Probabilistic Sobolev embeddings, applications to eigenfunctions estimates

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Abstract. The purpose of this article is to present some probabilistic versions of Sobolev embeddings and an application to the growth rate of the L^p norms of Spherical harmonics on spheres. More precisely, we prove that (for natural probability measures), almost every Hilbert base of $L^2(\mathbb{S}^d)$ made of spherical harmonics has all its elements uniformly bounded in any L^p space ($p < +\infty$). We also show that most of the analysis extends to the case of Riemanian manifolds with groups of isometries acting transitively.

1. Introduction

Consider (M, g) a compact Riemanian manifold of dimension d and $\mathbf{\Delta}$ the Laplace operator on (M, g). Since the works by Hörmander **[10]** and Sogge **[14]**, it is known that the eigenfunctions satisfy

Theorem. *For any $2 \leq p \leq +\infty$, there exists $C > 0$ such that for any eigenfunctions of the Laplace operator u, $-\mathbf{\Delta}_g u = \lambda^2 u$, one has*

$$\|u\|_{L^p(M)} \leq C\lambda^{\delta(p)}\|u\|_{L^2} \tag{1.1}$$

with

$$\delta(p) = \begin{cases} \frac{(d-1)}{2} - \frac{d}{p} \text{ if } p \geq \frac{2(d+1)}{d-1} \\ \frac{(d-1)}{2}\left(\frac{1}{2} - \frac{1}{p}\right) \text{ if } p \leq \frac{2(d+1)}{d-1} \end{cases} \tag{1.2}$$

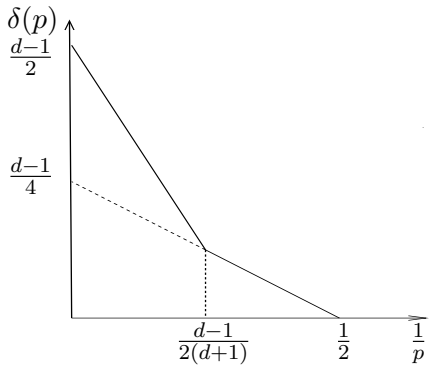

FIGURE 1.1. Eigenfunction estimates

These estimates are also known to be optimal on the spheres (endowed with their standard metric). In the first regime, the so called zonal spherical harmonics (which concentrate on two opposite points on a diameter of the sphere) are optimizing (1.1) while in the second regime, the highest weight spherical harmonics ($u(x_1, \cdots x_{d+1}) = (x_1 + ix_2)^n$) which concentrate on the equator $(x_3, \cdot, x_{d+1}) = 0$ are saturating (1.1). On the other hand, the case of the tori shows that there exists manifolds on which these estimates can be improved. Indeed, on tori, we have by simple number theory arguments

$$\delta(p) \leq \frac{d-2}{2}\Big(1 - \frac{2}{p}\Big) + \epsilon \tag{1.3}$$

(for $d \geq 5$, the ϵ can be dropped). On the other hand, it is standard and can easily be shown (see e.g. Section 4.1 and the works by Bourgain **[2, 3]** and Bourgain-Rudnick **[4]** for many more results on eigenfunctions on tori) that on tori

$$\delta(p) \geq \frac{d-2}{2} - \frac{d}{p}, \qquad p \geq \frac{2d}{d-2} \tag{1.4}$$

and consequently, as soon as $d \geq 3$, even on the torus, there exists sequences of L^2-normalized eigenfunctions having unbounded L^p norms. On the other hand, of course, the usual eigenbasis of eigenfunctions

$$e_n(x) = e^{in\cdot x}, n \in \mathbb{Z}^d$$

has all its L^p norms bounded and consequently a natural question is to ask whether this phenomenon is unique or there exists other manifolds exhibiting the same kind of behaviour (existence of both sequences having unbounded L^p norms and sequences having bounded L^p norms). Our result is that it is true on all spheres $\mathbb{S}^d$ endowed with their standard metric (for any $p < +\infty$), despite the optimality of (1.1). We actually prove a stronger result. Let us recall that the eigenfunctions of the Laplace operator on the spheres $\mathbb{S}^d$ (with its standard metric) are the spherical harmonics of degree k (restrictions to the sphere $\mathbb{S}^d$ of the harmonic polynomials of degree k) and they satisfy

