(t, z')\, dz'\, d\zeta \tag{2.64}$$

with
(2.65)

$$\begin{aligned}\widetilde{p}(t, z, z', \zeta, \widetilde{h}) = \frac{1}{2}\int \widehat{\chi}\big(\mu, M(t, \widetilde{h}z, \widetilde{h}z')\zeta\big)\big[\gamma_\delta\big(t, X(t, \widetilde{h}z) - \widetilde{h}^2\mu, M(t, \widetilde{h}z, \widetilde{h}z')\zeta\big) \\ +\gamma_\delta\big(t, X(t, \widetilde{h}z') - \widetilde{h}^2\mu, M(t, \widetilde{h}z, \widetilde{h}z')\zeta\big)\big]d\mu \times \varphi_1(M(t, \widetilde{h}z, \widetilde{h}z')\zeta)J(t, \widetilde{h}z, \widetilde{h}z').\end{aligned}$$

We shall set in what follows

$$p(t,z,\zeta,\widetilde{h}) = \widetilde{p}(t,z,z,\zeta,\widetilde{h}).$$

Since $M(t,\widetilde{h}z,\widetilde{h}z) = M_0(t,\widetilde{h}z)$ and $J(t,\widetilde{h}z,\widetilde{h}z) = 1$ we obtain

$$p(t,z,\zeta,\widetilde{h}) = \int \widehat{\chi}\big(\mu, M_0(t,\widetilde{h}z)\zeta\big)\gamma_\delta\big(t, X(t,\widetilde{h}z) - \widetilde{h}^2\mu, M_0(t,\widetilde{h}z)\zeta\big)\, d\mu \cdot \varphi_1(M_0(t,\widetilde{h}z)\zeta). \tag{2.66}$$

Since the function χ is even with respect to its first variable the symbol p is real. Summing up we have proved that

$$\widetilde{h}\{\partial_t + \frac{1}{2}\big(S_{j\delta}(V)\cdot\nabla + \nabla\cdot S_{j\delta}(V)\big) + i\mathfrak{T}_{\gamma_\delta}\}U(t,x) = \big(\widetilde{h}\partial_t + \widetilde{h}c + iP\big)w_h(t,z) \tag{2.67}$$

where

$$x = X(t,\widetilde{h}z), \quad c(t,z,\widetilde{h}) = \frac{1}{2}S_{j\delta}(\mathrm{div}V)(t,X(t,\widetilde{h}z)), \quad w_h(t,z) = U(t,X(t,\widetilde{h}z)) \tag{2.68}$$

and the self-adjoint operator P is given by (2.64)

2.6.1 Estimates on the pseudo-differential symbol

Let $I_{\widetilde{h}} := [0,\widetilde{h}^\delta]$. We introduce the following norms on the paradifferential symbol γ. For $k \in \mathbf{N}$ we set

$$\mathcal{N}_k(\gamma) = \sum_{|\alpha|\le k}\sup_{\xi\in\mathcal{C}_3}\|D_\xi^\alpha\gamma\|_{L^\infty(I_{\widetilde{h}},L^\infty(\mathbf{R}^d))} + \sum_{|\alpha|\le k}\sup_{\xi\in\mathcal{C}_3}\|D_\xi^\alpha\gamma\|_{L^p(I_{\widetilde{h}},W^{\frac{1}{2},\infty}(\mathbf{R}^d))}, \tag{2.69}$$

where $p = 4$ if $d = 1$ and $p = 2$ if $d \geq 2$. Recall moreover that

$$\|V\|_{E_0} = \|V\|_{L^\infty(I_{\widetilde{h}},L^\infty(\mathbf{R}^d))} + \|V\|_{L^p(I_{\widetilde{h}},W^{1,\infty}(\mathbf{R}^d))} \tag{2.70}$$

We estimate now the derivatives of the symbol of the operator appearing in the right hand side of (2.67).

Lemma 2.18. *For any $\alpha \in \mathbf{N}^d$ there exists $\mathcal{F}_\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ such that for $t \in I_{\widetilde{h}}$*

$$\|(D_z^\alpha c)(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \le \mathcal{F}_\alpha(\|V\|_{E_0})\widetilde{h}^{|\alpha|(1-2\delta)}\|V(t,\cdot)\|_{W^{1,\infty}(\mathbf{R}^d)}.$$

Proof. By the Faa-di-Bruno formula $D_z^\alpha c$ is a finite linear combination of terms of the form

$$(1) = \widetilde{h}^{|\alpha|}D_x^a\big[S_{j\delta}(\mathrm{div}V)\big](t,X(t,\widetilde{h}z))\prod_{j=1}^r\Big(\big(D_x^{l_j}X\big)(t,\widetilde{h}z)\Big)^{p_j} \tag{2.71}$$

where $1 \leq |a| \leq |\alpha|$, $|l_j| \geq 1$, $\sum_{j=1}^r |p_j| l_j = \alpha$, $\sum_{j=1}^r p_j = a$. For fixed t we have

$$|D_x^a \big[S_{j\delta}(\mathrm{div} V)\big](t,\cdot)| \leq C_\alpha \widetilde{h}^{-2\delta|a|} \|V(t,\cdot)\|_{W^{1,\infty}(\mathbf{R}^d)} \tag{2.72}$$

and by Proposition 2.16

$$|(D_x^{l_j} X)(t,\cdot)| \leq \mathcal{F}_\alpha(\|V\|_{E_0}) \widetilde{h}^{-2\delta(|l_j|-1)}.$$

The product appearing in the term (1) is bounded by $\mathcal{F}_\alpha(\|V\|_{E_0})\widetilde{h}^M$ where $M = -2\delta \sum_{j=1}^r |p_j|(|l_j|-1) = -2\delta|\alpha| + 2\delta|a|$. The lemma follows from (2.71) and (2.72). □

Lemma 2.19. *For every $k \in \mathbf{N}$ there exist $\mathcal{F}_k : \mathbf{R}^+ \to \mathbf{R}^+$ such that,*

$$|D_z^\alpha D_\zeta^\beta p(t,z,\zeta,\widetilde{h})| \leq \mathcal{F}_k(\|V\|_{E_0}) \sum_{|a|\leq k} \sup_{\xi \in \mathcal{C}_3} \|D_\xi^a \gamma(t,\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \widetilde{h}^{|\alpha|(1-2\delta)}$$

for all $|\alpha| + |\beta| \leq k$ and all $(t,z,\zeta,\widetilde{h}) \in I_{\widetilde{h}} \times \mathbf{R}^d \times \mathcal{C}_1 \times (0,\widetilde{h}_0]$.

Corollary 2.20. *For every $k \in \mathbf{N}$ there exist $\mathcal{F}_k : \mathbf{R}^+ \to \mathbf{R}^+$ such that,*

$$\int_0^s |D_z^\alpha D_\zeta^\beta p(t,z,\zeta,\widetilde{h})|\, dt \leq \mathcal{F}_k(\|V\|_{E_0}) \mathcal{N}_k(\gamma))\, \widetilde{h}^{|\alpha|(1-2\delta)+\delta}$$

for all $|\alpha| + |\beta| \leq k$ and all $(s,z,\zeta,\widetilde{h}) \in I_{\widetilde{h}} \times \mathbf{R}^d \times \mathcal{C}_1 \times (0,\widetilde{h}_0]$.

Proof of Lemma 2.19. Here t is considered as a parameter which will be skipped, keeping in mind that the estimates should be uniform with respect to $t \in [0, \widetilde{h}^\delta]$. On the other hand we recall that, by Proposition 2.16 and Lemma 2.10, we have (since $h = \widetilde{h}^2$)

$$|D_x^\alpha X(x)| \leq \mathcal{F}_\alpha(\|V\|_{E_0})\, \widetilde{h}^{-2\delta(|\alpha|-1)}, \quad |\alpha| \geq 1, \beta \in \mathbf{N}^d \tag{2.73}$$

$$|D_x^\alpha D_\xi^\beta \gamma_\delta(x,\xi)| \leq C_{\alpha,\beta}\, \widetilde{h}^{-2\delta|\alpha|} \|D_\xi^\beta \gamma(\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)}, \quad \alpha,\beta \in \mathbf{N}^d. \tag{2.74}$$

Set

$$F(\mu,z,\zeta,\widetilde{h}) = \widehat{\chi}\big(\mu, M_0(z)\zeta\big)\varphi_1(M_0(z)\zeta)\gamma_\delta\big(X(z) - \widetilde{h}^2\mu, M_0(z)\zeta\big),$$

the lemma will follow immediately from the fact that for every $N \in \mathbf{N}$ we have

$$\begin{aligned} &|D_z^\alpha D_\zeta^\beta F(\mu,z,\zeta,\widetilde{h})| \\ &\qquad \leq \mathcal{F}_{\alpha,\beta}(\|V\|_{E_0}) \sum_{|a|\leq|\alpha|+|\beta|} \sup_{\xi \in \mathcal{C}_3} \|D_\xi^a \gamma(\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \widetilde{h}^{-2\delta|\alpha|} C_N \langle\mu\rangle^{-N}. \end{aligned} \tag{2.75}$$

If we call $m_{ij}(z)$ the entries of the matrix $M_0(z)$ we see easily that $D_\zeta^\beta F$ is a finite linear combination of terms of the form

$$(D_\xi^{\beta_1}(\widehat{\chi}\varphi_1))(\mu, M_0(z)\zeta) \cdot (D_\xi^{\beta_2}\gamma_\delta)(X(z) - \widetilde{h}^2\mu, M_0(z)\zeta) \cdot P_{|\beta|}(m_{ij}(z)) := G_1 \cdot G_2 \cdot G_3 \tag{2.76}$$

where $P_{|\beta|}$ is a polynomial of order $|\beta|$.

The estimate (2.75) will follow from the following ones.

$$|D_z^\alpha G_1| \leq \mathcal{F}_{\alpha,\beta}(\|V\|_{E_0})\widetilde{h}^{-2\delta|\alpha|} C_N \langle \mu \rangle^{-N}, \tag{2.77}$$

$$|D_z^\alpha G_2| \leq \mathcal{F}_{\alpha,\beta}(\|V\|_{E_0})) \sum_{|a|\leq|\alpha|+|\beta|} \sup_{\xi\in\mathcal{C}_3} \|D_\xi^a \gamma(\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \widetilde{h}^{-2\delta|\alpha|} \tag{2.78}$$

$$|D_z^\alpha G_3| \leq \mathcal{F}_{\alpha,\beta}(\|V\|_{E_0})\widetilde{h}^{-2\delta|\alpha|}. \tag{2.79}$$

Using the equality ${}^t\big(\frac{\partial X}{\partial z}\big)(z)M_0(z)\zeta = \zeta$, Proposition 2.16 and an induction we see that

$$|D_z^\alpha m_{ij}(z)| \leq \mathcal{F}_\alpha(\|V\|_{E_0})\widetilde{h}^{-2\delta|\alpha|+\frac{\delta}{2}} \tag{2.80}$$

from which (2.79) follows since G_3 is polynomial. Now according to the Faa-di-Bruno formula $D_z^\alpha G_1$ is a finite linear combination of terms of the form

$$D_\xi^{\beta_1+b}(\widehat{\chi}\varphi_1))(\mu, M_0(z)\zeta) \prod_{j=1}^r \Big(D_z^{l_j} M_0(z)\zeta\Big)^{p_j},$$
$$1 \leq |b| \leq |\alpha|, |l_j| \geq 1, \sum_{j=1}^r |p_j| l_j = \alpha, \sum_{j=1}^r p_j = b.$$

Then (2.77) follows immediately from Lemma 2.14 and (2.80). By the same formula we see that $D_z^\alpha G_2$ is a linear combination of terms of the form

$$(D_z^a D_\xi^{\beta_2+b}\gamma_\delta)(X(z) - \widetilde{h}^2\mu, M_0(z)\zeta) \prod_{j=1}^r \Big(D_z^{l_j} X(z)\Big)^{p_j} \Big(D_z^{l_j} M_0(z)\zeta\Big)^{q_j}$$

where $1 \leq |a| + |b| \leq |\alpha|$, $\sum_{j=1}^r (|p_j| + |q_j|) l_j = \alpha$, $\sum_{j=1}^r p_j = a$, $\sum_{j=1}^r q_j = b$. Then (2.78) follows (2.73), (2.74) and (2.80). The proof is complete. □

Remark 2.21. By exactly the same method we can show that we have the estimate

$$|D_z^{\alpha_1} D_{z'}^{\alpha_2} D_\zeta^\beta \widetilde{p}(t, z, z', \zeta, \widetilde{h})| \leq \mathcal{F}_k(\|V\|_{E_0}\mathcal{N}_k(\gamma))\, \widetilde{h}^{(|\alpha_1|+|\alpha_2|)(1-2\delta)} \tag{2.81}$$

for all $|\alpha_1| + |\alpha_2| + |\beta| \leq k$ and all $(t, z, z', \zeta, \widetilde{h}) \in I_{\widetilde{h}} \times \mathbf{R}^d \times \mathbf{R}^d \times \mathcal{C}_1 \times (0, \widetilde{h}_0]$.

Proposition 2.22. *There exist $T_0 > 0, c_0 > 0, \widetilde{h}_0 > 0$ such that*

$$\left|\det\Big(\frac{\partial^2 p}{\partial\zeta_j\partial\zeta_k}(t, z, \zeta, \widetilde{h})\Big)\right| \geq c_0$$

for any $t \in [0, T_0], z \in \mathbf{R}^d, \zeta \in \mathcal{C}_0 = \{\frac{1}{2} \leq |\zeta| \leq 2\}, 0 < \widetilde{h} \leq \widetilde{h}_0$.

Proof. By Proposition 2.16, (2.60) and (2.55) we have $\varphi_1(M_0(t,\widetilde{h}z)\zeta) = 1$. Let us set

$$M_0(t,\widetilde{h}z) = (m_{ij}) \text{ and } M_0(t,\widetilde{h}z)\zeta = \rho.$$

Then we have,

$$\frac{\partial^2 p}{\partial\zeta_j\partial\zeta_k}(t,z,\zeta,\widetilde{h}) = A_1 + A_2 + A_3,$$

where

$$\begin{aligned}
A_1 &= \sum_{l,r=1}^{d} \int \frac{\partial^2\widehat{\chi}}{\partial\zeta_l\partial\zeta_r}(\mu,\rho) m_{lj} m_{rk} \gamma_\delta\big(t, X(t,\widetilde{h}z) - \widetilde{h}^2\mu, \rho\big)\, d\mu, \\
A_2 &= 2\sum_{l,r=1}^{d} \int \frac{\partial\widehat{\chi}}{\partial\zeta_l}(\mu,\rho) \frac{\partial\gamma_\delta}{\partial\zeta_r}\big(t, X(t,\widetilde{h}z) - \widetilde{h}^2\mu, \rho\big) m_{lj} m_{rk}\, d\mu, \\
A_3 &= \sum_{l,r=1}^{d} \int \widehat{\chi}(\mu,\rho) \frac{\partial^2\gamma_\delta}{\partial\zeta_l\partial\zeta_r}\big(t, X(t,\widetilde{h}z) - \widetilde{h}^2\mu, \rho\big) m_{lj} m_{rk}\, d\mu.
\end{aligned} \tag{2.82}$$

Now we notice that by (2.37) we have

$$\int (\partial_\zeta^\alpha \widehat{\chi})(\mu,\rho) d\mu = (2\pi)^d (\partial_\zeta^\alpha \chi)(0,\rho) = \begin{cases} 0, & \alpha \neq 0 \\ (2\pi)^d, & \alpha = 0. \end{cases}$$

Using this remark we can write

$$A_1 = \sum_{l,r=1}^{d} \int \frac{\partial^2\widehat{\chi}}{\partial\zeta_l\partial\zeta_r}(\mu,\rho) m_{lj} m_{rk} \Big[\gamma_\delta\big(t, X(t,\widetilde{h}z) - \widetilde{h}^2\mu, \rho\big) - \gamma_\delta\big(t, X(t,\widetilde{h}z), \rho\big)\Big]\, d\mu.$$

Now recall (see (2.23) and Lemma 2.8) that for bounded $|\zeta|$ (considered as a parameter) we have for all $\alpha \in \mathbf{N}^d$

$$\partial_\zeta^\alpha \gamma_\delta \in L^\infty(I; H^{s-\frac{1}{2}}(\mathbf{R}^d)) \subset L^\infty(I; W^{s_0,\infty}(\mathbf{R}^d)), \quad s_0 > 0,$$

uniformly in ζ. Since, by Proposition 2.16, $\|M_0(t,\widetilde{h}z)\|$ is uniformly bounded we can write

$$|A_1| \leq C\widetilde{h}^{2s_0} \sum_{l,r=1}^{d} \int |\mu|^{s_0} \left| \frac{\partial^2\widehat{\chi}}{\partial\zeta_l\partial\zeta_r}(\mu,\rho) \right| d\mu,$$

the integral in the right hand side being bounded by Lemma 2.14.

By exactly the same argument we see that we have the following inequality

$$|A_2| \leq C\widetilde{h}^{2s_0}.$$

Moreover one can write

$$A_3 = \sum_{l,r=1}^{d} \frac{\partial^2\gamma_\delta}{\partial\zeta_l\partial\zeta_r}\big(t, X(t,\widetilde{h}z), \rho\big) m_{lj} m_{rk} \Big(\int \widehat{\chi}(\mu,\rho)\, d\mu\Big) + \mathcal{O}(\widetilde{h}^{2s_0}).$$

Gathering the estimates we see that

$$(2\pi)^{-d}\Big(\frac{\partial^2 p}{\partial\zeta_j\partial\zeta_k}(t,z,\zeta,\widetilde{h})\Big) = {}^tM_0(t,\widetilde{h}z)\mathrm{Hess}_\zeta(\gamma_\delta)(t,z,\zeta,\widetilde{h})M_0(t,\widetilde{h}z) + \mathcal{O}(\widetilde{h}^{2s_0}).$$

Then our claim follows from Proposition 2.11 and Proposition 2.16 if $\widetilde{h}_0$ is small enough. □

2.7 The parametrix

Our aim is to construct a parametrix for the operator $L = \widetilde{h}\partial_t + \widetilde{h}c + iP$ on a time interval of size $\widetilde{h}^\delta$ where $\delta = \frac{2}{3}$. This parametrix will be of the following form

$$\mathcal{K}v(t,z) = (2\pi\widetilde{h})^{-d}\iint e^{i\widetilde{h}^{-1}(\phi(t,z,\xi,\widetilde{h})-y\cdot\xi)}\widetilde{b}(t,z,y,\xi,\widetilde{h})v(y)\,dyd\xi.$$

Here ϕ is a real valued phase such that $\phi|_{t=0} = z\cdot\xi$, $\widetilde{b}$ is of the form

$$\widetilde{b}(t,z,y,\xi,\widetilde{h}) = b(t,z,\xi,\widetilde{h})\Psi_0\Big(\frac{\partial\phi}{\partial\xi}(t,z,\xi,\widetilde{h}) - y\Big) \tag{2.83}$$

where $b|_{t=0} = \chi(\xi), \chi \in C_0^\infty(\mathbf{R}^d\setminus\{0\})$ and $\Psi_0 \in C_0^\infty(\mathbf{R}^d)$ is such that $\Psi_0(t) = 1$ if $|t| \leq 1$.

More precisely, we shall define in Proposition 2.23 a phase ϕ and in (2.120) a symbol b such that we have

$$e^{-i\widetilde{h}^{-1}\phi}(\widetilde{h}\partial_t + \widetilde{h}c + iP)\big(e^{i\widetilde{h}^{-1}\phi}\widetilde{b}\big) = \mathfrak{R}_N$$

where $\mathfrak{R}_N$ is a negligible remainder.

2.7.1 Preliminaries

An important step in this construction is to compute the expression

$$J(t,z,y,\xi,\widetilde{h}) = e^{-i\widetilde{h}^{-1}\phi(t,z,\xi,\widetilde{h})}P(t,z,D_z)\big(e^{i\widetilde{h}^{-1}\phi(t,\cdot,\xi,\widetilde{h})}\widetilde{b}(t,\cdot,y,\xi,\widetilde{h})\big). \tag{2.84}$$

In this computation since $(t,y,\xi,\widetilde{h})$ are fixed we shall skip them and write $\phi = \phi(z), \widetilde{b} = \widetilde{b}(z)$.

Using (2.64) we obtain

$$J = (2\pi\widetilde{h})^{-d}\iint e^{i\widetilde{h}^{-1}(\phi(z')-\phi(z)+(z-z')\cdot\zeta)}\widetilde{p}(z,z',\zeta)\widetilde{b}(z')\,dz'd\zeta.$$

Then we write

$$\phi(z')-\phi(z) = \theta(z,z')\cdot(z'-z), \quad \theta(z,z') = \int_0^1 \frac{\partial\phi}{\partial z}(\lambda z+(1-\lambda)z')\,d\lambda. \tag{2.85}$$

Using this equality and setting $\zeta - \theta(z,z') = \eta$ in the integral we obtain

$$J = (2\pi\widetilde{h})^{-d} \iint e^{i\widetilde{h}^{-1}(z-z')\cdot\eta} \widetilde{p}(z,z',\eta + \theta(z,z'))\widetilde{b}(z')\, dz' d\eta.$$

The phase that we will obtain will be uniformly bounded, say $|\frac{\partial\phi}{\partial z}| \leq C_0$. It also can be seen that, due to the cut-off φ_1 in the expression of $\widetilde{p}$ and to Proposition 2.16, we also have $|\eta + \theta(z,z')| \leq C_0$. Therefore $|\eta| \leq 2C_0$. Let $\kappa \in C_0^\infty(\mathbf{R}^d)$ be such that $\kappa(\eta) = 1$ if $|\eta| \leq 2C_0$. Then we can write

$$J = (2\pi\widetilde{h})^{-d} \iint e^{i\widetilde{h}^{-1}(z-z')\cdot\eta} \kappa(\eta)\widetilde{p}(z,z',\eta + \theta(z,z'))\widetilde{b}(z')\, dz'\, d\eta.$$

By the Taylor formula we can write

$$\widetilde{p}(z,z',\eta + \theta(z,z')) = \sum_{|\alpha| \leq N-1} \frac{1}{\alpha!} (\partial_\eta^\alpha \widetilde{p})(z,z',\theta(z,z'))\eta^\alpha + r_N$$

$$r_N = \sum_{|\alpha| = N} \frac{N}{\alpha!} \int_0^1 (1-\lambda)^{N-1} (\partial_\eta^\alpha \widetilde{p})(z,z',\eta + \lambda\theta(z,z'))\eta^\alpha\, d\lambda.$$

It follows that
(2.86)

$$\begin{cases} J = J_N + R_N \\ J_N = \displaystyle\sum_{|\alpha| \leq N-1} \frac{(2\pi\widetilde{h})^{-d}}{\alpha!} \iint e^{i\widetilde{h}^{-1}(z-z')\cdot\eta} \kappa(\eta)(\partial_\eta^\alpha \widetilde{p})(z,z',\theta(z,z'))\eta^\alpha \widetilde{b}(z')\, dz' d\eta \\ R_N = (2\pi\widetilde{h})^{-d} \displaystyle\iint e^{i\widetilde{h}^{-1}(z-z')\cdot\eta} \kappa(\eta) r_N(z,z',\eta)\widetilde{b}(z')\, dz' d\eta. \end{cases}$$

Using the fact that $\eta^\alpha e^{i\widetilde{h}^{-1}(z-z')\cdot\eta} = (-\widetilde{h}D_{z'})^\alpha e^{i\widetilde{h}^{-1}(z-z')\cdot\eta}$ and integrating by parts in the integral with respect to z we get

$$J_N = (2\pi\widetilde{h})^{-d} \sum_{|\alpha| \leq N-1} \frac{\widetilde{h}^{|\alpha|}}{\alpha!} \iint e^{i\widetilde{h}^{-1}(z-z')\cdot\eta} \kappa(\eta) D_{z'}^\alpha \big[(\partial_\eta^\alpha \widetilde{p})(z,z',\theta(z,z'))\widetilde{b}(z')\big]\, dz' d\eta.$$

Therefore we can write

$$J_N = (2\pi\widetilde{h})^{-d} \sum_{|\alpha| \leq N-1} \frac{\widetilde{h}^{|\alpha|}}{\alpha!} \int \widehat{\kappa}\Big(\frac{z'-z}{\widetilde{h}}\Big) D_{z'}^\alpha \big[(\partial_\eta^\alpha \widetilde{p})(z,z',\theta(z,z'))\widetilde{b}(z')\big]\, dz'.$$

Let us set

$$f_\alpha(z,z',\widetilde{h}) = D_{z'}^\alpha \big[(\partial_\eta^\alpha \widetilde{p})(z,z',\theta(z,z'))\widetilde{b}(z')\big] \tag{2.87}$$

and then, $z' - z = \widetilde{h}\mu$ in the integral. We obtain

$$J_N = (2\pi)^{-d} \sum_{|\alpha| \leq N-1} \frac{\widetilde{h}^{|\alpha|}}{\alpha!} \int \widehat{\kappa}(\mu) f_\alpha(z, z + \widetilde{h}\mu, \widetilde{h})\, d\mu.$$

By the Taylor formula we can write

$$J_N = (2\pi)^{-d} \sum_{|\alpha|\le N-1} \frac{\widetilde{h}^{|\alpha|}}{\alpha!} \sum_{|\beta|\le N-1} \frac{\widetilde{h}^{|\beta|}}{\beta!} \Big(\int \mu^\beta \widehat{\kappa}(\mu)\, d\mu \Big) \big(\partial_{z'}^\beta f_\alpha\big)(z, z, \widetilde{h}) + S_N,$$

with
(2.88)

$$S_N = (2\pi)^{-d} \sum_{\substack{|\alpha|\le N-1\\ |\beta|=N}} N \frac{\widetilde{h}^{|\alpha|+|\beta|}}{\alpha!\beta!} \int\int_0^1 (1-\lambda)^{N-1} \mu^\beta \widehat{\kappa}(\mu) \big(\partial_{z'}^\beta f_\alpha\big)(z, z+\lambda\widetilde{h}\mu, \widetilde{h})\, d\lambda\, d\mu.$$

Noticing that

$$\int \mu^\beta \widehat{\kappa}(\mu)\, d\mu = (2\pi)^d (D^\beta \kappa)(0) = \begin{cases} 0 & \text{if } \beta \neq 0 \\ (2\pi)^d & \text{if } \beta = 0 \end{cases}$$

we conclude that

$$J_N = \sum_{|\alpha|\le N-1} \frac{\widetilde{h}^{|\alpha|}}{\alpha!} f_\alpha(z, z, \widetilde{h}) + S_N.$$

It follows from (2.86), (2.87) and (2.88) that

$$J = \sum_{|\alpha|\le N-1} \frac{\widetilde{h}^{|\alpha|}}{\alpha!} D_{z'}^\alpha \big[(\partial_\eta^\alpha \widetilde{p})(z, z', \theta(z, z')) \widetilde{b}(z') \big] |_{z'=z} + R_N + S_N \tag{2.89}$$

where R_N and S_N are defined in (2.86) and (2.88).

Reintroducing the variable $(t, y, \xi, \widetilde{h})$ we conclude from (2.84) that
(2.90)

$$e^{-i\widetilde{h}^{-1}\phi(t,z,\xi,\widetilde{h})} (\widetilde{h}\partial_t + \widetilde{h}c + iP) \big(e^{i\widetilde{h}^{-1}\phi(t,z,\xi,\widetilde{h})} \widetilde{b} \big) = \Big[i \frac{\partial \phi}{\partial t} \widetilde{b} + iJ + \widetilde{h} \frac{\partial \widetilde{b}}{\partial t} + \widetilde{h} c \widetilde{b} \Big] (t, z, y, \xi, \widetilde{h}).$$

We shall gather the terms the right hand side of (2.90) according to the power of $\widetilde{h}$. The term corresponding to $\widetilde{h}^0$ leads to the eikonal equation.

2.7.2 The eikonal equation

It is the equation

$$\frac{\partial \phi}{\partial t} + p\Big(t, z, \frac{\partial \phi}{\partial z}, \widetilde{h}\Big) = 0 \quad \phi(0, z, \xi, \widetilde{h}) = z \cdot \xi \tag{2.91}$$

where p is defined by the formula
(2.92)

$$p(t, z, \zeta, \widetilde{h}) = \int \widehat{\chi}\big(\mu, M_0(t, \widetilde{h}z)\zeta\big) \gamma_\delta\big(t, X(t, \widetilde{h}z) - \widetilde{h}^2\mu, M_0(t, \widetilde{h}z)\zeta\big)\, d\mu \cdot \varphi_1(M_0(t, \widetilde{h}z)\zeta).$$

We set

$$q(t, z, \tau, \zeta, \widetilde{h}) = \tau + p(t, z, \zeta, \widetilde{h})$$

and for $j \geq 1$ we denote by $\mathcal{C}_j$ the ring

$$\mathcal{C}_j = \{\xi \in \mathbf{R}^d : 2^{-j} \leq |\xi| \leq 2^j\}.$$

Moreover in all what follows we shall have

$$\delta = \frac{2}{3}. \tag{2.93}$$

The solution of the eikonal equation

Recall that $I_{\widetilde{h}} = [0, \widetilde{h}^\delta]$ is the time interval, (where $\delta = \frac{2}{3}$) and $\mathcal{C}_j$ the ring $\{2^{-j} \leq |\xi| \leq 2^j\}$. Consider the null-bicharacteristic flow of $q(t, z, \tau, \zeta, \widetilde{h}) = \tau + p(t, z, \zeta, \widetilde{h})$. It is defined by the system

$$\begin{cases} \dot{t}(s) = 1, \quad t(0) = 0, \\ \dot{z}(s) = \dfrac{\partial p}{\partial \zeta}\big(t(s), z(s), \zeta(s), \widetilde{h}\big), \quad z(0) = z_0, \\ \dot{\tau}(s) = -\dfrac{\partial p}{\partial t}\big(t(s), z(s), \zeta(s), \widetilde{h}\big), \quad \tau(0) = -p(0, z_0, \xi, \widetilde{h}), \\ \dot{\zeta}(s) = -\dfrac{\partial p}{\partial z}\big(t(s), z(s), \zeta(s), \widetilde{h}\big), \quad \zeta(0) = \xi. \end{cases} \tag{2.94}$$

Then $t(s) = s$ and this system has a unique solution defined on $I_{\widetilde{h}}$, depending on $(s, z_0, \xi, \widetilde{h})$.

We claim that for all fixed $s \in I_{\widetilde{h}}$ and $\xi \in \mathbf{R}^d$, the map

$$z_0 \mapsto z(s; z_0, \xi, \widetilde{h})$$

is a global diffeomorphism from $\mathbf{R}^d$ to $\mathbf{R}^d$. This will follow from the facts that this map is proper and the matrix $\big(\frac{\partial z}{\partial z_0}(s; z_0, \xi, \widetilde{h})\big)$ is invertible. Let us begin by the second point.

Let us set $m(s) = (s, z(s), \zeta(s), \widetilde{h})$. Differentiating System (2.94) with respect to z_0, we get

$$\begin{aligned} &\frac{d}{ds}\left(\frac{\partial z}{\partial z_0}\right)(s) = p''_{z\zeta}(m(s))\frac{\partial z}{\partial z_0}(s) + p''_{\zeta\zeta}(m(s))\frac{\partial \zeta}{\partial z_0}(s), \quad \frac{\partial z}{\partial z_0}(0) = Id \\ &\frac{d}{ds}\left(\frac{\partial \zeta}{\partial z_0}\right)(s) = -p''_{zz}(m(s))\frac{\partial z}{\partial z_0}(s) - p''_{z\zeta}(m(s))\frac{\partial \zeta}{\partial z_0}(s), \quad \frac{\partial \zeta}{\partial z_0}(0) = 0. \end{aligned} \tag{2.95}$$

Setting $U(s) = (\frac{\partial z}{\partial z_0}(s), \frac{\partial \zeta}{\partial z_0}(s))$ and

$$\mathcal{A}(s) = \begin{pmatrix} p''_{z\zeta}(m(s)) & p''_{\zeta\zeta}(m(s)) \\ -p''_{zz}(m(s)) & -p''_{z\zeta}(m(s)) \end{pmatrix}. \tag{2.96}$$

The system (2.95) can be written as $\dot{U}(s) = \mathcal{A}(s)U(s), U(0) = (Id, 0)$. Lemma 2.19 gives

$$
\begin{aligned}
(2.97)\quad & |p''_{\zeta\zeta}(m(s))| + |p''_{z\zeta}(m(s))| + |p''_{zz}(m(s))| \\
& \qquad \leq \mathcal{F}(\|V\|_{E_0}) \sum_{|\beta|\leq 2} \sup_{\xi\in\mathcal{C}_3} \|D_\xi^\beta \gamma(t,\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \widetilde{h}^{2(1-2\delta)},
\end{aligned}
$$

therefore

$$
\|\mathcal{A}(s)\| \leq \mathcal{F}(\|V\|_{E_0}) \sum_{|\beta|\leq 2} \sup_{\xi\in\mathcal{C}_3} \|D_\xi^\beta \gamma(t,\cdot,\xi)\|_{L^\infty(\mathbf{R}^d)} \widetilde{h}^{2(1-2\delta)}.
$$

Using the equality $2(1-2\delta)+\delta = 0$, we deduce that for $s \in I_{\widetilde{h}} = (0, \widetilde{h}^\delta)$ we have

$$
\int_0^s \|\mathcal{A}(\sigma)\|\, d\sigma \leq \mathcal{F}(\|V\|_{E_0})\mathcal{N}_2(\gamma). \tag{2.98}
$$

The Gronwall inequality shows that $\|U(s)\|$ is uniformly bounded on $I_{\widetilde{h}}$. Coming back to (2.95) we see that we have

$$
\Big|\frac{\partial \zeta}{\partial z_0}(s)\Big| \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_2(\gamma)\big), \quad \Big|\frac{\partial z}{\partial z_0}(s) - Id\Big| \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_2(\gamma)\big)\widetilde{h}^{\frac{\delta}{2}}. \tag{2.99}
$$

Taking $\widetilde{h}$ small enough we obtain the invertibility of the matrix $\big(\frac{\partial z}{\partial z_0}(s; z_0, \xi, \widetilde{h})\big)$.
Now we have

$$
|z(s; z_0, \xi, \widetilde{h}) - z_0| \leq \int_0^s |\dot{z}(\sigma, x_0, \xi, \widetilde{h})|\, d\sigma.
$$

Since the right hand side is uniformly bounded for $s \in \big[0, \widetilde{h}^\delta\big]$, we see that our map is proper. Therefore we can write

$$
z(s; z_0, \xi, \widetilde{h}) = z \Longleftrightarrow z_0 = \kappa(s; z, \xi, \widetilde{h}). \tag{2.100}
$$

Let us set for $t \in \big[0, \widetilde{h}^\delta\big]$

$$
\phi(t, z, \xi, \widetilde{h}) = z\cdot\xi - \int_0^t p\big(\sigma, z, \zeta(\sigma; \kappa(\sigma; z, \xi, \widetilde{h}), \xi, \widetilde{h}), \widetilde{h}\big)\, d\sigma. \tag{2.101}
$$

Proposition 2.23. *The function ϕ defined in (2.101) is the solution of the eikonal equation (2.91).*

Proof. The initial condition is trivially satisfied. Moreover we have

$$
\frac{\partial \phi}{\partial t}\big(t, z, \xi, \widetilde{h}\big) = -p\big(t, z, \zeta(t; \kappa\big(t; z, \xi, \widetilde{h}\big), \xi, \widetilde{h}), \tilde{h}\big).
$$

Therefore it is sufficient to prove that

$$
\frac{\partial \phi}{\partial z}\big(t, z, \xi, \widetilde{h}\big) = \zeta\big(t; \kappa\big(t; z, \xi, \widetilde{h}\big), \xi, \widetilde{h}\big). \tag{2.102}
$$

Let us consider the Lagrangean manifold

$$\Sigma = \Big\{ \big(t, z(t;z_0,\xi,\widetilde{h}), \tau(t;z_0,\xi,\widetilde{h}), \zeta(t;z_0,\xi,\widetilde{h})\big) : t \in I_{\widetilde{h}}, (z_0,\xi) \in \mathbf{R}^{2d} \Big\}. \tag{2.103}$$

According to (2.100) we can write

$$\Sigma = \Big\{ (t, z, \tau(t;\kappa(t;z,\xi,\widetilde{h}),\xi,\widetilde{h}), \zeta(t;\kappa(t;z,\xi,\widetilde{h}),\xi,\widetilde{h})) : t \in I_{\widetilde{h}}, (z,\xi) \in \mathbf{R}^{2d} \Big\}.$$

Let us set

$$\begin{aligned} F_0\big(t,z,\xi,\widetilde{h}\big) &= \tau\big(t;\kappa(t;z,\xi,\widetilde{h}),\xi,\widetilde{h}\big),\\ F_j\big(t,z,\xi,\widetilde{h}\big) &= \zeta_j\big(t;\kappa(t;z,\xi,\widetilde{h}),\xi,\widetilde{h}\big). \end{aligned}$$

Since the symbol q is constant along its bicharacteristic and $q(0,z(0),\tau(0),\zeta(0),\widetilde{h}) = 0$ we have

$$F_0\big(t,z,\xi,\widetilde{h}\big) = -p\big(t,z,\zeta(t;\kappa(\sigma;z,\xi,\widetilde{h}),\xi,\widetilde{h}),\widetilde{h}\big).$$

Now Σ being Lagrangean we have

$$\mathrm{d}\, t \wedge \mathrm{d}\, F_0 + \mathrm{d}\, z \wedge \mathrm{d}\, F = 0.$$

Thus $\partial_{z_j} F_0 - \partial_t F_j = 0$ since it is the coefficient of $\mathrm{d}\, t \wedge \mathrm{d}\, z_j$ in the above expression. Therefore using (2.101) we can write

$$\begin{aligned} \frac{\partial \phi}{\partial z_j}(t,z,\xi,\widetilde{h}) &= \xi_j - \int_0^t \frac{\partial}{\partial z_j}\big[p\big(\sigma,z,\zeta(\sigma;\kappa(\sigma;z,\xi,\widetilde{h}),\xi,\widetilde{h})\big)\big]\, d\sigma \\ &= \xi_j + \int_0^t \frac{\partial}{\partial \sigma}\big[\zeta_j\big(\sigma;\kappa(\sigma;z,\xi,\widetilde{h}),\xi,\widetilde{h}\big)\big]\, d\sigma \\ &= \zeta_j\big(t;\kappa(t;z,\xi,\widetilde{h}),\xi,\widetilde{h}\big). \end{aligned}$$

□

The Hessian of the phase

Let us recall that the phase ϕ is the solution of the problem

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(t,z,\xi,\widetilde{h}) + p\Big(t,z,\dfrac{\partial \phi}{\partial z}(t,z,\xi,\widetilde{h}),\widetilde{h}\Big) = 0 \\ \phi|_{t=0} = z\cdot\xi. \end{cases} \tag{2.104}$$

On the other hand the map $(t,z,\xi) \mapsto \phi(t,z,\xi,\widetilde{h})$ is C^1 in time and C^∞ in (x,ξ). Differentiating twice, with respect to ξ, the above equation we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\Big(\frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j}\Big) = &- \sum_{k,l=1}^{d} \frac{\partial^2 p}{\partial \zeta_k \partial \zeta_l}\Big(t,z,\frac{\partial \phi}{\partial z},\widetilde{h}\Big) \frac{\partial^2 \phi}{\partial z_k \partial \xi_i}\frac{\partial^2 \phi}{\partial z_l \partial \xi_j} \\ &- \sum_{k=1}^{d} \frac{\partial p}{\partial \zeta_k}\Big(t,z,\frac{\partial \phi}{\partial z},\widetilde{h}\Big) \frac{\partial^3 \phi}{\partial z_k \partial \xi_i \partial \xi_j}. \end{aligned}$$

By the initial condition in (2.104) we have

$$\frac{\partial^2\phi}{\partial z_k\partial\xi_i}\Big|_{t=0}=\delta_{ki}\quad \frac{\partial^2\phi}{\partial z_l\partial\xi_j}\Big|_{t=0}=\delta_{lj},\quad \frac{\partial^3\phi}{\partial z_k\partial\xi_i\partial\xi_j}\Big|_{t=0}=0,\quad \frac{\partial^2\phi}{\partial\xi_i\partial\xi_j}\Big|_{t=0}=0.$$

It follows that

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2\phi}{\partial\xi_i\partial\xi_j}\Big)\Big|_{t=0}=-\frac{\partial^2 p}{\partial\xi_i\partial\xi_j}(0,z,\xi,\widetilde{h})$$

from which we deduce that

$$\frac{\partial^2\phi}{\partial\xi_i\partial\xi_j}(t,z,\xi,\widetilde{h})=-t\frac{\partial^2 p}{\partial\xi_i\partial\xi_j}(0,z,\xi,\widetilde{h})+o(t).$$

It follows from Proposition 2.22 that one can find $M_0>0$ such that

$$\Big|\det\Big(\frac{\partial^2\phi}{\partial\xi_i\partial\xi_j}(t,z,\xi,\widetilde{h})\Big)\Big|\geq M_0 t^d, \tag{2.105}$$

for $t\in I_{\widetilde{h}}, z\in\mathbf{R}^d, \xi\in\mathcal{C}_0, 0<\widetilde{h}\leq\widetilde{h}_0$.

Our goal now is to prove estimates of higher order on the phase (see Corollary 2.28 below.)

Classes of symbol and symbolic calculus

Recall here that $\delta=\frac{2}{3}$ and that $\mathcal{N}_k(\gamma)$ has been defined in (2.69).

Definition 2.24. *Let $m\in\mathbf{R},\mu_0\in\mathbf{R}^+$ and $a=a(t,z,\xi,\widetilde{h})$ be a smooth function defined on $\Omega=[0,\widetilde{h}^\delta]\times\mathbf{R}^d\times\mathcal{C}_0\times(0,\widetilde{h}_0]$. We shall say that*

(i) $a\in S^m_{\mu_0}$ if for every $k\in\mathbf{N}$ one can find $\mathcal{F}_k:\mathbf{R}^+\to\mathbf{R}^+$ such that for all $(t,z,\xi,\widetilde{h})\in\Omega$

$$|D_z^\alpha D_\xi^\beta a(t,z,\xi,\widetilde{h})|\leq\mathcal{F}_k(\|V\|_{E_0}+\mathcal{N}_{k+1}(\gamma))\,\widetilde{h}^{m-|\alpha|\mu_0},\quad |\alpha|+|\beta|=k, \tag{2.106}$$

(ii) $a\in\dot{S}^m_{\mu_0}$ if (2.106) holds for every $k\geq 1$.

Remark 2.25. 1. If $m\geq m'$ then $S^m_{\mu_0}\subset S^{m'}_{\mu_0}$ and $\dot{S}^m_{\mu_0}\subset\dot{S}^{m'}_{\mu_0}$.

2. Let $a(t,z,\xi,\widetilde{h})=z$ and $b(t,z,\xi,\widetilde{h})=\xi$. Then $a\in\dot{S}^{\delta/2}_{2\delta-1}$, $b\in\dot{S}^0_{2\delta-1}$.

3. If $a\in S^m_{\mu_0}$ with $m\geq 0$ then $b=e^a\in S^0_{\mu_0}$.

We study now the composition of such symbols.

Proposition 2.26. *Let $m\in\mathbf{R}, f\in S^m_{2\delta-1}(resp. \dot{S}^m_{2\delta-1}), U\in\dot{S}^{\delta/2}_{2\delta-1}, V\in\dot{S}^0_{2\delta-1}$ and assume that $V\in\mathcal{C}_0$. Set*

$$F(t,z,\xi,\widetilde{h})=f(t,U(t,z,\xi,\widetilde{h}),V(t,z,\xi,\widetilde{h}),\widetilde{h}).$$

Then $F\in S^m_{2\delta-1}(resp.\ \dot{S}^m_{2\delta-1})$.

Proof. Let $\Lambda=(\alpha,\beta)\in\mathbf{N}^d\times\mathbf{N}^d, |\Lambda|=k$. If $k=0$ the estimate of F follows easily from the hypothesis on f. Assume $k\geq 1$. Then $D^\Lambda F$ is a finite linear combination of terms of the form

$$(1)=(D^Af)(\cdots)\prod_{j=1}^{r}(D^{L_j}U)^{p_j}(D^{L_j}V)^{q_j}$$

where $A=(a,b),\quad 1\leq|A|\leq|\Lambda|,\quad L_j=(l_j,m_j)$ and

$$\sum_{j=1}^{r}p_j=a,\quad\sum_{j=1}^{r}q_j=b,\quad\sum_{j=1}^{r}(|p_j|+|q_j|)L_j=\Lambda$$

By the hypothesis on f we have

$$|D^Af(\cdots)|\leq\mathcal{F}_k(\|V\|_{E_0}+\mathcal{N}_{k+1}(\gamma))\,\widetilde{h}^{m-|a|(2\delta-1)}. \tag{2.107}$$

By the hypotheses on U,V, the product occuring in the definition of (1) is bounded by $\mathcal{F}_k(\|V\|_{E_0}+\mathcal{N}_{k+1}(\gamma))\,\widetilde{h}^M$ where

$$M=\sum_{j=1}^{r}|p_j|\big(\frac{\delta}{2}-|l_j|(2\delta-1)\big)-\sum_{j=1}^{r}|q_j|(|l_j|(2\delta-1))=-|\alpha|(2\delta-1)+\frac{\delta}{2}|a|.$$

Using (2.107) and the fact that $1-2\delta+\frac{\delta}{2}=0$ we obtain the desired conclusion. $\square$

Further estimates on the flow

We shall denote by $z(s)=z(s;z,\xi,\widetilde{h}),\zeta(s)=\zeta(s;z,\xi,\widetilde{h})$ the solution of (2.94) with $z(0)=z,\zeta(0)=\xi$. Recall that $\delta=\frac{2}{3}$.

Proposition 2.27. *There exists $\mathcal{F}:\mathbf{R}^+\to\mathbf{R}^+$ non decreasing such that*

$$\begin{aligned}&(i)\quad\Big|\frac{\partial z}{\partial z}(s)-Id\Big|\leq\mathcal{F}\big(\|V\|_{E_0}+\mathcal{N}_2(\gamma)\big)\widetilde{h}^{\frac{\delta}{2}},\quad\Big|\frac{\partial\zeta}{\partial z}(s)\Big|\leq\mathcal{F}\big(\|V\|_{E_0}+\mathcal{N}_2(\gamma)\big),\\&(ii)\quad\Big|\frac{\partial z}{\partial\xi}(s)\Big|+\Big|\frac{\partial\zeta}{\partial\xi}(s)-Id\Big|\leq\mathcal{F}\big(\|V\|_{E_0}+\mathcal{N}_2(\gamma)\big)\,\widetilde{h}^{\frac{\delta}{2}}.\end{aligned}$$

for all $s\in I_{\widetilde{h}}=\big[0,\widetilde{h}^\delta\big],z\in\mathbf{R}^d,\xi\in\mathcal{C}_0$.

For any $k\geq 1$ there exists $\mathcal{F}_k:\mathbf{R}^+\to\mathbf{R}^+$ non decreasing such that for $\alpha,\beta\in\mathbf{N}^d$ with $|\alpha|+|\beta|=k$

$$\begin{cases}\Big|D_z^\alpha D_\xi^\beta z(s)\Big|\leq\mathcal{F}_k\big(\|V\|_{E_0}+\mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{|\alpha|(1-2\delta)+\frac{\delta}{2}},\\\Big|D_z^\alpha D_\xi^\beta\zeta(s)\Big|\leq\mathcal{F}_k\big(\|V\|_{E_0}+\mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{|\alpha|(1-2\delta)}.\end{cases}\tag{2.108}$$

Proof. The estimates of the first terms in (i) and (ii) have been proved in (2.99). By exactly the same argument one deduces the estimates on the second terms.

We shall prove (2.108) by induction on k. According to (i) and (ii) it is true for $k=1$. Assume it is true up to the order k and let $|\alpha|+|\beta| = k+1 \geq 2$. Let us set $\Lambda = (\alpha, \beta)$, $D^\Lambda = D_z^\alpha D_\zeta^\beta$ and $m(s) = (s, z(s), \zeta(s), \widetilde{h})$. By the Faa-di-Bruno formula we have

$$\begin{cases} D^\Lambda[p'_\zeta(m(s))] = p''_{z\zeta}(m(s))D^\Lambda z(s) + p''_{\zeta\zeta}(m(s))D^\Lambda \zeta(s) + F_1(s), \\ D^\Lambda[p'_z(m(s))] = p''_{zz}(m(s))D^\Lambda z(s) + p''_{z\zeta}(m(s))D^\Lambda \zeta(s) + F_2(s). \end{cases} \tag{2.109}$$

It follows that $U(s) = (D^\Lambda z(s), D^\Lambda \zeta(s)$ is the solution of the problem

$$\dot{U}(s) = \mathcal{A}(s)U(s) + F(s), \quad U(0) = 0$$

where $\mathcal{A}(s)$ has been defined in (2.96) and $F(s) = (F_1(s), F_2(s))$.

According to the estimates of the symbol p given in Lemma 2.19 the worse term is F_2. By the formula mentionned above we see that F_1 is a finite linear combination of terms of the form

$$(D^A p'_z)(m(s)) \prod_{i=1}^r (D^{L_i} z(s))^{p_i} \prod_{i=1}^r (D^{L_i} \zeta(s))^{q_i},$$

where $A = (a, b), \quad 2 \leq |A| \leq |\Lambda|$ and

$$L_i = (l_i, l'_i), \quad 1 \leq |L_i| \leq |\Lambda| - 1, \quad \sum_{i=1}^r p_i = a, \quad \sum_{i=1}^r q_i = b, \quad \sum_{i=1}^k (|p_i| + |q_i|) L_i = \Lambda.$$

It follows from Corollary 2.20 that for s in $[0, \widetilde{h}^\delta]$ we have,

$$\int_0^s \left|(D^A p'_z)(m(\sigma))\right| d\sigma \leq \mathcal{N}_{|A|+1}(\gamma) \widetilde{h}^{(|a|+1)(1-2\delta)+\delta} \leq \mathcal{N}_{|A|+1}(\gamma) \widetilde{h}^{|a|(1-2\delta)+\frac{\delta}{2}}. \tag{2.110}$$

since $1 - 2\delta + \delta = \frac{\delta}{2}$. Now since $1 \leq |L_i| \leq |\Lambda| - 1 = k$ we have, by the induction,

$$\begin{aligned} |D^{L_i} z(s)| &\leq \widetilde{h}^{|l_i|(1-2\delta)+\frac{\delta}{2}} \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big), \\ |D^{L_i} \zeta(s)| &\leq \widetilde{h}^{|l_i|(1-2\delta)} \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big). \end{aligned}$$

It follows that

$$\int_0^s |F_2(\sigma)| d\sigma \leq \Big(\int_0^s \left|(D^A p'_z)(m(\sigma))\right| d\sigma\Big) \mathcal{F}_{k+1}\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big) \widetilde{h}^M$$

where $M = \sum_{i=1}^r \big((|p_i| + |q_i|)|l_i|(1-2\delta)| + |p_i|\frac{\delta}{2}\big) = |\alpha|(1-2\delta) + |a|\frac{\delta}{2}$. It follows from (2.110) and the fact that $1 - \frac{3\delta}{2} = 0$ that

$$\int_0^s |F_2(\sigma)| d\sigma \leq \mathcal{F}_{k+1}\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big) \widetilde{h}^{|\alpha|(1-2\delta)+\frac{\delta}{2}}.$$

Since $D^A p'_\zeta$ has even a better estimate, the same computation shows that

$$\int_0^s |F_1(\sigma)| d\sigma \leq \mathcal{F}_{k+1}\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{|\alpha|(1-2\delta)+\frac{\delta}{2}}.$$

Then we write

$$U(s) = \int_0^s F(\sigma) d\sigma + \int_0^s \mathcal{A}(\sigma) U(\sigma) d\sigma$$

and we use the above estimates on F_1, F_2, (2.98) and the Gronwall lemma to see that the step $k+1$ of the induction is achieved. This completes the proof of Proposition 2.27. □

Corollary 2.28. *For every $k \geq 1$ there exists $\mathcal{F}_k : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for every $(\alpha, \beta) \in \mathbf{N}^d \times \mathbf{N}^d$ with $|\alpha| + |\beta| = k$ we have*

$$\begin{aligned}
&(i) \qquad |D_z^\alpha D_\xi^\beta \kappa(s, z, \xi, \widetilde{h})| \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{|\alpha|(1-2\delta)+\frac{\delta}{2}},\\
&(ii) \qquad \Big|D_z^\alpha D_\xi^\beta \big(\frac{\partial \phi}{\partial z}\big)(s, z, \xi, \widetilde{h})\Big| \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{|\alpha|(1-2\delta)},\\
&(iii) \qquad |D_\xi^\beta \phi(s, z, \xi, \widetilde{h})| \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)|s|, \quad |\beta| \geq 2,
\end{aligned}$$

for all $s \in I_{\widetilde{h}}, z \in \mathbf{R}^d, \xi \in \mathcal{C}_0$. This implies that $\kappa \in \dot{S}^{\delta/2}_{2\delta-1}$ and $\frac{\partial \phi}{\partial z} \in S^0_{2\delta-1}$.

Proof. We first show (ii) and (iii). Recall that

$$\frac{\partial \phi}{\partial z}(s, z, \xi, \widetilde{h}) = \zeta(s; \kappa(s; z, \xi, \widetilde{h}), \xi, \widetilde{h}).$$

By Proposition 2.27 (since ζ is bounded) we have $\zeta \in S^0_{2\delta-1}$. By (i) we have $\kappa \in \dot{S}^{\delta/2}_{2\delta-1}$ and by Remark 2.25 we have $\xi \in \dot{S}^0_{2\delta-1}$. Then Proposition 2.26 implies that $\frac{\partial \phi}{\partial z} \in \dot{S}^0_{2\delta-1}$. Moreover $\frac{\partial \phi}{\partial z}$ is bounded since $|\zeta(s) - \xi| \leq \int_0^s |\frac{\partial p}{\partial z}(t, \ldots)| dt \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_2(\gamma)\big)\widetilde{h}^{\frac{\delta}{2}}$ and $\xi \in \mathcal{C}_0$. Now (iii) follows from the definition (2.101) of the phase, the facts that $p \in S^0_{2\delta-1}, z \in \dot{S}^{\frac{1}{2}}_{2\delta-1}, \zeta(s; \kappa(s; z, \xi, \widetilde{h}), \xi, \widetilde{h}) \in \dot{S}^0_{2\delta-1}$ and Proposition 2.26.

We are left with the proof of (i). We proceed by induction on $|\alpha| + |\beta| = k \geq 1$. Recall that by definition of κ we have the equality $z(s; \kappa(s; z, \xi, \widetilde{h}), \xi, \widetilde{h}) = z$. It follows that

$$\frac{\partial z}{\partial z} \cdot \frac{\partial \kappa}{\partial z} = Id, \quad \frac{\partial z}{\partial z} \cdot \frac{\partial \kappa}{\partial \xi} = -\frac{\partial z}{\partial \xi}.$$

Then the estimate for $k = 1$ follows from (i) in Proposition 2.27. Assume the estimate true up to the order k and let $\Lambda = (\alpha, \beta), |\Lambda| = k + 1 \geq 2$. Then differentiating $|\Lambda|$ times the first above equality we see that $\frac{\partial z}{\partial z} \cdot D^\Lambda \kappa$ is a finite linear combination of terms of the form

$$(2) = D^A z(\cdots) \prod_{j=1}^r (D^{L_j}\kappa)^{p_j} \prod_{j=1}^r (D^{L_j}\xi)^{q_j}$$

where $A = (a, b)$ $\quad 2 \leq |A| \leq |\Lambda|, \quad L_j = (l_j, m_j), \quad 1 \leq |L_j| \leq k$ and

$$\sum_{j=1}^{r} p_j = \alpha, \quad \sum_{j=1}^{r} q_j = \beta, \quad \sum_{j=1}^{r} (|p_j| + |q_j|) L_j = (\alpha, \beta).$$

We use the estimate (given by Proposition 2.27)

$$|D^A z(\cdots)| \leq \mathcal{F}_{k+1}\big(\|V\|_{E_0} + \mathcal{N}^0_{k+2}(\gamma)\big) \widetilde{h}^{|a|(1-2\delta)+\frac{\delta}{2}},$$

the induction, the fact that $\xi \in \dot{S}^0_{2\delta-1}$ and the equality $1 - 2\delta + \frac{\delta}{2} = 0$ to see that

$$|(2)| \leq \mathcal{F}_{k+1}\big(\|V\|_{E_0} + \mathcal{N}^0_{k+2}(\gamma)\big) \widetilde{h}^{|\alpha|(1-2\delta)+\frac{\delta}{2}}.$$

Then we use Proposition 2.27 (i) to conclude the induction. $\square$

Remark 2.29. Since $\theta(t, z, z', \xi, \widetilde{h}) = \int_0^1 \frac{\partial \phi}{\partial z}(t, \lambda z + (1-\lambda) z', \xi, \widetilde{h})\, d\lambda$ we have also the estimate

$$\Big| D_z^{\alpha_1} D_{z'}^{\alpha_2} D_\xi^{\beta} \theta(s, z, z', \xi, \widetilde{h}) \Big| \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big) \widetilde{h}^{(|\alpha_1|+|\alpha_2|)(1-2\delta)}. \tag{2.111}$$

for $|\alpha_1| + |\alpha_2| + |\beta| = k$

2.7.3 The transport equations

According to (2.89) and (2.90) if ϕ satisfies the eikonal equation we have

$$\begin{aligned} &e^{-i\widetilde{h}^{-1}\phi}(\widetilde{h}\partial_t + \widetilde{h}c + iP)\big(e^{i\widetilde{h}^{-1}\phi}\widetilde{b}\big) \\ &= \widetilde{h}\partial_t \widetilde{b} + \widetilde{h}c\widetilde{b} + i \sum_{|\alpha|=1}^{N-1} \frac{\widetilde{h}^{|\alpha|}}{\alpha!} D_{z'}^{\alpha} \Big[(\partial_\eta^\alpha \widetilde{p})\big(t, z, z', \theta(t, z, z', \widetilde{h})\big), \widetilde{h}\big) \widetilde{b}(z') \Big] \Big|_{z'=z} \\ &+ R_N + S_N \end{aligned} \tag{2.112}$$

Recall (see (2.83)) that $\widetilde{b} = b\Psi_0$. Let us set
(2.113)

$$T_N = \partial_t b + cb + i \sum_{1 \leq |\alpha| \leq N-1} \frac{\widetilde{h}^{|\alpha|-1}}{\alpha!} D_{z'}^{\alpha} \Big[(\partial_\eta^\alpha \widetilde{p})\big(t, z, z', \theta(t, z, z', \widetilde{h}), \widetilde{h}\big) b(z') \Big] \Big|_{z'=z}.$$

Then

$$e^{-i\widetilde{h}^{-1}\phi}(\widetilde{h}\partial_t + \widetilde{h}c + iP)\big(e^{i\widetilde{h}^{-1}\phi}\widetilde{b}\big) = \widetilde{h} T_N \Psi_0 + U_N + R_N + S_N, \tag{2.114}$$

where

$$\begin{aligned} U_N &= \widetilde{h} b(\partial_t \Psi_0) \\ &+ i \sum_{|\alpha|=1}^{N-1} \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} \frac{\widetilde{h}^{|\alpha|-1}}{\alpha!} D_{z'}^{\beta} \Big[(\partial_\eta^\alpha \widetilde{p})\big(t, z, z', \theta(t, z, z', \widetilde{h}), \widetilde{h}\big) b(z') \Big] D_{z'}^{\alpha-\beta} \Psi_0 \Big|_{z'=z}. \end{aligned}$$

Our purpose is to show that one can find a symbol b such that, in a sense to be explained,

$$T_N = \mathcal{O}(\widetilde{h}^M), \quad \forall M \in \mathbf{N}. \tag{2.115}$$

We set

$$\mathcal{L}b = \partial_t b + c + \sum_{i=1}^{n} \frac{\partial}{\partial z'_i}\Big[\frac{\partial \widetilde{p}}{\partial \zeta_i}\big(t, z, z', \theta(t, z, z', \widetilde{h}), \widetilde{h}\big) b(t, z', \xi, \widetilde{h})\Big]\Big|_{z'=z}$$

Then we can write

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} a_j(t, z, \xi, \widetilde{h}) \frac{\partial}{\partial z_i} + c_0(t, z, \xi, \widetilde{h}) \tag{2.116}$$

where

$$\begin{cases} a_i(t, z, \xi, \widetilde{h}) = \dfrac{\partial p}{\partial \zeta_i}\Big(t, z, \dfrac{\partial \phi}{\partial z}(t, z, \xi, \widetilde{h}), \widetilde{h}\Big), \\ c_0(t, z, \xi, \widetilde{h}) = \displaystyle\sum_{i=1}^{d} \frac{\partial}{\partial z'_i}\Big[\frac{\partial \widetilde{p}}{\partial \zeta_i}\big(t, z, z', \theta(t, z, z', \xi, \widetilde{h}), \widetilde{h}\big)\Big]\Big|_{z'=z} + c(t, z, \widetilde{h}), \\ \theta(t, z, z', \xi, \widetilde{h}) = \displaystyle\int_0^1 \frac{\partial \phi}{\partial z}\big(t, \lambda z + (1-\lambda) z', \xi, \widetilde{h}\big)\, d\lambda \end{cases} \tag{2.117}$$

and c has been defined in (2.68).

Notice that, with $m = (t, z, \xi, \widetilde{h})$ we have

$$c_0(m) = \sum_{i=1}^{d} \frac{\partial^2 \widetilde{p}}{\partial \zeta_i \partial z'_i}(t, z, z, \frac{\partial \phi}{\partial z}(m), \widetilde{h}) + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 p}{\partial \zeta_i \partial \zeta_j}(t, z, \frac{\partial \phi}{\partial z}(m), \widetilde{h})) \frac{\partial^2 \phi}{\partial z_i \partial z_j}(m) + c. \tag{2.118}$$

Then we can write

$$T_N = \mathcal{L}b + i \sum_{2 \leq |\alpha| \leq N-1} \frac{\widetilde{h}^{|\alpha|-1}}{\alpha!} D^\alpha_{z'}\Big[(\partial^\alpha_\zeta \widetilde{p})\big(t, z, z', \theta(t, z, z', \widetilde{h}), \widetilde{h}\big) b(t, z', \xi, \widetilde{h})\Big]\Big|_{z'=z}. \tag{2.119}$$

We shall seek b on the form

$$b = \sum_{j=0}^{N} \widetilde{h}^j\, b_j. \tag{2.120}$$

Including this expression of b in (2.119) after a change of indices we obtain

$$T_N = \sum_{k=0}^{N} \widetilde{h}^k \mathcal{L} b_k + i \sum_{k=1}^{N+1} \widetilde{h}^k \sum_{2 \leq |\alpha| \leq N-1} \frac{\widetilde{h}^{|\alpha|-2}}{\alpha!} D^\alpha_{z'}\big[(\partial^\alpha_\zeta \widetilde{p})(\cdots) b_{k-1}\big]\big|_{z'=z}.$$

We will take b_j for $j = 0, \dots, N$, as solutions of the following problems

(2.121)
$$\begin{cases} \mathcal{L}b_0 = 0, \quad b_0|_{t=0} = \chi, \quad \chi \in C_0^\infty(\mathbf{R}^d), \\ \mathcal{L}b_j = F_{j-1} := -i \displaystyle\sum_{2\leq|\alpha|\leq N-1} \frac{\widetilde{h}^{|\alpha|-2}}{\alpha!} D_{z'}^\alpha \left[(\partial_\zeta^\alpha \widetilde{p})(\cdots) b_{j-1}\right] \big|_{z'=z}, \quad b_j|_{t=0} = 0. \end{cases}$$

This choice will imply that

$$T_N = i\widetilde{h}^{N+1} \sum_{2\leq|\alpha|\leq N-1} \frac{\widetilde{h}^{|\alpha|-2}}{\alpha!} D_{z'}^\alpha \left[(\partial_\zeta^\alpha \widetilde{p})(\cdots) b_N\right] \big|_{z'=z}. \tag{2.122}$$

Proposition 2.30. *The system (2.121) has a unique solution with $b_j \in S^0_{2\delta-1}$.*

We prove this result by induction. To solve these equations we use the method of characteristics and we begin by preliminaries.

Lemma 2.31. *We have $a_i \in S^0_{2\delta-1}$ for $i = 1, \dots, d$.*

Proof. This follows from Lemma 2.19, Proposition 2.26 with $f = \frac{\partial p}{\partial \zeta_i}, U(t, z, \xi, \widetilde{h}) = z, V(\cdots) = \frac{\partial \phi}{\partial z}$ and Corollary 2.28. $\square$

Consider now the system of differential equations

$$\dot{Z}_j(s) = a_j(s, Z(s), \xi, \widetilde{h}), \quad Z_j(0) = z_j, \quad 1 \leq j \leq d.$$

By Lemma 2.31 a_j is bounded. Therefore this system has a unique solution defined on $I_{\widetilde{h}}$. Differentiating with respect to z we obtain

$$\left|\frac{\partial Z}{\partial z}(s)\right| \leq C + \mathcal{F}_2(\|V\|_{E_0} + \mathcal{N}_2(\gamma)) \int_0^s \widetilde{h}^{1-2\delta} \left|\frac{\partial Z}{\partial z}(\sigma)\right| d\sigma, \quad 0 < s \leq \widetilde{h}^\delta,$$

since $|s|\,\widetilde{h}^{1-2\delta} \leq \widetilde{h}^{1-\delta} = \widetilde{h}^{\frac{\delta}{2}}$, the Gronwall inequality shows that $\left|\frac{\partial Z}{\partial z}(s)\right|$ is uniformly bounded. Using again the equation satisfied by $\frac{\partial Z}{\partial z}(s)$ we deduce that

$$\left|\frac{\partial Z}{\partial z}(s) - Id\right| \leq \mathcal{F}_2(\|V\|_{E_0} + \mathcal{N}_2(\gamma))\,\widetilde{h}^{\frac{\delta}{2}}, \quad 0 < s \leq \widetilde{h}^\delta. \tag{2.123}$$

This shows that the map $z \mapsto Z(s; z, \xi, \widetilde{h})$ is a global diffeomorphism from $\mathbf{R}^d$ to itself so

$$Z(s; z, \xi, \widetilde{h}) = z \Longleftrightarrow z = \omega(s; Z; \xi, \widetilde{h}). \tag{2.124}$$

An analogue computation shows that

$$\left|\frac{\partial Z}{\partial \xi}(s)\right| \leq \mathcal{F}_2(\|V\|_{E_0} + \mathcal{N}_2(\gamma))\,\widetilde{h}^\delta, \quad 0 < s \leq \widetilde{h}^\delta. \tag{2.125}$$

Lemma 2.32. *The function* $(s, z, \xi, \widetilde{h}) \mapsto Z(s, z, \xi, \widetilde{h})$ *belongs to* $\dot{S}^{\delta/2}_{2\delta-1}$.

Proof. We have to prove that for $|\alpha| + |\beta| = k \geq 1$ we have the estimate

$$|D^\alpha_z D^\beta_\xi Z(s; z, \xi, \widetilde{h})| \leq \mathcal{F}_k(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma))\, \widetilde{h}^{|\alpha|(1-2\delta)+\frac{\delta}{2}}. \tag{2.126}$$

Indeed this is true for $k = 1$ by (2.123), (2.125). Assume this is true up to the order k and let $|\alpha| + \beta| = k + 1 \geq 2$. Set $U(s) = D^\Lambda Z(s; z, \xi, \widetilde{h})$ where $\Lambda = (\alpha, \beta)$. It satifies the system $\dot{U}(s) = \frac{\partial a}{\partial z}(s; Z(s), \xi, \widetilde{h})U(s) + F(s), U(0) = 0$ where $F(s)$ is a finite linear combination of terms of the form

$$(1) = (D^A a)(\cdots) \prod_{j=1}^{r} (\partial^{L_j} Z(s))^{p_j} (\partial^{L_j} \xi)^{q_j}$$

where $A = (a, b), 2 \leq |A| \leq |\Lambda|, L_j = (l_j, m_j), 1 \leq |L_j| \leq k$ and

$$\sum_{j=1}^{r} (|p_j| + |q_j|) L_j = (\alpha, \beta), \quad \sum_{j=1}^{r} p_j = \alpha, \quad \sum_{j=1}^{r} q_j = \beta.$$

First of all, by Lemma 2.31 we can write

$$|(D^A a)(\cdots)| \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{|a|(1-2\delta)}.$$

Using the induction and the fact that $\xi \in \dot{S}^0_{2\delta-1}$ we can estimate the product occuring in (1) by $\mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^M$ where

$$M = \sum_{j=1}^{r} \Big\{ |p_j|\big(|l_j|(1-2\delta) + \frac{\delta}{2}\big) + |q_j||l_j|(1-2\delta) \Big\} = |\alpha|(1-2\delta) + |a|\frac{\delta}{2}.$$

It follows that $\int_0^s |F(t)|dt \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{|\alpha|(1-2\delta)+\delta}$ and we conclude by the Gronwall inequality. $\square$

Corollary 2.33. *The function* ω *defined in* (2.124) *belongs to* $\dot{S}^{\delta/2}_{2\delta-1}$.

Proof. The proof is the same as that of Corollary 2.28. $\square$

Proof of Proposition 2.30. Now, with the notations in (2.117) and (2.121) we have

$$\frac{d}{ds}\big[b_j(s, Z(s))\big] = \big(\frac{\partial u}{\partial t} + a \cdot \nabla u\big)(s, Z(s)) = -(c_0 u)(s, Z(s)) + F_{j-1}(s, z(s)), \quad j \geq 0$$

with $F_{-1} = 0$. It follows that

$$\frac{d}{ds}\Big[e^{\int_0^s c_0(\sigma, Z(\sigma))d\sigma} b_j(s, Z(s))\Big] = e^{\int_0^s c_0(\sigma, Z(\sigma))\,d\sigma} F_{j-1}(s, Z(s)),$$

Using (2.124) we see that the unique solution of (2.121) is given by

(2.127)
$$\begin{cases} b_0(s,z,\xi,\widetilde{h}) = \chi(\xi)\exp\Big(\int_0^s c_0(t,Z(t;\omega(s,z,\xi,\widetilde{h}),\xi,\widetilde{h}))\,dt\Big), \\ b_j(s,z,\xi,\widetilde{h}) = \int_0^s e^{\int_s^\sigma c_0(t,Z(t;\omega(s,z,\xi,\widetilde{h}),\xi,\widetilde{h})\,dt} F_{j-1}(\sigma,Z(\sigma;\omega(s,z,\xi,\widetilde{h}),\xi,\widetilde{h})\,d\sigma. \end{cases}$$

The last step in the proof of Proposition 2.30 is contained in the following lemma.

Lemma 2.34. *We have* $b_j \in S^0_{2\delta-1}$.

Proof. Step1: we show that

$$e^{\int_s^\sigma c_0(t,Z(t;\omega(s,z,\xi,\widetilde{h}),\xi,\widetilde{h})\,dt} \in S^0_{2\delta-1}. \tag{2.128}$$

According to Remark 2.25 this will be implied by $\int_s^\sigma c_0(t,Z(t;\omega(s,z,\xi,\widetilde{h}),\xi,\widetilde{h})\,dt \in S^{\delta/2}_{2\delta-1}$. By Lemma 2.32 we have $Z \in \dot{S}^{\delta/2}_{2\delta-1}$ and $\omega \in \dot{S}^{\delta/2}_{2\delta-1}$. Moreover $\xi \in \dot{S}^0_{2\delta-1}$. By Proposition 2.26 the function $Z(t;\omega(s;z,\xi,\widetilde{h}),\xi,\widetilde{h})$ belongs to $\dot{S}^{\delta/2}_{2\delta-1}$. Now by Corollary 2.28 we have $\frac{\partial\phi}{\partial z} \in \dot{S}^0_{2\delta-1}$ and $\frac{\partial^2\phi}{\partial z^2} \in S^{-\delta/2}_{2\delta-1}$ (since $1-2\delta = -\delta/2$.) It follows from Proposition 2.26 that for $s \in [0,\widetilde{h}^\delta]$

$$\begin{cases} U_1(t;z,\xi,\widetilde{h}) = \dfrac{\partial\phi}{\partial z}(t,Z(t;\omega(s,z,\xi,\widetilde{h}),\xi,\widetilde{h}),\xi,\widetilde{h}) \in \dot{S}^0_{2\delta-1}, \\ U_2(t;z,\xi,\widetilde{h}) = \dfrac{\partial^2\phi}{\partial z^2}(t,Z(t;\omega(s,z,\xi,\widetilde{h}),\xi,\widetilde{h}),\xi,\widetilde{h}) \in S^{-\delta/2}_{2\delta-1}. \end{cases} \tag{2.129}$$

Now by Lemma 2.19 the functions $\frac{\partial^2\widetilde{p}}{\partial\zeta\partial z'}(t,z,z,\zeta,\widetilde{h})$ (resp. $\frac{\partial^2 p}{\partial\zeta\partial\zeta}(t,z,\zeta,\widetilde{h})$) satisfy the condition of Proposition 2.26 with $m = 1-2\delta$ (resp. $m=0$.) Using (2.129) and the fact that $z \in \dot{S}^{\delta/2}_{2\delta-1}$ we deduce that

$$\int_s^\sigma \frac{\partial^2\widetilde{p}}{\partial\zeta\partial z'}\big(t,z,z,U_1\big(t,z,\xi,\widetilde{h}\big),\widetilde{h}\big)\,dt \in S^{\frac{\delta}{2}}_{2\delta-1},$$
$$\int_s^\sigma \frac{\partial^2 p}{\partial\zeta^2}\big(t,z,U_1\big(t,z,\xi,\widetilde{h}\big),\widetilde{h}\big)U_2\big(t;z,\xi,\widetilde{h}\big)\,dt \in S^{\frac{\delta}{2}}_{2\delta-1}.$$

This shows that $\int_s^\sigma c_0\big(t,Z(t;\omega\big(s,z,\xi,\widetilde{h}\big),\xi,\widetilde{h}\big)\,dt \in S^{\frac{\delta}{2}}_{2\delta-1}$ as claimed.

Step 2: we show that for $|a|+|b| = k \geq 0$ we have, with $\Lambda = (a,b) \in \mathbf{N}^d \times \mathbf{N}^d$

(2.130)
$$\int_0^s \Big|D^\Lambda\big[G_{j-1}\big(\sigma;Z\big(\sigma;\omega\big(s;z,\xi,\widetilde{h}\big),\xi,\widetilde{h}\big)\big)\big]\Big|\,d\sigma \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{|a|(1-2\delta)}$$

where for $|\rho| \geq 2$,

(2.131)
$$G_{j-1}(\sigma,z,\xi,\widetilde{h}) = \widetilde{h}^{|\rho|-2}D^\rho_{z'}\big[(\partial^\rho_\zeta\widetilde{p})\big(\sigma;z,z',\theta\big(\sigma;z,z',\xi,\widetilde{h}\big),\widetilde{h}\big)b_{j-1}(\sigma;z',\xi,\widetilde{h})\big]\,\big|_{z'=z}.$$

We claim that for $\Lambda = (\alpha, \beta), |\alpha| + |\beta| = k \geq 0$,

$$\left| D^\Lambda G_{j-1}(\sigma, z, \xi, \widetilde{h}) \right| \leq \widetilde{h}^{-\delta + |\alpha|(1-2\delta)} \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big). \tag{2.132}$$

Indeed $D^\Lambda G_{j-1}$ is a finite sum of terms of the form $H_1 \times H_2$ with

$$\begin{aligned} H_1 &= \widetilde{h}^{|\rho|-2} D^{\Lambda_1} D_{z'}^{\rho_1} \big[(\partial_\zeta^\rho \widetilde{p})(\sigma, , z, z', \theta(\sigma; z, z', \xi, \widetilde{h}), \widetilde{h}))\big]|_{z'=z}, \\ H_2 &= D^{\Lambda_2} D_z^{\rho_2} b_{j-1}(\sigma, z, \xi, \widetilde{h}), \end{aligned}$$

where $\Lambda_i = (\alpha_i, \beta_i) \quad |\Lambda_1| + |\Lambda_2| = |\Lambda|, \quad |\rho_1| + |\rho_2| = |\rho|$.

By the induction we have

$$|H_2| \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\, \widetilde{h}^{(|\alpha_2|+|\rho_2|)(1-2\delta)}$$

and since $z' \in \dot{S}_{2\delta-1}, \theta \in \dot{S}^0_{2\delta-1}$ using Proposition 2.19 we see that

$$|H_1| \leq \mathcal{F}_k\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\, \widetilde{h}^{|\rho|-2+(|\alpha_1|+|\rho_1|)(1-2\delta)}.$$

Now since $|\rho| \geq 2$ and $\delta = \frac{2}{3}$, we have $|\rho|-2+|\alpha|(1-2\delta)+|\rho|(1-2\delta) \geq |\alpha|(1-2\delta)-\delta$ which proves (2.132).

Eventually since the function $Z\big(t; \omega(s; z, \xi, \widetilde{h}), \xi, \widetilde{h}\big)$ belongs to $\dot{S}^{\delta/2}_{2\delta-1}$ (see Step 1), we deduce from Proposition 2.26, with $m = -\delta$, that (2.130) holds. Then Step 1 and Step 2 prove the lemma. Notice that b_j can be written $\chi(\xi) b_j^0$. □

Thus the proof of Proposition 2.30 is complete. □

Summing up we have proved that with the choice of ϕ and b made in Proposition 2.23 and in (2.120) we have

$$e^{-i\widetilde{h}^{-1}\phi}(\widetilde{h}\partial_t + \widetilde{h}c + iP)\big(e^{i\widetilde{h}^{-1}\phi}\widetilde{b}\big) = \widetilde{h}T_N\Psi_0 + U_N + R_N + S_N \tag{2.133}$$

where R_N is defined in (2.86), S_N in (2.88), T_N in (2.122) and U_N in (2.114).

2.8 The dispersion estimate

The purpose of this section is to prove the following result. Recall that $\delta = \frac{2}{3}$.

Theorem 2.35. *Let $\chi \in C_0^\infty(\mathbf{R}^d)$ be such that $\operatorname{supp}\chi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Let $t_0 \in \mathbf{R}$, $u_0 \in L^1(\mathbf{R}^d)$ and set $u_{0,h} = \chi(hD_x)u_0$. Denote by $S(t, t_0)u_{0,h}$ the solution of the problem*

$$\Big(\partial_t + \frac{1}{2}(S_{j\delta}(V) \cdot \nabla + \nabla \cdot S_{j\delta}(V)) + i\mathfrak{T}_{\gamma_\delta}\Big) U_h(t, x) = 0, \quad U_h(t_0, x) = u_{0,h}(x).$$

Then there exist $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ $k = k(d) \in \mathbf{N}$ and $h_0 > 0$ such that

$$\|S(t, t_0)u_{0,h}\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_k(\gamma)\big)\, h^{-\frac{3d}{4}} |t - t_0|^{-\frac{d}{2}} \|u_{0,h}\|_{L^1(\mathbf{R}^d)},$$

for all $0 < |t - t_0| \leq h^{\frac{\delta}{2}}$ and all $0 < h \leq h_0$.

This result will be a consequence of the following one in the variables (t, z).

Theorem 2.36. *Let $\chi \in C_0^\infty(\mathbf{R}^d)$ be such that $\operatorname{supp}\chi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Let $t_0 \in \mathbf{R}$, $w_0 \in L^1(\mathbf{R}^d)$ and set $w_{0,\widetilde{h}} = \chi(\widetilde{h}D_z)w_0$. Denote by $\widetilde{S}(t, t_0)w_{0,\widetilde{h}}$ the solution of the problem*

$$\big(\widetilde{h}\partial_t + \widetilde{h}c + iP\big)\widetilde{U}(t, z) = 0, \quad \widetilde{U}(t_0, z) = w_{0,\widetilde{h}}(z).$$

Then there exist $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$, $k = k(d) \in \mathbf{N}$ and $h_0 > 0$ such that

$$\|\widetilde{S}(t, t_0)w_{0,\widetilde{h}}\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_k(\gamma)\big)\, \widetilde{h}^{-\frac{d}{2}}|t - t_0|^{-\frac{d}{2}}\|w_{0,\widetilde{h}}\|_{L^1(\mathbf{R}^d)}$$

for all $0 < |t - t_0| \leq \widetilde{h}^\delta$ and $0 < \widetilde{h} \leq \widetilde{h}_0$.

Indeed suppose that Theorem 2.36 is proved. We can assume $t_0 = 0$. According to the two change of variables $x = X(t, y)$ and $z = \widetilde{h}^{-1}y$ we have for any smooth function W (see (2.67) and (2.68))

$$\big(\widetilde{h}\partial_t+\widetilde{h}c+iP\big)\big[W(t, X(t, \widetilde{h}z)\big] = \widetilde{h}\Big(\big(\partial_t+\frac{1}{2}(S_{j\delta}(V)\cdot\nabla+\nabla\cdot S_{j\delta}(V))+i\mathfrak{T}_{\gamma_\delta}\big)W\Big)\big(t, X(t, \widetilde{h}z)\big).$$

It follows that

$$\big(\widetilde{S}(t, 0)w_{0,\widetilde{h}}\big)(t, z) = \big(S(t, 0)u_{0,h}\big)(t, X(t, \widetilde{h}z)).$$

Moreover since $w_0(z) = u_0(\widetilde{h}z)$ we have

$$w_{0,\widetilde{h}}(z) = (\chi(\widetilde{h}D_z)w_0)(z) = (\chi(hD_x)u_0)(\widetilde{h}z) = u_{0,h}(\widetilde{h}z).$$

Therefore using Theorem 2.36 we obtain

$$\begin{aligned}\|S(t, 0)u_{0,h}\|_{L^\infty(\mathbf{R}^d)} &= \|\tilde{S}(t, 0)w_{0,\widetilde{h}}\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}(\cdots)\,\widetilde{h}^{-\frac{d}{2}}t^{-\frac{d}{2}}\|w_{0,\widetilde{h}}\|_{L^1(\mathbf{R}^d)} \\ &\leq \mathcal{F}(\cdots)\,\widetilde{h}^{-\frac{d}{2}}t^{-\frac{d}{2}}\widetilde{h}^{-d}\|u_{0,h}\|_{L^1(\mathbf{R}^d)} \\ &\leq \mathcal{F}(\cdots)\,h^{-\frac{3d}{4}}t^{-\frac{d}{2}}\|u_{0,h}\|_{L^1(\mathbf{R}^d)}\end{aligned}$$

since $\widetilde{h} = h^{\frac{1}{2}}$. Thus Theorem 2.35 is proved.