- For any $k \in \mathbb{N}$, the vector space of spherical harmonics of degree k, E_k, has dimension

$$N_k = \binom{k+d}{d} - \binom{k+d-2}{d} \sim_{k\to+\infty} \frac{2e}{(d-1)!}\Big(\frac{k}{e}\Big)^{d-1} \tag{1.5}$$

- For any $e \in E_k$, $-\mathbf{\Delta}_{\mathbb{S}^d} e = -k(k+d-1)e$.
- The vector space spanned by the spherical harmonics is dense in $L^2(\mathbb{S}^d)$

As a consequence, any family $B = (B_k)_{k\in\mathbb{N}} = ((b_{k,l})_{l=1,\dots,N_k})_{k\in\mathbb{N}}$ of orthonormal bases of (E_{N_k}) is a Hilbert base of $L^2(\mathbb{S}^d)$. Our result is the following (with $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$ the classical Gamma function).

THEOREM 1. *For any $d \geq 2$, consider the space of spherical harmonics of degree k, E_k endowed with the $L^2(\mathbb{S}^d)$ norm and $\mathbb{S}_k$ its unit sphere endowed with the uniform probability measure P_k. Consider the space of Hilbert bases of $L^2(\mathbb{S}^d)$ having all elements harmonic polynomials, with real coefficients, endowed with its natural probability measure, ν (see Section 3 for a precise definition). Then for any $2 \leq q < +\infty$, we have*

$$\mathcal{A}^q_{q,k} = \mathbb{E}(\|u\|^q_{L^q(\mathbb{S}^d)}) = \int_{u\in\mathbb{S}_k} \|u\|^q_{L^q(\mathbb{S}^d)} dP_k = \frac{N_k^{q/2}}{Vol(\mathbb{S}^d)^{\frac{q}{2}-1}} \Big(\frac{\Gamma\big(\frac{q+1}{2}\big)\Gamma\big(\frac{N_k}{2}\big)}{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{N_k+q}{2}\big)} \Big), \tag{1.6}$$

$$\mathcal{A}_{q,k} = \frac{e^{-e}}{Vol(\mathbb{S}^d)^{\frac{1}{2}}} \sqrt{q} \Big(1 + \mathcal{O}(\frac{q}{N_k}) + \mathcal{O}(\frac{1}{q}) \Big)$$

and there exists $\mathcal{M}_{q,k} > 0$ such that

$$\Big| \mathcal{M}_{q,k} - \mathcal{A}_{q,k} \Big| \leq \begin{cases} C \frac{\sqrt{q}}{N_k^{\frac{d}{(d-1)q}}} & \text{if } \frac{2(d+1)}{d-1} \leq q < +\infty \\ C \frac{\sqrt{q}}{N_k^{\frac{1}{4}+\frac{1}{2q}}} & \text{if } 2 \leq q \leq \frac{2(d+1)}{d-1} \end{cases} \tag{1.7}$$

$$P_k(u \in S_k; \Big| \|u\|_{L^q} - \mathcal{M}_{q,k} \Big| > \Lambda) \leq \begin{cases} 2e^{-N_k^{\frac{2d}{(d-1)q}} r^2} & \text{if } \frac{2(d+1)}{d-1} \leq q < +\infty \\ 2e^{-N_k^{\frac{1}{2}+\frac{1}{q}} r^2} & \text{if } 2 \leq q \leq \frac{2(d+1)}{d-1} \end{cases} \tag{1.8}$$

and

$$\nu(\{B = (b_{k,l})_{k\in\mathbb{N}, l=1,\dots N_k} \in \mathcal{B}; \exists k,l; \Big| \|b_{k,l}\|_{L^q(\mathbb{S}^d)} - \mathcal{M}_{q,k} \Big| > r\}) \leq Ce^{-cr^2} \tag{1.9}$$

Furthermore, there exists $C, c, c_0 > 0$ such that

$$\nu(\{B = (b_{k,l})_{k\in\mathbb{N}, l=1,\dots N_k} \in \mathcal{B}; \exists k,l; \|b_{k,l}\|_{L^\infty(\mathbb{S}^d)} > (c_0 + r)\sqrt{\log(k)}\}) \leq Ce^{-cr^2} \tag{1.10}$$