Proof of Theorem 2.36. We set

(2.134)
$$\mathcal{K}w_{0,\widetilde{h}}(t, z) = (2\pi\widetilde{h})^{-d}\iint e^{i\widetilde{h}^{-1}(\phi(t,z,\xi,\widetilde{h})-y\cdot\xi)}\widetilde{b}\big(t, z, y, \xi, \widetilde{h}\big)\chi_1(\xi)w_{0,\widetilde{h}}(y)\,dy\,d\xi$$

where χ_1 belongs to $C_0^\infty(\mathbf{R}^d)$ with $\chi_1 \equiv 1$ on the support of χ and $\tilde{b}$ is defined in (2.83). We can write

$$(2.135)\qquad \begin{aligned}&\mathcal{K}w_{0,\widetilde{h}}(t, z) = \int K\big(t, z, y, \widetilde{h}\big)w_{0,\widetilde{h}}(y)dy \quad \text{with} \\ &K(t, z, y, \widetilde{h}) = (2\pi\widetilde{h})^{-d}\int e^{i\widetilde{h}^{-1}(\phi(t,z,\xi,\widetilde{h})-y\cdot\xi)}\widetilde{b}(t, z, y, \xi, \widetilde{h})\chi_1(\xi)\,d\xi.\end{aligned}$$

Recall from (2.120) that

$$b = \sum_{j=0}^{N} \widetilde{h}^j \, b_j,$$

where the b_j's are given by Proposition 2.30 with $b_j \in S^0_{2\delta-1}$. On the other hand, it follows from the (2.105) that the hessian of the phase satisfies

$$\left| \det \Big(\frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j}(t, z, \xi, \widetilde{h}) \Big) \right| \geq M_0 t^d,$$

for some $M_0 > 0$ and all $t \in [0, h^{\delta/2}], z \in \mathbf{R}^d, \xi \in \mathcal{C}_0, 0 < \widetilde{h} \leq \widetilde{h}_0$. Then we have the following estimate.

Lemma 2.37. *There holds*

$$\big| K(t, z, y, \widetilde{h}) \big| \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma) \big) \widetilde{h}^{-\frac{d}{2}} t^{-\frac{d}{2}}$$

for all $0 < t \leq \widetilde{h}^\delta$, $z, y \in \mathbf{R}^d$ *and* $0 < \widetilde{h} \leq \widetilde{h}_0$.

Proof. See the Appendix E. □

Using this lemma we obtain

$$\big\| \mathcal{K} w_{0,\widetilde{h}}(t, \cdot) \big\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}(\cdots) \, \widetilde{h}^{-\frac{d}{2}} t^{-\frac{d}{2}} \big\| w_{0,\widetilde{h}} \big\|_{L^1(\mathbf{R}^d)} \tag{2.136}$$

for all $0 < t \leq \widetilde{h}^\delta$ and $0 < \widetilde{h} \leq \widetilde{h}_0$.

We can state now the following result.

Proposition 2.38. *Let σ_0 be an integer such that $\sigma_0 > \frac{d}{2}$. Set*

$$\big(\widetilde{h} \partial_t + \widetilde{h} c + iP \big) \Big(\mathcal{K} w_{0,\widetilde{h}} \Big)(t, z) = F_{\widetilde{h}}(t, z).$$

Then there exists $k = k(d) \in \mathbf{N}$ and for any $N \in \mathbf{N}$, $\mathcal{F}_N : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\sup_{0 < t \leq \widetilde{h}^\delta} \big\| F_{\widetilde{h}}(t, \cdot) \big\|_{H^{\sigma_0}(\mathbf{R}^d)} \leq \mathcal{F}_N \big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma) \big) \, \widetilde{h}^N \|w_{0,\widetilde{h}}\|_{L^1(\mathbf{R}^d)}.$$

We shall use the following result.

Lemma 2.39. *Let $k_0 > \frac{d}{2}$. Let us set $m(t, z, y, \xi, \widetilde{h}) = \frac{\partial \phi}{\partial \xi}(t, z, \xi, \widetilde{h}) - y$ and*

$$\Sigma = \Big\{ (t, y, \xi, \widetilde{h}) : 0 < \tilde{h} \leq \tilde{h}_0, 0 \leq t \leq \widetilde{h}^\delta, y \in \mathbf{R}^d, |\xi| \leq C \Big\}.$$

Then

$$\sup_{(t,y,\xi,\widetilde{h}) \in \Sigma} \int_{\mathbf{R}^d} \frac{dz}{\langle m(t, z, y, \xi, \widetilde{h}) \rangle^{2k_0}} \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_2(\gamma) \big).$$

Proof. By (2.102) we have $\frac{\partial\phi}{\partial z}\big(t,z,\xi,\widetilde{h}\big)=\zeta\big(t;\kappa(t;z,\xi,\widetilde{h}),\xi,\widetilde{h}\big)$ so $\frac{\partial^2\phi}{\partial\xi\partial z}=\frac{\partial\zeta}{\partial z}\frac{\partial\kappa}{\partial\xi}+\frac{\partial\zeta}{\partial\xi}$. We deduce from Proposition 2.27 and Corollary 2.28 that $|\frac{\partial^2\phi}{\partial\xi\partial z}-Id|\leq\mathcal{F}\big(\|V\|_{E_0}+\mathcal{N}_2(\gamma)\big)\,\widetilde{h}^{\frac{\delta}{2}}$. It follows that the map $z\mapsto\frac{\partial\phi}{\partial\xi}(t,z,\xi,\widetilde{h})$ is proper and therefore is a global diffeomorphism from $\mathbf{R}^d$ to itself. Consequently, we can perform the change of variable $X=\frac{\partial\phi}{\partial\xi}(t,z,\xi,\widetilde{h})$ and the lemma follows. □

Proof of Proposition 2.38. According to (2.135) Proposition 2.38 will be proved if we have

$$\sup_{(t,y,\tilde{h})\in\Sigma}\big\|\big(\tilde{h}\partial_t+\widetilde{h}c+iP\big)K(t,\cdot,y,\tilde{h})\big\|_{H^{\sigma_0}}\leq\mathcal{F}_N(\cdots)\,\tilde{h}^N. \tag{2.137}$$

Now, setting $L=\tilde{h}\partial_t+\widetilde{h}c+iP(t,z,D_z)$, we have

$$LK\big(t,z,y,\tilde{h}\big)=\big(2\pi\tilde{h}\big)^{-d}\int e^{-i\widetilde{h}^{-1}y\cdot\xi}L\Big(e^{i\widetilde{h}^{-1}\phi(t,z,\xi,\tilde{h})}\widetilde{b}\big(t,z,y,\xi,\tilde{h}\big)\Big)\chi_1(\xi)\,d\xi \tag{2.138}$$

and according to (2.112), (2.113), (2.122) we have,

$$L\Big(e^{i\widetilde{h}^{-1}\phi(t,z,\xi,\tilde{h})}\widetilde{b}\big(t,z,y,\xi,\tilde{h}\big)\Big)=e^{i\widetilde{h}^{-1}\phi(t,z,\xi,\tilde{h})}\big(R_N+S_N+\tilde{h}T_N\Psi_0+U_N\big)(t,z,y,\xi,\widetilde{h}) \tag{2.139}$$

where R_N, S_N, T_N, U_N are defined in (2.86), (2.88), (2.113), (2.114).

Lemma 2.40. *Let σ_0,k_0 be integers, $\sigma_0>\frac{d}{2}$, $k_0>\frac{d}{2}$. There exists a fixed integer $N_0(d)$ such that for any $N\in\mathbf{N}$ there exists $C_N>0$ such that, if we set $\Xi=(t,z,y,\xi,\widetilde{h})$, then*

$$\big\langle m(\Xi)\big\rangle^{k_0}\Big\{|D_z^\beta R_N(\Xi)|+|D_z^\beta S_N(\Xi)|+|D_z^\beta(\tilde{h}(T_N\Psi_0)(\Xi))|\Big\}\leq\mathcal{F}_N(\cdots)\,\tilde{h}^{\delta N-N_0}, \tag{2.140}$$

for all $|\beta|\leq\sigma_0$, all $(t,y,\xi,\tilde{h})\in\Sigma$ and all $z\in\mathbf{R}^d$.

Proof. According to (2.86), $D_z^\beta R_N(\Xi)$ is a finite linear combination of terms of the form

$$R_{N,\beta}(\Xi)=\tilde{h}^{-d-|\beta_1|}\iint e^{i\widetilde{h}^{-1}(z-z')\cdot\eta}\eta^{\beta_1}\kappa(\eta)D_z^{\beta_2}r_N(t,z,z',\eta,\xi,\widetilde{h})$$
$$b(t,z',\xi,\tilde{h})\Psi_0\Big(\frac{\partial\phi}{\partial\xi}(t,z',\xi,\tilde{h})-y\Big)\,dz'\,d\eta$$

where $\beta_1+\beta_2=\beta$ and

$$D_z^{\beta_2}r_N(\cdots)=\sum_{|\alpha|=N}\frac{N}{\alpha!}\int_0^1(1-\lambda)^{N-1}D_z^{\beta_2}\Big[\big(\partial_\eta^\alpha\widetilde{p}\big)(t,z,z',\eta+\lambda\theta(t,z,z',\xi,\widetilde{h}))\Big]\eta^\alpha\,d\lambda.$$

Using the equality $\eta^{\alpha+\beta_1}e^{i\tilde{h}^{-1}(z-z')\cdot\eta} = (-\tilde{h}D_{z'})^{\alpha+\beta_1}e^{i\tilde{h}^{-1}(z-z')\cdot\eta}$ we see that $R_{N,\beta}$ is a linear combination of terms of the form

$$R'_{N,\beta}(\Xi) = \tilde{h}^{N-d}\int_0^1 \iint e^{i\tilde{h}^{-1}(z-z')\cdot\eta}\kappa(\eta)\times \\ D_{z'}^{\alpha+\beta_1}D_z^{\beta_2}\Big[(\partial_\eta^\alpha\tilde{p})(t,z,z',\eta+\lambda\theta(t,z,z',\xi,\tilde{h})) \\ b(t,z',\xi,\tilde{h})\Psi_0\big(\frac{\partial\phi}{\partial\xi}(t,z'\xi,\tilde{h})-y\big)\Big]\,d\lambda\,dz'\,d\eta.$$

Then we insert in the integral the quantity $\big(\frac{\partial\phi}{\partial\xi}(t,z,\xi,\tilde{h})-y\big)^\gamma$ where $|\gamma| = k_0$. It is a finite linear combination of terms of the form

$$\Big(\frac{\partial\phi}{\partial\xi}(t,z,\xi,\tilde{h})-\frac{\partial\phi}{\partial\xi}(t,z',\xi,\tilde{h})\Big)^{\gamma_1}\Big(\frac{\partial\phi}{\partial\xi}(t,z',\xi,\tilde{h})-y\Big)^{\gamma_2}.$$

Using the Taylor formula we see that $\langle m(\Xi)\rangle^{k_0}R'_{N,\beta}(\Xi)$ is a finite linear combination of terms of the form

$$\tilde{h}^{N-d}\int_0^1\iint (z-z')^\nu e^{i\tilde{h}^{-1}(z-z')\cdot\eta}F(t,z,z',\xi,\tilde{h})\kappa(\eta)\Big(\frac{\partial\phi}{\partial\xi}(t,z',\xi,\tilde{h})-y\Big)^l \\ D_{z'}^{\alpha+\beta_1}D_z^{\beta_2}\Big[(\partial_\eta^\alpha\tilde{p})(\cdots)b(t,z',\xi,\tilde{h})\Psi_0\big(\frac{\partial\phi}{\partial\xi}(t,z',\xi,\tilde{h})-y\big)\Big]\,d\lambda\,dy\,d\eta,$$

where, by Corollary 2.28 (ii), F is a bounded function.

Eventually we use the identity $(z-z')^\nu e^{i\tilde{h}^{-1}(z-z')\cdot\eta} = (\widetilde{h}D_\eta)^\nu e^{i\tilde{h}^{-1}(z-z')\cdot\eta}$, we integrate by parts in the integral with respect to η and we use Remark 2.21, Remark 2.29, the estimate (ii) in Corollary 2.28, the fact that $b\in S^0_{2\delta-1}$ and the fact that $N+N(1-2\delta)=\delta N$ to deduce that

$$\big\langle m(\Xi)\big\rangle^{k_0}|D_z^\beta R_N(\Xi)| \le \mathcal{F}_N(\cdots)\,\widetilde{h}^{\delta N-N_d} \tag{2.141}$$

where N_d is a fixed number depending only on the dimension.

Let us consider the term S_N. Recall that $D_z^\beta S_N$ is a finite linear combination for $|\alpha|\le N-1$ and $|\gamma| = N$ of terms of the form

$$S_{N,\alpha,\beta,\gamma} = \widetilde{h}^{N+|\alpha|}\int_0^1\int(1-\lambda)^{N-1}\mu^\gamma\widehat{\kappa}(\mu)D_{z'}^{\alpha+\beta+\gamma}\Big[\big(\partial_\zeta^\alpha\widetilde{p}\big)\big(t,z,z',\theta(t,z,z',\xi,\widetilde{h}),\widetilde{h}\big) \\ b\big(t,z',\xi,\widetilde{h}\big)\Psi_0\Big(\frac{\partial\phi}{\partial\xi}\big(t,z',\xi,\widetilde{h}\big)-y\Big)\Big]\Big|_{z'=z+\lambda\widetilde{h}\mu}\,d\lambda\,d\mu.$$

Then we multiply $S_{N,\alpha,\beta,\gamma}$ by $\langle m(\Xi)\rangle^{k_0}$ and we write

$$\frac{\partial\phi}{\partial\xi}\big(t,z,\xi,\widetilde{h}\big)-y = \frac{\partial\phi}{\partial\xi}\big(t,z,\xi,\widetilde{h}\big)-\frac{\partial\phi}{\partial\xi}\big(t,z+\lambda\widetilde{h}\mu,\xi,\widetilde{h}\big)+\frac{\partial\phi}{\partial\xi}\big(t,z+\lambda\widetilde{h}\mu,\xi,\widetilde{h}\big)-y.$$

By the Taylor formula the first term will give rise to a power of $\lambda\widetilde{h}\mu$ which will be absorbed by $\hat{\kappa}(\mu)$ and the second term will be absorbed by Ψ_0. Then we use again Remark 2.21, Remark 2.29 to conclude that

$$\big\langle m(\Xi)\big\rangle^{k_0}|D_z^\beta S_N(\Xi)| \le \mathcal{F}_N(\cdots)\,\widetilde{h}^{\delta N-N'_d}. \tag{2.142}$$

Let us look now to the term $\widetilde{h}D_z^\beta T_N\Psi_0$. According to (2.122) this expression is a linear combination for $2 \leq |\alpha| \leq N$ of terms of the form

$$\widetilde{h}^{|\alpha|+N} D_z^\beta \Big\{ D_{z'}^\alpha \Big[(\partial_\zeta^\alpha \widetilde{p}(t,x,z,z',\theta(t,z,z',\xi,\widetilde{h}),\widetilde{h}) b_N(t,z',\xi,\widetilde{h}) \Big] \Big|_{z'=z} \\ \Psi_0\Big(\frac{\partial \phi}{\partial \xi}(t,z',\xi,\widetilde{h}) - y\Big)\Big\}.$$

It follows from Remark 2.21, Remark 2.29 and Corollary 2.28 that we have

$$\langle m(\Xi)\rangle^{k_0} \big|\widetilde{h}D_z^\beta\big[T_N\Psi_0\big]\big| \leq \mathcal{F}_N(\cdots)\,\widetilde{h}^{\delta N}. \tag{2.143}$$

Lemma 2.40 follows from (2.141), (2.142), (2.143). □

¿From Lemma 2.40 we can write

$$\widetilde{h}^{-d}\int \big\| e^{i\widetilde{h}^{-1}(\phi(t,z,\xi,\widetilde{h})-y\cdot\xi)}\big(R_N+S_N+\widetilde{h}T_N\Psi_0\big)\big\|_{H_z^{\sigma_0}} |\chi_1(\xi)|\,d\xi \leq \mathcal{F}_N(\cdots)\,\widetilde{h}^{\delta N - N_1(d)} \tag{2.144}$$

where $N_1(d)$ is a fixed number depending only on the dimension.

To conclude the proof of Proposition 2.38 we have to estimate the integrals

$$I_{N,\beta} = (2\pi\widetilde{h})^{-d}\int D_z^\beta\big[e^{i\widetilde{h}^{-1}(\phi(t,z,\xi,\widetilde{h})-y\cdot\xi)} U_N\big(t,z,y,\xi,\widetilde{h}\big)\big]\chi_1(\xi)\,d\xi.$$

Now according to (2.112), (2.113) and (2.114) on the support of U_N the function Ψ_0 is differentiated at least one time. Thus on this support one has $\big|\frac{\partial\phi}{\partial\xi}(t,z,\xi,\widetilde{h})-y\big| \geq 1$. Then we can use the vector field

$$X = \frac{\widetilde{h}}{\big|\frac{\partial\phi}{\partial\xi}(t,z,\xi,\widetilde{h})-y\big|^2}\sum_{j=1}^d \Big(\frac{\partial\phi}{\partial\xi_j}(t,z,\xi,\widetilde{h})-y_j\Big)D_{\xi_j}$$

to integrate by parts in $I_{N,\beta}$ to obtain

$$|I_{N,\beta}| \leq \mathcal{F}_N(\cdots)\,\widetilde{h}^{\delta N}.$$

The proof of Proposition 2.38 is complete. □

We show now that

$$\mathcal{K}w_{0,\widetilde{h}}(0,z) = w_{0,\widetilde{h}}(z) + r_{\widetilde{h}}(z) \tag{2.145}$$

with

$$\big\|r_{\widetilde{h}}\big\|_{H^{\sigma_0}(\mathbf{R}^d)} \leq \mathcal{F}_N(\cdots)\,\widetilde{h}^N \big\|w_{0,\widetilde{h}}\big\|_{L^1(\mathbf{R}^d)} \quad \forall N\in\mathbf{N}. \tag{2.146}$$

It follows from the initial condition on ϕ given in (2.91) and the initial condition on b that (2.145) is true with

$$r_{\widetilde{h}}(z) = (2\pi\widetilde{h})^{-d}\iint e^{i\widetilde{h}^{-1}(z-y)\cdot\xi}\chi_1(\xi)(1-\Psi_0(z-y))w_{0,\widetilde{h}}(y)\,dy\,d\xi.$$

We see easily that for $|\beta| \leq \sigma_0$, $D_z^\beta r_{\widetilde{h}}(z)$ is a finite linear combination of terms of the form

$$r_{\widetilde{h},\beta}(z) = \widetilde{h}^{-d-|\beta_1|} \iint e^{i\widetilde{h}^{-1}(z-y)\cdot\xi} \xi^{\beta_1} \chi_1(\xi) \Psi_\beta(z-y) w_{0,\widetilde{h}}(y)\, dy\, d\xi, \quad |\beta_1| \leq |\beta|$$

where $\Psi_\beta \in C_b^\infty(\mathbf{R}^d)$ and $|z-y| \geq 1$ on the support of Ψ_β. Then one can write

$$r_{\widetilde{h},\beta}(z) = \big(F_{\widetilde{h}} * w_{0,\widetilde{h}}\big)(z)$$

where

$$F_{\widetilde{h}}(X) = \widetilde{h}^{-d-|\beta_1|} \int e^{i\widetilde{h}^{-1} X\cdot\xi} \xi^{\beta_1} \chi_1(\xi) \Psi_\beta(X)\, d\xi$$

and $|X| \geq 1$ on the support of $\Psi_\beta(X)$. Then we remark that if we set $L = \frac{1}{|X|^2} \sum_{j=1}^d X_j \frac{\partial}{\partial \xi_j}$ we have $\widetilde{h} L e^{i\widetilde{h}^{-1}X\cdot\xi} = e^{i\widetilde{h}^{-1}X\cdot\xi}$. Therefore one can write

$$F_{\widetilde{h}}(X) = \widetilde{h}^{M-d-|\beta_1|} \int e^{i\widetilde{h}^{-1}X\cdot\xi} (-L)^M \big[\xi^{\beta_1} \chi_1(\xi)\big] \Psi_\beta(X)\, d\xi$$

from which we deduce

$$|F_{\widetilde{h}}(X)| \leq \mathcal{F}_M(\cdots)\, \widetilde{h}^{M-d-|\beta_1|} \frac{|\tilde{\Psi}(X)|}{|X|^M}, \quad \forall M \in \mathbf{N}$$

where $\tilde{\Psi} \in C_b^\infty(\mathbf{R}^d)$ is equal to 1 on the support of Ψ_β.

It follows then that $\|F_{\widetilde{h}}\|_{L^2(\mathbf{R}^d)} \leq \mathcal{F}_M(\cdots)\, \widetilde{h}^{M-d-|\beta_1|}$ from which we deduce that for $|\beta| \leq \sigma_0$,

$$\begin{aligned} \big\|D^\beta r_{\widetilde{h}}\big\|_{L^2(\mathbf{R}^d)} &\leq \mathcal{F}_N(\cdots)\, \big\|F_{\widetilde{h}}\big\|_{L^2(\mathbf{R}^d)} \big\|w_{0,\widetilde{h}}\big\|_{L^1(\mathbf{R}^d)} \\ &\leq \mathcal{F}_N(\cdots)\, \widetilde{h}^{M-d-|\beta_1|} \big\|w_{0,\widetilde{h}}\big\|_{L^1(\mathbf{R}^d)} \end{aligned}$$

which proves (2.146).

Using Proposition 2.38 and the Duhamel formula one can write

$$\tilde{S}(t,0) w_{0,\widetilde{h}}(z) = \mathcal{K} w_{0,\widetilde{h}}(t,z) - \tilde{S}(t,0) r_{\widetilde{h}}(z) - \int_0^t \tilde{S}(t,s)[F_{\widetilde{h}}(s,z)]\, ds.$$

Now we can write

$$\begin{aligned} \Big\| \int_0^t \tilde{S}(t,s)\big[F_{\widetilde{h}}(s,z)\big]\, ds \Big\|_{L^\infty(\mathbf{R}^d)} &\leq \int_0^t \Big\| \tilde{S}(t,s)\big[F_{\widetilde{h}}(s,z)\big] \Big\|_{H^{\sigma_0}(\mathbf{R}^d)}\, ds \\ &\leq C \int_0^t \big\|F_{\widetilde{h}}(s,z)\big\|_{H^{\sigma_0}(\mathbf{R}^d)}\, ds \\ &\leq \mathcal{F}_N(\cdots)\, \widetilde{h}^N \big\|w_{0,\widetilde{h}}\big\|_{L^1(\mathbf{R}^d)} \end{aligned}$$

and, for every $N \in \mathbf{N}$,

$$\begin{aligned} \big\|\tilde{S}(t,0) r_{\widetilde{h}}\big\|_{L^\infty(\mathbf{R}^d)} \leq C \big\|\tilde{S}(t,0) r_{\widetilde{h}}\big\|_{H^{\sigma_0}(\mathbf{R}^d)} &\leq C' \big\|r_{\widetilde{h}}\big\|_{H^{\sigma_0}(\mathbf{R}^d)} \\ &\leq \mathcal{F}_N(\cdots)\, \widetilde{h}^N \big\|w_{0,\widetilde{h}}\big\|_{L^1(\mathbf{R}^d)}. \end{aligned}$$

Then Theorem 2.36 follows from these estimates and (2.136). □

2.9 The Strichartz estimates

Theorem 2.41. *Consider the problem*

$$\Big(\partial_t + \frac{1}{2}(S_{j\delta}(V)\cdot\nabla + \nabla\cdot S_{j\delta}(V)) + i\mathfrak{T}_{\gamma_\delta}\Big)u_h(t,x) = f_h(t,x), \quad u_h(t_0,x) = u_{0,h}(x).$$

where $u_h, u_{0,h}$ and f_h have spectrum in $\{\xi : c_1 h^{-1} \leq |\xi| \leq c_2 h^{-1}\}$. Let $I_h = (0, h^{\frac{\delta}{2}})$. Then there exists $k = k(d), h_0 > 0$ such that for any $s \in \mathbf{R}$ and $\varepsilon > 0$ there exists $\mathcal{F}, \mathcal{F}_\varepsilon : \mathbf{R}^+ \to \mathbf{R}^+$, such that, with $N := \|V\|_{E_0} + \mathcal{N}_k(\gamma)$,

(*i*) *if $d = 1$:*

$$\|u_h\|_{L^4(I_h, W^{s-\frac{3}{8},\infty}(\mathbf{R}))} \leq \mathcal{F}(N)\Big(\|u_{0,h}\|_{H^s(\mathbf{R})} + \|f_h\|_{L^1(I_h,H^s(\mathbf{R}))}\Big),$$

(*ii*) *if $d \geq 2$:*

$$\|u_h\|_{L^{2+\varepsilon}(I_h, W^{s-\frac{d}{2}+\frac{1}{4}-\varepsilon,\infty}(\mathbf{R}^d))} \leq \mathcal{F}_\varepsilon(N)\Big(\|u_{0,h}\|_{H^s(\mathbf{R}^d)} + \|f_h\|_{L^1(I_h,H^s(\mathbf{R}^d))}\Big),$$

for any $0 < h \leq h_0$

Proof. If $d = 1$, by the TT^* argument we deduce from the dispersive estimate given in Theorem 2.35 that

$$\|u_h\|_{L^4(I_h,L^\infty(\mathbf{R}))} \leq \mathcal{F}(\cdots)\,h^{-\frac{3}{8}}\Big(\|u_{0,h}\|_{L^2(\mathbf{R})} + \|f_h\|_{L^1(I_h,L^2(\mathbf{R}))}\Big).$$

Then multiplying this estimate by h^s and using the fact that $u_h, u_{0,h}, f_h$ are spectrally supported in $\{\xi : c_1 h^{-1} \leq |\xi| \leq c_2 h^{-1}\}$ we deduce (*i*).

If $d \geq 2$ we use the same argument. Then if $(q,r) \in \mathbf{R}^2$ is such that $q > 2$ and $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$ we obtain

$$\|u_h\|_{L^q(I_h,L^r(\mathbf{R}^d))} \leq \mathcal{F}(\cdots)\,h^{-\frac{3}{2q}}\Big(\|u_{0,h}\|_{L^2(\mathbf{R}^d)} + \|f_h\|_{L^1(I_h,L^2(\mathbf{R}^d))}\Big). \tag{2.147}$$

Taking $q = 2+\varepsilon$ we find $r = 2 + \frac{8}{(2+\varepsilon)d-4}$. Moreover $h^{-\frac{3}{2q}} \leq h^{-\frac{3}{4}}$. Then multiplying both members of (2.147) by h^s we obtain

$$\|u_h\|_{L^{2+\varepsilon}(I_h,W^{s-\frac{3}{4},r}(\mathbf{R}^d))} \leq \mathcal{F}(\cdots)\Big(\|u_{0,h}\|_{H^s(\mathbf{R}^d)} + \|f_h\|_{L^1(I_h,H^s(\mathbf{R}^d))}\Big).$$

On the other hand the Sobolev embedding shows that $W^{a+b,r}(\mathbf{R}^d) \subset W^{b,\infty}(\mathbf{R}^d)$ provided that $a > \frac{d}{r} = \frac{d}{2} - 1 + \frac{\varepsilon}{2+\varepsilon}$. In particular we can take $a = \frac{d}{2} - 1 + \varepsilon$. Taking $\frac{d}{2} - 1 + \varepsilon + b = s - \frac{3}{4}$ we obtain the conclusion of the Theorem. ☐

Corollary 2.42. *With the notations in Theorem 2.41 and $\delta = \frac{2}{3}, I = [0,T]$ we have*

(*i*) *if $d = 1$:*

$$\|u_h\|_{L^4(I;W^{s-\frac{3}{8}-\frac{\delta}{8},\infty}(\mathbf{R}))} \leq \mathcal{F}(N)\Big(\|f_h\|_{L^4(I;H^{s-\frac{\delta}{2}}(\mathbf{R}))} + \|u_h\|_{C^0(I;H^s(\mathbf{R}))}\Big),$$

(*ii*) *if* $d \geq 2$*:*

$$\|u_h\|_{L^2(I;W^{s-\frac{d}{2}+\frac{1}{4}-\frac{\delta}{4}-\varepsilon,\infty}(\mathbf{R}^d))} \leq \mathcal{F}_\varepsilon(N)\Big(\|f_h\|_{L^2(I;H^{s-\frac{\delta}{2}}(\mathbf{R}^d))} + \|u_h\|_{C^0(I;H^s(\mathbf{R}^d))}\Big)$$

for any $\varepsilon > 0$.

Proof. Let $T > 0$ and $\chi \in C_0^\infty(0,2)$ equal to one on $[\frac{1}{2},\frac{3}{2}]$. For $0 \leq k \leq [Th^{-\frac{\delta}{2}}] - 2$ define

$$I_{h,k} = [kh^{\frac{\delta}{2}},(k+2)h^{\frac{\delta}{2}}], \quad \chi_{h,k}(t) = \chi\Big(\frac{t-kh^{\frac{\delta}{2}}}{h^{\frac{\delta}{2}}}\Big), \quad u_{h,k} = \chi_{h,k}(t)u_h.$$

Then

$$\Big(\partial_t + \frac{1}{2}(S_{j\delta}(V)\cdot\nabla + \nabla\cdot S_{j\delta}(V)) + iT_{\gamma_\delta}\Big)u_{h,k} = \chi_{h,k}f_h + h^{-\frac{\delta}{2}}\chi'\Big(\frac{t-kh^{\frac{\delta}{2}}}{h^{\frac{\delta}{2}}}\Big)u_h$$

and $u_{h,k}(kh^{\frac{\delta}{2}},\cdot) = 0$.

Consider first the case $d = 1$. Applying Theorem 2.41, (i) to each $u_{h,k}$ on the interval $I_{h,k}$ we obtain, since $\chi_{h,k}(t) = 1$ for $(k+\frac{1}{2})h^{\frac{\delta}{2}} \leq t \leq (k+\frac{3}{2})h^{\frac{\delta}{2}}$,

$$\begin{aligned}
&\|u_h\|_{L^4((k+\frac{1}{2})h^{\frac{\delta}{2}},(k+\frac{3}{2})h^{\frac{\delta}{2}});W^{s-\frac{3}{8},\infty}(\mathbf{R}))}\\
&\leq \mathcal{F}(\cdots)\Big(\|f_h\|_{L^1((kh^{\frac{\delta}{2}},(k+2)h^{\frac{\delta}{2}});H^s(\mathbf{R}))} + h^{-\frac{\delta}{2}}\|\chi'\Big(\frac{t-kh^{\frac{\delta}{2}}}{h^{\frac{\delta}{2}}}\Big)u_h\|_{L^1(\mathbf{R};H^s(\mathbf{R}))}\Big)\\
&\leq \mathcal{F}(\cdots)\Big(h^{\frac{3\delta}{8}}\|f_h\|_{L^4((kh^{\frac{\delta}{2}},(k+2)h^{\frac{\delta}{2}});H^s(\mathbf{R}))} + \|u_h\|_{L^\infty(I;H^s(\mathbf{R}))}\Big).
\end{aligned}$$

Multiplying both members of the above inequality by $h^{\frac{\delta}{8}}$ and taking into account that u_h and f_h are spectrally supported in a ring of size h^{-1} we obtain

$$\begin{aligned}
&\|u_h\|_{L^4((k+\frac{1}{2})h^{\frac{\delta}{2}},(k+\frac{3}{2})h^{\frac{\delta}{2}});W^{s-\frac{3}{8}-\frac{\delta}{8},\infty}(\mathbf{R}))}\\
&\leq \mathcal{F}(\cdots)\Big(\|f_h\|_{L^4((kh^{\frac{\delta}{2}},(k+2)h^{\frac{\delta}{2}});H^{s-\frac{\delta}{2}}(\mathbf{R}))} + h^{\frac{\delta}{8}}\|u_h\|_{L^\infty(I;H^s(\mathbf{R}))}\Big).
\end{aligned} \tag{2.148}$$

Taking the power 4 of (2.148), summing in k from 0 to $[Th^{-\frac{\delta}{2}}] - 2$ we obtain (since there are $\approx Th^{-\frac{\delta}{2}}$ intervals)

$$\|u_h\|_{L^4(I;W^{s-\frac{3}{8}-\frac{\delta}{8},\infty}(\mathbf{R}))} \leq \mathcal{F}(\cdots)\Big(\|f_h\|_{L^4(I;H^{s-\frac{\delta}{2}}(\mathbf{R}))} + \|u_h\|_{C^0(I;H^s(\mathbf{R}))}\Big).$$

This completes the proof of (i).

The proof of (ii) follows exactly the same path. We apply Theorem 2.41, (ii) to each $u_{h,k}$ on the interval $I_{h,k}$. The only difference with the case $d = 1$ is that, passing from the L^1 norm in t of f_h to the L^2 norm, it gives rise to a $h^{\frac{\delta}{4}}$ factor. Therefore we multiply the inequality by $h^{\frac{\delta}{4}}$, we take the square of the new inequality and we sum in k. □

2.9.1 End of the proof of Theorem 1.6

Let u be a solution of equation (1.8). We have proved in (2.45) that $\Delta_j u$ satifies the equation

$$\big(\partial_t + \frac{1}{2}(S_{j\delta}(V)\cdot\nabla + \nabla\cdot S_{j\delta}(V)) + i\mathfrak{T}_{\gamma_\delta}\big)\Delta_j u = f_{4,j}$$

where

$$f_{4,j} = f_{2,j} + \big\{(S_{j\delta}(V) - S_j(V))\cdot\nabla + i(T_{\gamma_\delta} - T_\gamma) + \frac{1}{2}S_{j\delta}(\mathrm{div}\ V) + \frac{i}{2}(T_{\gamma_\delta} - T^*_{\gamma_\delta})\big\}\Delta_j u$$

and with $p = 4$ if $d = 1$, $p = 2$ if $d \geq 2$ we have

$$\|f_{2,j}\|_{L^p(I,H^s(\mathbf{R}^d))} \leq \|\Delta_j f\|_{L^p(I,H^s(\mathbf{R}^d))} + \mathcal{F}(N)\|u\|_{C^0(I,H^s(\mathbf{R}^d))}$$

where $N(V,\gamma) = \|V\|_{E_0} + \mathcal{N}_k(\gamma)$.

We can therefore apply Corollary 2.42 to $\Delta_j u$.

Since $\delta = \frac{2}{3}$ we have when $d = 1$, $s - \frac{3}{8} - \frac{\delta}{8} = s - \frac{1}{2} + \frac{1}{24}$ and when $d \geq 2$, $s - \frac{d}{2} + \frac{1}{4} - \frac{\delta}{4} - \varepsilon = s - \frac{d}{2} + \mu$ with $\mu < \frac{1}{12}$. Therefore in all cases the left hand side is $\|\Delta_j u\|_{L^p(I,C_*^{s-\frac{d}{2}+\mu'}(\mathbf{R}^d))}$ where $\mu' < \frac{1}{24}$ id $d = 1$, $\mu' < \frac{1}{12}$ if $d \geq 2$.

It remains to estimate the quantities

$$\begin{aligned}
(1) &= \|(S_{j\delta}(V) - S_j(V))\cdot\nabla\Delta_j u\|_{L^p(I,H^{s-\frac{\delta}{2}}(\mathbf{R}^d))},\\
(2) &= \|(T_{\gamma_\delta} - T_\gamma)\Delta_j u\|_{L^p(I,H^{s-\frac{\delta}{2}}(\mathbf{R}^d))},\\
(3) &= \|S_{j\delta}(\mathrm{div}\ V)\Delta_j u\|_{L^p(I,H^{s-\frac{\delta}{2}}(\mathbf{R}^d))},\\
(4) &= \|(T_{\gamma_\delta} - T^*_{\gamma_\delta})\Delta_j u\|_{L^p(I,H^{s-\frac{\delta}{2}}(\mathbf{R}^d))}.
\end{aligned}$$

We first estimate the quantities $(1), (2), (3)$ for fixed t which will be skipped.

Consider the term (1). Set $A_j = (S_{j\delta}(V) - S_j(V))\cdot\nabla\Delta_j u$.

Since the spectrum of A_j is contained in a ball of radius $C\,2^j$, we can write

$$\begin{aligned}
&\|A_j\|_{H^{s-\frac{\delta}{2}}(\mathbf{R}^d)}\\
&\quad\leq C\,2^{j(s-\frac{\delta}{2})}\|(S_j(V) - S_{j\delta}(V))\cdot\nabla\Delta_j u\|_{L^2(\mathbf{R}^d)}\\
&\quad\leq C\,2^{j(s-\frac{\delta}{2})}\|S_j(V) - S_{j\delta}(V)\|_{L^\infty(\mathbf{R}^d)}2^{j(1-s)}\|\Delta_j u\|_{H^s(\mathbf{R}^d)}.
\end{aligned}$$

Now we can write

$$(S_j(V) - S_{j\delta}(V))(t,x) = \int_{\mathbf{R}^d}\hat{\psi}(z)\big(V(t,x-2^{-j}z) - V(t,x-2^{-j\delta}z)\big)\,dz.$$

where $\psi \in C_0^\infty(\mathbf{R}^d)$ has its support contained in a ball of radius 1. It follows easily, since $0 < \delta < 1$, that

$$\|(S_j(V) - S_{j\delta}(V))(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}(\cdots)\,2^{-j\delta}\|V(t,\cdot)\|_{W^{1,\infty}(\mathbf{R}^d)}.$$

Therefore we obtain with $p = 2, 4$ and $I = [0, T]$,

$$\|A_j\|_{L^p(I;H^{s-\frac{\delta}{2}}(\mathbf{R}^d))} \leq \mathcal{F}(\cdots)\, 2^{j(1-\frac{3\delta}{2})} \|V\|_{L^p(I;W^{1,\infty}(\mathbf{R}^d))} \|\Delta_j u\|_{L^\infty(I;H^s(\mathbf{R}^d))}.$$

Since $1 - \frac{3\delta}{2} = 0$ we deduce that

$$(1) \leq \mathcal{F}(N(V,\gamma)) \|u\|_{L^\infty(I;H^s(\mathbf{R}^d))}. \tag{2.149}$$

Consider the term (2). Since the spectrum of $(T_{\gamma_\delta} - T_\gamma)\Delta_j u$ is contained in a ball of radius $C2^j$ one can write (for fixed t) using moreover the symbolic calculus,

$$\begin{aligned} \|(T_{\gamma_\delta} - T_\gamma)\Delta_j u\|_{H^{s-\frac{\delta}{2}}(\mathbf{R}^d)} &\leq C 2^{\frac{1-\delta}{2}j} \|T_{\gamma_\delta} - T_\gamma)\Delta_j u\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)}, \\ &\leq C' 2^{\frac{1-\delta}{2}j} M_0^{\frac{1}{2}}((\gamma_\delta - \gamma)\varphi(2^{-j}\cdot)) \|u\|_{H^s(\mathbf{R}^d)}. \end{aligned} \tag{2.150}$$

Recall that for $a \in \Gamma^m_\rho, m \in \mathbf{R}, \rho \geq 0$ we have

$$M^m_\rho(a) = \sup_{|\alpha| \leq k(d)} \sup_{|\xi| \geq \frac{1}{2}} \|\langle\xi\rangle^{|\alpha|-m} D^\alpha_\xi a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)}.$$

Set $\gamma_j = \varphi(2^{-j}\cdot)\gamma, \gamma_{\delta,j} = \varphi(2^{-j}\cdot)\gamma_\delta$. We have

$$D^\alpha_\xi \gamma_{\delta,j}(x,\xi) = (2\pi)^{-d} 2^{j\delta d}\, \widehat{\psi}(2^{j\delta}\cdot) \star D^\alpha_\xi \gamma_j(\cdot,\xi).$$

Since $(2\pi)^{-d}\, 2^{j\delta d} \int \widehat{\psi}(2^{j\delta} y)\, dy = 1$ one can write

$$D^\alpha_\xi(\gamma_{\delta,j}(x,\xi) - \gamma_j(x,\xi)) = (2\pi)^{-d} 2^{j\delta d} \int \widehat{\psi}(2^{j\delta}y)\big(D^\alpha_\xi \gamma_j(x-y,\xi) - D^\alpha_\xi \gamma_j(x,\xi)\big)\, dy.$$

It follows that

$$|D^\alpha_\xi(\gamma_{\delta,j}(x,\xi) - \gamma_j(x,\xi))| \leq C 2^{j\delta d} \int \widehat{\psi}(2^{j\delta}y)|y|^{\frac{1}{2}}\, dy\ \|D^\alpha_\xi(\gamma_j(\cdot,\xi)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)}.$$

Setting $2^{j\delta}y = z$ in the integral we see that

$$M_0^{\frac{1}{2}}(\gamma_{\delta,j} - \gamma_j) \leq C 2^{-j\frac{\delta}{2}} M^{\frac{1}{2}}_{\frac{1}{2}}(\gamma_j).$$

Using (2.150) we deduce that

$$\|(T_{\gamma_\delta} - T_\gamma)\Delta_j u\|_{H^{s-\frac{\delta}{2}}(\mathbf{R}^d)} \leq 2^{j(\frac{1}{2}-\delta)} M^{\frac{1}{2}}_{\frac{1}{2}}(\gamma_j) \|u\|_{H^s(\mathbf{R}^d)}.$$

Taking the L^p norm in t and using the fact that $\frac{1}{2} - \delta = \frac{1}{2} - \frac{2}{3} < 0$ we deduce that

$$(2) \leq C N(V,\gamma) \|u\|_{L^\infty(I;H^s(\mathbf{R}^d))}. \tag{2.151}$$

Consider the term (3). Since $S_{j\delta}(\text{div } V)\Delta_j u$ has its spectrum contained in a ball of radius $C2^j$ on can write

$$\begin{aligned}\|S_{j\delta}(\text{div } V)\Delta_j u\|_{H^s(\mathbf{R}^d)} &\leq C2^{js}\|S_{j\delta}(\text{div}\, V)\Delta_j u\|_{L^2(\mathbf{R}^d)},\\ &\leq C2^{js}\|S_{j\delta}(\text{div } V)\|_{L^\infty(\mathbf{R}^d)}\|\Delta_j u\|_{L^2(\mathbf{R}^d)},\\ &\leq C\|S_{j\delta}(\text{div } V)\|_{L^\infty(\mathbf{R}^d)}\|\Delta_j u\|_{H^s(\mathbf{R}^d)},\\ &\leq C'\|V\|_{W^{1,\infty}(\mathbf{R}^d)}\|\Delta_j u\|_{H^s(\mathbf{R}^d)}.\end{aligned}$$

It follows that

$$(3) \leq \mathcal{F}(N(V,\gamma))\|u\|_{L^\infty(I;H^s(\mathbf{R}^d))}. \tag{2.152}$$

Finally by the symbolic calculus (see Theorem A.5) we can write

$$(4) \leq \mathcal{F}(N(V,\gamma))\|u\|_{L^\infty(I;H^s(\mathbf{R}^d))}. \tag{2.153}$$

Summing up, it follows from the estimates in (2.149) to (2.153) that one can apply Corollary 2.42 to $\Delta_j u$ to deduce that

$$\|\Delta_j u\|_{L^p(I,C_*^{s-\frac{d}{2}+\mu'}(\mathbf{R}^d))} \leq \mathcal{F}(N(V,\gamma))\big\{\|\Delta_j f\|_{L^p(I,H^s(\mathbf{R}^d))} + \|u\|_{L^\infty(I,H^s(\mathbf{R}^d))}\big\}.$$

Now for all $\mu < \mu'$ we can write

$$\begin{aligned}\|\Delta_j u\|_{L^p(I,C_*^{s-\frac{d}{2}-\mu}(\mathbf{R}^d))} &\leq 2^{-j(\mu'-\mu)}\|\Delta_j u\|_{L^p(I,C_*^{s-\frac{d}{2}-\mu'}(\mathbf{R}^d))},\\ &\leq 2^{-j(\mu'-\mu)}\mathcal{F}(N(V,\gamma))\big\{\|\Delta_j f\|_{L^p(I,H^s(\mathbf{R}^d))} + \|u\|_{L^\infty(I,H^s(\mathbf{R}^d))}\big\},\\ &\leq 2^{-j(\mu'-\mu)}\mathcal{F}(N(V,\gamma)))\big\{\|f\|_{L^p(I,H^s(\mathbf{R}^d))} + \|u\|_{L^\infty(I,H^s(\mathbf{R}^d))}\big\}.\end{aligned}$$

Summing in j and using the fact that $\sum_{j=-1}^{+\infty} 2^{-j(\mu'-\mu)} < +\infty$ we obtain the conclusion of Theorem 1.6. The proof is complete.

Chapter 3

Cauchy problem

In this chapter we complete the proof of our main result, which is Theorem 1.2 stated in the introduction. We begin in Section 3.1 by combining Sobolev and Strichartz estimates to obtain *a priori* estimates. Then, in Section 3.2, we obtain an estimate for the difference of two solutions. This estimate will be used to prove the uniqueness of solutions as well as to prove that a family of approximate solutions is a Cauchy sequence, in some larger space, and then converges strongly. In Section 3.3 we prove that one can pass to the limit in the equations under weak assumptions. In Section 3.4 we briefly recall how to complete the proof from these three technical ingredients.

3.1 A priori estimates

3.1.1 Notations

For the sake of clarity we recall here our assumptions and notations. We work with the Craig-Sulem-Zakharov formulation of the water-waves equations:

$$\begin{cases} \partial_t\eta - G(\eta)\psi = 0, \\ \partial_t\psi + g\eta + \frac{1}{2}\left|\nabla\psi\right|^2 - \frac{1}{2}\frac{\left(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\right)^2}{1+|\nabla\eta|^2} = 0. \end{cases} \tag{3.1}$$

Assumption 3.1. *We consider smooth solutions of* (3.1) *such that*

i) (η,ψ) *belongs to* $C^1([0,T_0];H^{s_0}(\mathbf{R}^d)\times H^{s_0}(\mathbf{R}^d))$ *for some* T_0 *in* $(0,1]$ *and some* s_0 *large enough;*

ii) there exists $h>0$ *such that* (1.3) *holds for any* t *in* $[0,T_0]$ *(this is the assumption that there exists a curved strip of width* h *separating the free surface from the bottom);*

iii) there exists $c>0$ *such that the Taylor coefficient* $\mathfrak{a}(t,x) = -\partial_y P|_{y=\eta(t,x)}$ *is bounded from below by* c *for any* (t,x) *in* $[0,T_0]\times\mathbf{R}^d$.

We work with the horizontal and vertical traces of the velocity on the free boundary, namely $B = (\partial_y \phi)|_{y=\eta}$ and $V = (\nabla_x \phi)|_{y=\eta}$, which can be defined in terms of η and ψ by means of

$$B := \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1 + |\nabla \eta|^2}, \qquad V := \nabla \psi - B \nabla \eta. \tag{3.2}$$

Let s and r be two positive real numbers such that

$$s > \frac{3}{4} + \frac{d}{2}, \quad s + \frac{1}{4} - \frac{d}{2} > r > 1, \quad r \notin \frac{1}{2}\mathbf{N}. \tag{3.3}$$

Define, for T in $(0, T_0]$, the norms

$$\begin{aligned} M_s(T) &:= \left\| (\psi, \eta, B, V) \right\|_{C^0([0,T];H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)}, \\ Z_r(T) &:= \left\| \eta \right\|_{L^p([0,T];W^{r+\frac{1}{2},\infty})} + \left\| (B, V) \right\|_{L^p([0,T];W^{r,\infty} \times W^{r,\infty})}, \end{aligned} \tag{3.4}$$

where $p = 4$ if $d = 1$ and $p = 2$ for $d \geq 2$.

Our goal is to estimate $M_s(T) + Z_r(T)$ in terms of

$$M_{s,0} := \left\| (\psi(0), \eta(0), B(0), V(0)) \right\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}. \tag{3.5}$$

In Appendix D, using results already proved in [3], we prove that, for any s and r satisfying (3.3), there exists a continuous non-decreasing function $\mathcal{F} \colon \mathbf{R}^+ \to \mathbf{R}^+$ such that, for all smooth solution (η, ψ) of (3.1) defined on the time interval $[0, T_0]$ and satisfying Assumption 3.1 on this time interval, for any $T \in (0, T_0]$,

$$M_s(T) \leq \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T) + Z_r(T)\big)\big). \tag{3.6}$$

If $s > 1 + d/2$, then one can apply the previous inequality with $r = s - d/2$. Then $Z_r(T) \lesssim M_s(T)$ by Sobolev embedding and one deduces from (3.6) an estimate which involves only $M_s(T)$. Thus we recover the *a priori* estimate in Sobolev spaces proved in [3] under the assumption that $s > 1 + d/2$. Using classical inequalities, this implies that for any $A > 0$ there exist $B > 0$ and $T_1 > 0$ such that

$$M_{s,0} \leq A \Rightarrow M_s(T_1) \leq B.$$

We shall prove that a stronger *a priori* estimate holds. We extend the previous estimate in two directions. Firstly, we prove that one can control Sobolev norms for some $s < 1 + d/2$. Secondly, we prove that one can control Strichartz norms even for rough solutions.

Proposition 3.2. *Let μ be such that $\mu < \frac{1}{24}$ if $d = 1$ and $\mu < \frac{1}{12}$ for $d \geq 2$. Consider two real numbers s and r satisfying*

$$s > 1 + \frac{d}{2} - \mu, \quad 1 < r < s + \mu - \frac{d}{2}, \quad r \notin \frac{1}{2}\mathbf{N}. \tag{3.7}$$

For any $A > 0$ there exist $B > 0$ and $T_1 > 0$ such that, for all $0 < T_0 \leq T_1$ and all smooth solution (η, ψ) of (1.5) defined on the time interval $[0, T_0]$ satisfying Assumption 2.1 on this time interval, then the solution satisfies the a priori *bound*

$$M_{s,0} \leq A \Rightarrow M_s(T_0) + Z_r(T_0) \leq B.$$

3.1.2 Reduction

In this section we show that one can reduce the proof of Proposition 3.2 to the proof of an *a priori* estimate for $Z_r(T)$.

To prove Proposition 3.2, the key point is to prove that there exists a continuous non-decreasing function $\mathcal{F}\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$M_s(T) + Z_r(T) \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T) + Z_r(T)\big)\big). \tag{3.8}$$

Since $\mu < 1/4$ and since the estimate (3.6) is proved under the general assumption (3.3), it remains only to prove that $Z_r(T)$ is bounded by the right-hand side of (3.8).

Proposition 3.3. *Let $d \ge 1$ and consider s, r, μ satisfying (3.7). There exists a continuous non-decreasing function $\mathcal{F}\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that, for all $T_0 \in (0,1]$ and all smooth solution (η,ψ) of (3.1) defined on the time interval $[0,T_0]$ and satisfying Assumption 3.1 on this time interval, there holds*

$$Z_r(T) \le \mathcal{F}\big(T\mathcal{F}\big(M_s(T) + Z_r(T)\big)\big), \tag{3.9}$$

for any T in $[0,T_0]$.

Let us admit this result and deduce Proposition 3.2.

Proof of Proposition 3.2 given Proposition 3.3. Introduce for T in $[0,T_0]$, $f(T) = M_s(T) + Z_r(T)$. It follows from (3.6) and (3.9) that (3.8) holds. This means that there exists a continuous non-decreasing function $\mathcal{F}\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that, for all $T \in (0,T_0]$,

$$f(T) \le \mathcal{F}\big(\mathcal{F}(A) + T\mathcal{F}\big(f(T)\big)\big). \tag{3.10}$$

Now fix B such that $B > \max\big\{A, \mathcal{F}\big(\mathcal{F}(A)\big)\big\}$ and then chose $T_1 \in (0,T_0]$ such that

$$\mathcal{F}\big(\mathcal{F}(A) + T_1\mathcal{F}(B)\big) < B.$$

We claim that $f(T) < B$ for any T in $[0,T_1]$. Indeed, since $f(0) = M_{s,0} \le A < B$, assume that there exists $T' \in (0,T_1]$ such that $f(T') > B$. Since f is continuous, this implies that there is $T'' \in (0,T_1)$ such that $f(T'') = B$. Now it follows from (3.10), the fact that $\mathcal{F}$ is increasing, and the definition of T_1 that

$$B = f(T'') \le \mathcal{F}\big(\mathcal{F}(A) + T''\mathcal{F}\big(f(T'')\big)\big) \le \mathcal{F}\big(\mathcal{F}(A) + T_1\mathcal{F}\big(B\big)\big) < B.$$

Hence the contradiction which proves that $f(T) \le B$ for any T in $[0,T_1]$. □

It remains to prove Proposition 3.3. This will be the purpose of the end of this chapter.

We begin by using an interpolation inequality to reduce the proof of Proposition 3.3 to the proof of the following proposition.

Proposition 3.4. *Let $d \geq 1$ and consider μ, s, r as in (3.7). Consider in addition r' such that*

$$r < r' < s + \mu - \frac{d}{2}, \quad r' \notin \frac{1}{2}\mathbf{N}$$

and set

$$Z_{r'}(T) := \|\eta\|_{L^p([0,T];W^{r'+\frac{1}{2},\infty})} + \|(B,V)\|_{L^p([0,T];W^{r',\infty}\times W^{r',\infty})}$$

where $p = 4$ if $d = 1$ and $p = 2$ for $d \geq 2$. There exists a continuous non-decreasing function $\mathcal{F}\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that, for all $T_0 \in (0,1]$ and all smooth solution (η, ψ) of (3.1) defined on the time interval $[0, T_0]$ and satisfying Assumption 3.1 on this time interval, there holds

$$Z_{r'}(T) \leq \mathcal{F}\big(M_s(T) + Z_r(T)\big), \tag{3.11}$$

for any T in $[0, T_0]$.

We prove in this paragraph that Proposition 3.4 implies Proposition 3.3. Proposition 3.4 will be proved in the next paragraph.

Proof of Proposition 3.3 given Proposition 3.4. Consider a function $v = v(t,x)$. By interpolation, since $1 - \mu < 1 < r < r'$, there exists a real number $\theta \in (0,1)$ such that

$$\|v(t,\cdot)\|_{W^{r,\infty}} \lesssim \|v(t,\cdot)\|_{W^{1-\mu,\infty}}^{\theta} \|v(t,\cdot)\|_{W^{r',\infty}}^{1-\theta}.$$

This implies that

$$\int_0^T \|v(t,\cdot)\|_{W^{r,\infty}}^p \, dt \lesssim \|v\|_{C^0([0,T];W^{1-\mu,\infty})}^{p\theta} \int_0^T \|v(t,\cdot)\|_{W^{r',\infty}}^{p(1-\theta)} \, dt.$$

The Hölder inequality then implies that

$$\|v\|_{L^p([0,T];W^{r,\infty})} \lesssim T^{\frac{\theta}{p}} \|v\|_{C^0([0,T];W^{1-\mu,\infty})}^{\theta} \|v\|_{L^p([0,T];W^{r',\infty})}^{1-\theta}.$$

By the same way, there holds

$$\|v\|_{L^p([0,T];W^{r+\frac{1}{2},\infty})} \lesssim T^{\frac{\theta'}{p}} \|v\|_{C^0([0,T];W^{1-\mu+\frac{1}{2},\infty})}^{\theta'} \|v\|_{L^p([0,T];W^{r'+\frac{1}{2},\infty})}^{1-\theta'}.$$

Since $s > (1-\mu) + d/2$, the Sobolev embedding implies that

$$\|v\|_{C^0([0,T];W^{1-\mu,\infty})} \lesssim \|v\|_{C^0([0,T];H^s)}, \quad \|v\|_{C^0([0,T];W^{1-\mu+\frac{1}{2},\infty})} \lesssim \|v\|_{C^0([0,T];H^{s+\frac{1}{2}})}.$$

This proves that

$$Z_r(T) \leq T^{\frac{\theta}{p}} M_s(T)^{\theta} (Z_{r'}(T))^{1-\theta}$$

for some $\theta > 0$. This in turn proves that (3.11) implies (3.9). □

3.1.3 Proof of Proposition 3.4

Recall that the positive real number μ has been chosen such that $\mu < 1/24$ if $d = 1$ and $\mu < 1/12$ for $d \geq 2$, and s, r, r' are such that

$$s > 1 + \frac{d}{2} - \mu, \quad 1 < r < r' < s + \mu - \frac{d}{2}, \quad r \notin \frac{1}{2}\mathbf{N}.$$

Let $T > 0$ and set $I = [0, T]$.

The proof of Proposition 3.4 is based on Corollary 2.7 and Theorem 1.6. By combining these two results we shall deduce in the first step of the proof that

$$\|u\|_{L^p(I;W^{r',\infty})} \leq \mathcal{F}\big(M_s(T) + Z_r(T)\big) \tag{3.12}$$

where u is defined in terms of (η, V, B) by (see (2.17))

$$\begin{aligned} &u = \langle D_x\rangle^{-s}(U_s - i\theta_s),\\ &U_s := \langle D_x\rangle^{s} V + T_\zeta \langle D_x\rangle^{s} B \qquad (\zeta = \nabla\eta),\\ &\theta_s := T_{\sqrt{\mathfrak{a}/\lambda}}\langle D_x\rangle^{s}\nabla\eta. \end{aligned} \tag{3.13}$$

In the next steps of the proof we show how to recover estimates for the original unknowns (η, V, B) in $L^p([0,T]; W^{r'+\frac{1}{2}} \times W^{r',\infty} \times W^{r',\infty})$.

Step 1: proof of (3.12). It follows from Theorem 1.6 that

$$\|u\|_{L^p(I;W^{r',\infty}(\mathbf{R}^d))} \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_k(\gamma)\big)\Big\{\|f\|_{L^p(I;H^s(\mathbf{R}^d))} + \|u\|_{C^0(I;H^s(\mathbf{R}^d))}\Big\}.$$

Clearly we have

$$\|V\|_{E_0} \leq Z_1(T) \leq Z_r(T), \quad \mathcal{N}_k(\gamma) \lesssim M_s(T).$$

Moreover, (2.19) and (2.22) imply that

$$\|u\|_{C^0(I;H^s(\mathbf{R}^d))} \leq \mathcal{F}(M_s(T)), \quad \|f\|_{L^p(I;H^s(\mathbf{R}^d))} \leq \mathcal{F}\big(M_s(T) + Z_r(T)\big).$$

By combining the previous estimates we deduce the desired estimate (3.12).

Step 2: estimate for η. Separating real and imaginary parts, directly from the definition (3.13) of u, we get

$$\big\|\langle D_x\rangle^{-s} T_{\sqrt{a/\lambda}}\langle D_x\rangle^{s}\nabla\eta\big\|_{W^{r',\infty}} \leq \|u\|_{W^{r',\infty}}\,.$$

We shall make repeated uses of the following elementary result.

Lemma 3.5. *Consider $m \in \mathbf{R}$ and ρ in $[0,1]$. Let $(r, r_1, r_2) \in [0, +\infty)^3$ be such that $r \le \min(r_1 + \rho, r_2 + m)$, $r \notin \mathbf{N}$. Consider the equation $T_\tau v = f$ where $\tau = \tau(x, \xi)$ is a symbol such that τ (resp. $1/\tau$) belongs to Γ^m_ρ (resp. Γ^{-m}_ρ). Then*

$$\|v\|_{W^{r,\infty}} \le K \|v\|_{W^{r_1,\infty}} + K \|f\|_{W^{r_2,\infty}}$$

for some constant K depending only on $\mathcal{M}^m_\rho(\tau) + \mathcal{M}^{-m}_\rho(1/\tau)$.

Proof. Write

$$v = \big(I - T_{1/\tau} T_\tau\big) v + T_{1/\tau} f$$

and use (A.4) (resp. (A.5)) to estimate the first (resp. second) term. □

Now write

$$\langle D_x \rangle^{-s} T_{\sqrt{a/\lambda}} \langle D_x \rangle^s \nabla = T_{\sqrt{a/\lambda}} \nabla + R$$

where $R = \big[\langle D_x \rangle^{-s}, T_{\sqrt{a/\lambda}} \langle D_x \rangle^s \nabla\big]$. Since $\sqrt{a/\lambda}$ is a symbol of order $-1/2$ in ξ, it follows from (A.5) that, for any $\rho \in (0,1)$,

$$\|R\eta\|_{W^{r',\infty}} \le K \mathcal{M}^{-\frac{1}{2}}_\rho \Big(\sqrt{\frac{a}{\lambda}}\Big) \|\eta\|_{W^{r'+\frac{1}{2}-\rho,\infty}}$$

and hence

$$\|R\eta\|_{W^{r',\infty}} \le \mathcal{F}\big(\|\nabla\eta\|_{W^{\rho,\infty}}, \|a\|_{W^{\rho,\infty}}\big) \|\eta\|_{W^{r'+\frac{1}{2}-\rho,\infty}}.$$

Now by assumption on s and r' we can chose ρ (say $\rho = 1/4$) so that

$$\rho < s - \frac{1}{2} - \frac{d}{2}, \qquad r' + \frac{1}{2} - \rho < s + \frac{1}{2} - \frac{d}{2}$$

and hence

$$\|R\eta\|_{W^{r',\infty}} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}}, \|a - g\|_{H^{s-\frac{1}{2}}}\big).$$

Recalling (see (C.1)) that $\|a - g\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}(M_s)$ for any $s > 3/4 + d/2$, we obtain $\|R\eta\|_{W^{r',\infty}} \le \mathcal{F}(M_s)$.

We thus deduce that

$$\Big\|T_{\sqrt{a/\lambda}} \nabla\eta\Big\|_{W^{r',\infty}} \le \|u\|_{W^{r',\infty}} + \mathcal{F}(M_s).$$

Now, Lemma 3.5, applied with $m = -1/2$ and $\rho = 1/4$, yields an estimate for the $W^{r'-\frac{1}{2},\infty}$-norm of $\nabla\eta$ which implies that

$$\|\eta\|_{W^{r'+\frac{1}{2},\infty}} \le K \|u\|_{W^{r',\infty}} + K \|\eta\|_{W^{r'+\frac{1}{2}-\rho,\infty}} + K\mathcal{F}(M_s)$$

for some constant K depending only on $\mathcal{M}^{-\frac{1}{2}}_\rho\Big(\sqrt{\frac{a}{\lambda}}\Big)$. As already seen, $K \le \mathcal{F}(M_s)$ and $\|\eta\|_{W^{r'+\frac{1}{2}-\rho,\infty}} \le M_s$ (using the Sobolev embedding). We conclude that

$$\|\eta\|_{W^{r'+\frac{1}{2},\infty}} \le \mathcal{F}(M_s) \|u\|_{W^{r',\infty}} + \mathcal{F}(M_s).$$

Therefore (3.12) implies that

$$\|\eta\|_{L^p(I; W^{r'+\frac{1}{2},\infty})} \le \mathcal{F}\big(M_s(T) + Z_r(T)\big). \tag{3.14}$$

Step 3: estimate for $V + T_\zeta B$. We proceed as above: starting from (3.12) one deduces an estimate for the $W^{r',\infty}$-norm of $\langle D_x\rangle^{-s}\big(\langle D_x\rangle^s V + T_\zeta\langle D_x\rangle^s B\big)$. One rewrite this term as $V + T_\zeta B$ plus a commutator which is estimated by means of (A.5) and the Sobolev embedding. It is find that

$$\|V + T_\zeta B\|_{W^{r',\infty}} \le \mathcal{F}(M_s)\,\|u\|_{W^{r',\infty}} + \mathcal{F}(M_s)$$

so that (3.12) implies that

$$\|V + T_\zeta B\|_{L^p(I;W^{r',\infty})} \le \mathcal{F}\big(M_s(T) + Z_r(T)\big). \tag{3.15}$$

Step 4: estimate for V and B. We shall estimate the $L^p(I;W^{r',\infty})$-norm of B. The estimate for the $L^p(I;W^{r',\infty})$-norm of V will follow from $V = (V + T_\zeta B) - T_\zeta B$ since the first term $V + T_\zeta B$ is already estimated (see (3.15)) and since, for the second term, one can use the rule (A.4) to obtain $\|T_\zeta B\|_{W^{r',\infty}} \lesssim \|\zeta\|_{L^\infty}\|B\|_{W^{r',\infty}} \lesssim \|\eta\|_{H^{s+\frac12}}\|B\|_{W^{r',\infty}}$.

To estimate the $W^{r',\infty}$-norm of B, as above, we use the identity $G(\eta)B = -\operatorname{div} V + \widetilde{\gamma}$ where (see (2.5))

$$\|\widetilde{\gamma}\|_{W^{r'-1,\infty}} \le \|\widetilde{\gamma}\|_{H^{s-\frac12}} \le \mathcal{F}\big(\,\|(\eta,V,B)\|_{H^{s+\frac12}\times H^{\frac12}\times H^{\frac12}}\,\big).$$

In order to use this identity, write

$$\begin{aligned}
\operatorname{div}\big(V + T_\zeta B\big) &= \operatorname{div} V + T_{\operatorname{div}\zeta} B + T_\zeta\cdot\nabla B\\
&= -G(\eta)B + T_{\operatorname{div}\zeta}B + T_\zeta\cdot\nabla B + \widetilde{\gamma}\\
&= T_{-\lambda + i\zeta\cdot\xi}B + r
\end{aligned} \tag{3.16}$$

where

$$r = T_{\operatorname{div}\zeta}B + \widetilde{\gamma} + (T_\lambda - G(\eta))B.$$

The first term in the right-hand side is estimated by means of

$$\begin{aligned}
\|T_{\operatorname{div}\zeta}B\|_{W^{r'-1,\infty}} &\lesssim \|T_{\operatorname{div}\zeta}B\|_{H^{s-1+\mu}} && \text{since } r' < s + \mu - \frac{d}{2}\\
&\lesssim \|\operatorname{div}\zeta\|_{C^{\mu-1}_*}\|B\|_{H^s} && \text{(see (A.12))}\\
&\lesssim \|\eta\|_{W^{1+\mu,\infty}}\|B\|_{H^s} && \text{since } \operatorname{div}\zeta = \Delta\eta\\
&\lesssim \|\eta\|_{H^{s+\frac12}}\|B\|_{H^s} && \text{since } 1 + \mu < 1 + \frac14 < s + \frac12 - \frac{d}{2}.
\end{aligned}$$

The key point is to estimate $(T_\lambda - G(\eta))B$. To do so we use Proposition D.4 with (μ,σ,ε) replaced by $(s+\frac12, s, \frac14)$ and the Sobolev embedding $H^{s-\frac34}(\mathbf{R}^d) \subset W^{r'-1,\infty}(\mathbf{R}^d)$. This implies that

$$\|(T_\lambda - G(\eta))B\|_{W^{r'-1,\infty}} \lesssim \|(T_\lambda - G(\eta))B\|_{H^{s-\frac34}} \le \mathcal{F}\big(\,\|\eta\|_{H^{s+\frac12}}\,\big)\,\|B\|_{H^s}\,.$$

We end up with

$$\|r\|_{W^{r'-1,\infty}} \le \mathcal{F}\big(\,\|(\eta,V,B)\|_{H^{s+\frac12}\times H^s\times H^s}\,\big).$$

Writing (see (3.16))

$$T_{-\lambda+i\zeta\cdot\xi}B = \operatorname{div}\big(V + T_{\zeta}B\big) - r,$$

it follows from (3.15) and Lemma 3.5 that

$$\|B\|_{L^p(I;W^{r',\infty})} \le \mathcal{F}\big(M_s(T) + Z_r(T)\big).$$

This completes the proof of Proposition 3.4 and hence the proof of Proposition 3.2.

3.2 Contraction estimates

Our goal in this section is to prove the following estimate for the difference of two solutions.

Theorem 3.6. *Let μ be such that $\mu < \frac{1}{24}$ if $d = 1$ and $\mu < \frac{1}{12}$ for $d \ge 2$. Consider two real numbers s and r satisfying*

$$s > 1 + \frac{d}{2} - \mu, \quad 1 < r < s + \mu - \frac{d}{2}, \quad r \notin \frac{1}{2}\mathbf{N}.$$

Let (η_j, ψ_j), $j = 1, 2$, be two solutions such that

$$(\eta_j, \psi_j, V_j, B_j) \in C^0([0,T]; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s),$$
$$(\eta_j, V_j, B_j) \in L^p([0,T]; W^{r+\frac{1}{2},\infty} \times W^{r,\infty} \times W^{r,\infty}),$$

for some fixed $T > 0$, $d \ge 1$ with $p = 4$ if $d = 1$ and $p = 2$ otherwise. We also assume that the condition (B.1) holds for $0 \le t \le T$ and that there exists $c > 0$ such that for all $0 \le t \le T$ and $x \in \mathbf{R}^d$, we have $\mathfrak{a}_j(t,x) \ge c$ for $j = 1, 2$. Set

$$M_j := \sup_{t\in[0,T]} \|(\eta_j, \psi_j, V_j, B_j)(t)\|_{H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s},$$
$$+ \|(\eta_j, V_j, B_j)\|_{L^p([0,T];W^{r+\frac{1}{2},\infty}\times W^{r,\infty}\times W^{r,\infty})}.$$

Set

$$\eta := \eta_1 - \eta_2, \quad \psi := \psi_1 - \psi_2, \quad V := V_1 - V_2, \quad B := B_1 - B_2,$$

and

$$N(T) := \sup_{t\in[0,T]} \|(\eta, \psi, V, B)(t)\|_{H^{s-\frac{1}{2}}\times H^{s-\frac{1}{2}}\times H^{s-1}\times H^{s-1}}$$
$$+ \|(\eta, V, B)\|_{L^p([0,T];W^{r-\frac{1}{2},\infty}\times W^{r-1,\infty}\times W^{r-1,\infty})}.$$

Then we have

$$N(T) \le \mathcal{K}\|(\eta, \psi, V, B)\,|_{t=0}\,\|_{H^{s-\frac{1}{2}}\times H^{s-\frac{1}{2}}\times H^{s-1}\times H^{s-1}}, \tag{3.17}$$

where $\mathcal{K}$ is a positive constant depending only on T, M_1, M_2, r, s, d, c.

Remark 3.7. To prove this theorem, we shall follow closely the analysis in [3]. However, there are quite difficulties which appear for $s < 1 + d/2$, in particular for $d = 1$ and with a general bottom. For instance, one has to estimate the $H^{s-\frac{3}{2}}$-norm of various products of the form uv with $u \in H^{s-1}$ and $v \in H^{s-\frac{3}{2}}$. For $s < 1 + d/2$, the product is no longer bounded from $H^{s-1} \times H^{s-\frac{3}{2}}$ to $H^{s-\frac{3}{2}}$ and, clearly, one has to further assume some control in Hölder or Zygmund norms. Namely, we assume that $u \in H^{s-1} \cap L^\infty$ and $v \in H^{s-\frac{3}{2}} \cap C_*^{-\frac{1}{2}}$. Then, paralinearizing the product $uv = T_u v + T_v u + R(u, v)$ and using the usual estimate for paraproducts (see (A.4) and (A.12)), one obtains

$$\|T_u v\|_{H^{s-\frac{3}{2}}} \lesssim \|u\|_{L^\infty} \|v\|_{H^{s-\frac{3}{2}}}, \quad \|T_v u\|_{H^{s-\frac{3}{2}}} \lesssim \|v\|_{C_*^{-\frac{1}{2}}} \|u\|_{H^{s-1}}$$

so the only difficulty is to estimate the $H^{s-\frac{3}{2}}$-norm of the remainder $R(u, v)$. However, the estimate (A.11) requires that $s - \frac{3}{2} > 0$, which does not hold in general under the assumption (3.7). To circumvent this problem, each times that we shall need to estimate the $H^{s-\frac{3}{2}}$-norm of such remainders, we shall prove that one can factor out some derivative exploiting the structure of the water waves equations. This means that one can replace $R(u, v)$ by $\partial_x R(\tilde{u}, \tilde{v})$ for some functions, say, $\tilde{u} \in L^\infty$ and $\tilde{v} \in H^{s-\frac{1}{2}}$. Now one can estimate the $H^{s-\frac{1}{2}}$-norm of $R(\tilde{u}, \tilde{v})$ by means of (A.11) which immediately implies the desired estimate for the $H^{s-\frac{3}{2}}$-norm of $\partial_x R(\tilde{u}, \tilde{v})$.

3.2.1 Contraction for the Dirichlet-Neumann

A key step in the proof of Theorem 3.6 is to prove a Lipschitz property for the Dirichlet-Neumann operator.

Proposition 3.8. *Assume $d \geq 1$, $s > \frac{3}{4} + \frac{d}{2}$, $s + \frac{1}{4} - \frac{d}{2} > r > 1$. Then there exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\begin{aligned} &\|G(\eta_1)f - G(\eta_2)f\|_{H^{s-\frac{3}{2}}} \\ &\leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}}}) \Big\{ \|\eta_1 - \eta_2\|_{W^{r-\frac{1}{2},\infty}} \|f\|_{H^s} \\ &\qquad + \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \big(\|f\|_{H^s} + \|f\|_{W^{r,\infty}} \big) \Big\}. \end{aligned} \tag{3.18}$$

In the proof of Proposition 3.8 we shall use the following classical lemma.

Lemma 3.9. *Let $I = (-1, 0)$ and $\sigma \in \mathbf{R}$. Let $u \in L^2_z(I, H^{\sigma+\frac{1}{2}}(\mathbf{R}^d))$ such that $\partial_z u \in L^2_z(I, H^{\sigma-\frac{1}{2}}(\mathbf{R}^d))$. Then $u \in C^0([-1, 0], H^\sigma(\mathbf{R}^d))$ and there exists an absolute constant $C > 0$ such that*

$$\sup_{z \in [-1,0]} \|u(z, \cdot)\|_{H^\sigma(\mathbf{R}^d)} \leq C\big(\|u\|_{L^2(I, H^{\sigma+\frac{1}{2}}(\mathbf{R}^d))} + \|\partial_z u\|_{L^2(I, H^{\sigma-\frac{1}{2}}(\mathbf{R}^d))}\big).$$

Proof of Proposition 3.8. In the sequel we shall denote by RHS the right hand side of (3.18) where $\mathcal{F}$ may vary from line to line.

We want to apply changes of variable as in (B.4). However, here, we have an additional constraint. Indeed, after this change of variables, we want to get the same domain for η_1 and η_2 to be able to compare the variational solutions. For this purpose, we need to modify slightly the change of variables in (B.4).

To prove the theorem, we can assume without loss of generality that $\|\eta_1 - \eta_2\|_{L^\infty}$ is small enough. Then there exists $\widetilde{\eta} \in C_b^\infty$ such that for $j = 1; 2$,

$$\eta_j - \frac{3h}{4} \leq \widetilde{\eta} \leq \eta_j - \frac{2h}{3}.$$

Let

$$\begin{cases} \Omega_{1,j} = \{(x,y) : x \in \mathbf{R}^d, \eta_j(x) - \frac{h}{2} < y < \eta_j(x)\}, \\ \Omega_{2,j} = \{(x,y) \in \mathcal{O} : \widetilde{\eta}(x) < y \leq \eta_j(x) - \frac{h}{2}\}, \\ \Omega_3 = \{(x,y) \in \mathcal{O} : y < \widetilde{\eta}(x)\} \\ \Omega_j = \Omega_{1,j} \cup \Omega_{2,j} \cup \Omega_3 \end{cases} \tag{3.19}$$

and

$$\begin{cases} \widetilde{\Omega}_1 = \mathbf{R}^d_x \times [-1,0)_z \\ \widetilde{\Omega}_2 = \mathbf{R}^d_x \times [-2,-1)_z \\ \widetilde{\Omega}_3 = \{(x,z) \in \mathbf{R}^d \times (-\infty,-2) : (x, z+2+\widetilde{\eta}(x)) \in \Omega_3\}, \\ \widetilde{\Omega} = \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2 \cup \widetilde{\Omega}_3. \end{cases} \tag{3.20}$$

We define Lipschitz diffeomorphisms from $\widetilde{\Omega}$ to Ω_j of the form $(x,z) \mapsto (x, \rho_j(x,z))$ where the map $(x,z) \mapsto \rho_j(x,z)$ from $\widetilde{\Omega}$ to $\mathbf{R}$ is defined as follows

$$\begin{cases} \rho_j(x,z) = (1+z)e^{\delta z\langle D_x\rangle}\eta_j(x) - ze^{-(1+z)\delta\langle D_x\rangle}\big(\eta_j(x) - \frac{h}{2}\big) & \text{if } (x,z) \in \widetilde{\Omega}_1, \\ \rho_j(x,z) = (2+z)e^{\delta z\langle D_x\rangle}\big(\eta_j(x) - \frac{h}{2}\big) - (1+z)\widetilde{\eta} & \text{if } (x,z) \in \widetilde{\Omega}_2, \\ \rho_j(x,z) = z + 2 + \widetilde{\eta}(x) & \text{if } (x,z) \in \widetilde{\Omega}_3 \end{cases} \tag{3.21}$$

for some small enough positive constant δ. Notice that, since for $z \in I = (-2,0)$ we kept essentially for ρ_j the same expression as in (B.4), we get the same estimates as in Lemma B.1.

Recall that according to (B.13) we have for $j = 1,2$

$$G(\eta_j)f = U_j|_{z=0}, \quad U_j = \frac{1 + |\nabla_x \rho_j|^2}{\partial_z \rho_j}\partial_z \widetilde{\phi}_j - \nabla_x \rho_j \cdot \nabla_x \widetilde{\phi}_j,$$

where $\widetilde{\phi}_j$ is the variational solution of the problem

$$\widetilde{P}_j \widetilde{\phi}_j = 0, \quad \widetilde{\phi}_j|_{z=0} = f.$$

Set $U = U_1 - U_2$. According to Lemma 3.9 with $\sigma = s - \frac{3}{2}$, the theorem will be a consequence of the following estimate

$$\|U\|_{L^2(I,H^{s-1})} + \|\partial_z U\|_{L^2(I,H^{s-2})} \le \text{RHS}, \quad I = (-1,0). \tag{3.22}$$

Set $\widetilde{\phi} = \widetilde{\phi}_1 - \widetilde{\phi}_2$. We claim that (3.22) is a consequence of the following estimate

$$\|\nabla_{x,z}\widetilde{\phi}\|_{L^2(I,H^{s-1})} \le \text{RHS}. \tag{3.23}$$

Indeed assume that (3.23) is proved. One term in U can be written as

$$\nabla_x\rho_1 \cdot \nabla_x\widetilde{\phi}_1 - \nabla_x\rho_2 \cdot \nabla_x\widetilde{\phi}_2 = \nabla_x\rho_1 \cdot \nabla_x\widetilde{\phi} + \nabla_x(\rho_1 - \rho_2) \cdot \nabla_x\widetilde{\phi}_2.$$

Now, since $s > \frac{3}{4} + \frac{d}{2}$, we can apply (A.16) with $s_0 = s-1, s_1 = s - \frac{1}{2}, s_2 = s-1$. It follows that

$$\begin{aligned}\|\nabla_x\rho_1 \cdot \nabla_x\widetilde{\phi}\|_{L^2(I,H^{s-1})} &\lesssim \|\nabla_x\rho_1\|_{L^\infty(I,H^{s-\frac{1}{2}})}\|\nabla_x\widetilde{\phi}\|_{L^2(I,H^{s-1})} \\ &\lesssim \|\eta_1\|_{H^{s+\frac{1}{2}}}\text{RHS}.\end{aligned}$$

For the second term, since $s > \frac{3}{4} + \frac{d}{2} > 1$, we can write

$$\begin{aligned}\|\nabla_x(\rho_1 - \rho_2) \cdot \nabla_x\widetilde{\phi}_2\|_{L^2(I,H^{s-1})} \lesssim &\|\nabla_x(\rho_1-\rho_2)\|_{L^2(I,H^{s-1})}\|\nabla_x\widetilde{\phi}_2\|_{L^\infty(I,L^\infty)} \\ &+ \|\nabla_x(\rho_1-\rho_2)\|_{L^2(I,L^\infty)}\|\nabla_x\widetilde{\phi}_2\|_{L^\infty(I,H^{s-1})}.\end{aligned}$$

Now we have (as in (B.6) and (B.7))

$$\begin{aligned}\|\nabla_x(\rho_1-\rho_2)\|_{L^2(I,H^{s-1})} &\lesssim \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}}, \\ \|\nabla_x(\rho_1-\rho_2)\|_{L^2(I,L^\infty)} &\lesssim \|\eta_1 - \eta_2\|_{W^{r-\frac{1}{2},\infty}}.\end{aligned}$$

Also it follows from (B.18) that

$$\left\|\nabla_x\widetilde{\phi}_2\right\|_{L^\infty(I,H^{s-1})} \le \mathcal{F}\big(\|\eta_2\|_{H^{s+\frac{1}{2}}}\big)\|f\|_{H^s}$$

and it follows from Proposition B.7 that

$$\|\nabla_x\widetilde{\phi}_2\|_{L^\infty(I,L^\infty)} \le \mathcal{F}\big(\|\eta_2\|_{H^{s+\frac{1}{2}}}\big)\big\{\|f\|_{H^s} + \|f\|_{W^{r,\infty}}\big\}.$$

Therefore

$$\|\nabla_x(\rho_1-\rho_2) \cdot \nabla_x\widetilde{\phi}_2\|_{L^2(I,H^{s-1})} \le \text{RHS}. \tag{3.24}$$

We have thus completed the estimate of the first term in U. For the second term, one checks that, similarly, the $L^2(I,H^{s-1})$-norm of

$$\frac{1+|\nabla_x\rho_1|^2}{\partial_z\rho_1}\partial_z\widetilde{\phi}_1 - \frac{1+|\nabla_x\rho_2|^2}{\partial_z\rho_2}\partial_z\widetilde{\phi}_2$$

is bounded by RHS. This completes the proof of the fact $\|U\|_{L^2(I,H^{s-1})} \leq$ RHS provided that (3.23) is granted.

Let us now prove that, similarly, (3.23) implies that $\|\partial_z U\|_{L^2(I,H^{s-2})} \leq$ RHS. We begin by claiming that we have, for $j = 1, 2$,

$$\partial_z U_j = \operatorname{div}(\nabla_x \rho_j \partial_z \widetilde{\phi}_j) - \operatorname{div}(\partial_z \rho_j \nabla_x \widetilde{\phi}_j). \tag{3.25}$$

This follows from the fact (see (B.12)) that we can write

$$0 = \widetilde{P}_j \widetilde{\phi}_j = -\operatorname{div}(\nabla_x \rho_j \partial_z \widetilde{\phi}_j) + \operatorname{div}(\partial_z \rho_j \nabla_x \widetilde{\phi}_j) + \partial_z U_j.$$

Therefore we have

$$\begin{aligned}\|\partial_z U\|_{L^2(I,H^{s-2})} \leq \|\nabla_x \rho_1 \partial_z \widetilde{\phi}_1 - \nabla_x \rho_2 \partial_z \widetilde{\phi}_2\|_{L^2(I,H^{s-1})} \\ + \|\partial_z \rho_1 \nabla_x \widetilde{\phi}_1 - \partial_z \rho_2 \nabla_x \widetilde{\phi}_2\|_{L^2(I,H^{s-1})}.\end{aligned}$$

Since $s - 1 > 0$, we can argue as above and conclude the estimate by means of product rules.