REMARK 1.1. *VanderKam* [**15**] *(see also Zelditch* [**16**]*) proves an estimate for the L^∞-norm of spherical harmonics on $\mathbb{S}^2$ which involves a logarithmic loss ($\log^2(k)$), in the same context. Though the results are in spirit very close, the methods of proof appear to differ significantly: VanderKam's approach (which do not appear to be flexible enough to tackle the case $p < +\infty$, or to give lower bounds) is based upon the precise descriptions of some particular spherical harmonics, implied by Ramanujan's conjectures proved by Deligne* [**7**]*. As explained above, our proof is very general (see Section 4) and the only sphere-specific fact we use is the exact spectral clustering*

$$\sigma(-\mathbf{\Delta}) = \{k(k+d-1), k \in \mathbb{N}^*\}.$$

REMARK 1.2. *We actually prove a stronger result. Indeed, when proving the first part of Theorem 1 (equations* (1.6), (1.7), (1.8)*), modulo a small change in the estimates, we only use the fact that the group of isometries on the manifold M acts transitively, whereas for* (1.9) *and* (1.10)*, we use only the fact that the dimension of the eigenspaces is bounded from below by a positive power of the eigenvalue (see Theorem 3 and Remark 3.2)*

Let us end this introduction by mentioning that most of the work presented here is extracted from more general results on probabilistic Sobolev embeddings and applications to the study of the behaviour of solutions to Partial Differential Equations [**5**].

2. Probabilistic estimates

2.1. The spectral function. In this section, we consider a compact Riemanian manifold (M, g). We assume that the group of isometries of M acts transitively. Denote by $E_\lambda = \mathrm{Ker}(-\mathbf{\Delta}_g - \lambda^2 \mathrm{Id}), \lambda \in \sigma(\sqrt{-\mathbf{\Delta}})$, an eigenspace of the Laplace operator of dimension N_λ. Let $(e_{j,\lambda})_{j=1}^{N_\lambda}$ be an orthonormal basis of the space E_λ, and denote by $F_\lambda = (e_{j,\lambda})_{j=1}^{N_\lambda}$, and

$$f_\lambda(x) = \big(\sum_{j=1}^{N_\lambda} e_{j,\lambda}^2(x)\big)^{1/2}.$$

We can identify a point $U = (u_j)_{j=1}^{N_\lambda} \in \mathbb{S}^{N_\lambda - 1}$ on the unit sphere in $\mathbb{R}^{N_\lambda}$ with the function

$$u(x) = \sum_{j=1}^{N_\lambda} u_j e_{j,\lambda}(x) = U \cdot F_\lambda(x) \in \mathbb{S}_k, \tag{2.1}$$

the sphere of functions in E_λ having $L^2(M)$-norm equal to 1.

LEMMA 2.1. *Under the assumptions above, we have*

$$f_\lambda^2(x) = \sum_{j=1}^{N_\lambda} e_{j,\lambda}^2(x) = \frac{N_\lambda}{Vol(\mathbb{S}^d)}.$$

PROOF. We notice that

$$K_\lambda(x, y) = \sum_{j=1}^{N_\lambda} e_{j,\lambda}(x) e_{j,\lambda}(y)$$

is the kernel of the orthogonal projector on the space E_λ which is invariant by the action of any isometry R. As a consequence, we deduce $K_\lambda(x, R^{-1}y) = K_\lambda(Rx, y)$, which implies that $f_\lambda^2(x)$ is invariant by the isometries of the manifold M, hence constant on M, with integral N_λ. □

LEMMA 2.2. *Under the same assumptions as in Lemma 2.1, there exists $C > 0$ such that for any $\lambda \in \sigma(-\mathbf{\Delta})$, for any $u \in E_\lambda$ we have*

$$\|u\|_{L^\infty(M)} \leq \Big(\frac{N_\lambda}{Vol(M)}\Big)^{\frac{1}{2}} \|u\|_{L^2(M)} \tag{2.2}$$

PROOF. Let us denote by T the orthogonal projector on E_k. By the usual TT^* argument the norm of T seen as an operator from $L^2(M)$ to $L^\infty(M)$ is equal to the square root of the norm of $TT^* = T$ seen as an operator from $L^1(M)$ to $L^\infty(M)$. This norm is equal to the $L^\infty(M \times M)$-norm of its kernel $K(x, y)$. By Cauchy-Schwartz inequality, we have

$$|K_\lambda(x,y)| \leq \big(K_\lambda(x,x)K_\lambda(y,y)\big)^{1/2} = \frac{N_\lambda}{\mathrm{Vol}(M)} \tag{2.3}$$
$$\Rightarrow \|K_\lambda\|_{L^\infty(M\times M)} = \|f_\lambda^2(x)\|_{L^\infty(M)} = \frac{N_\lambda}{\mathrm{Vol}(M)}$$

which proves (2.2). Notice that actually the proof also gives that there exists a (non trivial) function $u \in E_\lambda$ for which

$$\|u\|_{L^\infty(M)} = \Big(\frac{N_\lambda}{\mathrm{Vol}(M)}\Big)^{\frac{1}{2}} \|u\|_{L^2(M)}.$$

□

2.2. Almost sure L^q estimates. In this section, we prove the main estimate on the L^q norms. The proof is very much inspired from Shiffman-Zelditch [**13**]. The main step in our proof is the proof of the estimates (1.8).