Therefore we are left with the proof of (3.23). Since $\widetilde{P}_j \widetilde{\phi}_j = 0, j = 1, 2$ we can write

$$\begin{aligned}&\widetilde{P}_1 \widetilde{\phi} = (\widetilde{P}_2 - \widetilde{P}_1)\widetilde{\phi}_2 = F + \partial_z G \\ &F = \operatorname{div}\big(\partial_z(\rho_2 - \rho_1)\nabla_x \widetilde{\phi}_2\big) + \operatorname{div}\big(\nabla_x(\rho_1 - \rho_2)\partial_z \widetilde{\phi}_2\big) \\ &G = \nabla_x(\rho_1 - \rho_2) \cdot \nabla_x \widetilde{\phi}_2 + \Big(\frac{1 + |\nabla_x \rho_2|^2}{\partial_z \rho_2} - \frac{1 + |\nabla_x \rho_1|^2}{\partial_z \rho_1}\Big)\partial_z \widetilde{\phi}_2.\end{aligned} \tag{3.26}$$

Arguing exactly as in the proof of (3.24) we can write

$$\|F\|_{L^2(I,H^{s-2})} + \|G\|_{L^2(I,H^{s-1})} \leq \text{RHS}. \tag{3.27}$$

It follows from Proposition B.21 that

$$\|\nabla_{x,z} \widetilde{\phi}\|_{L^2(I,H^{s-1})} \leq \mathcal{F}(\|\eta_1\|_{H^{s+\frac{1}{2}}})\big(\text{RHS} + \|\nabla_{x,z} \widetilde{\phi}\|_{X^{-\frac{1}{2}}(I)}\big). \tag{3.28}$$

Then (3.23) will be proved if we show that

$$\|\nabla_{x,z} \widetilde{\phi}\|_{X^{-\frac{1}{2}}(I)} \leq \text{RHS}. \tag{3.29}$$

We begin by proving the following estimate.

$$\|\nabla_{x,z} \widetilde{\phi}\|_{L^2(J,L^2)} \leq \text{RHS}. \tag{3.30}$$

For this purpose we use the variational characterization of the solutions given in [1, 3]. It is sufficient to know that $\widetilde{\phi}_j = \widetilde{u}_j + \underline{f}$ where $\underline{f} = e^{z\langle D_x \rangle} f$ and $\widetilde{u}_j$ is such that, with the notations

$$\begin{aligned}&X = (x, z) \in \widetilde{\Omega} = \{(x, z) : x \in \mathbf{R}^d, -1 < z < 0\}, \\ &\Lambda^j = (\Lambda_1^j, \Lambda_2^j), \qquad \Lambda_1^j = \frac{1}{\partial_z \rho_j}\partial_z, \quad \Lambda_2^j = \nabla_x - \frac{\nabla_x \rho_j}{\partial_z \rho_j}\partial_z,\end{aligned}$$

we have

$$\int_{\widetilde{\Omega}} \Lambda^j \widetilde{u}_j \cdot \Lambda^j \theta \, J_j \, dX = -\int_{\widetilde{\Omega}} \Lambda^j \underline{\widetilde{f}} \cdot \Lambda^j \theta \, J_j \, dX \tag{3.31}$$

for all $\theta \in H^{1,0}(\widetilde{\Omega})$, where $J_j = |\partial_z \rho_j|$.

Making the difference between the two equations (3.31), and taking $\theta = \widetilde{\phi} = \widetilde{u}_1 - \widetilde{u}_2$ one can find a positive constant C such that

$$\int_{\widetilde{\Omega}} |\Lambda^1 \widetilde{\phi}|^2 \, dX \le C(A_1 + \cdots + A_6)$$

where

$$\begin{cases} A_1 = \int_{\widetilde{\Omega}} |(\Lambda^1 - \Lambda^2)\widetilde{u}_2||\Lambda^1 \widetilde{\phi}| \, J_1 \, dX, & A_2 = \int_{\widetilde{\Omega}} |(\Lambda^1 - \Lambda^2)\widetilde{\phi}||\Lambda^2 \widetilde{u}_2| \, J_1 dX, \\ A_3 = \int_{\widetilde{\Omega}} |\Lambda^2 \widetilde{u}_2||\Lambda^2 \widetilde{\phi}| \, |J_1 - J_2| \, dX, & A_4 = \int_{\widetilde{\Omega}} |(\Lambda^1 - \Lambda^2)\underline{\widetilde{f}}||\Lambda^1 \widetilde{u}| \, J_1 \, dX, \\ A_5 = \int_{\widetilde{\Omega}} |(\Lambda^1 - \Lambda^2)\widetilde{\phi}||\Lambda^2 \underline{\widetilde{f}}| \, J_1 \, dX, & A_6 = \int_{\widetilde{\Omega}} |\Lambda^2 \underline{\widetilde{f}}||\Lambda^2 \widetilde{\phi}| \, |J_1 - J_2| \, dX. \end{cases} \tag{3.32}$$

Noticing that $\Lambda^1 - \Lambda^2 = \beta \partial_z$ we deduce from Proposition B.7 that

$$\begin{aligned} A_1 &\le \|\Lambda^1 \widetilde{\phi}\|_{L^2(\widetilde{\Omega})} \|\beta\|_{L^2(\widetilde{\Omega})} \|\partial_z \widetilde{u}_2\|_{L^\infty(I, L^\infty)} \\ &\le \|\Lambda^1 \widetilde{\phi}\|_{L^2(\widetilde{\Omega})} \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac12} \times H^{s+\frac12}}) \|\eta_1 - \eta_2\|_{H^{\frac12}} \{\|f\|_{H^s} + \|f\|_{W^{r,\infty}}\}. \end{aligned}$$

Now

$$A_2 \le \|\beta\|_{L^2(\widetilde{\Omega})} \|\Lambda^2 \widetilde{u}_2\|_{L^\infty(\widetilde{\Omega})} \|\partial_z \widetilde{\phi}\|_{L^2(\widetilde{\Omega})}.$$

Using Proposition B.7 we obtain

$$\begin{aligned} A_2 &\le \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{H^{s+\frac12} \times H^{s+\frac12}}\big) \|\eta_1 - \eta_2\|_{H^{\frac12}} \{\|f\|_{H^s} + \|f\|_{W^{r,\infty}}\} \|\Lambda^1 \widetilde{\phi}\|_{L^2(\widetilde{\Omega})} \\ &\le \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{H^{s+\frac12} \times H^{s+\frac12}}\big) \|\eta_1 - \eta_2\|_{H^{s-\frac12}} \{\|f\|_{H^s} + \|f\|_{W^{r,\infty}}\} \|\Lambda^1 \widetilde{\phi}\|_{L^2(\widetilde{\Omega})}. \end{aligned}$$

Now we estimate A_3 as follows. We have

$$A_3 \le \|\Lambda^2 \widetilde{u}_2\|_{L^\infty(\widetilde{\Omega})} \|\Lambda^2 \widetilde{\phi}\|_{L^2(\widetilde{\Omega})} \|J_1 - J_2\|_{L^2(\widetilde{\Omega})}.$$

Then we observe that $\|J_1 - J_2\|_{L^2(\widetilde{\Omega})} \lesssim \|\eta_1 - \eta_2\|_{H^{\frac12}} \lesssim \|\eta_1 - \eta_2\|_{H^{s-\frac12}}$ and we use the elliptic regularity to obtain

$$A_3 \le \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{H^{s+\frac12} \times H^{s+\frac12}}\big) \|f\|_{W^{r,\infty}} \|\eta_1 - \eta_2\|_{H^{s-\frac12}} \|\Lambda^2 \widetilde{\phi}\|_{L^2(\widetilde{\Omega})}.$$

To estimate A_4 and A_5 we recall that $\underline{\widetilde{f}} = e^{z\langle D_x \rangle} f$. Then we have

$$\|\beta \partial_z \underline{\widetilde{f}}\|_{L^2(I \times \mathbf{R}^d)} \le \|\beta\|_{L^2(I \times \mathbf{R}^d)} \|\partial_z \underline{\widetilde{f}}\|_{L^\infty(I \times \mathbf{R}^d)}.$$

Since $\|\partial_z \underline{\widetilde{f}}\|_{L^\infty(I\times\mathbf{R}^d)} \le \|f\|_{W^{r,\infty}}$ we obtain

$$A_4 + A_5 \le \|\Lambda^1 \widetilde{\phi}\|_{L^2(\widetilde{\Omega})} \mathcal{F}\big(\|(\eta_1,\eta_2)\|_{H^{s+\frac12}}\big) \|\eta_1-\eta_2\|_{H^{s-\frac12}\times H^{s-\frac12}} \|f\|_{W^{r,\infty}} .$$

The term A_6 is estimated like A_3. This proves (3.30).

To complete the proof of (3.29), in view of (3.30), it remains only to prove that $\|\nabla_{x,z}\widetilde{\phi}\|_{L^\infty(I,H^{-\frac12})} \le \text{RHS}$. First of all by Lemma 3.9 we have

$$\|\nabla_x\widetilde{\phi}\|_{L^\infty(I,H^{-\frac12})} \le C\big(\|\nabla_x\widetilde{\phi}\|_{L^2(I,L^2)} + \|\partial_z\nabla_x\widetilde{\phi}\|_{L^2(I,H^{-1})}\big) \le C'\|\nabla_{x,z}\widetilde{\phi}\|_{L^2(I,L^2)}$$

and we use (3.30) to deduce that $\|\nabla_x\widetilde{\phi}\|_{L^\infty(I,H^{-\frac12})} \le \text{RHS}$. So it remains to prove that, similarly, $\|\partial_z\widetilde{\phi}\|_{L^\infty(I,H^{-\frac12})} \le \text{RHS}$. Here, by contrast with the estimate for $\nabla_x\widetilde{\phi}$, one cannot obtain the desired result from Lemma 3.9, exploiting the previous bound (3.30). Indeed, one cannot use the equation satisfied by $\widetilde{\phi}$ to estimate the $L^2(I,H^{-1})$-norm of $\partial_z^2\widetilde{\phi}$. As above, we shall exploit the fact that one can factor out a spatial derivative by working with U instead of $\widetilde{\phi}$. We shall prove that $\|U\|_{L^\infty(I,H^{-\frac12})} \le \text{RHS}$ and then relate $\partial_z\widetilde{\phi}$ and U to complete the proof.

Lemma 3.9 implies that

$$\|U\|_{L^\infty(I,H^{-\frac12})} \lesssim \|U\|_{L^2(I,L^2)} + \|\partial_z U\|_{L^2(I,H^{-1})} .$$

The $L^2(L^2)$-norm of U is easily estimated using the bound (3.30) for the $L^2(L^2)$-norm of $\nabla_{x,z}\widetilde{\phi}$. To estimate $\|\partial_z U\|_{L^2(I,H^{-1})}$, write

$$\partial_z U = \operatorname{div}(\nabla\rho_1\partial_z\widetilde{\phi}) + \operatorname{div}(\nabla\rho\partial_z\widetilde{\phi}_2) - \operatorname{div}(\partial_z\rho_1\nabla\widetilde{\phi}) - \operatorname{div}(\partial_z\rho\nabla\widetilde{\phi}_2)$$

so

$$\begin{aligned}\|\partial_z U\|_{L^2(I,H^{-1})} \le{}& \|\nabla_{x,z}\rho_1\|_{L^\infty(I,L^\infty)} \big\|\nabla_{x,z}\widetilde{\phi}\big\|_{L^2(I,L^2)} \\ &+ \|\nabla_{x,z}\rho\|_{L^2(I,L^2)} \big\|\nabla_{x,z}\widetilde{\phi}_2\big\|_{L^\infty(I,L^\infty)}\end{aligned}$$

which implies that $\|U\|_{L^\infty(I,H^{-\frac12})} \le \text{RHS}$. Now, directly from the definition of U one has

$$\partial_z\widetilde{\phi} = \frac{\partial_z\rho_1}{1+|\nabla\rho_1|^2}\Big[U + \Big(\frac{1+|\nabla\rho_2|^2}{\partial_z\rho_2} - \frac{1+|\nabla\rho_1|^2}{\partial_z\rho_1}\Big)\partial_z\widetilde{\phi}_2 + \nabla\rho_1\cdot\nabla\widetilde{\phi} + \nabla\rho\cdot\nabla\widetilde{\phi}_2\Big].$$

The $L^\infty(I,H^{-\frac12})$-norm of the term between brackets is bounded by RHS, using the fact that the product is bounded from $H^{s-\frac12}\times H^{-\frac12}$ (resp. $L^2\times H^{s-1}$) to $H^{-\frac12}$ (resp. $H^{-\frac12}$) in order to estimate $\nabla\rho_1\cdot\nabla\widetilde{\phi}$ (resp. the other terms). Since the coefficient $\frac{\partial_z\rho_1}{1+|\nabla\rho_1|^2}$ belongs to $L^\infty(I,H^{s-\frac12})$ and since the product is bounded from $H^{s-\frac12}\times H^{-\frac12}$ to $H^{-\frac12}$, this concludes the proof. □

3.2.2 Paralinearization of the equations

Recall from Proposition 2.3 that

$$
(3.33)\qquad \begin{cases} (\partial_t + V_j\cdot\nabla)B_j = \mathfrak{a}_j - g,\\ (\partial_t + V_j\cdot\nabla)V_j + \mathfrak{a}_j\zeta_j = 0,\\ (\partial_t + V_j\cdot\nabla)\zeta_j = G(\eta_j)V_j + \zeta_j G(\eta_j)B_j + \gamma_j, \quad \zeta_j = \nabla\eta_j,\end{cases}
$$

where γ_j is the remainder term given by (2.8). We now compute and paralinearize the equations satisfied by the differences

$$
\zeta = \zeta_1 - \zeta_2, \quad V = V_1 - V_2, \quad B = B_1 - B_2.
$$

In [3], assuming that $s > 1 + d/2$, we deduced that

$$
(3.34)\qquad \begin{cases} (\partial_t + V_1\cdot\nabla)(V + \zeta_1 B) + \mathfrak{a}_2\zeta = F_1,\\ (\partial_t + V_2\cdot\nabla)\zeta - G(\eta_1)V - \zeta_1 G(\eta_1)B = F_2,\end{cases}
$$

for some remainders such that

$$
\|(F_1,F_2)\|_{L^p([0,T];H^{s-1}\times H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).
$$

However, the estimate for f_2 no longer hold when $s < 3/2$ for the reason explained in Remark 3.7. To overcome this problem, the key point is that one obtains the desired result by replacing $\partial_t + V_2\cdot\nabla$ (resp. $\zeta_1 G(\eta_1)B$) by $\partial_t + T_{V_2}\cdot\nabla$ (resp. $T_{\zeta_1}G(\eta_1)B$ in the equation for ζ (there is a cancellation when one adds the remainders).

Lemma 3.10. *The differences ζ, B, V satisfy a system of the form*

$$
(3.35)\qquad \begin{cases} (\partial_t + T_{V_1}\cdot\nabla)(V + \zeta_1 B) + \mathfrak{a}_2\zeta = f_1,\\ (\partial_t + T_{V_2}\cdot\nabla)\zeta - G(\eta_1)V - T_{\zeta_1} G(\eta_1)B = f_2,\end{cases}
$$

for some remainders such that

$$
\|(f_1,f_2)\|_{L^p([0,T];H^{s-1}\times H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).
$$

Proof. Directly from (3.33), it is easily verified that

$$
(3.36)\qquad \begin{cases} \partial_t B + V_1\cdot\nabla B = a + R_1,\\ \partial_t V + V_1\cdot\nabla V + \mathfrak{a}_2\zeta + a\zeta_1 = R_2,\end{cases}
$$

where $a = a_1 - a_2$ and

$$
R_1 = -V\cdot\nabla B_2, \quad R_2 = -V\cdot\nabla V_2.
$$

Now we use the first equation of (3.36) to express a in terms of $\partial_t B + V_1\cdot\nabla B$ in the last term of the left-hand side of the second equation of (3.36). Replacing $V_1\cdot\nabla$ in the right-hand side by $T_{V_1}\cdot\nabla$, we thus obtain the first equation of (3.35) with

$$
f_1 = B(\partial_t\zeta_1 + V_1\cdot\nabla\zeta_1) + R_1\zeta_1 + R_2 - T_{\nabla(V+\zeta_1 B)}\cdot V_1 - R\big(V_1, \nabla(V+\zeta_1 B)\big).
$$

To estimate $\partial_t\zeta_1 + V_1 \cdot \nabla\zeta_1$ we use the identity (2.10) to deduce

$$\partial_t\zeta_1 + V_1 \cdot \nabla\zeta_1 = G(\eta_1)V_1 - (\operatorname{div} V_1)\zeta_1.$$

Then, using (B.20) to estimate $G(\eta_1)V_1$ and the fact that the product is bounded from $H^{s-1} \times H^{s-\frac{1}{2}}$ into H^{s-1}, we get

$$\begin{aligned}\|\partial_t\zeta_1 + V_1 \cdot \nabla\zeta_1\|_{H^{s-1}} &\le \|G(\eta_1)V_1\|_{H^{s-1}} + \|\operatorname{div} V_1\|_{H^{s-1}} \|\zeta_1\|_{H^{s-\frac{1}{2}}}\\ &\le \mathcal{F}\big(\|\eta_1\|_{H^{s+\frac{1}{2}}}\big)\|V_1\|_{H^s}.\end{aligned}$$

Also, it follows from the L^∞-estimate (B.31) for $G(\eta)f$ that

$$\|\partial_t\zeta_1 + V_1 \cdot \nabla\zeta_1\|_{L^\infty} \le \mathcal{F}\big(\|\eta_1\|_{H^{s+\frac{1}{2}}}\big)\big\{\|V_1\|_{H^s} + \|V_1\|_{W^{r,\infty}}\big\}.$$

Then the estimate of the H^{s-1}-norm of f_1 follows from the usual tame estimate in Sobolev space (see (A.17)), the rule (A.12) applied with $m = 1$ to estimate $T_{\nabla(V+\zeta_1 B)} \cdot V_1$, the rule (A.11) applied with $\alpha = 1$ and $a = V_1$ to estimate the remainder $R\big(V_1, \nabla(V + \zeta_1 B)\big)$ as well as the estimates for the Dirichlet-Neumann operator given by Propositions B.2 and B.7.

To compute and to estimate f_2 we shall rewrite the equation for ζ_j, $j = 1, 2$, by using the identity (2.10) written under the form

$$\partial_t\zeta_j = G(\eta_j)V_j - \Theta_j$$

where Θ_j is the function with values in $\mathbf{R}^d$ whose coordinates Θ_j^k is given by $\Theta_j^k = \operatorname{div}(V_j\zeta_j^k)$. Now write

$$\Theta_j^k = \operatorname{div}(V_j\zeta_j^k) = \operatorname{div}(T_{V_j}\zeta_j^k) + \operatorname{div}(T_{\zeta_j^k}V_j) + \operatorname{div}(R(\zeta_j^k, V_j))$$

and use the Leibniz rule $\partial_\alpha T_a b = T_{\partial_\alpha a}b + T_a\partial_\alpha b$ to obtain

$$\Theta_j^k = T_{V_j}\nabla\zeta_j^k + T_{\zeta_j^k}\operatorname{div} V_j + F_j^k$$

with

$$F_j^k = T_{\operatorname{div} V_j}\zeta_j^k + T_{\nabla\zeta_j^k} \cdot V_j + \operatorname{div}(R(\zeta_j^k, V_j)).$$

Using the identity $G(\eta_j)B_j = -\operatorname{div} V_j + \widetilde{\gamma}_j$ where $\widetilde{\gamma}_j$ is estimated by means of (2.4), we obtain

$$\Theta_j = T_{V_j} \cdot \nabla\zeta_j - T_{\zeta_j}G(\eta_j)B_j + T_{\zeta_j}\widetilde{\gamma}_j + F_j$$

so

$$\partial_t\zeta_j + T_{V_j} \cdot \nabla\zeta_j = G(\eta_j)V_j + T_{\zeta_j}G(\eta_j)B_j - T_{\zeta_j}\widetilde{\gamma}_j - F_j.$$

Subtracting the equation for $j = 1$ and the one for $j = 2$ we obtain the desired equation for $\zeta = \zeta_1 - \zeta_2$ with

$$\begin{aligned}f_2 &:= (I) + (II) + (III) - F\\ (I) &:= \big(G(\eta_1) - G(\eta_2)\big)V_2 + T_{\zeta_2}\big(G(\eta_1) - G(\eta_2)\big)B_2,\\ (II) &:= -T_V \cdot \nabla\zeta_1 + T_\zeta G(\eta_1)B_2,\\ (III) &:= -T_{\zeta_1}\widetilde{\gamma} - T_\zeta\widetilde{\gamma}_2,\\ F &:= F_1 - F_2.\end{aligned}$$

The term (I) is estimated by means of Proposition 3.8. To estimate (II), write

$$\left\|T_V\cdot\nabla\zeta_1\right\|_{H^{s-\frac{3}{2}}}\lesssim\left\|V\right\|_{L^\infty}\left\|\zeta_1\right\|_{H^{s-\frac{1}{2}}}$$

and

$$\left\|T_\zeta G(\eta_1)B_2\right\|_{H^{s-\frac{3}{2}}}\lesssim\left\|\zeta\right\|_{C_*^{-\frac{1}{2}}}\left\|G(\eta_1)B_2\right\|_{H^{s-1}}\leq\left\|\zeta\right\|_{C_*^{-\frac{1}{2}}}\mathcal{F}\big(\left\|\eta_1\right\|_{H^{s+\frac{1}{2}}}\big)\left\|B_2\right\|_{H^s}$$

where we used Proposition B.2 to estimate $G(\eta_1)B_2$.

It remains to estimate F given by

$$F=T_{\operatorname{div}V_2}\zeta+T_{\operatorname{div}V}\zeta_1+T_{\nabla\zeta_2}\cdot V+T_{\nabla\zeta}\cdot V_1+\operatorname{div}\big(R(\zeta,V_2)\big)+\operatorname{div}\big(R(\zeta_1,V)\big).$$

Write

$$\begin{aligned}
&\left\|T_{\operatorname{div}V_2}\zeta\right\|_{H^{s-\frac{3}{2}}}\lesssim\left\|\operatorname{div}V_2\right\|_{L^\infty}\left\|\zeta\right\|_{H^{s-\frac{3}{2}}}\lesssim\left\|V_2\right\|_{W^{r,\infty}}\left\|\eta\right\|_{H^{s-\frac{1}{2}}},\\
&\left\|T_{\operatorname{div}V}\zeta_1\right\|_{H^{s-\frac{3}{2}}}\lesssim\left\|\operatorname{div}V\right\|_{C_*^{-1}}\left\|\zeta_1\right\|_{H^{s-\frac{1}{2}}}\lesssim\left\|V\right\|_{W^{r-1,\infty}}\left\|\eta_1\right\|_{H^{s\frac{1}{2}}},\\
&\left\|T_{\nabla\zeta_2}\cdot V\right\|_{H^{s-\frac{3}{2}}}\lesssim\left\|\nabla\zeta_2\right\|_{C_*^{-\frac{1}{2}}}\left\|V\right\|_{H^{s-1}}\lesssim\left\|\eta\right\|_{W^{r+\frac{1}{2},\infty}}\left\|V\right\|_{H^{s-1}}.
\end{aligned}$$

Now the key point of this proof is that one has the following obvious inequalities

$$\left\|\operatorname{div}\big(R(\zeta,V_2)\big)+\operatorname{div}\big(R(\zeta_1,V)\big)\right\|_{H^{s-\frac{3}{2}}}\leq\left\|R(\zeta,V_2)+R(\zeta_1,V)\right\|_{H^{s-\frac{1}{2}}}.$$

Since $s-1/2>0$ (by contrast with $s-3/2$ which might be negative), one can use (A.11) to deduce

$$\begin{aligned}
&\left\|R(\zeta,V_2)\right\|_{H^{s-\frac{1}{2}}}\lesssim\left\|\zeta\right\|_{C_*^{-\frac{1}{2}}}\left\|V_2\right\|_{H^s}\lesssim\left\|\eta\right\|_{W^{r-\frac{1}{2},\infty}}\left\|V_2\right\|_{H^s},\\
&\left\|R(\zeta_1,V)\right\|_{H^{s-\frac{1}{2}}}\lesssim\left\|\zeta_1\right\|_{H^{s-\frac{1}{2}}}\left\|V\right\|_{L^\infty}\lesssim\left\|\eta_1\right\|_{H^{s+\frac{1}{2}}}\left\|V\right\|_{W^{r-1,\infty}}.
\end{aligned}$$

This completes the proof. □

Once Proposition 3.8 and Lemma 3.10 are established, the end of the proof of the contraction estimate in Theorem 3.6 follows from the analysis in [3]. We shall recall the scheme of the proof for the sake of completeness.

Our goal is to prove an estimate of the form

$$N(T)\leq\mathcal{F}(M_1,M_2)N(0)+T^\delta\,\mathcal{F}(M_1,M_2)N(T),\tag{3.37}$$

for some $\delta>0$ and some function $\mathcal{F}$ depending only on s and d. This implies $N(T_1)\leq 2\mathcal{F}(M_1,M_2)N(0)$ for T_1 small enough (depending on T and $\frac{1}{2}\mathcal{F}(M_1,M_2)$), and iterating the estimate between $[T_1,2T_1],\dots,[T-T_1,T]$ implies Theorem 3.6.

Firstly, one symmetrize the equations (3.35).

Lemma 3.11. *Set $I = [0,T]$ and*

$$\lambda_1 := \sqrt{(1 + |\nabla\eta_1|^2)|\xi|^2 - (\nabla\eta_1 \cdot \xi)^2},$$

and

$$\ell := \sqrt{\lambda_1 \mathfrak{a}_2}, \quad \varphi := T_{\sqrt{\lambda_1}}(V + \zeta_1 B), \quad \vartheta := T_{\sqrt{\mathfrak{a}_2}}\zeta.$$

Then

$$(\partial_t + T_{V_1} \cdot \nabla)\varphi + T_\ell \vartheta = g_1, \tag{3.38}$$

$$(\partial_t + T_{V_2} \cdot \nabla)\vartheta - T_\ell \varphi = g_2, \tag{3.39}$$

where

$$\|(g_1, g_2)\|_{L^p(I;H^{s-\frac{3}{2}} \times H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1, M_2)N(T).$$

The previous result is proved following the proof of Lemma 5.6 in [3].

We then use the previous result and classical arguments to obtain a Sobolev estimate.

Lemma 3.12. *Set*

$$M'(T) := \sup_{t\in I}\left\{ \|\vartheta(t)\|_{H^{s-\frac{3}{2}}} + \|\varphi(t)\|_{H^{s-\frac{3}{2}}} \right\}.$$

We have

$$M'(T) \le \mathcal{K}(M_1, M_2)\big(N(0) + T^\delta N(T)\big) \tag{3.40}$$

for some $\delta > 0$.

This follows by using *mutatis mutandis* the arguments used in the proof of Lemma 5.7 in [3].

Then the end of the proof of Theorem 3.6 is in two steps. Firstly, using (3.40), one deduces a Sobolev estimate for the original unknown (η, ψ, V, B). Again, this follows from the analysis in [3]. Secondly, one has to estimate the Hölder norms. To do so, we use Theorem 1.6. To use this theorem, one has to reduce the analysis to a scalar equation. Notice that System (3.38)–(3.39) involves two different transport operators $\partial_t + V_1 \cdot \nabla$ and $\partial_t + V_2 \cdot \nabla$. This is used in the proof of Lemma 5.6 in [3] to bound the commutator $[T_{\sqrt{a_2}}, \partial_t + V_2 \cdot \nabla]$ in terms of the L^∞-norm of $\partial_t a_2 + V_2 \cdot \nabla a_2$. However, once this symmetrization is done and Lemma 3.11 is proved, one can freely replace $\partial_t + T_{V_2} \cdot \nabla$ by $\partial_t + T_{V_1} \cdot \nabla$ in the equation for ϑ. Indeed, this produces an error $T_{V_1 - V_2} \cdot \nabla\vartheta$ that we estimate writing

$$T_{V_1-V_2} \cdot \nabla\vartheta = \left\{T_{V_1-V_2} \cdot \nabla T_{\sqrt{a_2}}\zeta_1\right\} - \left\{T_{V_1-V_2} \cdot \nabla T_{\sqrt{a_2}}\zeta_2\right\}$$

and we estimate separately the contribution of each term, writing

$$\left\|T_{V_1-V_2} \cdot \nabla T_{\sqrt{a_2}}\zeta_1\right\|_{H^{s-\frac{3}{2}}} \lesssim \|V_1 - V_2\|_{L^\infty} \|\sqrt{a_2}\|_{L^\infty} \|\zeta_1\|_{H^{s-\frac{1}{2}}}$$

together with a similar estimate for the other term. By so doing, it follows from (3.39) that

$$(\partial_t + T_{V_1}\cdot\nabla)\vartheta - T_\ell\varphi = G_2,$$

where G_2 satisfies $\|G_2\|_{L^p(I;H^{s-\frac{3}{2}}\times H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T)$. Now we find that $u = \vartheta + i\varphi$ satisfies

$$\partial_t u + T_{V_1}\cdot\nabla u + iT_\ell u = g$$

with

$$\|g\|_{L^p(I;H^{s-\frac{3}{2}}\times H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).$$

Since the $L^p(I;H^{s-\frac{3}{2}}\times H^{s-\frac{3}{2}})$-norm of $T_{\operatorname{div} V_1}u$ is bounded by $\mathcal{K}(M_1,M_2)N(T)$, one can further reduce the analysis to

$$\partial_t u + \frac{1}{2}\big(T_{V_1}\cdot\nabla + \nabla\cdot T_{V_1}\big)u + iT_\ell u = g'$$

with

$$\|g'\|_{L^p(I;H^{s-\frac{3}{2}})} \le \mathcal{K}(M_1,M_2)N(T).$$

Then one is in position to apply Theorem 1.6 with s replaced by $s-3/2$ (notice that s in any real number in Theorem 1.6) to obtain

$$\begin{aligned}
&\|u\|_{L^p(I;W^{r-\frac{1}{2},\infty})}\\
&\quad\le \|u\|_{L^p(I;C_*^{s-\frac{3}{2}-\frac{d}{2}+\mu})}\\
&\quad\le \mathcal{F}\big(\|V_1\|_{E_0} + \mathcal{N}_k(\sqrt{\lambda_1\mathfrak{a}_2})\big)\Big\{\|g'\|_{L^p(I;H^{s-\frac{3}{2}})} + \|u\|_{C^0(I;H^{s-\frac{3}{2}})}\Big\}\\
&\quad\le \mathcal{K}(M_1,M_2)N(T).
\end{aligned}$$

Then, proceeding as above, we recover an estimate for the original unknowns, that is an estimate for $\|(\eta,V,B)\|_{L^p([0,T];W^{r-\frac{1}{2},\infty}\times W^{r-1,\infty}\times W^{r-1,\infty})}$. This completes the proof of Theorem 3.6.

3.3 Passing to the limit in the equations

Below we shall obtain rough solutions of the water waves system as limits of smoother solutions. This requires to prove that one can pass to the limit in the equations. Here we shall prove that it is possible to do so even under very mild assumptions.

Firstly, we prove a strong continuity property of the Dirichlet Neumann operator at the minimal level of regularity required to prove that $G(\eta)\psi$ is well-defined, that is for any Lipschitz function η and any ψ in $H^{\frac{1}{2}}(\mathbf{R}^d)$, recalling that

$$\|G(\eta)f\|_{H^{-\frac{1}{2}}} \le \mathcal{F}\big(\|\eta\|_{W^{1,\infty}}\big)\,\|f\|_{H^{\frac{1}{2}}}. \tag{3.41}$$

We have the following result which complements Proposition 3.8.

Proposition 3.13. *There exists a non decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that, for all $\eta_j \in W^{1,\infty}(\mathbf{R}^d)$, $j = 1, 2$ and all $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$,*

$$\left\| \big(G(\eta_1) - G(\eta_2)\big) f \right\|_{H^{-\frac{1}{2}}} \le \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{W^{1,\infty}\times W^{1,\infty}}\big) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}. \tag{3.42}$$

Proof. The proof follows the one of Proposition 3.8. In particular, we shall use the variational formulation of the harmonic extension of f used in the last part of the proof of Proposition 3.8.

Recall that, for $j = 1, 2$, we introduce $\rho_j(x, z)$ defined by (B.4) (replacing of course η by η_j). Recall also (see the paragraph below (3.30)) that we have set

$$\Lambda_1^j = \frac{1}{\partial_z \rho_j}\partial_z, \quad \Lambda_2^j = \nabla_x - \frac{\nabla_x \rho_j}{\partial_z \rho_j}\partial_z. \tag{3.43}$$

and $\widetilde{\phi}_j(x, z) = \phi_j(x, \rho_j(x, z))$ (where $\Delta_{x,y}\phi_j = 0$ in Ω_j, $\phi_j|_{\Sigma_j} = f$). As already seen, we then have

$$G(\eta_j) f = U_j|_{z=0}, \quad U_j = \Lambda_1^j \widetilde{\phi}_j - \nabla_x \rho_j \cdot \Lambda_2^j \widetilde{\phi}_j. \tag{3.44}$$

We shall make repeated use of the bound

$$\|\nabla_{x,z}(\rho_1 - \rho_2)\|_{L^\infty(I, L^\infty(\mathbf{R}^d))} \le C\|\eta_1 - \eta_2\|_{W^{1,\infty}(\mathbf{R}^d)}. \tag{3.45}$$

This implies that

$$\begin{cases} (i) \quad \Lambda_k^1 - \Lambda_k^2 = \beta_k \partial_z, \quad \text{with } \operatorname{supp} \beta_k \subset \mathbf{R}^d \times I, \text{where } I = (-1, 0), \\ (ii) \quad \|\beta_k\|_{L^\infty(I\times\mathbf{R}^d)} \le \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty}\times W^{1,\infty}})\|\eta_1 - \eta_2\|_{W^{1,\infty}}. \end{cases} \tag{3.46}$$

Lemma 3.14. *Set $I = (-1, 0)$, $v = \widetilde{\phi}_1 - \widetilde{\phi}_2$, and $\Lambda^j = (\Lambda_1^j, \Lambda_2^j)$. There exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\|\Lambda^j v\|_{L^2(I;L^2(\mathbf{R}^d))} \le \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty}\times W^{1,\infty}})\|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}. \tag{3.47}$$

Let us show how this Lemma implies Theorem 3.13. According to (3.44) we have

$$\begin{aligned} &U_1 - U_2 = (1) + (2) + (3) + (4) + (5) \quad \text{where} \\ &\qquad (1) = \Lambda_1^1 \widetilde{\phi}, \quad (2) = (\Lambda_1^1 - \Lambda_1^2)\widetilde{\phi}_2, \quad (3) = -\nabla_x(\rho_1 - \rho_2)\Lambda_2^1 \widetilde{\phi}_1 \\ &\qquad (4) = -(\nabla_x \rho_2)\Lambda_2^1 \widetilde{\phi}, \quad (5) = -(\nabla_x \rho_2)(\Lambda_2^1 - \Lambda_2^2)\widetilde{\phi}_2. \end{aligned} \tag{3.48}$$

The $L^2(I, L^2(\mathbf{R}^d))$ norms of (1) and (4) are estimated using (3.47). Also, the $L^2(I, L^2(\mathbf{R}^d))$ norms of (2) and (5) are estimated by the right hand side of (3.47) using (3.46) and (3.23). Eventually the $L^2(I, L^2(\mathbf{R}^d))$ norm of (3) is also estimated by the right hand side of (3.47) using (3.45) and (3.23). It follows that

$$\|U_1 - U_2\|_{L^2(I,L^2)} \le \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty}\times W^{1,\infty}})\|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}. \tag{3.49}$$

Now according to (3.25) we have

$$\partial_z(U_1 - U_2) = -\nabla_x\big(\partial_z(\rho_1-\rho_2)\Lambda_2^1\widetilde{\phi}_1 + (\partial_z\rho_2)(\Lambda_2^1-\Lambda_2^2)\widetilde{\phi}_1 + (\partial_z\rho_2)\Lambda_2^2\widetilde{\phi}\big). \tag{3.50}$$

Therefore using the same estimates as above we see easily that

$$\|\partial_z(U_1-U_2)\|_{L^2(I,H^{-1})} \leq \mathcal{F}(\|(\eta_1,\eta_2)\|_{W^{1,\infty}\times W^{1,\infty}})\|\eta_1-\eta_2\|_{W^{1,\infty}}\|f\|_{H^{\frac{1}{2}}}. \tag{3.51}$$

Then Theorem 3.13 follows from (3.49), (3.51) and Lemma 3.9. □

Proof of Lemma 3.14. We proceed as in the end of the proof of Proposition 3.8. Namely we use the inequality

$$\int_{\widetilde{\Omega}} |\Lambda^1\widetilde{\phi}|^2\,dX \leq C(A_1+\cdots+A_6)$$

where $A_1,\dots,A_6$ are given by (3.32), and we recall their expressions for the reader convenience:

$$\begin{cases} A_1 = \displaystyle\int_{\widetilde{\Omega}} |(\Lambda^1-\Lambda^2)\widetilde{u}_2||\Lambda^1\widetilde{\phi}|\,J_1\,dX, & A_2 = \displaystyle\int_{\widetilde{\Omega}} |(\Lambda^1-\Lambda^2)\widetilde{\phi}||\Lambda^2\widetilde{u}_2|\,J_1 dX,\\ A_3 = \displaystyle\int_{\widetilde{\Omega}} |\Lambda^2\widetilde{u}_2||\Lambda^2\widetilde{\phi}|\,|J_1-J_2|\,dX, & A_4 = \displaystyle\int_{\widetilde{\Omega}} |(\Lambda^1-\Lambda^2)\underline{\widetilde{f}}||\Lambda^1\widetilde{u}|\,J_1\,dX,\\ A_5 = \displaystyle\int_{\widetilde{\Omega}} |(\Lambda^1-\Lambda^2)\widetilde{\phi}||\Lambda^2\underline{\widetilde{f}}|\,J_1\,dX, & A_6 = \displaystyle\int_{\widetilde{\Omega}} |\Lambda^2\underline{\widetilde{f}}||\Lambda^2\widetilde{\phi}|\,|J_1-J_2|\,dX, \end{cases}$$

where $\underline{\widetilde{f}}$ and $\widetilde{u}_j$ ($\widetilde{u} := \widetilde{u}_1 - \widetilde{u}_2$) are such that $\widetilde{\phi}_j = \widetilde{u}_j + \underline{\widetilde{f}}$ with $\underline{\widetilde{f}} = e^{z\langle D_x\rangle}f$.
Using (3.46), (3.23), (3.45) we can write

$$\begin{aligned} |A_1| &\leq \|\beta\|_{L^\infty(I\times\mathbf{R}^d)}\|J_1\|_{L^\infty(I\times\mathbf{R}^d)}\|\partial_z\widetilde{u}_2\|_{L^2(I\times\mathbf{R}^d)}\|\Lambda^1\widetilde{\phi}\|_{L^2(\widetilde{\Omega})}\\ &\leq \mathcal{F}(\|(\eta_1,\eta_2)\|_{W^{1,\infty}\times W^{1,\infty}})\|\eta_1-\eta_2\|_{W^{1,\infty}}\|f\|_{H^{\frac{1}{2}}}\|\Lambda^1\widetilde{\phi}\|_{L^2(\widetilde{\Omega})}. \end{aligned} \tag{3.52}$$

Since $\Lambda_j^1 - \Lambda_j^2 = \beta_j\partial_z\rho_1\Lambda_1^1$ the term A_2 can be bounded by the right hand side of (3.52).
Now we have $\|J_1-J_2\|_{L^\infty(I\times\mathbf{R}^d)} \leq C\|\eta_1-\eta_2\|_{W^{1,\infty}(\mathbf{R}^d)}$ and

$$\|\Lambda^2\widetilde{\phi}\|_{L^2(\widetilde{\Omega})} \leq \mathcal{F}(\|(\eta_1,\eta_2)\|_{W^{1,\infty}\times W^{1,\infty}})\|\Lambda^1\widetilde{\phi}\|_{L^2(\widetilde{\Omega})}.$$

So using (3.23) we see that the term A_3 can be also estimated by the right hand side of (3.52). To estimate the terms A_4 to A_6 we use the same arguments and also $\|\underline{\widetilde{f}}\|_{H^1(\widetilde{\Omega})} \lesssim \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$. This completes the proof. □

Notation 3.15. Given two functions η,ψ one writes

$$B(\eta)\psi = \frac{G(\eta)\psi + \nabla\eta\cdot\nabla\psi}{1+|\nabla\eta|^2}, \quad V(\eta)\psi = \nabla\psi - (B(\eta)\psi)\nabla\eta.$$

Corollary 3.16. *Fix $s > \frac{1}{2} + \frac{d}{2}$. Consider two functions η and ψ in $H^{s+\frac{1}{2}}(\mathbf{R}^d)$ as well as two sequences $(\eta_n)_{n\in\mathbf{N}}$ and $(\psi_n)_{n\in\mathbf{N}}$ such that the following properties hold:*

1. *$(\eta_n)_{n\in\mathbf{N}}$ and $(\psi_n)_{n\in\mathbf{N}}$ are bounded sequences in $H^{s+\frac{1}{2}}(\mathbf{R}^d)$;*
2. *$(\eta_n)_{n\in\mathbf{N}}$ converges to η in $W^{1,\infty}(\mathbf{R}^d)$;*
3. *$(\psi_n)_{n\in\mathbf{N}}$ converges to ψ in $H^{\frac{1}{2}}(\mathbf{R}^d)$.*

Then $G(\eta_n)\psi_n$ (resp. $B(\eta_n)\psi_n$, resp. $V(\eta_n)\psi_n$) converges in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ to $G(\eta)\psi$ (resp. $B(\eta)\psi$, resp. $V(\eta)\psi$).