According to Theorem 1, with $F_q(u) = \|u\|_{L^q(\mathbb{S}^d)}$,

$$|F_q(u) - F_q(v)| = |\|u\|_{L^q} - \|v\|_{L^q}| \leq \|u - v\|_{L^q} \tag{2.4}$$
$$\leq C N_k^{\frac{1}{2}-\gamma(q))} \|u - v\|_{L^2} \leq C N_k^{\frac{1}{2}-\gamma(q))} \mathrm{dist}(u, v),$$

with

$$\gamma(q) = \begin{cases} \frac{d}{(d-1)q} & \text{if } \frac{2(d+1}{d-1} \leq q < +\infty \\ \frac{1}{4} + \frac{1}{2q} & \text{if } 2 \leq q \leq \frac{2(d+1)}{d-1} \end{cases}$$

We deduce

$$\|F\|_{\mathrm{Lips}} \leq C N_k^{\frac{1}{2}-\gamma(q)}$$

and consequently, according to the concentration of measure property (see Proposition 5.2) applied to the function $F_q(u) = \|u\|_{L^q}$ on the N_k-1 dimensional sphere $\mathbb{S}_k$ we get

$$P_k(|F_q(u) - \mathcal{M}_{q,k}| > r) \leq 2e^{-cN_k^{2\gamma(q)} r^2} \tag{2.5}$$

where $\mathcal{M}_{q,k}$ is the median value of the function F on $\mathbb{S}_k$. This implies (1.8). To get (1.7), we need to estimate from below and above $\mathcal{M}_{q,k}$. According to (2.1), (5.1) and Lemma 2.1 (and the rotation invariance of the uniform measure on the sphere S_k), we have

$$P_k(|u(x)| > \lambda) = \frac{\Gamma\big(\frac{N_k}{2}\big)}{\Gamma\big(\frac{N_k-1}{2}\big)\Gamma\big(\frac{1}{2}\big)} \mathbf{1}_{\lambda \in \big(0, \frac{N_k^{1/2}}{\mathrm{Vol}(\mathbb{S}^d)^{1/2}}\big)} \int_0^\theta \sin^{N_k-2}(\varphi) d\varphi,$$

where the value of θ in the expression above is fied by the relation $\lambda = \frac{N_k^{1/2}}{\mathrm{Vol}(\mathbb{S}^2)^{1/2}} \cos(\theta)$.

Using

$$\mathbb{E}_k(|g|^q) = q \int_0^{+\infty} \lambda^{q-1} P_k(|g| > \lambda) d\lambda$$

we obtain

$$\mathcal{A}^q_{q,k} = \mathbb{E}_k(\|u\|^q_{L^q}) = \int_{S_k}\int_{\mathbb{S}^d} |u(x)|^q dx dp_k = q\int_{\mathbb{S}^d}\int_0^\infty \lambda^{q-1} P_k(|u(x)|>\lambda) d\lambda \tag{2.6}$$

$$= 2\frac{\Gamma\big(\frac{N_k}{2}\big)}{\Gamma\big(\frac{N_k-1}{2}\big)\Gamma\big(\frac12\big)}\Big(\frac{N_k}{\mathrm{Vol}(\mathbb{S}^d)}\Big)^{q/2} q\int_{\mathbb{S}^d}\int_0^{\frac\pi2}\cos(\theta)^{q-1}\sin(\theta)\int_0^\theta \sin^{N_k-2}(\varphi) d\varphi d\theta dx$$

$$= 2\frac{\Gamma\big(\frac{N_k}{2}\big)}{\Gamma\big(\frac{N_k-1}{2}\big)\Gamma\Big(\frac12\Big)}\big(\frac{N_k}{\mathrm{Vol}(\mathbb{S}^d)}\big)^{q/2}\mathrm{Vol}(\mathbb{S}^d)\int_0^{\frac\pi2}\sin^{N_k-2}(\varphi)\cos(\varphi)^q d\varphi$$

$$= \frac{\big(N_k\big)^{\frac q2}}{\big(\mathrm{Vol}(\mathbb{S}^d)\big)^{\frac q2-1}}\frac{\Gamma\big(\frac{N_k}{2}\big)\Gamma\big(\frac{q+1}{2}\big)}{\Gamma\big(\frac{N_k+q}{2}\big)\Gamma\Big(\frac12\Big)}.$$

We deduce

$$\mathcal{A}_{q,k} = \frac{\big(N_k\big)^{\frac12}}{\big(\mathrm{Vol}(\mathbb{S}^d)\big)^{\frac12-\frac1q}}\Big(\frac{\Gamma\big(\frac{N_k}{2}\big)\Gamma\big(\frac{q+1}{2}\big)}{\Gamma\big(\frac{N_k+q}{2}\big)\Gamma\Big(\frac12\Big)}\Big)^{\frac1q} = \frac{\sqrt{2e}}{\big(\mathrm{Vol}(\mathbb{S}^d)\big)^{\frac12-\frac1q}}\Big(\frac{\Gamma\big(\frac{q+1}{2}\big)}{\Gamma\Big(\frac12\Big)}\Big)^{\frac1q}e^{-e}\Big(1+\mathcal{O}\big(\frac{q}{N_k}\big)\Big) \tag{2.7}$$

$$= \frac{e^{-e}}{\big(\mathrm{Vol}(\mathbb{S}^d)\big)^{\frac12}}\sqrt{q}\Big(1+\mathcal{O}\big(\frac{q}{N_k}\big)+o(1)_{q\to+\infty}\Big)$$

We also have

$$\big|\mathcal{A}_{q,k}-\mathcal{M}_{q,k}\big|^q = \big|\|F\|_{L^q(\mathbb{S}_k)}-\|\mathcal{M}_{q,k}\|_{L^q(\mathbb{S}_k)}\big|^q \tag{2.8}$$

$$\le \|F-\mathcal{M}_{q,k}\|^q_{L^q(\mathbb{S}_k)} = q\int_0^{+\infty}\lambda^{q-1}P_k(|F-\mathcal{M}_{q,k}|>\lambda)d\lambda$$

$$\le q\int_0^{+\infty}\lambda^{q-1}e^{-c_1N_k^{2\gamma(q)}\lambda^2}d\lambda = \frac{q}{2N_k^{q\gamma(q)}c_1^{q/2}}\Gamma(q/2).$$

We deduce

$$\big|\mathcal{A}_{q,k}-\mathcal{M}_{q,k}\big| \le \frac{C}{N_k^{\gamma(q)}}\sqrt{q} \tag{2.9}$$

which, according to (2.7), implies (1.7).

We can now deduce an estimation of the L^∞ norm

THEOREM 2. *Denote by $\mathcal{M}_{\infty,k}$ the median value of the function $F_\infty(u)=\|u\|_{L^\infty}$ on the unit sphere S_k (endowed with its uniform probability measure). Then there exists $c_0, c_1 > 0$ for any $k\in\mathbb{N}^*$ we have*

$$\mathcal{M}_{\infty,k} \in [c_0\sqrt{\log(k)}, c_1\sqrt{\log(k)}] \tag{2.10}$$

and consequently according to (2.4) *and Proposition 5.2,*

$$\begin{aligned} P_k(u\in S_k;\Big|\|u\|_{L^\infty} > c_1\sqrt{\log(k)}+\Lambda) &\le 2e^{-c\Lambda^2},\\ P_k(u\in S_k;\Big|\|u\|_{L^\infty} < c_0\sqrt{\log(k)}-\Lambda) &\le 2e^{-c\Lambda^2}\end{aligned} \tag{2.11}$$

PROOF. According to Sobolev embeddings, there exists $C_0 > 0$ such that for any k, any $q > 2$, and any $u \in E_k$

$$\|u\|_q \leq \|u\|_{L^\infty} \leq C_0 k^{-d/q} \|u\|_q$$

and choosing $q_k = a\varepsilon_0 \log(k)$, we obtain with $C_1 = C_0 e^{d/a\varepsilon_0}$ independent of k

$$\mathbb{E}_k(\|u\|_{q_k}) \leq \mathbb{E}_k(\|u\|_{L^\infty}) \leq C_1 \mathbb{E}_k(\|u\|_{q_k}) \tag{2.12}$$