Proof. The proof is straightforward. Write

$$G(\eta_n)\psi_n - G(\eta)\psi = \big(G(\eta_n) - G(\eta)\big)\psi_n + G(\eta)(\psi_n - \psi).$$

The inequality (3.41) (resp. (3.42)) then implies that the second (resp. first) term in the right-hand side converges to 0 in $H^{-\frac{1}{2}}(\mathbf{R}^d)$.

To study the limit of $B(\eta_n)\psi_n$, we first prove that $\nabla\eta_n \cdot \nabla\psi_n$ converges to $\nabla\eta\cdot\nabla\psi$ in $H^{-\frac{1}{2}}$. To do so, one makes the difference and then use the fact that the product is bounded from $H^{s-\frac{1}{2}} \times H^{-\frac{1}{2}}$ to $H^{-\frac{1}{2}}$ to obtain

$$\left\| \nabla\eta_n \cdot \nabla(\psi_n - \psi)\right\|_{H^{-\frac{1}{2}}} \lesssim \left\|\eta_n\right\|_{H^{s+\frac{1}{2}}} \left\|\psi_n - \psi\right\|_{H^{\frac{1}{2}}}.$$

On the other hand

$$\left\|\nabla(\eta_n - \eta)\cdot\nabla\psi\right\|_{H^{-\frac{1}{2}}} \le \left\|\nabla(\eta_n-\eta)\cdot\nabla\psi\right\|_{L^2} \le \left\|\eta_n - \eta\right\|_{W^{1,\infty}} \left\|\psi\right\|_{H^{s+\frac{1}{2}}}.$$

This proves that $\nabla\eta_n\cdot\nabla\psi_n$ converges to $\nabla\eta\cdot\nabla\psi$ in $H^{-\frac{1}{2}}$.

Now set $a_n = (1+|\nabla\eta_n|^2)^{-1}$, $b_n = G(\eta_n)\psi_n + \nabla\eta_n\cdot\nabla\psi_n$. We have proved that b_n converges to its limit $b = G(\eta)\psi + \nabla\eta\cdot\nabla\psi$ in $H^{-\frac{1}{2}}$. It is easily checked that a_n converges to $a = (1+|\nabla\eta|^2)^{-1}$ in L^∞ and that $a-1$ belongs to $H^{s-\frac{1}{2}}$. So, as above, one easily verify that $B(\eta_n)\psi_n = a_n b_n$ converges to $B(\eta)\psi = ab$ in $H^{-\frac{1}{2}}$. This in turn easily implies that $V(\eta_n)\psi_n$ converges to $V(\eta)\psi$ in $H^{-\frac{1}{2}}$. □

3.4 Existence and uniqueness

We have already proved the uniqueness of solutions (which is a straightforward consequence of Theorem 3.6) so, to complete the proof of Theorem 1.2, it remains to prove the existence. This is done by means of standard arguments together with a sharp blow up criterion proved by de Poyferré [30]. Namely, it follows from his result that, if the lifespan T^* of a smooth solution of the water waves system is finite, then

$$\lim_{T\to T^*}\big(M_s(T) + Z_r(T)\big) = +\infty, \tag{3.53}$$

with notations as above (with the same assumptions on s and r, see (3.7)).

We use this criterion to obtain solutions to the water waves system as limits of smooth solutions. Namely, consider a family of initial data $(\psi_0^\varepsilon, \eta_0^\varepsilon)$ in $H^\infty(\mathbf{R}^d)^2$ converging to (ψ_0, η_0). It follows that the Cauchy problem has a unique smooth solution $(\psi^\varepsilon, \eta^\varepsilon)$ defined on some time interval $[0, T_\varepsilon^*)$ (this follows from the Cauchy result in [3] (see also [64, 65] and [43]). The question is to prove that this family of smooth solutions exists on a uniform time interval and that, in addition, it converges to a solution of the water waves system. By applying our a priori estimate (3.8), it follows that there exists a function $\mathcal{F}$ such that, for all $\varepsilon \in (0,1]$ and all $T < T_\varepsilon$, we have

$$M_s^\varepsilon(T) + Z_r^\varepsilon(T) \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s^\varepsilon(T) + Z_r^\varepsilon(T)\big)\big), \tag{3.54}$$

with obvious notations.

Then, by standard arguments and using the blow up criterion (3.53), we infer that the lifespan is bounded from below by a positive time T independent of ε and that we have uniform estimates on $[0,T]$ for $M_s^\varepsilon(T) + Z_r^\varepsilon(T)$.

For $\sigma \in \mathbf{R}$ and $a \in [0, +\infty]$, set

$$\mathcal{H}^\sigma := H^{\sigma+\frac{1}{2}} \times H^{\sigma+\frac{1}{2}} \times H^\sigma \times H^\sigma, \quad \mathcal{W}^a := W^{a+\frac{1}{2},\infty} \times W^{a+\frac{1}{2},\infty} \times W^{a,\infty} \times W^{a,\infty}.$$

Since $(\eta_\varepsilon, \psi_\varepsilon, V_\varepsilon, B_\varepsilon)$ is uniformly bounded in $X := L^\infty([0,T]; \mathcal{H}^s) \cap L^p([0,T]; \mathcal{W}^r)$ and since X is the dual of a Banach space, it has (after extraction of a subsequence) a weak limit (η, ψ, V, B) in X. Moreover, the contraction estimate (3.17) implies that $(\eta_\varepsilon, \psi_\varepsilon, B_\varepsilon, V_\varepsilon)$ is a Cauchy sequence in $L^\infty([0,T]; \mathcal{H}^{s-1}) \cap L^p([0,T]; \mathcal{W}^{r-1})$. Therefore $(\eta_\varepsilon, \psi_\varepsilon)$ converges to its limit (η, ψ) strongly in $L^\infty([0,T]; H^{s-\frac{1}{2}} \times H^{s-1})$. Since $(\eta_\varepsilon, \psi_\varepsilon)$ is uniformly bounded in $L^\infty([0,T]; H^{s+\frac{1}{2}} \times H^s)$, by interpolation, $(\eta_\varepsilon, \psi_\varepsilon)$ converges also strongly in $L^\infty([0,T]; H^{s'+\frac{1}{2}} \times H^{s'})$ for any $s' < s$. In particular, $(\eta_\varepsilon, \psi_\varepsilon)$ converges strongly to (η, ψ) in $L^\infty([0,T]; W^{1,\infty} \times H^{\frac{1}{2}})$. As a result (see Corollary 3.16)

$$G(\eta_\varepsilon)\psi_\varepsilon, \quad B_\varepsilon = \frac{G(\eta_\varepsilon)\psi_\varepsilon + \nabla\eta_\varepsilon \cdot \nabla\psi_\varepsilon}{1 + |\nabla\eta_\varepsilon|^2}, \quad V_\varepsilon = \nabla\psi_\varepsilon - B_\varepsilon\nabla\eta_\varepsilon$$

converge, respectively, to

$$G(\eta)\psi, \quad \frac{G(\eta)\psi + \nabla\eta \cdot \nabla\psi}{1 + |\nabla\eta|^2}, \quad \nabla\psi - B\nabla\eta.$$

This proves that the weak limits B, V of $B_\varepsilon, V_\varepsilon$ satisfy

$$B = \frac{G(\eta)\psi + \nabla\eta \cdot \nabla\psi}{1 + |\nabla\eta|^2}, \quad V = \nabla\psi - B\nabla\eta,$$

as well as the fact that one can pass to the limit in the equations. We thus obtain a solution (η, ψ) such that (η, ψ, V, B) is in $L^\infty([0,T]; \mathcal{H}^s) \cap L^p([0,T]; \mathcal{W}^r)$. By interpolation, (η, ψ, V, B) is continuous in time with values in $\mathcal{H}^{s'}$ for any $s' < s$. It remains to prove that the solution is continuous in time with values in $\mathcal{H}^s$. This was done in details in [1] for the case with surface tension. For the case without surface tension, this is done by Nguyen [53] following the Bona-Smith' strategy.

Appendix A

Paradifferential calculus

A.1 Notations and classical results

For the reader convenience, we recall notations as well as estimates for Bony's paradifferential operators (following [16, 49, 51, 61]). We also gather various estimates in Hölder or Zygmund spaces.

For $k \in \mathbf{N}$, we denote by $W^{k,\infty}(\mathbf{R}^d)$ the usual Sobolev spaces. For $\rho = k + \sigma$, $k \in \mathbf{N}, \sigma \in (0,1)$ denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the space of functions whose derivatives up to order k are bounded and uniformly Hölder continuous with exponent σ.

Definition A.1. *Given $\rho \in [0,1]$ and $m \in \mathbf{R}$, $\Gamma^m_\rho(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ functions of ξ outside the origin and such that, for any $\alpha \in \mathbf{N}^d$ and any $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ is in $W^{\rho,\infty}(\mathbf{R}^d)$ and there exists a constant C_α such that,*

$$\forall\, |\xi| \geq \frac{1}{2}, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

Given a symbol a in one such symbol class, one defines the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\psi(\eta)\widehat{u}(\eta)\,d\eta, \tag{A.1}$$

where $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta} a(x,\xi)\,dx$ is the Fourier transform of a with respect to the first variable; χ and ψ are two fixed C^∞ functions such that:

$$\psi(\eta) = 0 \quad \text{for } |\eta| \leq 1, \qquad \psi(\eta) = 1 \quad \text{for } |\eta| \geq 2, \tag{A.2}$$

and $\chi(\theta,\eta)$ satisfies, for some small enough positive numbers $\varepsilon_1 < \varepsilon_2$,

$$\chi(\theta,\eta) = 1 \quad \text{if} \quad |\theta| \leq \varepsilon_1 |\eta|, \qquad \chi(\theta,\eta) = 0 \quad \text{if} \quad |\theta| \geq \varepsilon_2 |\eta|,$$

and

$$\forall (\theta,\eta) : \qquad \left| \partial_\theta^\alpha \partial_\eta^\beta \chi(\theta,\eta) \right| \leq C_{\alpha,\beta}(1+|\eta|)^{-|\alpha|-|\beta|}.$$

Given a symbol $a \in \Gamma^m_\rho(\mathbf{R}^d)$, we set

$$M^m_\rho(a) = \sup_{|\alpha| \le 1+2d+\rho} \sup_{|\xi| \ge 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)}. \tag{A.3}$$

Notice that the cut-off function χ can be so chosen that the definition of T_a coincides with the usual one of a paraproduct (in terms of Littlewood-Paley decomposition) when the symbol a depends on x only.

A.2 Symbolic calculus

We shall use quantitative results from [49] about operator norms estimates in symbolic calculus. To do so, introduce the following semi-norms.

Definition A.2 (Zygmund spaces)**.** *Consider a dyadic decomposition of the identity: $I = \Delta_{-1} + \sum_{q=0}^{\infty} \Delta_q$. If s is any real number, the Zygmund class $C^s_*(\mathbf{R}^d)$ is the space of tempered distributions u such that*

$$\|u\|_{C^s_*} := \sup_q 2^{qs} \|\Delta_q u\|_{L^\infty} < +\infty.$$

Remark A.3. It is known that $C^s_*(\mathbf{R}^d)$ is the usual Hölder space $W^{s,\infty}(\mathbf{R}^d)$ if $s > 0$ is not an integer.

Definition A.4. *Let $m \in \mathbf{R}$. An operator T is said to be of order m if, for all $\mu \in \mathbf{R}$, it is bounded from H^μ to $H^{\mu-m}$ and from C^μ_* to $C^{\mu-m}_*$.*

The main features of symbolic calculus for paradifferential operators are given by the following theorem.

Theorem A.5. *Let $m \in \mathbf{R}$ and $\rho \in [0,1]$.*

(i) If $a \in \Gamma^m_0(\mathbf{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \le K M^m_0(a), \qquad \|T_a\|_{C^\mu_* \to C^{\mu-m}_*} \le K M^m_0(a). \tag{A.4}$$

(ii) If $a \in \Gamma^m_\rho(\mathbf{R}^d), b \in \Gamma^{m'}_\rho(\mathbf{R}^d)$ then $T_a T_b - T_{ab}$ is of order $m + m' - \rho$. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\begin{aligned} &\|T_a T_b - T_{ab}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \le K M^m_\rho(a) M^{m'}_0(b) + K M^m_0(a) M^{m'}_\rho(b), \\ &\|T_a T_b - T_{ab}\|_{C^\mu_* \to C^{\mu-m-m'+\rho}_*} \le K M^m_\rho(a) M^{m'}_0(b) + K M^m_0(a) M^{m'}_\rho(b). \end{aligned} \tag{A.5}$$

(iii) Let $a \in \Gamma^m_\rho(\mathbf{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of T_a and by $\overline{a}$ the complex conjugate of a. Then $(T_a)^* - T_{\overline{a}}$ is of order $m - \rho$. Moreover, for all μ there exists a constant K such that

$$\begin{aligned} &\|(T_a)^* - T_{\overline{a}}\|_{H^\mu \to H^{\mu-m+\rho}} \le K M^m_\rho(a), \\ &\|(T_a)^* - T_{\overline{a}}\|_{C^\mu_* \to C^{\mu-m+\rho}_*} \le K M^m_\rho(a). \end{aligned} \tag{A.6}$$

We also need in this article to consider paradifferential operators with negative regularity. As a consequence, we need to extend our previous definition.

Definition A.6. *For $m \in \mathbf{R}$ and $\rho \in (-\infty, 0)$, $\Gamma^m_\rho(\mathbf{R}^d)$ denotes the space of distributions $a(x,\xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $C^\rho_*(\mathbf{R}^d)$ and there exists a constant C_α such that,*

$$\forall\, |\xi| \geq \frac{1}{2}, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{C^\rho_*} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}. \tag{A.7}$$

For $a \in \Gamma^m_\rho$, we define

$$M^m_\rho(a) = \sup_{|\alpha| \leq \frac{3d}{2}+\rho+1} \ \sup_{|\xi|\geq 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot,\xi) \right\|_{C^\rho_*(\mathbf{R}^d)}. \tag{A.8}$$

A.3 Paraproducts and product rules

We recall here some properties of paraproducts (a paraproduct is a paradifferential operator T_a whose symbol $a = a(x)$ is a function of x only). For our purposes, a key feature is that one can define paraproducts T_a for rough functions a which do not belong to $L^\infty(\mathbf{R}^d)$ but merely $C^{-m}_*(\mathbf{R}^d)$ with $m > 0$.

Definition A.7. *Given two functions a, b defined on $\mathbf{R}^d$ we define the remainder*

$$R(a,u) = au - T_a u - T_u a.$$

We record here various estimates about paraproducts (see chapter 2 in [12]).

Theorem A.8. *i) Let $\alpha, \beta \in \mathbf{R}$. If $\alpha + \beta > 0$ then*

$$\left\| R(a,u) \right\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)} \leq K \left\| a \right\|_{H^\alpha(\mathbf{R}^d)} \left\| u \right\|_{H^\beta(\mathbf{R}^d)}, \tag{A.9}$$

$$\left\| R(a,u) \right\|_{C^{\alpha+\beta}_*(\mathbf{R}^d)} \leq K \left\| a \right\|_{C^\alpha_*(\mathbf{R}^d)} \left\| u \right\|_{C^\beta_*(\mathbf{R}^d)}, \tag{A.10}$$

$$\left\| R(a,u) \right\|_{H^{\alpha+\beta}(\mathbf{R}^d)} \leq K \left\| a \right\|_{C^\alpha_*(\mathbf{R}^d)} \left\| u \right\|_{H^\beta(\mathbf{R}^d)}. \tag{A.11}$$

ii) Let $m > 0$ and $s \in \mathbf{R}$. Then

$$\left\| T_a u \right\|_{H^{s-m}} \leq K \left\| a \right\|_{C^{-m}_*} \left\| u \right\|_{H^s}, \tag{A.12}$$

$$\left\| T_a u \right\|_{C^{s-m}_*} \leq K \left\| a \right\|_{C^{-m}_*} \left\| u \right\|_{C^s_*}. \tag{A.13}$$

$$\left\| T_a u \right\|_{C^s_*} \leq K \left\| a \right\|_{L^\infty} \left\| u \right\|_{C^s_*}. \tag{A.14}$$

iii) Let s_0, s_1, s_2 be such that $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$, then

$$\left\| T_a u \right\|_{H^{s_0}} \leq K \left\| a \right\|_{H^{s_1}} \left\| u \right\|_{H^{s_2}}. \tag{A.15}$$

By combining the two previous points with the embedding $H^\mu(\mathbf{R}^d) \subset C_*^{\mu-d/2}(\mathbf{R}^d)$ (for any $\mu \in \mathbf{R}$) we immediately obtain the following results.

Proposition A.9. *Let $r, \mu \in \mathbf{R}$ be such that $r + \mu > 0$. If $\gamma \in \mathbf{R}$ satisfies*

$$\gamma \le r \quad \text{and} \quad \gamma < r + \mu - \frac{d}{2},$$

then there exists a constant K such that, for all $a \in H^r(\mathbf{R}^d)$ and all $u \in H^\mu(\mathbf{R}^d)$, we have

$$\|au - T_a u\|_{H^\gamma} \le K \|a\|_{H^r} \|u\|_{H^\mu}.$$

Corollary A.10. *i) If $u_j \in H^{s_j}(\mathbf{R}^d)$ $(j = 1, 2)$ with $s_1 + s_2 > 0$ then $u_1 u_2 \in H^{s_0}(\mathbf{R}^d)$ and*

$$\|u_1 u_2\|_{H^{s_0}} \le K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}}, \tag{A.16}$$

if

$$s_0 \le s_j, \quad j = 1, 2, \quad \text{and} \quad s_0 < s_1 + s_2 - d/2.$$

ii) (Tame estimate in Sobolev spaces) If $s \ge 0$ then

$$\|u_1 u_2\|_{H^s} \le K\big(\|u_1\|_{H^s} \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty} \|u_2\|_{H^s}\big). \tag{A.17}$$

iii) (Tame estimate in Zygmund spaces) If $s \ge 0$ then

$$\|u_1 u_2\|_{C_*^s} \le K\big(\|u_1\|_{C_*^s} \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty} \|u_2\|_{C_*^s}\big). \tag{A.18}$$

iv) Let $\mu, m \in \mathbf{R}$ be such that $\mu, m > 0$ and $m \notin \mathbf{N}$. Then

$$\|u_1 u_2\|_{H^\mu} \le K\big(\|u_1\|_{L^\infty} \|u_2\|_{H^\mu} + \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}}\big). \tag{A.19}$$

v) Let $\beta > \alpha > 0$. Then

$$\|u_1 u_2\|_{C_*^{-\alpha}} \le K \|u_1\|_{C_*^\beta} \|u_2\|_{C_*^{-\alpha}}. \tag{A.20}$$

vi) Let $s > d/2$ and consider $F \in C^\infty(\mathbf{C}^N)$ such that $F(0) = 0$. Then there exists a non-decreasing function $\mathcal{F} \colon \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\|F(U)\|_{H^s} \le \mathcal{F}\big(\|U\|_{L^\infty}\big) \|U\|_{H^s}, \tag{A.21}$$

for any $U \in H^s(\mathbf{R}^d)^N$.

vii) Let $s \ge 0$ and consider $F \in C^\infty(\mathbf{C}^N)$ such that $F(0) = 0$. Then there exists a non-decreasing function $\mathcal{F} \colon \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\|F(U)\|_{C_*^s} \le \mathcal{F}\big(\|U\|_{L^\infty}\big) \|U\|_{C_*^s}, \tag{A.22}$$

for any $U \in C_*^s(\mathbf{R}^d)^N$.

Proof. The first three estimates are well-known, see Hörmander [38] or [12]. To prove *iv*) and *v*) we write

$$u_1 u_2 = T_{u_1} u_2 + T_{u_2} u_1 + R(u_1, u_2).$$

Then (A.19) follows from

$$\begin{aligned} &\|T_{u_1} u_2\|_{H^\mu} \lesssim \|u_1\|_{L^\infty} \|u_2\|_{H^\mu} && \text{(see (A.4))},\\ &\|T_{u_2} u_1\|_{H^\mu} \lesssim \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}} && \text{(see (A.12))},\\ &\|R(u_1, u_2)\|_{H^\mu} \lesssim \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}} && \text{(see (A.11))}, \end{aligned}$$

while similarly (A.20) follows from

$$\begin{aligned} &\|T_{u_1} u_2\|_{C_*^{-\alpha}} \lesssim \|u_1\|_{L^\infty} \|u_2\|_{C_*^{-\alpha}} \lesssim \|u_1\|_{C_*^\beta} \|u_2\|_{C_*^{-\alpha}},\\ &\|T_{u_2} u_1\|_{C_*^{-\alpha}} \lesssim \|u_2\|_{C_*^{-\alpha}} \|u_1\|_{C_*^0} \leq \|u_2\|_{C_*^{-\alpha}} \|u_1\|_{C_*^\beta},\\ &\|R(u_1, u_2)\|_{C_*^{-\alpha}} \leq \|R(u_1, u_2)\|_{C_*^{\beta-\alpha}} \lesssim \|u_2\|_{C_*^{-\alpha}} \|u_1\|_{C_*^\beta}. \end{aligned}$$

(With regards to the last inequality, to apply (A.10) we do need $\beta > \alpha > 0$.) Finally, *vi*) and *vii*) are due to Meyer [51, Théorème 2.5 and remarque], in the line of the work by Bony [16]. □

Finally, we recall Prop. 2.12 in [3] which is a generalization of (A.12).

Proposition A.11. *Let $\rho < 0$, $m \in \mathbf{R}$ and $a \in \dot{\Gamma}^m_\rho$. Then the operator T_a is of order $m - \rho$:*

$$\text{(A.23)} \qquad \begin{aligned} \|T_a\|_{H^s \to H^{s-(m-\rho)}} &\leq C M^m_\rho(a),\\ \|T_a\|_{C_*^s \to C_*^{s-(m-\rho)}} &\leq C M^m_\rho(a). \end{aligned}$$

We also need the following technical result.

Proposition A.12. *Set $\langle D_x \rangle = (I - \Delta)^{1/2}$.*

i) Let $s > \frac{1}{2} + \frac{d}{2}$ and $\sigma \in \mathbf{R}$ be such that $\sigma \leq s$. Then there exists $K > 0$ such that for all $V \in W^{1,\infty}(\mathbf{R}^d) \cap H^s(\mathbf{R}^d)$ and $u \in H^{\sigma - \frac{1}{2}}(\mathbf{R}^d)$ one has

$$\|[\langle D_x \rangle^\sigma, V] u\|_{L^2(\mathbf{R}^d)} \leq K \big\{ \|V\|_{W^{1,\infty}(\mathbf{R}^d)} + \|V\|_{H^s(\mathbf{R}^d)} \big\} \|u\|_{H^{\sigma - \frac{1}{2}}(\mathbf{R}^d)}.$$

ii) Let $s > 1 + \frac{d}{2}$ and $\sigma \in \mathbf{R}$ be such that $\sigma \leq s$. Then there exists $K > 0$ such that for all $V \in H^s(\mathbf{R}^d)$ and $u \in H^{\sigma-1}(\mathbf{R}^d)$ one has

$$\|[\langle D_x \rangle^\sigma, V] u\|_{L^2(\mathbf{R}^d)} \leq K \|V\|_{H^s(\mathbf{R}^d)} \|u\|_{H^{\sigma-1}(\mathbf{R}^d)}.$$

iii) Let $s > \frac{1}{2} + \frac{d}{2}$ and $V \in H^s(\mathbf{R}^d)$. Then

$$\|[\langle D_x \rangle^{\frac{1}{2}}, V] u\|_{L^\infty(\mathbf{R}^d)} \leq K \|V\|_{H^s(\mathbf{R}^d)} \|u\|_{L^\infty(\mathbf{R}^d)}.$$

Proof. The first two statements are proved in [3]. To prove *iii*) we use (A.5) with $m = \frac{1}{2}, m' = 0, \rho = \frac{1}{2} + \epsilon$ to obtain

$$\|[\langle D_x\rangle^{\frac{1}{2}}, T_V]u\|_{C^{\epsilon}_{*}} \le C\|V\|_{H^s}\|u\|_{C^0_*} \le C\|V\|_{H^s}\|u\|_{L^\infty}.$$

On the other hand,

$$[\langle D_x\rangle^{\frac{1}{2}}, V - T_V]u = \langle D_x\rangle^{\frac{1}{2}}(V - T_V)u - (V - T_V)\langle D_x\rangle^{\frac{1}{2}}u.$$

Let $\frac{1}{2} < r < s - \frac{d}{2}$ so that

$$\|V\|_{C^r_*} \le C\|V\|_{H^s}.$$

According to (A.14) and (A.10), $V - T_V$ is bounded from L^∞ to C^r_* by $K\|V\|_{C^r_*}$ and according to (A.13) and (A.10), from $C_*^{-\frac{1}{2}}$ to $C_*^{r-\frac{1}{2}}$ by $K\|V\|_{C^r_*}$, which implies

$$\|[\langle D_x\rangle^{\frac{1}{2}}, V - T_V]u\|_{C_*^{r-\frac{1}{2}}} \le K\|V\|_{H^s}\|u\|_{L^\infty}.$$

This completes the proof. □

We need elementary estimates on the solutions of transport equations that we recall now.

Proposition A.13. *Let $I = [0,T]$ and consider the Cauchy problem*

$$\begin{cases} \partial_t u + V\cdot\nabla u = f, & t \in I, \\ u|_{t=0} = u_0. \end{cases} \tag{A.24}$$

We have the following estimates

$$\|u(t)\|_{L^\infty(\mathbf{R}^d)} \le \|u_0\|_{L^\infty(\mathbf{R}^d)} + \int_0^t \|f(\sigma,\cdot)\|_{L^\infty(\mathbf{R}^d)}d\sigma. \tag{A.25}$$

There exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\|u(t)\|_{L^2(\mathbf{R}^d)} \le \mathcal{F}\big(\|V\|_{L^1(I;W^{1,\infty}(\mathbf{R}^d))}\big)\big(\|u_0\|_{L^2(\mathbf{R}^d)} + \int_0^t \|f(t',\cdot)\|_{L^2(\mathbf{R}^d)}\,dt'\big). \tag{A.26}$$

If $s > 1 + \frac{d}{2}$ and $\sigma \le s$ there exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\|u(t)\|_{H^\sigma(\mathbf{R}^d)} \le \mathcal{F}\big(\|V\|_{L^1(I;H^s(\mathbf{R}^d))}\big)\big(\|u_0\|_{H^\sigma(\mathbf{R}^d)} + \int_0^t \|f(t',\cdot)\|_{H^\sigma(\mathbf{R}^d)}\,dt'\big). \tag{A.27}$$

Appendix B

Tame estimates for the Dirichlet-Neumann operator

In this appendix we prove Theorem 1.4 on the paralinearization of the Dirichlet-Neumann operator.

B.1 Scheme of the analysis

We shall revisit the approach given in [8, 1, 3] using tame estimates at each step. In this section, we recall the scheme of the analysis and indicate the points at which the argument must be adapted.

Hereafter, we consider a time-independent fluid domain Ω satisfying the assumptions given in Section 1.1, which we recall here. We assume that

$$\Omega = \{(x,y) \in \mathcal{O} \,:\, y < \eta(x)\},$$

for some Lipschitz function η and a given open domain $\mathcal{O}$. We denote by Σ (resp. Γ) the free surface (resp. the bottom). They are defined by

$$\Sigma = \{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, y = \eta(x)\}, \qquad \Gamma = \partial\Omega \setminus \Sigma.$$

We assume that the domain $\mathcal{O}$ contains a fixed strip separating the free surface and the bottom. This implies that there exists $h > 0$ such that

$$\Big\{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, \eta(x) - h < y < \eta(t,x)\Big\} \subset \Omega. \tag{B.1}$$

We also assume that the domain $\mathcal{O}$ (and hence the domain Ω) is connected. Without loss of generality we assume below that $h > 1$.

The fact that the Dirichlet-Neumann operator $G(\eta)$ is well-defined in such domains is proved in [1, 3].

In the analysis of free boundary problems it is classical to begin by reducing the analysis to a domain with a fixed boundary. We flatten the free surface by using a diffeomorphism introduced in [3] whose definition is here recalled. Set

$$\text{(B.2)} \qquad \begin{cases} \Omega_1 = \{(x,y) : x \in \mathbf{R}^d, \eta(x) - h < y < \eta(x)\}, \\ \Omega_2 = \{(x,y) \in \mathcal{O} : y \leq \eta(x) - h\}, \\ \ \Omega = \Omega_1 \cup \Omega_2, \end{cases}$$

and

$$\text{(B.3)} \qquad \begin{cases} \widetilde{\Omega}_1 = \{(x,z) : x \in \mathbf{R}^d, z \in I\}, \quad I = (-1,0), \\ \widetilde{\Omega}_2 = \{(x,z) \in \mathbf{R}^d \times (-\infty,-1] : (x, z + 1 + \eta(x) - h) \in \Omega_2\}, \\ \widetilde{\Omega} = \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2. \end{cases}$$

Guided by Lannes ([43]), we consider a Lipschitz diffeomorphism from $\widetilde{\Omega}_1$ to Ω_1 of the form $(x,z) \mapsto (x, \rho(x,z))$ where the map $(x,z) \mapsto \rho(x,z)$ from $\widetilde{\Omega}$ to $\mathbf{R}$ is defined as follows

$$\text{(B.4)} \qquad \begin{cases} \rho(x,z) = (1+z)e^{\delta z \langle D_x \rangle}\eta(x) - z\big\{e^{-(1+z)\delta\langle D_x\rangle}\eta(x) - h\big\} \quad \text{if } (x,z) \in \widetilde{\Omega}_1, \\ \rho(x,z) = z + 1 + \eta(x) - h \quad \text{if } (x,z) \in \widetilde{\Omega}_2 \end{cases}$$

for some small enough positive constant δ.

Lemma B.1. *Assume $\eta \in W^{1,\infty}(\mathbf{R}^d)$.*

1. *There exists $C > 0$ such that for every $(x,z) \in \widetilde{\Omega}$ we have*

$$|\nabla_x \rho(x,z)| \leq C\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}.$$

2. *There exists $K > 0$ such that, if $\delta\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \leq \frac{h}{2K}$ we have*

$$\text{(B.5)} \qquad \textit{min}\ \big(1, \frac{h}{2}\big) \leq \partial_z \rho(x,z) \leq \textit{max}\ (1, \frac{3h}{2}), \quad \forall (x,z) \in \widetilde{\Omega}.$$

3. *The map $(x,z) \mapsto (x, \rho(x,z))$ is a Lipschitz diffeomorphism from $\widetilde{\Omega}_1$ to Ω_1.*
4. *Let $I = (-1,0)$ and s be a real number. There exists $C > 0$ such that for every $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ we have*

$$\text{(B.6)} \qquad \begin{aligned} &\|\partial_z \rho - h\|_{C^0_z(I;H^{s-\frac{1}{2}}(\mathbf{R}^d)) \cap L^2_z(I;H^s(\mathbf{R}^d))} \leq C\sqrt{\delta}\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}, \\ &\|\nabla_x \rho\|_{C^0_z(I;H^{s-\frac{1}{2}}(\mathbf{R}^d)) \cap L^2_z(I;H^s(\mathbf{R}^d))} \leq \frac{C}{\sqrt{\delta}}\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}. \end{aligned}$$

5. *Assume that $\eta \in W^{r+\frac{1}{2},\infty}(\mathbf{R}^d)$ with $r > 1/2$. Then, for any r' in $[1/2, r]$,*

$$\text{(B.7)} \quad \|\nabla_{x,z}\rho\|_{C^0([-1,0];W^{r'-\frac{1}{2},\infty})} + \|\nabla_{x,z}\rho\|_{L^2([-1,0];W^{r',\infty})} \leq C(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}).$$

Proof. The first four statements are proved in [3]. The last one follows from the fact that, for any $\mu' > \mu \geq 0$, $|z|^{\mu'}\langle D_x\rangle^{\mu} e^{z\langle D_x\rangle}$ is bounded from L^∞ to L^∞, uniformly in $z \in [-1, 0]$ and therefore

$$\begin{aligned}\|\Delta_j e^{z\langle D_x\rangle} u\|_{L^\infty} &= |z|^{-\mu'} \| |z|^{\mu'} \langle D_x\rangle^{\mu} e^{z\langle D_x\rangle} \langle D_x\rangle^{-\mu} \Delta_j u\|_{L^\infty} \\ &\leq C|z|^{-\mu'} \|\langle D_x\rangle^{-\mu}\Delta_j u\|_{L^\infty}\end{aligned}$$

which easily implies the desired result in view of Remark A.3. □

Flattening the free surface. In [1, 3] we proved that the problem

$$\Delta_{x,y}\phi = 0, \quad \phi|_{y=\eta} = f, \quad \partial_n \phi = 0 \text{ on } \Gamma,$$

has a unique variational solution. Then we introduce the following function

$$v(x, z) = \phi(x, \rho(x, z))$$

where (x, z) belongs to the 'flattened' domain $\widetilde{\Omega}$ (notice that we flatten only the free surface).

The equation satisfied by v in $\widetilde{\Omega}$ can be written in three forms. Firstly,

$$(\partial_z^2 + \alpha\Delta_x + \beta\cdot\nabla_x\partial_z - \gamma\partial_z)v = 0, \tag{B.8}$$

where

$$\alpha := \frac{(\partial_z\rho)^2}{1+|\nabla\rho|^2}, \quad \beta := -2\frac{\partial_z\rho\nabla_x\rho}{1+|\nabla_x\rho|^2}, \quad \gamma := \frac{1}{\partial_z\rho}\big(\partial_z^2\rho + \alpha\Delta_x\rho + \beta\cdot\nabla_x\partial_z\rho\big). \tag{B.9}$$

Secondly, one has

$$(\Lambda_1^2 + \Lambda_2^2)v = 0, \tag{B.10}$$

where

$$\Lambda_1 = \frac{1}{\partial_z\rho}\partial_z \quad \Lambda_2 = \nabla_x - \frac{\nabla_x\rho}{\partial_z\rho}\partial_z. \tag{B.11}$$

Eventually,

$$\widetilde{P}v := \operatorname{div}\big(\partial_z\rho\nabla_x v\big) - \operatorname{div}\big(\nabla_x\rho\,\partial_z v\big) - \partial_z\big(\nabla_x\rho\cdot\nabla_x v\big) + \partial_z\Big(\frac{1+|\nabla_x\rho|^2}{\partial_z\rho}\partial_z v\Big) = 0, \tag{B.12}$$

as can be verified starting from (B.10) by a direct calculation. Moreover,

$$v|_{z=0} = \phi|_{y=\eta(x)} = f,$$

and

$$G(\eta)f = \Big(\frac{1+|\nabla\rho|^2}{\partial_z\rho}\partial_z v - \nabla_x\rho\cdot\nabla_x v\Big)\Big|_{z=0} = (\Lambda_1 v - \nabla_x\rho\cdot\Lambda_2 v)|_{z=0}. \tag{B.13}$$

The analysis of the Dirichlet-Neumann operator is then divided into three steps.

First step. We paralinearize the equation. That is we write the equation for $v = \phi(x, \rho(x, z))$ in the form

$$\partial_z^2 v + T_\alpha \Delta v + T_\beta \cdot \nabla \partial_z v = F_1 + F_2, \tag{B.14}$$

where

$$\begin{aligned} F_1 &= \gamma \partial_z v, \\ F_2 &= (T_\alpha - \alpha)\Delta v + (T_\beta - \beta) \cdot \nabla \partial_z v. \end{aligned} \tag{B.15}$$

We are going to estimate F_1 by product rules in Sobolev spaces and F_2 by using results recalled in Appendix A.

Second step. We factor out the elliptic equation as the product of a forward and a backward parabolic evolution equations. We write, for some symbols a, A and a remainder F_3,

$$(\partial_z - T_a)(\partial_z - T_A)v = F_1 + F_2 + F_3. \tag{B.16}$$

Namely

$$a = \frac{1}{2}\big(-i\beta\cdot\xi - \sqrt{4\alpha\left|\xi\right|^2 - (\beta\cdot\xi)^2}\big), \qquad A = \frac{1}{2}\big(-i\beta\cdot\xi + \sqrt{4\alpha\left|\xi\right|^2 - (\beta\cdot\xi)^2}\big). \tag{B.17}$$

The term F_3 is estimated by means of the symbolic calculus rules recalled in §A.2.

Third step. Let us view z as a time variable. Then $\partial_z u - T_a u = F$ is a parabolic equation (since $\operatorname{Re}(-a) \geq c\left|\xi\right|$). On the other hand, $\partial_z u - T_A u = F$ is a *backward* parabolic evolution equation (by definition $\operatorname{Re} A \geq c\left|\xi\right|$). We shall use parabolic estimates twice to deduce from the previous step estimates for $\nabla_{x,z} v$ and $(\partial_z - T_A)v$.

Previous results. Let $I = [-1, 0]$. By using the approach explained above, we proved in [3] that, for any $s > 1/2 + d/2$,

$$\left\|\nabla_{x,z} v\right\|_{C^0_z(I;H^{s-1}(\mathbf{R}^d))\cap L^2_z(I;H^{s-\frac{1}{2}}(\mathbf{R}^d))} \leq \mathcal{F}(\left\|\eta\right\|_{H^{s+\frac{1}{2}}}) \left\|f\right\|_{H^s}. \tag{B.18}$$

Moreover, for any $0 < \varepsilon \leq \frac{1}{2}$ such that $\varepsilon < s - \frac{1}{2} - \frac{d}{2}$, we have

$$\left\|\partial_z v - T_A v\right\|_{C^0_z(I;H^{s-1+\varepsilon}(\mathbf{R}^d))\cap L^2_z(I;H^{s-\frac{1}{2}+\varepsilon}(\mathbf{R}^d))} \leq \mathcal{F}(\left\|\eta\right\|_{H^{s+\frac{1}{2}}}) \left\|f\right\|_{H^s}. \tag{B.19}$$

The key point to prove Theorem 1.4 will be to prove an estimate analogous to (B.19) with $\varepsilon = 1/2$ and $s < 1 + d/2$, assuming an extra control of η and f in Hölder spaces.

Actually, concerning elliptic regularity, in [3] we proved more general results than (B.18) and we record here two statements for later references.

Proposition B.2. *Let $d \geq 1$, $s > \frac{1}{2} + \frac{d}{2}$ and $\frac{1}{2} \leq \sigma \leq s + \frac{1}{2}$. Then there exists a non-decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that, for all $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ and all $f \in H^\sigma(\mathbf{R}^d)$, we have $G(\eta)f \in H^{\sigma-1}(\mathbf{R}^d)$, together with the estimate*

$$\left\|G(\eta)f\right\|_{H^{\sigma-1}(\mathbf{R}^d)} \leq \mathcal{F}\big(\left\|\eta\right\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}\big) \left\|f\right\|_{H^\sigma(\mathbf{R}^d)}. \tag{B.20}$$

Given $\mu \in \mathbf{R}$ we define the spaces

$$X^\mu(I) = C^0_z(I; H^\mu(\mathbf{R}^d)) \cap L^2_z(I; H^{\mu+\frac{1}{2}}(\mathbf{R}^d)),$$
$$Y^\mu(I) = L^1_z(I; H^\mu(\mathbf{R}^d)) + L^2_z(I; H^{\mu-\frac{1}{2}}(\mathbf{R}^d))$$

and we consider the problem

$$\partial_z^2 v + \alpha\Delta v + \beta\cdot\nabla\partial_z v - \gamma\partial_z v = F_0 + \partial_z G_0, \quad v|_{z=0} = f, \tag{B.21}$$

where $f = f(x), F_0 = F_0(x,z), G_0 = G_0(x,z)$ are given functions. Then we have,

Proposition B.3. *Let $d \geq 1$ and*

$$s > \frac{1}{2} + \frac{d}{2}, \quad -\frac{1}{2} \leq \sigma \leq s - \frac{1}{2}.$$

Consider $f \in H^{\sigma+1}(\mathbf{R}^d)$, $F_0 \in Y^\sigma([-1,0])$, $G_0 \in Y^{\sigma+1}([-1,0])$ and v a solution to (B.21) such that $\nabla_{x,z}v \in X^{-\frac{1}{2}}([-1,0])$. Then for any $z_0 \in (-1,0)$, $\nabla_{x,z}v \in X^\sigma([z_0,0])$, and

$$\|\nabla_{x,z}v\|_{X^\sigma([z_0,0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\Big\{ \|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma([-1,0])} + \|G_0\|_{Y^{\sigma+1}([-1,0])} + \|\nabla_{x,z}v\|_{X^{-\frac{1}{2}}([-1,0])} \Big\}$$

for some non-decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ depending only on σ and d.

B.2 Parabolic evolution equation

As explained above, we need estimates for paradifferential parabolic equations of the form

$$\partial_z w + T_p w = f, \quad w|_{z=z_0} = w_0,$$

where p is an elliptic symbol and $z \in \mathbf{R}$ plays the role of a time variable.