On the other hand for any $q \in [2, \infty]$, by Proposition 5.2 we have

$$\begin{aligned} |\mathbb{E}_k(\|u\|_q) - \mathcal{M}_{q,h}| &\leq \int |\|u\|_q - \mathcal{M}_{q,h}| dP = \int_0^\infty P_k(|\|u\|_q - \mathcal{M}_{q,h}| > \lambda) d\lambda \\ &\leq 2 \int_0^\infty e^{-c_1 \lambda^2} = \sqrt{\pi / c_1} \end{aligned} \tag{2.13}$$

Since $q_k \leq \varepsilon_0 \log(N_k)$, (1.7) imply $\mathcal{M}_{q_k,h} \simeq \sqrt{q_k}$. We deduce consequently from (2.13) $\mathbb{E}_k(\|u\|_{q_k}) \simeq \sqrt{q_k}$, and also according to (2.12), $\mathbb{E}_k(\|u\|_{L^\infty}) \simeq \sqrt{q_k}$, hence, again according to (2.13) $\mathcal{M}_{\infty,k} \simeq \sqrt{q_k}$. □

Let us now remark that in the proof above, we used only Lemma 2.1 and the L^p estimates on eigenfunctions given by Theorem 1. However, in the case of a riemanian manifold with a group of isometries acting transitively, we can use Lemma 2.2 and interpolation to obtain that there exists $C > 0$ such that for any $\lambda \in \sigma(\sqrt{-\mathbf{\Delta}})$ and any $u \in E_\lambda$,

$$\|u\|_{L^p(M)} \leq C N_\lambda^{\frac{1}{2} - \frac{1}{p}} \|u\|_{L^2(M)}$$

Plugging this estimate instead of Theorem 1 in the proof above gives

THEOREM 3. *Consider (M, g) a riemanian manifold on which the group of isometries act transitively. for any $\lambda \in \sigma(\sqrt{-\mathbf{\Delta}}$, denote by E_λ the eigenspace of the Laplace operator of dimension $N_\lambda > 0$, endowed with the $L^2(M)$ norm and $\mathbb{S}_\lambda$ its unit sphere endowed with the uniform probability measure P_λ. Then for any $2 \leq q < +\infty$, we have*

$$\mathcal{A}_{q,\lambda}^q = \mathbb{E}(\|u\|_{L^q(\mathbb{S}^d)}^q) = \int_{u \in \mathbb{S}_\lambda} \|u\|_{L^q(\mathbb{S}^d)}^q dP_\lambda = \frac{N_\lambda^{q/2}}{Vol(\mathbb{S}^d)^{\frac{q}{2}-1}} \Big(\frac{\Gamma\big(\frac{q+1}{2}\big)\Gamma\big(\frac{N_\lambda}{2}\big)}{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{N_\lambda+q}{2}\big)} \Big), \tag{2.14}$$

$$\mathcal{A}_{q,\lambda} = \frac{1}{Vol(\mathbb{S}^d)^{\frac{1}{2}}} \sqrt{\frac{q}{e}} \Big(1 + \mathcal{O}(\frac{q}{N_\lambda}) + \mathcal{O}(\frac{1}{q}) \Big)$$

and there exists $\mathcal{M}_{q,\lambda} > 0$ such that

$$\Big| \mathcal{M}_{q,\lambda} - \mathcal{A}_{q,\lambda} \Big| \leq C \frac{\sqrt{q}}{N_\lambda^{1/q}}, \tag{2.15}$$

$$P_\lambda(u \in S_\lambda; \Big| \|u\|_{L^q} - \mathcal{M}_{q,\lambda} \Big| > \Lambda) \leq 2 e^{-N_\lambda^{\frac{2}{q}} r^2} \tag{2.16}$$

.

3. Eigenbases

We come back to the case of the sphere and spherical harmonic. We can identify the space of orthonormal basis of E_k (endowed with the L^2-norm) with the orthogonal group, $O(N_k)$ and endow this space with its Haar measure, ν_k. Let us recall that

$$\oplus_{k\in\mathbb{N}} E_k \tag{3.1}$$

is dense in $L^2(\mathbb{S}^d)$. The set of Hilbert bases of L^2 compatible with the decomposition (3.1) is

$$\mathcal{B} = \times_{k\in\mathbb{N}} O(N_k))$$

and is endowed with the product probability measure

$$\nu = \otimes_k \nu_k.$$

Now, we can proceed to prove (1.9) and (1.10). According to (1.8), we get

PROPOSITION 3.1. *There exists $C, c > 0$ such that for any $k \in \mathbb{N}$,*

$$\nu_k(\{B = (b_l)_{l=1}^{N_k} \in SU(N_k); \exists 1 \leq l \leq N_k; \left| \|b_l\|_{L^q} - \mathcal{M}_{q,k} \right| > \Lambda) \leq c_0 e^{-c_1 k^{2(d-1)\gamma(q)}\Lambda^2} k^{d-1} \tag{3.2}$$

Indeed, the map

$$B = (b_l)_{l=1}^{N_k} \in SU(N_k) \mapsto b_1 \in S_k$$

sends the measure ν_k to the measure P_k and consequently, according to (1.8) for any $1 \leq l_0 \leq N_k$

$$\nu_k(\{B = (b_l)_{l=1}^{N_k} \in SU(N_k); \left| \|b_{l_0}\|_{L^q} - \mathcal{M}_{q,k} \right| > \Lambda) \leq c_0 e^{-c_1 N_k^{-\gamma(q)}\Lambda^2}.$$

We deduce

$$\nu_k(\{B = (b_l)_{l=1}^{N_k} \in SU(N_k); \exists l_0 \in \{1, \dots, N_k\}; \left| \|b_{l_0}\|_{L^q} - \mathcal{M}_{q,k} \right| > \Lambda) \leq c_0 e^{-c_1 k^{2(d-1)\gamma(q)q}\Lambda^2} k^{(d-1)}, \tag{3.3}$$

which implies (1.9). Let us define now

$$F_{k,r} = \{B = (b_k) \in SU(N_k); \forall l_0; \left| \|b_{k,l_0}\|_{L^q} - \mathcal{M}_{q,k} \right| \leq r\}$$

and

$$F_r = \cap_{k\geq 0} F_{r,k} = \{B = (b_{k,l}); k \in \mathbb{N}, l = 1, \cdots N_k; \forall k_0, l_0; \left| \|b_{k_0,l_0}\|_{L^q} - \mathcal{M}_{q,k} \right| \leq r\}$$

We have

$$\nu(F_r)^c \leq \sum_k \nu_k(F_{k,r}^c) \leq \sum_k c_0 e^{-c_1 k^{2(d-1)\gamma(q)} r^2} k^{d-1} \leq \sum_k c e^{-c' k^{2(d-1)\gamma(q)} r^2} \leq C e^{-c' r^2}. \tag{3.4}$$

This ends the proof of (1.9). To prove (1.10), it sufficies to follow the same strategy and use Theorem 2 instead of (1.8).

REMARK 3.2. *In the more general case of a riemanian manifold having a group of isometries acting transitively, the results above on Hilbert bases extend straightforwardly under the additional assumption that*

$$\liminf_{\lambda\in\sigma(\sqrt{-\Delta})} \log(N_\lambda) > 0.$$

4. Other manifolds

4.1. Tori. A natural class of examples for which the group of isometries acts transitively are the tori. As a consequence, as already mentionned, we deduce that Theorem 3 holds. Finally, Theorem 2 holds also with $\log(k)$ replaced by $\log(N_k)$, with the notable difference that the multiplicity N_k do not tend to $+\infty$ if $d \leq 4$. On the other hand, the proofs of (1.9) and (1.10) use the fact that the multiplicity tends to infinity at least polynomially, and it consequently holds on rational tori only if $d \geq 5$ (see Grosswald [**8**, Chapter 12])). For dimensions $d \leq 4$, we only obtain the weaker

$$\forall A > 0, \nu(\{B = (b_{k,l})_{k\in\mathbb{N},l=1,\cdots N_k} \in \mathcal{B}; \forall k, l; \|b_{k,l}\|_{L^\infty(\mathbb{S}^d)} \leq A\}) = 0. \tag{4.1}$$

Finally, to complete our probabilistic upper bounds, we can also get lower bounds on tori: For $n \in \mathbb{N}$ Let us denote by $E_{n,d}$ the eigenspace of the Laplace operator associated with the eigenvalue λ and $r_k(\lambda)$ the number of presentations of the integer n as a sum of d squares of integers:

$$r_d(n) = \#E_{\lambda,d}, \qquad E_{\lambda,d} = \{(n_1, \ldots, n_d) \in \mathbb{Z}^d; \sum_{j=1}^{d} n_j^2 = \lambda\}.$$