Given $J \subset \mathbf{R}$, $z_0 \in J$ and $\varphi = \varphi(x,z)$ defined on $\mathbf{R}^d \times J$, we denote by $\varphi(z_0)$ the function $x \mapsto \varphi(x,z_0)$. When a and u are symbols and functions depending on z, we still denote by $T_a u$ the function defined by $(T_a u)(z) = T_{a(z)}u(z)$ where $z \in J$ is seen as a parameter. $\Gamma^m_\rho(\mathbf{R}^d \times J)$ denotes the space of symbols $a = a(z;x,\xi)$ such that $z \mapsto a(z;\cdot)$ is bounded from J into the space $\Gamma^m_\rho(\mathbf{R}^d)$ introduced in Definition A.1. This space is equipped with the semi-norm

$$\mathcal{M}^m_\rho(a) = \sup_{z\in J} \sup_{|\alpha|\leq \frac{3d}{2}+\rho+1} \sup_{|\xi|\geq 1/2} \left\| (1+|\xi|)^{|\alpha|-m}\partial_\xi^\alpha a(z;\cdot,\xi)\right\|_{W^{\rho,\infty}(\mathbf{R}^d)}. \tag{B.22}$$

The next proposition is a parabolic estimate in Zygmund spaces $C^r_*(\mathbf{R}^d)$ with $r \in \mathbf{R}$ (see Definition A.2 for the definition of these spaces). Recall that $C^r_*(\mathbf{R}^d)$ is the usual Hölder space $W^{r,\infty}(\mathbf{R}^d)$ if $r > 0$ is not an integer (since we shall need to consider various negative indexes, we shall often prefer to use the notation $C^r_*(\mathbf{R}^d)$ instead of $W^{r,\infty}(\mathbf{R}^d)$ even when $r > 0$ is not an integer).

Proposition B.4. *Let* $\rho \in (0,1)$, $J = [z_0, z_1] \subset \mathbf{R}$, $p \in \Gamma^1_\rho(\mathbf{R}^d \times J)$ *with the assumption that*

$$\operatorname{Re} p(z; x, \xi) \geq c\,|\xi|\,,$$

for some positive constant c. *Assume that* w *solves*

$$\partial_z w + T_p w = F_1 + F_2, \quad w|_{z=z_0} = w_0.$$

Then for any $q \in [1, +\infty]$, $(r_0, r) \in \mathbf{R}^2$ *with* $r_0 < r$, *if*

$$w \in L^\infty(J; C^{r_0}_*),\ F_1 \in L^1(J; C^r_*),\ F_2 \in L^q(J; C_*^{r-1+\frac{1}{q}+\delta}) \text{ with } \delta > 0,$$

and $w_0 \in C^r_*(\mathbf{R}^d)$, *we have* $w \in C^0(J; C^r_*)$ *and*

$$\|w\|_{C^0(J;C^r_*)} \leq K \left\{ \|w_0\|_{C^r_*} + \|F_1\|_{L^1(J;C^r_*)} + \|F_2\|_{L^q(J;C_*^{r-1+\frac{1}{q}+\delta})} + \|w\|_{L^\infty(J;C^{r_0}_*)} \right\},$$

for some positive constant K *depending only on* $r_0, r, \rho, c, \delta, q$ *and* $\mathcal{M}^1_\rho(p)$.

Proof. We follow a classical strategy (see [62, 45, 8, 1]).

For this proof, we denote by K various constants which depend only on r_0, r, ρ, c and $\mathcal{M}^1_\rho(p)$. Given $y \in J$ introduce the symbol $e = e(y, z; x, \xi)$ defined by

$$e(y, z; x, \xi) = \exp\big(-\int_z^y p(s; x, \xi)\, ds\big) \quad (z \in [z_0, y]).$$

This symbol satisfies $\partial_z e = ep$, so that

$$\partial_z(T_e w) = (T_{ep} - T_e T_p) w + T_e F, \quad F = F_1 + F_2.$$

Integrating on $[z_0, y]$ the function $\frac{d}{dz} T_{e(y,z,x,\xi)} w(z)$, we find

$$T_1 w(y) = T_{e|z=z_0} w_0 + \int_{z_0}^y (T_e F)(z)\, dz + \int_{z_0}^y (T_{ep} - T_e T_p) w(z)\, dz. \tag{B.23}$$

(Notice that the paraproduct T_1 differs from the identity I only by a smoothing operator.) Introduce $G(y) = T_{e|z=z_0} w_0 + \int_{z_0}^y (T_e F)(z)\, dz$ and the operator R defined on functions $u\colon J \to C^m_*(\mathbf{R}^d)$ by

$$(Ru)(y) = u(y) - T_1 u(y) + \int_{z_0}^y (T_{ep} - T_e T_p) u(z)\, dz$$

so that $w = G + Rw$. Now, by a bootstrap argument, to complete the proof it is enough to prove that the function G belongs to $L^\infty(J; C^r_*)$ and that R is a smoothing operator of order $-a$ for some $a > 0$, which means that R maps $L^\infty(J; C^t_*)$ to $L^\infty(J; C^{t+a}_*)$. Indeed, by writing

$$w = (I + R + \cdots R^N) G - R^{N+1} w,$$

and choosing N large enough, we can estimate the second term in the right-hand side in $L^\infty(J;C^r_*)$ by means of any $L^\infty(J;C^{r_0}_*)$-norm of w.

In the analysis, we need to take into account how the semi-norms $M^{-m}_\rho(e(z))$ (see Definition A.1) depend on z. Then the key estimates are stated in the following lemma.

Lemma B.5. *For any $m \geq 0$ there exists a positive constant K depending only on $\sup_J M^1_\rho(p(\cdot;x,\xi))$ such that, for all $y \in (0,-z_1]$ and all $z \in [0,y)$,*

$$M^{-m}_\rho(e(z)) \leq \frac{K}{(y-z)^m}. \tag{B.24}$$

This follows easily from the assumptions $p \in \Gamma^1_\rho$, $\operatorname{Re} p(s;x,\xi) \geq c\,|\xi|$, and the elementary inequalities (valid for any $a \geq 0$)

$$(y-z)^a\,|\xi|^a \exp\bigl((z-y)\,|\xi|\bigr) \lesssim 1.$$

By using the bound (B.24), applied with $m=0$, it follows from the operator norm estimate (A.5) that, for any $z \leq y$ and any function $f = f(x)$, we have

$$\left\lVert T_{e(y,z)} f\right\rVert_{C^r_*} \lesssim M^0_0(e(y,z))\,\lVert f\rVert_{C^r_*} \leq K\,\lVert f\rVert_{C^r_*}. \tag{B.25}$$

This implies that

$$\left\lVert T_{e|z=z_0} w_0 + \int_{z_0}^y (T_e F_1)(z)\,dz \right\rVert_{L^\infty(J;C^r_*)} \leq K\,\lVert w_0\rVert_{C^r_*} + K\,\lVert F_1\rVert_{L^1(J;C^r_*)}.$$

On the other hand, by using the bound (B.24), applied with $m = 1 - \frac{1}{q} - \delta$, we obtain that

$$\left\lVert \int_{z_0}^y (T_e F_2)(z)\,dz \right\rVert_{L^\infty(J;C^r_*)} \leq K \int_{z_0}^y \frac{1}{|y-z|^m} \lVert F_2(z)\rVert_{C^{r-m}_*} dz,$$

which implies by Hölder inequality that

$$\lVert G\rVert_{L^\infty(J;C^r_*)} \leq K\,\lVert w_0\rVert_{C^r_*} + K\,\lVert F_1\rVert_{L^1(J;C^r_*)} + \lVert F_2\rVert_{L^q(J;C^{r-1+\frac{1}{q}+\delta}_*)}.$$

It remains to show that R is a smoothing operator. To do that, we first use the operator norm estimate (A.5) (applied with (m,m',ρ) replaced with $(-m,1,\rho)$) to obtain

$$\lVert (T_{ep} - T_e T_p)(z)\rVert_{C^t_* \to C^{t+m-1+\rho}_*} \lesssim M^{-m}_\rho(e(z)) M^1_\rho(p(z)).$$

Taking $m = 1 - \rho/2$, it follows from the previous bound and Lemma B.5 that

$$\lVert (T_{ep} - T_e T_p) v(z)\rVert_{C^{t+\rho/2}_*} \leq \frac{K}{(y-z)^m}\,\lVert v(z)\rVert_{C^t_*}.$$

Since $0 \leq m < 1$ we have $\int_0^y (y-z)^{-m}\,dz < +\infty$ and hence

$$\lVert Ru(y)\rVert_{C^{t+\rho/2}_*} \leq \int_0^y \lVert (T_{ep} - T_e T_p) u(z)\rVert_{C^{t+\rho/2}_*} \leq K\,\lVert u\rVert_{L^\infty(J;C^t_*)}, \tag{B.26}$$

which completes the proof. □

We shall also need the following estimate in Sobolev spaces. Given $\mu \in \mathbf{R}$, recall that we define the spaces

$$X^\mu(I) = C^0_z(I; H^\mu(\mathbf{R}^d)) \cap L^2_z(I; H^{\mu+\frac{1}{2}}(\mathbf{R}^d)),$$
$$Y^\mu(I) = L^1_z(I; H^\mu(\mathbf{R}^d)) + L^2_z(I; H^{\mu-\frac{1}{2}}(\mathbf{R}^d)). \tag{B.27}$$

Proposition B.6 (from [3])**.** *Let $r \in \mathbf{R}$, $\rho \in (0,1)$, $J = [z_0, z_1] \subset \mathbf{R}$ and let $p \in \Gamma^1_\rho(\mathbf{R}^d \times J)$ satisfying*

$$\operatorname{Re} p(z; x, \xi) \geq c\,|\xi|\,,$$

for some positive constant c. Then for any $f \in Y^r(J)$ and $w_0 \in H^r(\mathbf{R}^d)$, there exists $w \in X^r(J)$ solution of the parabolic evolution equation

$$\partial_z w + T_p w = f, \quad w|_{z=z_0} = w_0, \tag{B.28}$$

satisfying

$$\|w\|_{X^r(J)} \leq K\left\{\|w_0\|_{H^r} + \|f\|_{Y^r(J)}\right\},$$

for some positive constant K depending only on r, ρ, c and $\mathcal{M}^1_\rho(p)$. Furthermore, this solution is unique in $X^s(J)$ for any $s \in \mathbf{R}$.

B.3 Paralinearization

We are now ready to prove Theorem 1.4. Recall that we consider the elliptic equation

$$\partial_z^2 v + \alpha\Delta v + \beta\cdot\nabla\partial_z v - \gamma\partial_z v = 0, \quad v|_{z=0} = f, \tag{B.29}$$

where $f = f(x)$ is a given function and the coefficients α, β, γ are given by (B.9) (these coefficients depend on the variable ρ which is given by (B.4)). In the sequel we fix indexes $\delta, s, r, \varepsilon$ in $\mathbf{R}$ such that

$$0 < \delta < \frac{1}{4}, \quad s > 1 + \frac{d}{2} - \delta, \quad r > 1, \quad \frac{1}{4} < \varepsilon = \frac{1}{2} - \delta < \min(\frac{1}{2}, s - \frac{1}{2} - \frac{d}{2}). \tag{B.30}$$

It follows from (B.18) and the Sobolev embedding that we have

$$\|\nabla_{x,z} v\|_{C^0([-1,0];C^{s-1-d/2}_*)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\,\|f\|_{H^s}\,,$$

for some non-decreasing function $\mathcal{F}$. Since we only assume that $s > 3/4 + d/2$, this is not enough to control the L^∞ norm of $\nabla_{x,z} v$. The purpose of the next result is to provide such control under the additional assumption that f belongs to C^r_* for some $r > 1$.

Proposition B.7. *Let $r > 1$ and $s > 3/4 + d/2$. For any $-1 < z_1 < 0$, we have*

$$\|\nabla_{x,z} v\|_{C^0\cap L^\infty(\mathbf{R}^d\times[z_1,0])} \leq \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}}\big)\left\{\|f\|_{H^s} + \|f\|_{W^{r,\infty}}\right\},$$

for some non-decreasing function $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$.

Remark B.8. *i)* Since $v|_{z=0} = f \in L^\infty(\mathbf{R}^d)$, we have also

$$\begin{aligned}\|v\|_{L^\infty(\mathbf{R}^d\times[z_1,0])} &\le \|f\|_{L^\infty(\mathbf{R}^d)} + |z_1|\,\|\partial_z v\|_{L^\infty(\mathbf{R}^d\times[z_1,0])}\\ &\le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})\left\{\|f\|_{H^s} + \|f\|_{C^r_*}\right\}.\end{aligned}$$

ii) Let $\mu > 1 + d/2$. Since

$$\left\|\frac{1+|\nabla\rho|^2}{\partial_z\rho}\right\|_{L^\infty} \lesssim 1 + \|\eta\|_{H^\mu}, \quad \|\partial_z\rho\|_{L^\infty} \lesssim 1 + \|\eta\|_{H^\mu},$$

it follows from the previous proposition and the definition of the Dirichlet-Neumann operator that, for any $r > 1$ and $s > 3/4 + d/2$, we have

$$\|G(\eta)f\|_{L^\infty} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big)\big\{\|f\|_{H^s} + \|f\|_{W^{r,\infty}}\big\}. \tag{B.31}$$

Other estimates are known which involve only Hölder norms (see Hu-Nicholls [36]), but they did not apply directly to our case with arbitrary bottoms. The fact that the previous bound involves a Sobolev estimate is harmless for our purposes.

Proof. Recall that the space $X^\mu(I)$ is defined by (B.27). Recall also that (see (B.18) and (B.19))

$$\|\nabla_{x,z} v\|_{X^{s-1}([-1,0])} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})\,\|f\|_{H^s} \tag{B.32}$$

and

$$\|\partial_z v - T_A v\|_{X^{s-1+\varepsilon}([-1,0])} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})\,\|f\|_{H^s}. \tag{B.33}$$

Since $v\,|_{z=0}= f$, writing $v(z) = v(0) + \int_0^z \partial_z v$, this implies that

$$\|v\|_{X^{s-1}([-1,0])} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})\,\|f\|_{H^s}. \tag{B.34}$$

Introduce a cutoff function χ such that

$$\chi(-1) = 0, \quad \chi(z) = 1 \quad \text{for } z \ge z_1,$$

and set $w := \chi(z)(\partial_z - T_A)v$. We shall use the fact that w is already estimated by means of (B.33) together with the parabolic estimate in Hölder spaces established above to deduce an estimate for v.

Since it is convenient to work with forward evolution equation, define the function $\widetilde{v}$ by $\widetilde{v}(x,z) = v(x,-z)$, so that

$$\partial_z\widetilde{v} + T_{\widetilde{A}}\widetilde{v} = -\widetilde{w} \quad \text{for } z \in \widetilde{I}_1 := [0,-z_1].$$

We split $\widetilde{v}$ as $\widetilde{v} = \widetilde{v}_1 + \widetilde{v}_2$ where $\widetilde{v}_1$ is the solution to the system

$$\partial_z\widetilde{v}_1 + T_{\widetilde{A}}\widetilde{v}_1 = 0 \quad \text{for } z \in \widetilde{I}_1, \qquad \widetilde{v}_1|_{z=0} = \widetilde{v}|_{z=0} = f$$

given by Proposition B.6, while $\widetilde{v}_2 = \widetilde{v} - \widetilde{v}_1$ satisfies

$$\partial_z \widetilde{v}_2 + T_{\widetilde{A}} \widetilde{v}_2 = -\widetilde{w} \quad \text{for } z \in \widetilde{I}_1, \qquad \widetilde{v}_2|_{z=0} = 0.$$

According to (B.19) we have

$$\|\widetilde{w}\|_{Y^{s+\varepsilon}(\widetilde{I}_1)} \leq \|\widetilde{w}\|_{L^2(\widetilde{I}_1;H^{s+\varepsilon-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s},$$

which in turn implies, according to Proposition B.6,

$$\|\widetilde{v}_2\|_{X^{s+\varepsilon}(\widetilde{I}_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}. \tag{B.35}$$

Set $m = s - 1 + \varepsilon - \frac{d}{2}$. By the Sobolev embedding we have

$$\|\nabla_x \widetilde{v}_2\|_{L^\infty(\widetilde{I}_1;C^m_*)} \lesssim \|\nabla_x \widetilde{v}_2\|_{L^\infty(\widetilde{I}_1;H^{s-1+\varepsilon})} \leq \|\widetilde{v}_2\|_{X^{s+\varepsilon}(\widetilde{I}_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

Let us prove that $\partial_z \widetilde{v}_2$ satisfies the same estimate. Using the equation for $\widetilde{v}_2$, we obtain $\partial_z \widetilde{v}_2 = -T_{\widetilde{A}} \widetilde{v}_2 - \widetilde{w}$. Now, $\widetilde{w}$ is estimated by means of the bound (B.19). Moving to the estimate of $T_{\widetilde{A}} \widetilde{v}_2$, recall that $T_{\widetilde{A}}$ is an operator of order 1 whose operator norm is estimated by means of the first inequality in (A.4), to get

$$\begin{aligned}\|T_{\widetilde{A}} \widetilde{v}_2\|_{L^\infty(\widetilde{I}_1;C^m_*)} &\lesssim \|T_{\widetilde{A}} \widetilde{v}_2\|_{X^{s-1+\varepsilon}(\widetilde{I}_1)} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\widetilde{v}_2\|_{X^{s+\varepsilon}(\widetilde{I}_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.\end{aligned}$$

We conclude

$$\|\nabla_{x,z} \widetilde{v}_2\|_{L^\infty(\widetilde{I}_1;C^m_*)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

Now, by assumption $s > 1 + d/2 - \delta$ with $\delta < 1/4$ and $\varepsilon = 1/2 - \delta$, so that

$$m = s - 1 + \varepsilon - \frac{d}{2} = s - 1 + \frac{1}{2} - \delta - \frac{d}{2} > \frac{1}{2} - 2\delta > 0,$$

and hence

$$\|\nabla_{x,z} \widetilde{v}_2\|_{L^\infty(\mathbf{R}^d \times [0,-z_1])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

It remains to estimate $\widetilde{v}_1$. Using Proposition B.4 with $r = 1, r_0 = -1$, we obtain

$$\|\widetilde{v}_1\|_{C^0(\widetilde{I}_1;C^r_*)} \leq K\big(\|f\|_{C^r_*} + \|\widetilde{v}_1\|_{C^0(\widetilde{I}_1;C^{-1}_*)}\big). \tag{B.36}$$

To estimate the last term in the right-hand side above, write, according to (B.34) and (B.35),

$$\|\widetilde{v}_1\|_{C^0(\widetilde{I}_1;C^{-1}_*)} \leq C\big(\|\widetilde{v}\|_{C^0(\widetilde{I}_1;H^{s-1})} + \|\widetilde{v}_2\|_{C^0(\widetilde{I}_1;H^{s-1})}\big) \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|f\|_{H^s}.$$

Since $\partial_z \widetilde{v}_1 = -T_{\widetilde{A}} \widetilde{v}_1$, and since $T_{\widetilde{A}}$ is an operator of order 1 whose operator norm is estimated by means of the second inequality in (A.4), the previous inequality (B.36) implies also

$$\|\nabla_{x,z} \widetilde{v}_1\|_{C^0(\widetilde{I}_1;C^{r-1}_*)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|f\|_{H^s} + \|f\|_{C^r_*}).$$

This completes the proof of Proposition B.7. □

Gathering (B.18) and the previous estimate, for any z_0 in $(-1,0]$, we have

(B.37)
$$\begin{aligned}&\|\nabla_{x,z}v\|_{C^0([z_0,0];H^{s-1})\cap L^2((z_0,0);H^{s-\frac{1}{2}})}\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\,\|f\|_{H^s}\,,\\ &\|\nabla_{x,z}v\|_{C^0\cap L^\infty(\mathbf{R}^d\times[z_0,0])}\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\left\{\|f\|_{H^s}+\|f\|_{C^r_*}\right\}.\end{aligned}$$

The end of the proof of Theorem 1.4 is in four steps.

Step 1. Tame estimates in Zygmund spaces. (See Definition A.2 for the definition of Zygmund spaces.)

Recall the following bounds for the coefficients α,β,γ defined in (B.9) (see [3, Lemma 3.25]): for any $s>\frac{1}{2}+\frac{d}{2}$, we have

(B.38)
$$\left\|\alpha-h^2\right\|_{X^{s-\frac{1}{2}}([-1,0])}+\|\beta\|_{X^{s-\frac{1}{2}}([-1,0])}+\|\gamma\|_{X^{s-\frac{3}{2}}([-1,0])}\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}).$$

We need also estimates in Zygmund spaces.

Lemma B.9. *There holds*

(B.39)
$$\|(\alpha,\beta)\|_{C^0([-1,0];W^{\frac{1}{2},\infty})}+\|\gamma\|_{L^2([-1,0];L^\infty)}\le \mathcal{F}(\|\eta\|_{H^{s+1/2}})\{1+\|\eta\|_{W^{r+\frac{1}{2},\infty}}\}.$$

Proof. Recall that, according to Lemma B.1,

(B.40)
$$\begin{aligned}&\|\nabla_x\rho\|_{C^0([-1,0];W^{\frac{1}{2},\infty})}+\|\nabla_{x,z}\rho\|_{L^2([-1,0];W^{1,\infty})}\lesssim\|\eta\|_{W^{r+\frac{1}{2},\infty}},\\ &\|\partial_z\rho-h\|_{C^0([-1,0];W^{\frac{1}{2},\infty})}\lesssim 1+\|\eta\|_{W^{r+\frac{1}{2},\infty}}.\end{aligned}$$

and (since $s-1/2>d/2$), by Sobolev embedding,

(B.41)
$$\|\nabla_{x,z}\rho\|_{L^\infty(\mathbf{R}^d\times(-1,0))}\lesssim 1+\|\eta\|_{H^{s+\frac{1}{2}}}\,.$$

We deduce the estimates for $\alpha-1$ and β from the composition rule (A.22) and the equality $W^{1/2,\infty}=C^{1/2}_*$. The estimate for γ follows from (B.41) and the estimate

$$\left\|\nabla^2_{x,z}\rho\right\|_{L^2([-1,0];L^\infty)}\lesssim\|\eta\|_{W^{r+\frac{1}{2},\infty}}.$$

which follows from (B.40) and the equation satisfied by ρ (to estimate $\partial_z^2\rho$). □

Step 2. Estimates for the source terms. We now estimate the source terms F_1, F_2 and F_3 which appear in (B.15) and (B.16).

Lemma B.10. *For any $z_0\in(-1,0)$, and any $j=1,2,3$ we have,*

(B.42)
$$\|F_j\|_{L^2_z((z_0,0);H^{s-1})}\le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}},\|f\|_{H^s}\big)\left\{1+\|\eta\|_{W^{r+\frac{1}{2},\infty}}+\|f\|_{W^{r,\infty}}\right\}.$$

Proof. By using the tame product rule (see (A.17))

$$\|u_1 u_2\|_{H^{s-1}} \lesssim \|u_1\|_{L^\infty} \|u_2\|_{H^{s-1}} + \|u_2\|_{L^\infty} \|u_1\|_{H^{s-1}}$$

we find that $F_1 = \gamma \partial_z v$ satisfies

$$\begin{aligned}\|F_1\|_{L^2((z_0,0);H^{s-1})} \lesssim & \|\partial_z v\|_{C^0([z_0,0];L^\infty)} \|\gamma\|_{L^2((z_0,0);H^{s-1})} \\ & + \|\gamma\|_{L^2((z_0,0);L^\infty)} \|\partial_z v\|_{L^\infty((z_0,0);H^{s-1})}.\end{aligned}$$

The desired estimate for F_1 follows from Lemma B.9, (B.37) and (B.38).

Let us now study

$$F_2 = (T_\alpha - \alpha)\Delta v + (T_\beta - \beta)\cdot \nabla \partial_z v = -\big(T_{\Delta v}\alpha + R(\alpha, \Delta v) + T_{\nabla \partial_z v}\cdot \beta + R(\beta, \nabla \partial_z v)\big).$$

According to (A.12), we obtain

$$\begin{aligned}\big\|T_{\Delta v(z)}\alpha(z)\big\|_{H^{s-1}} &\lesssim \|\Delta v(z)\|_{C^{-1}_*} \|\alpha(z)\|_{H^s},\\ \big\|T_{\nabla \partial_z v(z)}\cdot \beta(z)\big\|_{H^{s-1}} &\lesssim \|\nabla \partial_z v(z)\|_{C^{-1}_*} \|\beta(z)\|_{H^s}.\end{aligned}$$

On the other hand, since $s-1>0$ we can apply (A.11) to obtain

$$\begin{aligned}\|R(\alpha, \Delta v)(z)\|_{H^{s-1}} &\lesssim \|\Delta v(z)\|_{C^{-1}_*} \|\alpha(z)\|_{H^s},\\ \|R(\beta, \nabla \partial_z v)(z)\|_{H^{s-1}} &\lesssim \|\nabla \partial_z v(z)\|_{C^{-1}_*} \|\beta(z)\|_{H^s}.\end{aligned}$$

Consequently we have proved

$$\begin{aligned}\text{(B.43)} \quad \|F_2\|_{L^2([z_0,0];H^{s-1})} \lesssim & \|\Delta v\|_{C^0([z_0,0];C^{-1}_*)} \|\alpha\|_{L^2([z_0,0];H^s)} \\ & + \|\nabla \partial_z v\|_{C^0([z_0,0];C^{-1}_*)} \|\beta\|_{L^2([z_0,0];H^s)}.\end{aligned}$$

Notice that

$$\|\Delta v\|_{C^{-1}_*} \lesssim \|\nabla v\|_{C^0_*} \lesssim \|\nabla v\|_{L^\infty}, \qquad \|\nabla \partial_z v\|_{C^{-1}_*} \lesssim \|\partial_z v\|_{C^0_*} \lesssim \|\partial_z v\|_{L^\infty}$$

and consequently, according to (B.37) and (B.38) we conclude the proof of the claim (B.42) for $j=2$.

It remains to estimate F_3. In light of (B.37) it is enough to prove that

$$\text{(B.44)} \qquad \|F_3\|_{L^2(I;H^{s-1})} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big) \|\eta\|_{W^{r+\frac12,\infty}} \|\nabla_{x,z} v\|_{L^2(I;H^{s-\frac12})},$$

for some non-decreasing function. Directly from the definition of α and β, by using the tame estimates in Hölder spaces (A.18), we verify that the symbols a, A (given by (B.17)) belong to $\Gamma^1_{1/2}(\mathbf{R}^d \times I)$ and that they satisfy

$$\text{(B.45)} \qquad \mathcal{M}^1_{1/2}(a) + \mathcal{M}^1_{1/2}(A) \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big) \|\eta\|_{W^{r+\frac12,\infty}}.$$

(For later purpose, notice that we used here only $s > \frac12 + \frac{d}{2}$.) Moreover,

$$\mathcal{M}^1_{-1/2}(\partial_z A) \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big) \|\eta\|_{W^{r+\frac12,\infty}}.$$

By definition (B.17), $a + A = -i\beta\cdot\xi$ so $T_a + T_A = -T_\beta\cdot\nabla$. It follows that

$$F_3 = \big(T_a T_A - T_\alpha\Delta\big)v - T_{\partial_z A}v.$$

Set

$$R_0(z) := T_{a(z)}T_{A(z)} - T_\alpha\Delta, \qquad R_1(z) := -T_{\partial_z A}.$$

Since $aA = -\alpha|\xi|^2$, we deduce, using Theorem A.5 (see (ii) applied with $\rho = 1/2$), that for any $\mu\in\mathbf{R}$,

$$\sup_{z\in[-1,0]}\left\Vert R_0(z)\right\Vert_{H^{\mu+\frac{3}{2}}\to H^\mu} \le \mathcal{F}\big(\left\Vert\eta\right\Vert_{H^{s+\frac12}}\big)\left\Vert\eta\right\Vert_{W^{r+\frac12,\infty}}. \tag{B.46}$$

On the other hand, Proposition A.11 (applied with $\rho = -1/2$) implies that

$$\sup_{z\in[-1,0]}\left\Vert R_1(z)\right\Vert_{H^{\mu+\frac{3}{2}}\to H^\mu} \le \mathcal{F}\big(\left\Vert\eta\right\Vert_{H^{s+\frac12}}\big)\left\Vert\eta\right\Vert_{W^{r+\frac12,\infty}}. \tag{B.47}$$

Using these inequalities for $\mu = s-1$, we obtain the desired result (B.44). This completes the proof of Lemma B.10. □

Step 3 : elliptic estimates Introduce a cutoff function $\kappa = \kappa(z), z\in[-1,0]$, such that $\kappa(z) = 1$ near $z = 0$ and such that $\kappa(z_1) = 0$ (recall that $I_1 = [z_1, 0]$ for some $z_1\in(-1,0)$). Set

$$W := \kappa(z)(\partial_z - T_A)v. \tag{B.48}$$

Now it follows from the paradifferential equation (B.16) for v that

$$\partial_z W - T_a W = F',$$

where

$$F' = \kappa(z)(F_1+F_2+F_3) + \kappa'(z)(\partial_z - T_A)v.$$

Our goal is to prove that

$$\left\Vert W\right\Vert_{L^\infty(I_1;H^{s-\frac12})} \le \mathcal{F}\big(\left\Vert\eta\right\Vert_{H^{s+\frac12}}, \left\Vert f\right\Vert_{H^s}\big)\left\{1 + \left\Vert\eta\right\Vert_{W^{r+\frac12,\infty}} + \left\Vert f\right\Vert_{W^{r,\infty}}\right\}. \tag{B.49}$$

We have already proved that

$$\left\Vert F_1+F_2+F_3\right\Vert_{L^2_z(I_1;H^{s-1})} \le \mathcal{F}\big(\left\Vert\eta\right\Vert_{H^{s+\frac12}}, \left\Vert f\right\Vert_{H^s}\big)\left\{1 + \left\Vert\eta\right\Vert_{W^{r+\frac12,\infty}} + \left\Vert f\right\Vert_{W^{r,\infty}}\right\}.$$

We now turn to an estimate for $(\partial_z - T_A)v$. To do that we estimate separately $\partial_z v$ and $T_A v$. Clearly, by definition of the space X^{s-1}, noting that $I_1\subset I_0 = [-1,0]$, we have

$$\left\Vert\partial_z v\right\Vert_{L^2(I_1;H^{s-\frac12})} \le \left\Vert\partial_z v\right\Vert_{L^2(I_0;H^{s-\frac12})} \le \left\Vert\nabla_{x,z}v\right\Vert_{X^{s-1}(I_0)}.$$

On the other hand, as in the previous step, since $\mathcal{M}^1_0(A)\le C\big(\left\Vert\eta\right\Vert_{H^{s+\frac12}}\big)$, we have

$$\begin{aligned}\left\Vert T_A v\right\Vert_{L^2(I_1;H^{s-\frac12})} &\le \mathcal{F}(\left\Vert\eta\right\Vert_{H^{s+\frac12}})\left\Vert\nabla_x v\right\Vert_{L^2(I_1;H^{s-\frac12})}\\ &\le \mathcal{F}(\left\Vert\eta\right\Vert_{H^{s+\frac12}})\left\Vert\nabla_{x,z}v\right\Vert_{X^{s-1}(I_0)}.\end{aligned}$$

Now recall from (B.18) that $\|\nabla_{x,z} v\|_{X^{s-1}(I_0)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|f\|_{H^s}$. Therefore,

$$\|(\partial_z - T_A)v\|_{L^2(I_0;H^{s-\frac12})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|f\|_{H^s},$$

and we end up with

$$\|F'\|_{L^2(I_1;H^{s-1})} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}, \|f\|_{H^s}\big) \Big\{1 + \|\eta\|_{W^{r+\frac12,\infty}} + \|f\|_{W^{r,\infty}}\Big\}.$$

Since $\partial_z W - T_a W = F'$ and $W(x, z_1) = 0$ (by definition of the cutoff function κ) and since $a \in \Gamma^1_\varepsilon$ satisfies $\mathrm{Re}(-a(x,\xi)) \ge c|\xi|$, by using Proposition B.6 applied with $J = I_1$, $\rho = \varepsilon$ and $r = s - 1/2$, we have

$$\|W\|_{X^{s-\frac12}(I_1)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|F'\|_{Y^{s-\frac12}(I_1)}.$$

Now, by definition, $\|F'\|_{Y^{s-\frac12}(I_1)} \le \|F'\|_{L^2(I_1;H^{s-1})}$, so we conclude that W satisfies the desired estimate (B.49).

Step 4 : paralinearization of the Dirichlet-Neumann. We shall only use the following obvious consequence of (B.49): $\|W|_{z=0}\|_{H^{s-\frac12}}$ is estimated by the right-hand side of (B.49) (we can take the trace on $z = 0$ since W belongs to $X^{s-\frac12} \subset C^0_z(H^{s-\frac12})$ and not only to $L^\infty_z(H^{s-\frac12})$, as follows from Proposition B.6). Since $W|_{z=0} = \partial_z v - T_A v|_{z=0}$, we thus have proved that

$$\|\partial_z v - T_A v|_{z=0}\|_{H^{s-\frac12}} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}, \|f\|_{H^s}\big) \Big\{1 + \|\eta\|_{W^{r+\frac12,\infty}} + \|f\|_{W^{r,\infty}}\Big\}. \tag{B.50}$$

Now, recall that

$$G(\eta)f = \zeta_1 \partial_z v - \zeta_2 \cdot \nabla v \,\big|_{z=0}$$

with

$$\zeta_1 := \frac{1 + |\nabla\rho|^2}{\partial_z \rho}, \quad \zeta_2 := \nabla\rho.$$

As for the coefficients α, β (see Lemma B.9), we have

$$\Big\|\zeta_1 - \frac{1}{h}\Big\|_{L^\infty([-1,0];W^{\frac12,\infty})} + \|\zeta_2\|_{L^\infty([-1,0];W^{\frac12,\infty})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})\|\eta\|_{W^{r+\frac12,\infty}}, \tag{B.51}$$

$$\Big\|\zeta_1 - \frac{1}{h}\Big\|_{L^\infty([-1,0];H^{s-1/2})} + \|\zeta_2\|_{L^\infty([-1,0];H^{s-1/2})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}). \tag{B.52}$$

Write

$$\zeta_1 \partial_z v - \zeta_2 \cdot \nabla v = T_{\zeta_1} \partial_z v - T_{\zeta_2} \nabla v + R',$$

with

$$R' = T_{\partial_z v} \zeta_1 - T_{\nabla v} \cdot \zeta_2 + R(\zeta_1, \partial_z v) - R(\zeta_2, \nabla v). \tag{B.53}$$

Since a paraproduct by an L^∞ function acts on any Sobolev spaces, according to Proposition B.7 and (B.52), we obtain

$$\|T_{\partial_z v}\zeta_1 - T_{\nabla v}\cdot\zeta_2\|_{L^\infty(I_1;H^{s-\frac{1}{2}})} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}\big)\left\{1 + \|f\|_{W^{r,\infty}}\right\}.$$

We estimate similarly the last two terms in the right-hand side of (B.53), so

$$\|R'\|_{L^\infty(I_1;H^{s-\frac{1}{2}})} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}\big)\left\{1 + \|f\|_{W^{r,\infty}}\right\}.$$

Furthermore, (B.50) implies that

$$T_{\zeta_1}\partial_z v - T_{\zeta_2}\nabla v\,\big|_{z=0} = T_{\zeta_1}T_A v - T_{i\zeta_2\cdot\xi}v\,\big|_{z=0} + R'',$$

where $\|R''\|_{H^{s-\frac{1}{2}}}$ satisfies

$$\|R''\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}\big)\left\{1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{W^{r,\infty}}\right\}.$$

Thanks to (A.5) we have

$$\begin{aligned}\|T_{\zeta_1}T_A - T_{\zeta_1 A}\|_{H^s\to H^{s-\frac{1}{2}}} &\lesssim \|\zeta_1\|_{L^\infty}\mathcal{M}^1_{1/2}(A) + \|\zeta_1\|_{W^{\frac{1}{2},\infty}}\mathcal{M}^1_0(A)\\ &\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\eta\|_{W^{r+\frac{1}{2},\infty}},\end{aligned}$$

where we used (B.45) and $\mathcal{M}^1_0(A) \le C\big(\|\eta\|_{H^{s+\frac{1}{2}}}\big)$. Therefore,

$$G(\eta)f = T_{\zeta_1 A}v - T_{i\zeta_2\cdot\xi}v\,\big|_{z=0} + R(\eta)f$$

where

$$\|R(\eta)f\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}\big)\left\{1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{W^{r,\infty}}\right\}.$$

Now by definition of A (see (B.17)) one has

$$\zeta_1 A v - i\zeta_2\cdot\xi = \sqrt{(1+|\nabla\rho|^2)|\xi|^2 - (\nabla\rho\cdot\xi)^2}$$

so $T_{\zeta_1 A}v - T_{i\zeta_2\cdot\xi}v\,\big|_{z=0} = T_\lambda f$ since, by definition of λ,

$$\lambda = \sqrt{(1+|\nabla\eta|^2)|\xi|^2 - (\nabla\eta\cdot\xi)^2}.$$

This proves that $G(\eta)f = T_\lambda f + R(\eta)f$ which concludes the proof of Theorem 1.4.

Appendix C

Estimates for the Taylor coefficient

Here we prove several estimates for the Taylor coefficient.

Proposition C.1. *Let $d \geq 1$ and consider s, r such that*

$$s > \frac{3}{4} + \frac{d}{2}, \quad r > 1.$$

For any $0 < \varepsilon < \min(r-1, s-\frac{3}{4}-\frac{d}{2})$, there exists a non-decreasing function $\mathcal{F}$ such that, for all $t \in [0,T]$,

$$\|\mathfrak{a}(t) - g\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}\Big(\|(\eta,\psi,V,B)(t)\|_{H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s}\Big), \tag{C.1}$$

and

$$\begin{aligned}\|\mathfrak{a}(t)\|_{W^{\frac{1}{2}+\varepsilon,\infty}} &+ \|(\partial_t\mathfrak{a} + V\cdot\nabla\mathfrak{a})(t)\|_{W^{\varepsilon,\infty}} \\ &\leq \mathcal{F}\big(\|(\eta,\psi)(t)\|_{H^{s+\frac{1}{2}}}, \|(V,B)(t)\|_{H^s}\big)\Big\{1 + \|\eta(t)\|_{W^{r+\frac{1}{2},\infty}} + \|(V,B)(t)\|_{W^{r,\infty}}\Big\}.\end{aligned} \tag{C.2}$$

The estimate of $\|\mathfrak{a}-g\|_{H^{s-\frac{1}{2}}}$ follows directly from the arguments in [3]. The estimate of the $W^{\varepsilon,\infty}$-norm of $\partial_t\mathfrak{a}+V\cdot\nabla\mathfrak{a}$ is the easiest one. The main new difficulty here is to prove a tame estimate for $\|\mathfrak{a}\|_{W^{\frac{1}{2}+\varepsilon,\infty}}$. Indeed, there are several further complications which appear in the analysis in Hölder spaces.

Hereafter, since the time variable is fixed, we shall skip it. To prove the above estimates on $\mathfrak{a}$, we form an elliptic equation for P. As explained in Appendix B we flatten the free surface by using the change of variables $(x,z)\mapsto(x,\rho(x,z))$ (see (B.4) and Lemma B.1). Set

$$v(x,z) = \phi(x,\rho(x,z)), \quad \wp(x,z) = P(x,\rho(x,z)) + g\rho(x,z),$$

and notice that

$$\mathfrak{a} - g = -\frac{1}{\partial_z\rho}\partial_z\wp\,|_{z=0}\,.$$

The first elementary step is to compute the equation satisfied by the new unknown as well as the boundary conditions on $\{z = 0\}$. As in [3], one computes that

$$\text{(C.3)}\qquad \begin{aligned} &\partial_z^2 \wp + \alpha \Delta \wp + \beta \cdot \nabla \partial_z \wp - \gamma \partial_z \wp = F_0(x,z) && \text{for } z < 0,\\ &\wp = g\eta && \text{on } z = 0, \end{aligned}$$

where α, β, γ are as above (see (B.9)) and where

$$\text{(C.4)}\qquad F_0 = -\alpha \left|\Lambda^2 v\right|^2, \quad \Lambda = (\Lambda_1, \Lambda_2), \quad \Lambda_1 = \frac{1}{\partial_z \rho}\partial_z, \quad \Lambda_2 = \nabla - \frac{\nabla \rho}{\partial_z \rho}\partial_z.$$

Our first task is to estimate the source term F_0.

Lemma C.2. *Let $d \geq 1$ and consider $s \in]1, +\infty[$ such that*

$$s > \frac{3}{4} + \frac{d}{2}.$$

Then there exists $z_0 < 0$ such that

$$\|F_0\|_{L^1([z_0,0];H^{s-\frac12})} + \|F_0\|_{L^2([z_0,0];C_*^{s+\frac14-\frac{d}{2}})} \leq \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s \times H^s}\big).$$

Proof. The first part of this result follows from the proof of [3, Lemma 4.7] (although this lemma is proved under the assumption that $s > 1 + d/2$, its proof shows that the results (C.5)–(C.6) we quote below hold for any $s > 1/2 + d/2$). We proved in [3] that

$$\text{(C.5)}\qquad \|F_0\|_{L^1([z_0,0];H^{s-\frac12})} \leq \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s \times H^s}\big),$$

together with

$$\text{(C.6)}\qquad \begin{aligned} \|\Lambda_j\Lambda_k v\|_{C^0([z_0,0];H^{s-1})} + \|\Lambda_j\Lambda_k v\|_{L^2([z_0,0];H^{s-\frac12})} \\ \leq \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s \times H^s}\big). \end{aligned}$$

By interpolation, (C.6) also implies that

$$\|\Lambda_j\Lambda_k v\|_{L^4([z_0,0];H^{s-\frac34})} \leq \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s \times H^s}\big).$$

Since $s > 3/4 + d/2$ by assumption, the Sobolev space $H^{s-\frac34}(\mathbf{R}^d)$ is an algebra and hence, according to (B.38),

$$\left\|\alpha \left|\Lambda_j\Lambda_k v\right|^2\right\|_{L^2([z_0,0];H^{s-\frac34})} \lesssim \big(1 + \left\|\alpha - h^2\right\|_{L^\infty([z_0,0];H^{s-\frac34})}\big) \left\|\Lambda_j\Lambda_k v\right\|^2_{L^4([z_0,0];H^{s-\frac34})}.$$

The Sobolev embedding $H^{s-\frac34} \subset C_*^{s-\frac34-\frac{d}{2}}$ then yields

$$\text{(C.7)}\qquad \|F_0\|_{L^2([z_0,0];C_*^{s-\frac34-\frac{d}{2}})} \leq \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s \times H^s}\big).$$

This completes the proof. □

The proof of the fact that $\|\mathfrak{a} - g\|_{H^{s-\frac12}}$ is bounded by a constant depending only on $\|(\eta,\psi)\|_{H^{s+\frac12}}$ and $\|(V,B)\|_{H^s}$ then follows from elliptic regularity which implies that

(C.8) $$\|\nabla_{x,z}\wp\|_{X^{s-\frac12}([z_0,0])} \le \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big),$$

as can be proved using the arguments given above the statement of Proposition 4.8 in [3]. We shall now prove the estimate of $\|\mathfrak{a}\|_{W^{1/2+\epsilon,\infty}}$. This estimate will follow directly from the following result.

Proposition C.3. *Let $d \ge 1$ and consider $(s,r,r') \in \mathbf{R}^3$ such that*

$$s > \frac34 + \frac d2, \quad s + \frac14 - \frac d2 > r > r' > 1.$$

Then there exists $z_0 < 0$ such that

(C.9)
$$\|\nabla_{x,z}\wp\|_{C^0([z_0,0];W^{r'-\frac12,\infty})} \le \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big)\left\{1 + \|\eta\|_{W^{r+\frac12,\infty}}\right\}$$

for some non-decreasing function $\mathcal{F}$ depending only on s,r,r'.

Proof. It follows from (C.8) and the Sobolev embedding $H^{s-\frac12}(\mathbf{R}^d) \subset W^{r-\frac34,\infty}(\mathbf{R}^d)$ that

(C.10) $$\|\nabla_{x,z}\wp\|_{L^\infty([z_0,0];W^{r-\frac34,\infty})} \le \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big).$$

The key point is that, since $r - 3/4 \ge 0$, we now have an L^∞-estimate for $\nabla_{x,z}\wp$ which does not depend on the hölder norms, which are the highest order norms for $s < 1 + d/2$ (compare with Proposition B.7).

To prove (C.9), let us revisit the proof of Theorem 1.4. With the notations of Appendix B (see (B.48)), $W = \kappa(z)(\partial_z - T_A)\wp$ satisfies a parabolic evolution equation of the form

(C.11) $$\partial_z W - T_a W = F_0 + F_1 + F_2 + F_3 + F_4,$$

where the symbols a and A are as defined in (B.17), F_0 is given by (C.4) and

$$\begin{aligned}
&F_1 = \gamma\partial_z\wp,\\
&F_2 = -\big(T_{\Delta\wp}\alpha + R(\alpha,\Delta\wp) + T_{\nabla\partial_z\wp}\cdot\beta + R(\beta,\nabla\partial_z\wp)\big),\\
&F_3 = (T_aT_A - T_\alpha\Delta)\wp - T_{\partial_z A}\wp,\\
&F_4 = \kappa'(z)(\partial_z - T_A)\wp.
\end{aligned}$$

Since $(\partial_z - T_A)\wp = W$ for z small enough (by definition of W), in light of (C.10) and Proposition B.4 (applied with $r_0 = 1/4$, $q = +\infty$), one can reduce the proof of (C.9) to proving that, for some $\delta > 0$, the $L^\infty([z_0,0];C_*^{r'-\frac12+\delta})$ norm of W is bounded by the right-hand side of (C.9). Again, since $W|_{z=z_1} = 0$, by using Proposition B.4 (with $q = 2$), the former estimate for W will be deduced from the equation (C.11) and the following lemma.

Lemma C.4. *There exists $z_0 < 0$ and $\epsilon > 0$ such that, for $i \in \{0, \dots, 4\}$,*

$$\|F_i\|_{L^2([z_0,0];C_*^{r'-1+\epsilon})} \leq \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big)\left\{1+\|\eta\|_{W^{r+\frac12,\infty}}\right\}.$$

To prove Lemma C.4 we begin by recording the following easy refinement of previous bounds on the coefficients α, β, γ. We shall use the following variant of Lemma B.9:

$$\text{(C.12)}\quad \|\alpha\|_{L^2([z_0,0];C_*^r)} + \|\beta\|_{L^2([z_0,0];C_*^r)} + \|\gamma\|_{L^2([z_0,0];C_*^{r-1})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac12}})(\|\eta\|_{C_*^{r+\frac12}} + 1).$$

As in the proof of Lemma B.9, such estimates follow from the definition of ρ (by means of the Poisson kernel), the Sobolev embedding $H^{s+\frac12} \subset W^{1,\infty}$ and tame estimates in Hölder spaces (A.22). We are now ready to conclude the proof of Lemma C.4.

Estimate of F_0. Since $C_*^{s-3/4-d/2} \subset C_*^{r-1}$, (C.7) implies that

$$\|F_0\|_{L^2([z_0,0];C_*^{r-1})} \leq \mathcal{F}\big(\|(\eta,\psi,V,B)\|_{H^{s+\frac12}\times H^{s+\frac12}\times H^s\times H^s}\big).$$

Estimate of F_1. Using (A.18), the term $F_1 = \gamma\partial_z\wp$ is estimated by

$$\|F_1\|_{L^2_z([z_0,0];C_*^{r-1})} \leq K\,\|\partial_z\wp\|_{L^\infty_z([z_0,0];C_*^{r-1})}\,\|\gamma\|_{L^2_z([z_0,0];C_*^{r-1})}\,.$$

The desired estimate then follows from (C.10) and (C.12).

Estimate of F_2. According to (A.13) with $m = 1, s = r$, we obtain

$$\left\|T_{\Delta\wp(z)}\alpha(z)\right\|_{C_*^{r-1}} \lesssim \|\Delta\wp(z)\|_{C_*^{-1}}\,\|\alpha(z)\|_{C_*^r}\,,$$
$$\left\|T_{\nabla\partial_z\wp(z)}\cdot\beta(z)\right\|_{C_*^{r-1}} \lesssim \|\nabla\partial_z\wp(z)\|_{C_*^{-1}}\,\|\beta(z)\|_{C_*^r}\,.$$

On the other hand, since $r > 1$ we can apply (A.10) to obtain

$$\|R(\alpha,\Delta\wp)(z)\|_{C_*^{r-1}} \lesssim \|\Delta\wp(z)\|_{C_*^{-1}}\,\|\alpha(z)\|_{C_*^r} \lesssim \|\nabla\wp(z)\|_{L^\infty}\,\|\alpha(z)\|_{C_*^r}\,,$$
$$\|R(\beta,\nabla\partial_z\wp)(z)\|_{C_*^{r-1}} \lesssim \|\nabla\partial_z\wp(z)\|_{C_*^{-1}}\,\|\beta(z)\|_{C_*^r} \lesssim \|\partial_z\wp(z)\|_{L^\infty}\,\|\beta(z)\|_{C_*^r}$$

By using (C.10) and (C.12), we conclude the proof of the claim in Lemma C.4 for $i = 2$.

Estimate of F_3. Using (B.46)–(B.47) with $\mu = s - 1/2$ we find

$$\|F_3(z)\|_{C_*^{r-1}} \lesssim \|F_3(z)\|_{H^{s-\frac12}} \leq \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big)\|\eta\|_{W^{r+\frac12,\infty}}\,\|\nabla_{x,z}\wp(z)\|_{H^s}\,,$$

and hence

$$\|F_3\|_{L^2([z_0,0];C_*^{r-1})} \leq \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big)\|\eta\|_{W^{r+\frac12,\infty}}\,\|\nabla_{x,z}\wp\|_{X^{s-\frac12}([z_0,0])}\,,$$

by definition of $X^{s-1/2}([z_0,0])$. The desired estimate follows from (C.8).

Estimate of F_4. This follows from (C.10) and (A.4).

This completes the proof of Lemma C.4 and hence the proof of Proposition C.3. □

Appendix D

Sobolev estimates

In this appendix, we prove sharp *a priori* estimates in Sobolev spaces.

D.1 Introduction

Our goal is to estimate, for T in $(0, T_0]$, the norm

$$M_s(T) := \|(\psi, \eta, B, V)\|_{C^0([0,T];H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s)},$$

in terms of the norm of the initial data

$$M_{s,0} := \|(\psi(0), \eta(0), B(0), V(0))\|_{H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s}$$

and in terms of a quantity which involves Hölder norms, that will be later estimated by means of a Strichartz estimate, defined by

$$Z_r(T) := \|\eta\|_{L^p([0,T];W^{r+\frac{1}{2},\infty})} + \|(B, V)\|_{L^p([0,T];W^{r,\infty}\times W^{r,\infty})},$$

where $p = 4$ if $d = 1$ and $p = 2$ for $d \geq 2$.

Our goal in this chapter is to prove the following result.

Theorem D.1. *Let $T_0 > 0$, $d \geq 1$ and consider $s, r \in]1, +\infty[$ such that*

$$s > \frac{3}{4} + \frac{d}{2}, \qquad s + \frac{1}{4} - \frac{d}{2} > r > 1.$$

There exists a non-decreasing function $\mathcal{F}\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that, for all smooth solution (η, ψ) of (1.5) defined on the time interval $[0, T_0]$ and satisfying Assumption 2.1 on that time interval, for any T in $[0, T_0]$, there holds

$$M_s(T) \leq \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T) + Z_r(T)\big)\big). \tag{D.1}$$

If $s > 1 + d/2$ then one can apply the previous inequality with $r = s - d/2$. Then $Z_r(T) \lesssim M_s(T)$ and one deduces from (D.1) an estimate which involves only $M_s(T)$. Thus we recover the *a priori* estimate in Sobolev spaces proved in [3] for $s > 1+d/2$. The proof of Theorem D.1 follows closely the proof of [3, Prop. 4.1]. For the reader convenience we shall recall the scheme of the analysis, but we shall only prove the points which must be adapted. The main new difficulty is to prove sharp Hölder estimates for the Taylor coefficient.

Hereafter, $\mathcal{F}$ always refers to a non-decreasing function $\mathcal{F}\colon \mathbf{R}^+ \to \mathbf{R}^+$ depending only on s, r, d and h, c_0 (being of course independent of the time T and the unknowns).

D.2 Symmetrization of the equations

As already used in the analysis of the Strichartz estimate, the key point is to symmetrize the equations. To ease the readability, we recall this result here. Recall that we introduced the following notations

$$\zeta = \nabla\eta, \quad U_s := \langle D_x\rangle^s V + T_\zeta \langle D_x\rangle^s B, \quad \zeta_s := \langle D_x\rangle^s \zeta, \quad q = \sqrt{\frac{\mathfrak{a}}{\lambda}}, \quad \theta_s = T_q \zeta_s.$$

As already mentioned in Section 2.1, by combining the analysis done in [3] combined with the improved tame estimates established in Appendix B, one obtains the following result:

$$\partial_t U_s + T_V \cdot \nabla U_s + T_\gamma \theta_s = F_1, \tag{D.2}$$

$$\partial_t \theta_s + T_V \cdot \nabla \theta_s - T_\gamma U_s = F_2, \tag{D.3}$$

for some source terms F_1, F_2 satisfying

$$\begin{aligned} &\left\| (F_1(t), F_2(t)) \right\|_{L^2\times L^2} \\ &\quad \le C\left(\left\|(\eta,\psi)(t)\right\|_{H^{s+\frac{1}{2}}}, \left\|(V,B)(t)\right\|_{H^s}\right) \left\{ 1 + \left\|\eta(t)\right\|_{W^{r+\frac{1}{2},\infty}} + \left\|(V,B)(t)\right\|_{W^{r,\infty}} \right\}, \end{aligned} \tag{D.4}$$

for any real numbers s and r such that $s > \frac{3}{4} + \frac{d}{2}$, $r > 1$.

D.3 Sobolev estimates

We now explain how to deduce Theorem D.1 from (D.2)–(D.4).

Notice that, by definition of $M_s(T)$ and $Z_r(T)$,

$$\begin{aligned} &\left\| (F_1, F_2) \right\|_{L^1([0,T];L^2\times L^2)} \\ &\quad \le \mathcal{F}(M_s(T)) \left\{ T + \left\|\eta\right\|_{L^1([0,T];W^{r+\frac{1}{2},\infty})} + \left\|(V,B)\right\|_{L^1([0,T];W^{r,\infty})} \right\} \\ &\quad \le \sqrt{T}\mathcal{F}(M_s(T)) Z_r(T), \end{aligned}$$

for $T \leq 1$. Then it follows from the previous estimate and energy estimates (see §4.4 in [3]) that we have the following $L^\infty_t(L^2_x)$ estimate for (U_s, θ_s).

Lemma D.2. *There exists a non-decreasing function $\mathcal{F}$ such that*

$$\|U_s\|_{L^\infty([0,T];L^2)} + \|\theta_s\|_{L^\infty([0,T];L^2)} \leq \mathcal{F}(M_{s,0}) + \sqrt{T}\mathcal{F}(M_s(T))(1 + Z_r(T)). \tag{D.5}$$

It remains to deduce from this lemma estimates for the Sobolev norms of η, ψ, V, B. Recall that the functions U_s and θ_s are obtained from (η, V, B) through:

$$U_s := \langle D_x\rangle^s V + T_\zeta \langle D_x\rangle^s B,$$
$$\theta_s := T_{\sqrt{\mathfrak{a}/\lambda}} \langle D_x\rangle^s \nabla\eta.$$

We begin with the following result. It gives the desired estimate for η but only weaker estimates for (V, B) and the Taylor coefficient $\mathfrak{a}$.

Lemma D.3. *There exists a non-decreasing function $\mathcal{F}$ such that for any $r > 0$,*

$$\begin{aligned}\|\eta\|_{L^\infty([0,T];H^{s+\frac{1}{2}})} &+ \|(B,V)\|_{L^\infty([0,T];H^{s-\frac{1}{2}})} \\ &\leq \mathcal{F}(M_{s,0}) + \sqrt{T}\mathcal{F}(M_s(T))(Z_r(T) + 1),\end{aligned} \tag{D.6}$$

and, for any $1 < r' < r$,

$$\|\mathfrak{a}\|_{L^\infty([0,T];C^{r'-1}_*)} \leq \mathcal{F}(M_{s,0}) + \sqrt{T}\mathcal{F}(M_s(T))(Z_r(T) + 1). \tag{D.7}$$

We omit the proof since this lemma follows directly from the proof of Lemma 4.13 and Lemma 4.14 in [3].

Once η is estimated in $L^\infty([0,T]; H^{s+\frac{1}{2}})$, by using the estimate for U_s, we are going to estimate (B, V) in $L^\infty([0,T]; H^s)$. Here we shall make an essential use of the following result about the paralinearization of the Dirichlet-Neumann operator for domains whose boundary is in H^μ for some $\mu > 1 + d/2$.

Proposition D.4 (from [3])**.** *Let $d \geq 1$ and $\mu > 1 + \frac{d}{2}$. For any $\frac{1}{2} \leq \sigma \leq \mu - \frac{1}{2}$ and any*

$$0 < \varepsilon \leq \frac{1}{2}, \qquad \varepsilon < \mu - 1 - \frac{d}{2},$$

there exists a non-decreasing function $\mathcal{F}\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that $R(\eta)f := G(\eta)f - T_\lambda f$ satisfies

$$\|R(\eta)f\|_{H^{\sigma-1+\varepsilon}(\mathbf{R}^d)} \leq \mathcal{F}\big(\|\eta\|_{H^\mu(\mathbf{R}^d)}\big) \|f\|_{H^\sigma(\mathbf{R}^d)}.$$

Lemma D.5. *There exists a non-decreasing function $\mathcal{F}$ such that*

$$\|(V,B)\|_{L^\infty([0,T];H^s)} \leq \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T) + Z_r(T)\big)\big). \tag{D.8}$$

Proof. STEP 1. Recall that $U = V + T_\zeta B$. We begin by proving that there exists a non-decreasing function $\mathcal{F}$ such that

$$\|U\|_{L^\infty([0,T];H^{s-\frac{1}{4}})} \leq \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T) + Z_r(T)\big)\big). \tag{D.9}$$

Since $U_s = \langle D_x\rangle^s V + T_\zeta \langle D_x\rangle^s B$ by definition, we have

$$\langle D_x\rangle^{s-\frac{1}{4}} U = \langle D_x\rangle^{-\frac{1}{4}} \big\{U_s + \big[\langle D_x\rangle^s, T_\zeta\big]B\big\}.$$

Theorem A.5 implies that

$$\big\|\big[\langle D_x\rangle^s, T_\zeta\big]\big\|_{H^\mu \to H^{\mu-s+\frac{1}{4}}} \lesssim \|\zeta\|_{W^{\frac{1}{4},\infty}} \tag{D.10}$$

so

$$\big\|\big[\langle D_x\rangle^s, T_\zeta\big]B\big\|_{H^{-\frac{1}{4}}} \lesssim \|\zeta\|_{W^{\frac{1}{4},\infty}} \|B\|_{H^{s-\frac{1}{2}}}.$$

Since $s > 3/4 + d/2$, we have

$$\|\zeta\|_{W^{\frac{1}{4},\infty}} \lesssim \|\zeta\|_{H^{s-\frac{1}{2}}} \leq \|\eta\|_{H^{s+\frac{1}{2}}}$$

and hence

$$\|U\|_{H^{s-\frac{1}{4}}} \lesssim \|U_s\|_{H^{-\frac{1}{4}}} + \|\eta\|_{H^{s+\frac{1}{2}}} \|B\|_{H^{s-\frac{1}{2}}}.$$

The three terms in the right-hand side of the above inequalities have been already estimated; indeed, Lemma D.2 gives an estimate for the $L^\infty_t(L^2_x)$-norm of U_s and *a fortiori* for its $L^\infty_t(H_x^{-\frac{1}{4}})$-norm, see also Lemma D.3 for B and Lemma D.3 for η. This proves (D.9).

STEP 2. Recall that we have already estimated the $L^\infty_t(H_x^{s-\frac{1}{2}})$-norms of B, V and that we want to estimate their $L^\infty_t(H^s_x)$-norms. As an intermediate step, we begin by proving that

$$\|B\|_{L^\infty([0,T];H^{s-\frac{1}{4}})} \leq \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T) + Z_r(T)\big)\big). \tag{D.11}$$

To do so, the two key points are the paralinearization estimate for $R(\eta) := G(\eta) - T_\lambda$ (see Proposition D.4) and the relation (2.5) between V and B: for any $s > \frac{1}{2} + \frac{d}{2}$ one has $G(\eta)B = -\operatorname{div} V + \widetilde{\gamma}$ where

$$\|\widetilde{\gamma}\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta, V, B)\|_{H^{s+\frac{1}{2}} \times H^{\frac{1}{2}} \times H^{\frac{1}{2}}}). \tag{D.12}$$

Taking the divergence in $U = V + T_\zeta B$, we get according to Lemma D.3 and the previous identity $G(\eta)B = -\operatorname{div} V + \widetilde{\gamma}$,

$$\begin{aligned}
\operatorname{div} U &= \operatorname{div} V + \operatorname{div} T_\zeta B = \operatorname{div} V + T_{\operatorname{div}\zeta} B + T_\zeta \cdot \nabla B \\
&= -G(\eta)B + T_{i\zeta\cdot\xi + \operatorname{div}\zeta} B + \widetilde{\gamma} \\
&= -T_\lambda B - R(\eta)B + T_{i\zeta\cdot\xi + \operatorname{div}\zeta} B + \widetilde{\gamma} \\
&= T_q B - R(\eta)B + T_{\operatorname{div}\zeta} B + \widetilde{\gamma}
\end{aligned}$$

where, by notation,

$$q := -\lambda + i\zeta \cdot \xi.$$

Now write

$$T_q B = \operatorname{div} U - T_{\operatorname{div}\zeta} B + R(\eta)B - \widetilde{\gamma}$$

and

$$B = T_{\frac{1}{q}} T_q B + \Big(I - T_{\frac{1}{q}} T_q \Big) B$$

to obtain

$$B = T_{\frac{1}{q}} \operatorname{div} U - T_{\frac{1}{q}} \widetilde{\gamma} + \mathcal{R}B \tag{D.13}$$

where

$$\mathcal{R} := T_{\frac{1}{q}} \Big(-T_{\operatorname{div}\zeta} + R(\eta) \Big) + \Big(I - T_{\frac{1}{q}} T_q \Big). \tag{D.14}$$

We now claim that $\mathcal{R}$ is of order $-1/4$ together with the following estimate: for any $1/2 \le \sigma \le s$, we have

$$\left\| \mathcal{R}B \right\|_{H^{\sigma+\frac{1}{4}}} \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac{1}{2}}} \big) \left\| B \right\|_{H^\sigma}. \tag{D.15}$$

Fix $\sigma \in [1/2, s]$. We begin by estimating $R(\eta)$. According to Proposition D.4 (applied with $\mu = s + \frac{1}{2}$ and $\varepsilon = \frac{1}{4}$), we have

$$\left\| R(\eta)B \right\|_{H^{\sigma-\frac{3}{4}}} \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac{1}{2}}} \big) \left\| B \right\|_{H^\sigma}.$$

On the other hand, since $\operatorname{div}\zeta = \Delta\eta$, the rule (A.12) implies that

$$\left\| T_{\operatorname{div}\zeta} \right\|_{H^\sigma \to H^{\sigma-\frac{3}{4}}} \lesssim \left\| \operatorname{div}\zeta \right\|_{C_*^{-\frac{3}{4}}} \lesssim \left\| \eta \right\|_{H^{s+\frac{1}{2}}}.$$

Finally, $q = -\lambda + i\zeta\cdot\xi \in \Gamma^1_{1/4}$ with $M^1_{1/4}(q) \le C(\left\| \eta \right\|_{H^{s+\frac{1}{2}}})$ since $s > \frac{3}{4} + \frac{d}{2}$. Moreover, q^{-1} is of order -1 and we have

$$M^{-1}_{1/4}(q^{-1}) \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac{1}{2}}} \big).$$

Consequently, according to (A.4) and (A.5), we have

$$\Big\| T_{\frac{1}{q}} \Big\|_{H^{\sigma-\frac{3}{4}} \to H^{\sigma+\frac{1}{4}}} + \Big\| I - T_{\frac{1}{q}} T_q \Big\|_{H^\sigma \to H^{\sigma+\frac{1}{4}}} \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac{1}{2}}} \big). \tag{D.16}$$

By combining the previous bound, we obtain the desired estimate (D.15). Now, (D.15) applied with $\sigma = s - 1/2$ implies the following estimate for the last term in the right-hand side of (D.13),

$$\left\| \mathcal{R}B \right\|_{H^{s-\frac{1}{4}}} \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac{1}{2}}} \big) \left\| B \right\|_{H^{s-\frac{1}{2}}}.$$

To estimate the two other terms in the right-hand side of (D.13), we use the operator norm estimate (D.16) for $T_{1/q}$, to obtain

$$\Big\| T_{\frac{1}{q}} \operatorname{div} U - T_{\frac{1}{q}} \widetilde{\gamma} \Big\|_{H^{s-\frac{1}{4}}} \le \mathcal{F}\big(\left\| \eta \right\|_{H^{s+\frac{1}{2}}} \big) \big\{ \left\| U \right\|_{H^{s-\frac{1}{4}}} + \left\| \widetilde{\gamma} \right\|_{H^{s-\frac{5}{4}}} \big\}.$$

By combining the two previous estimates, it follows from (D.13) that

$$\|B\|_{H^{s-\frac14}} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big)\big\{ \|U\|_{H^{s-\frac14}} + \|\widetilde{\gamma}\|_{H^{s-\frac54}} + \|B\|_{H^{s-\frac12}} \big\}. \tag{D.17}$$

Taking the L^∞-norm in time, one obtains the claim (D.11) from the previous estimates, see (D.9), (D.12) and (D.6).

Step 3: Bootstrap. We now use the previous bound (D.11) for B to improve the estimate (D.9) for U, namely to prove that

$$\|U\|_{L^\infty([0,T];H^s)} \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T)+Z_r(T)\big)\big). \tag{D.18}$$

Firstly, writing $\langle D_x\rangle^s U = U_s + \big[\langle D_x\rangle^s, T_\zeta\big]B$ and using (D.10), one has

$$\|U\|_{H^s} \lesssim \|U_s\|_{L^2} + \|\eta\|_{H^{s+\frac12}} \|B\|_{H^{s-\frac14}}.$$

As above, the three terms in the right-hand side of the above inequalities have been already estimated; indeed, Lemma D.2 gives an estimate for the $L^\infty_t(L^2_x)$-norm of U_s, η is estimated by means of Lemma D.3 and we can now use (D.13) to estimate $\|B\|_{H^{s-\frac14}}$. This proves (D.18).

We next use (D.18) to improve the estimate (D.13) for B. Firstly, by using the estimate (D.15) with $\sigma = s-1/4$ instead of $s-1/2$, we obtain as above

$$\|B\|_{H^s} \le \mathcal{F}\big(\|\eta\|_{H^{s+\frac12}}\big)\big\{ \|U\|_{H^s} + \|\widetilde{\gamma}\|_{H^{s-\frac12}} + \|B\|_{H^{s-\frac14}} \big\}. \tag{D.19}$$

Taking the L^∞-norm in time, it follows from the previous estimates (see (D.18), (D.12) and (D.11)) that

$$\|B\|_{L^\infty([0,T];H^s)} \le \mathcal{F}\big(\mathcal{F}(M_{s,0}) + T\mathcal{F}\big(M_s(T)+Z_r(T)\big)\big). \tag{D.20}$$

Step 4: Estimate for V. Writing $V = U - T_\zeta B$, it easily follows from (D.18) and (D.20) that $\|V\|_{L^\infty([0,T];H^s)}$ is bounded by the right-hand side of (D.8).

This completes the proof of the lemma. □

It remains to estimate the $L^\infty([0,T];H^{s+\frac12})$-norm of ψ. This estimate is obtained in two steps. Firstly, since $\nabla\psi = V + B\nabla\eta$ and since the $L^\infty([0,T];H^{s-\frac12})$-norm of $(\nabla\eta, V, B)$ has been previously estimated, we obtain the desired estimate for the $L^\infty([0,T];H^{s-\frac12})$-norm of $\nabla\psi$. It remains to estimate $\|\psi\|_{L^\infty([0,T];L^2)}$. This in turn follows from the identity

$$\partial_t\psi + V\cdot\nabla\psi = -g\eta + \frac12 V^2 + \frac12 B^2$$

and classical L^2 estimate for hyperbolic equations (see (A.26)).

Appendix E

Proof of a technical result

Here we prove Lemma 2.37 using an inequality proved in [6]. Let

$$I(\lambda) = \int_{\mathbf{R}^d} e^{i\lambda\Phi(\xi)} b(\xi)\, d\xi$$

where $\Phi \in C^\infty(\mathbf{R}^d)$ is a real phase, $b \in C^\infty(\mathbf{R}^d)$ is a symbol with compact support. We shall set $K = \operatorname{supp} b$ and let V be a small open neighborhood of K. We shall assume that

$$\begin{aligned} &(i) \quad \mathcal{M}_k := \sum_{2\le|\alpha|\le k} \sup_{\xi\in V} |D_\xi^\alpha \Phi(\xi)| < +\infty, \quad 2 \le k \le d+2, \\ &(ii) \quad \mathcal{N}_l := \sum_{|\alpha|\le l} \sup_{\xi\in K} |D_\xi^\alpha b(\xi)| < +\infty, \quad l \le d+1, \\ &(iii) \quad |\det \text{ Hess } \Phi(\xi)| \ge a_0 > 0, \forall \xi \in V, \end{aligned} \tag{E.1}$$

where Hess Φ denotes the Hessian matrix of Φ. In [6], it is proved that, for all (Φ, b) satisfying the above assumptions (see [6] for another technical assumption which is easily checked for our purpose), there exists a constant C such that, for all $\lambda \ge 1$,

$$|I(\lambda)| \le C \frac{\max\left\{1, \mathcal{M}_{d+2}(\Phi)^{\frac{d}{2}+d^2}\right\}}{a_0^{1+d}} \mathcal{N}_{d+1} \lambda^{-\frac{d}{2}}. \tag{E.2}$$

We begin by recalling the notations. First of all

$$K(t,z,y,\widetilde{h}) = (2\pi\widetilde{h})^{-d} \int e^{i\widetilde{h}^{-1}(\phi(t,z,\xi,\widetilde{h}) - y\cdot\xi)} \widetilde{b}(t,z,y,\xi,\widetilde{h}) \chi_1(\xi)\, d\xi,$$

where $0 < t \le \widetilde{h}^\delta, (\delta = \frac{2}{3}), \chi_1 \in C_0^\infty(\mathbf{R}^d)$ $\widetilde{b}$ is given by (2.83). On the other hand (see (2.101))

$$\begin{aligned}\phi(t,z,\xi,\widetilde{h}) &= z\cdot\xi - \int_0^t p\big(\sigma,z,\zeta(\sigma;\kappa(\sigma;z,\xi,\widetilde{h}),\xi,\widetilde{h}),\tilde{h}\big)\,d\sigma,\\ &= z\cdot\xi - t\int_0^1 p\big(ts,z,\zeta(ts;\kappa(ts;z,\xi,\widetilde{h}),\xi,\widetilde{h}),\tilde{h}\big)\,ds,\\ &=: z\cdot\xi - t\,\theta(t,z,\xi,\widetilde{h}),\end{aligned}$$

and (see (2.105))

$$\Big|\det\Big(\frac{\partial^2\theta}{\partial\xi_i\partial\xi_j}(t,z,\xi,\widetilde{h})\Big)\Big| \geq M_0 > 0 \tag{E.3}$$

for all $0 < t \leq \widetilde{h}^{\delta}, z \in \mathbf{R}^d, \xi \in \operatorname{supp}\chi_1, 0 < \widetilde{h} \leq \widetilde{h}_0$. Recall that we want to prove that

$$\big|K(t,z,y,\widetilde{h})\big| \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{-\frac{d}{2}}t^{-\frac{d}{2}} \tag{E.4}$$

for all $0 < t \leq \widetilde{h}^{\delta}$, $z, y \in \mathbf{R}^d$ and $0 < \widetilde{h} \leq \widetilde{h}_0$.

Case 1. If $0 < t \leq \widetilde{h}$ then the estimate (E.4) follows imediatly from the fact that $\widetilde{h}^{-d} \leq \widetilde{h}^{-\frac{d}{2}}t^{-\frac{d}{2}}$.

Case 2. Let $\widetilde{h} \leq t \leq \widetilde{h}^{\delta}, (\delta = \frac{2}{3})$. Set $\lambda = \frac{t}{\widetilde{h}} \in [1, \widetilde{h}^{-\frac{1}{3}}]$ and let $Z = \frac{z}{t}, Y = \frac{y}{t}$. Then our kernel can be written as

$$K(t,z,y,\widetilde{h}) = (2\pi\widetilde{h})^{-d}\int e^{i\lambda\Phi(t,Z,Y,\xi,\widetilde{h})}\widetilde{b}(t,tZ,tY,\xi,\widetilde{h})\chi_1(\xi)\,d\xi$$

where

$$\Phi(t,Z,Y,\xi,\widetilde{h}) = (Z-Y)\cdot\xi - \theta(t,tZ,\xi,\widetilde{h}).$$

Now Corollary 2.28, Proposition 2.30 and (E.3) allow us to apply Theorem 1 in [6] where $(t,Z,Y,\widetilde{h})$ are considered as parameters. We obtain

$$|K(t,z,y,\widetilde{h})| \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)\big)\widetilde{h}^{-\frac{d}{2}}\widetilde{h}^{-d}\lambda^{-\frac{d}{2}}$$

which proves (E.4).

Bibliography

[1] Thomas Alazard, Nicolas Burq and Claude Zuily. On the water-wave equations with surface tension. *Duke Math. J.*, 158(3):413–499, 2011.

[2] Thomas Alazard, Nicolas Burq and Claude Zuily. Strichartz estimates for water waves. *Ann. Sci. Éc. Norm. Supér. (4)*, t.44, 855–903, 2011.

[3] Thomas Alazard, Nicolas Burq and Claude Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, 198, 71-163, 2014.

[4] Thomas Alazard, Nicolas Burq and Claude Zuily. The water-waves equations: from Zakharov to Euler. Studies in Phase Space Analysis with Applications to PDEs. Progress in Nonlinear Differential Equations and Their Applications Volume 84, 2013, pp 1-20.

[5] Thomas Alazard, Nicolas Burq and Claude Zuily. Cauchy theory for the gravity water waves system with non localized initial data. arXiv:1305.0457.

[6] Thomas Alazard, Nicolas Burq and Claude Zuily. A stationary phase type estimate. Preprint 2015.

[7] Thomas Alazard and Jean-Marc Delort. Global solutions and asymptotic behavior for two dimensional gravity water waves. *Ann. Sci. Norm. Supér.*, to appear.

[8] Thomas Alazard and Guy Métivier. Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves. *Comm. Partial Differential Equations*, 34(10-12):1632–1704, 2009.

[9] Serge Alinhac. Paracomposition et opérateurs paradifférentiels. *Comm. Partial Differential Equations*, 11(1):87–121, 1986.

[10] Serge Alinhac. Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Comm. Partial Differential Equations*, 14(2):173–230, 1989.

[11] Hajer Bahouri and Jean-Yves Chemin. Équations d'ondes quasilinéaires et estimations de Strichartz. *Amer. J. Math.*, 121(6):1337–1377, 1999.

[12] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[13] M.S Berger, Non linearity and functionnal analysis, *Academic Press New York*, 1977.

[14] T. Brooke Benjamin and Peter J. Olver. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.*, 125:137–185, 1982.

[15] Matthew Blair. Strichartz estimates for wave equations with coefficients of Sobolev regularity. *Comm. Partial Differential Equations*, 31(4-6):649–688, 2006.

[16] Jean-Michel Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2):209–246, 1981.

[17] Nicolas Burq, Patrick Gérard, and Nikolay Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.*, 126(3):569–605, 2004.

[18] Angel Castro and David Lannes. Well-posedness and shallow-water stability for a new Hamiltonian formulation of the water waves equations with vorticity. arXiv:1402.0464.

[19] Rémi Carles. Geometric optics and instability for semi-classical Schrödinger equations. *Arch. Ration. Mech. Anal.*, 183(3):525–553, 2007.

[20] Michael Christ, James Colliander, et Terence Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Amer. J. Math.*, 125(6):1235–1293, 2003.

[21] Robin Ming Chen, Jeremy L. Marzuola, Daniel Spirn, and J. Douglas Wright. On the regularity of the flow map for the gravity-capillary equations. *J. Funct. Anal.*, 264(3):752–782, 2013.

[22] Hans Christianson, Vera Mikyoung Hur, and Gigliola Staffilani. Strichartz estimates for the water-wave problem with surface tension. *Comm. Partial Differential Equations*, 35(12):2195–2252, 2010.

[23] Demetrios Christodoulou and Hans Lindblad. On the motion of the free surface of a liquid. *Comm. Pure Appl. Math.*, 53(12):1536–1602, 2000.

[24] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo. Interface evolution: water waves in 2-D. Adv. Math., 223(1):120173, 2010.

[25] Daniel Coutand and Steve Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.*, 20(3):829–930 (electronic), 2007.

[26] Walter Craig. An existence theory for water waves and the Boussinesq and Korteweg-deVries scaling limits. *Communications in Partial Differential Equations*, 10(8):787–1003, 1985.

[27] Walter Craig, Ulrich Schanz, and Catherine Sulem. The modulational regime of three-dimensional water waves and the Davey-Stewartson system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(5):615–667, 1997.

[28] Walter Craig and Catherine Sulem. Numerical simulation of gravity waves. *J. Comput. Phys.* 108(1):7383, 1993.

[29] Björn E.J. Dahlberg and Carlos E. Kenig, Harmonic Analysis and PDE's, 1985–1996, http://www.math.chalmers.se/Math/Research/GeometryAnalysis/Lecturenotes/

[30] Thibault de Poyferré Blow-up conditions for gravity water-waves. arXiv:1407.6881.

[31] Thibault de Poyferré, Quang Huy Nguyen Strichartz estimates and local existence for the capillary water waves with non-Lipschitz initial velocity. *J. Diff. Eq.*, 261(1), 396–438, 2016.

[32] Thibault de Poyferré, Quang Huy Nguyen A paradifferential reduction for the gravity-capillary waves system at low regularity and applications. arXiv:1508.00326.

[33] David, G. Ebin, The equations of motion of a perfect fluid with free boundary are not well posed, *Communications in Partial Differential Equations* 10 (12): 1175–1201,1987.

[34] Pierre Germain, Nader Masmoudi and Jalal Shatah. Global solutions for the gravity water waves equation in dimension 3. *Annals of Mathematics* (2) 175, no. 2, 691-754, 2012.

[35] Jean Ginibre and Giorgio Velo. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.* 133:5068, 1995.

[36] Bei Hu and David P. Nicholls. Analyticity of Dirichlet-Neumann operators on Hlder and Lipschitz domains. *SIAM J. Math. Anal.*, 37(1):302-320, 2005.

[37] John Hunter, Mihaela Ifrim, Daniel Tataru. Two dimensional water waves in holomorphic coordinates. arXiv:1401.1252.

[38] Lars Hörmander. *Lectures on linear hyperbolic differential equations*, *Mathématiques & Applications (Berlin) [Mathematics & Applications]*, volume 26. Springer-Verlag, Berlin, 1997.

[39] Alexandru Ionescu and Fabio Pusateri. Global solutions for the gravity water waves system in $2d$. *Invent. Math.*, 199 (3), 653-804, 2015.

[40] Markus Keel and Terence Tao. Endpoint Strichartz estimates. Amer. J. Math. 120(5):955980, 1998.

[41] Sergiu Klainerman, Igor Rodnianski, Jeremie Szeftel. Overview of the proof of the Bounded L^2 Curvature Conjecture. arXiv:1204.1772.

[42] David Lannes. A stability criterion for two-fluid interfaces and applications. *Arch. Ration. Mech. Anal.*, 208(2):481–567, 2013.

[43] David Lannes. Well-posedness of the water-waves equations. *J. Amer. Math. Soc.*, 18(3):605–654 (electronic), 2005.

[44] Gilles Lebeau. Singularités des solutions d'équations d'ondes semi-linéaires. *Ann. Sci. École Norm. Sup. (4)*, 25(2):201–231, 1992.

[45] Gilles Lebeau. Régularité du problème de Kelvin-Helmholtz pour l'équation d'Euler 2d. *ESAIM Control Optim. Calc. Var.*, 8:801–825 (electronic), 2002. A tribute to J. L. Lions.

[46] Gilles Lebeau. Contrôle de l'équation de Schrödinger. *J. Math. Pures Appl.*, (9) 71 (1992), no. 3, 267291.

[47] Hans Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. of Math. (2)*, 162(1):109–194, 2005.

[48] Nader Masmoudi and Frédéric Rousset. Uniform regularity and vanishing viscosity limit for the free surface Navier-Stokes equations. arXiv:1202.0657.

[49] Guy Métivier. *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, volume 5 of *Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series*. Edizioni della Normale, Pisa, 2008.

[50] Camil Muscalu and Wilhem Schlag. *Classical and Multilinear Harmonic Analysis* Vol 1, *Cambridge Studies in Advanced Mathematics.*

[51] Yves Meyer. Remarques sur un théorème de J.-M. Bony. In *Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980)*, number suppl. 1, pages 1–20, 1981.

[52] V. I. Nalimov. The Cauchy-Poisson problem. *Dinamika Splošn. Sredy*, (Vyp. 18 Dinamika Zidkost. so Svobod. Granicami):104–210, 254, 1974.

[53] Quang Huy Nguyen. Work in preparation.

[54] Luc Robbiano and Claude Zuily. Strichartz estimates for Schrödinger equations with variable coefficients. *Mém. Soc. Math. Fr. (N.S.)*, (101-102):vi+208, 2005.

[55] Mikhail V. Safonov. Boundary estimates for positive solutions to second order elliptic equations. arXiv:0810.0522

[56] Jalal Shatah and Chongchun Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. *Comm. Pure Appl. Math.*, 61(5):698–744, 2008.

[57] Hart F. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[58] Gigliola Staffilani and Daniel Tataru. Strichartz estimates for a Schrödinger operator with nonsmooth coefficients. *Comm. Partial Differential Equations*, 27(7-8):1337–1372, 2002.

[59] Daniel Tataru. Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation. *Amer. J. Math.*, 122(2):349–376, 2000.

[60] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. II. *Amer. J. Math.*, 123(3):385–423, 2001.

[61] Michael E. Taylor. *pseudo-differential operators and nonlinear PDE*, volume 100 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1991.

[62] Monique Sablé-Tougeron. Régularité microlocale pour des problèmes aux limites non linéaires. *Ann. Inst. Fourier (Grenoble)*, 36(1):39–82, 1986.

[63] Chao Wang and Zhifei Zhang. Break-down criterion for the water-wave equation. arXiv:1303.6029.

[64] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997.

[65] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12(2):445–495, 1999.

[66] Sijue Wu. Almost global wellposedness of the $2-D$ full water wave problem . *Invent. Math.*, 177: 45–135, 2009.

[67] Hideaki Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.*, 18(1):49–96, 1982.

[68] Vladimir E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194, 1968.

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