According to Weyl's formula,

$$\sum_{\frac{n}{2}<m\leq n} r_d(m) = Cn^{\frac{d}{2}} + O(n^{\frac{d-1}{2}})$$

From which we deduce that for any $d \geq 2$ there exists a sequence $\lambda_p \to +\infty$ such that

$$r_d(\lambda_p) \geq c\lambda_p^{\frac{d-2}{2}}$$

and by considering the following eigenfunction on $\mathbb{T}^d$ (having L^2-norm equal to 1),

$$u_p(x) = r_d(\lambda_p)^{-\frac{1}{2}} \sum_{N=(n_1,\ldots,n_d)\in E_{\lambda_p,d}} e^{iN\cdot x}$$

we have according to Sobolev embeddings

$$c\lambda_p^{\frac{d-2}{4}} \sim \|u_p\|_{L^\infty} \leq C\lambda_p^{\frac{k}{2r}} \|u_p\|_{L^r}$$

Hence

$$c\lambda_p^{\frac{d-2}{4}-\frac{k}{2r}} \leq \|u_p\|_{L^r}$$

which is (1.4).

4.2. Other manifolds. We can extend our results on arbitrary manifolds, to the price of considering only approximate eigenfunctions. Here the Weyl asymptotics are playing a crucial role. Let us consider $0 < a_h < 1 < b_h \leq c$ two functions defined for $(0, h_0)$ such that

$$\lim_{h\to 0} b_h = 1 = \lim_{h\to 0} a_h \tag{4.2}$$

We assume that

$$b_h - a_h \geq Mh \tag{4.3}$$

for a constant M to be precised later Let E_h be the subspace of $L^2(M)$

$$E_h = \{u = \sum_{k\in I_h} z_k e_k(x),\ z_k \in \mathbb{C}\}, \quad \widetilde{E}_h = \{u = \sum_{k\in I_h} z_k e_k(x),\ z_k \in \mathbb{R}\},$$
$$I_h = \{k \in \mathbb{N}; h\omega_k \in (a_h, b_h)\} \tag{4.4}$$

Let $N_h = dim(E_h)$. Let us recall that Weyl formula (with precise remainder) reads (see Hörmander [**10**])

$$N_h = (2\pi h)^{-d}\mathrm{Vol}(M)\mathrm{Vol}(S^{d-1})\int_{(a,b)} \rho^{d-1}d\rho + O(h^{-d+1}) \tag{4.5}$$

and consequently, we have

$$\exists C > 0; \forall \lambda > 0 \Big| \sharp\{k \in \mathbb{N}; \omega_k \le \lambda\} - c_d \frac{\mathrm{Vol}\ (M)}{(2\pi)^d}\lambda^d \Big| \le \lambda^{d-1} \tag{4.6}$$

and

$$\Big| \sharp\{k; \omega_k \in I_h\} - c_d \frac{\mathrm{Vol}\ (M)}{(2\pi)^d}\Big((h^{-1}b_h)^d - (h^{-1}a_h)^d\Big)\Big| \le Ch^{-d+1} \tag{4.7}$$

Which implies

$$\exists M_0 > 0; b_h - a_h \ge M_0 h \Rightarrow \exists \beta > \alpha > 0;$$
$$\alpha h^{-d}(b_h - a_h) \le N_h = \sharp\{k; \omega_k \in I_h\} \le \beta h^{-d}(b_h - a_h) \tag{4.8}$$

Let $(e_{h,j})_{j=1}^{N_h}$ be an orthonormal basis of the space E_h,

$$F_h(x) = \big(e_{h,j}(x)\big)_{j=1}^{N_h},$$

and $f_h(x) = \|F_h(x)\|_{l^2} = \big(\sum_{j=1}^{N_h} e_{h,j}^2(x)\big)^{1/2}$. We can identify a point $U = (u_j)_{j=1}^{N_h} \in \mathbb{S}_h$ with the function

$$u(x) = \sum_{j=1}^{N_h} u_j e_{h,j}(x) = U \cdot F_h(x). \tag{4.9}$$

We have the following other consequence of Weyl's formula (or rather a consequence of the proof by Hörmander [**10**])

LEMMA 4.1. *Let a_h, b_h as above. We assume that $b_h - a_h \ge M_0 h$ so that (4.8) is satisfied. We also assume if $a = b$ that their common value is positive.Then there exists $C_0 > 0$ such that for any $x \in M$ and any $h \in]0, 1]$ we have*

$$|e_{x,h}| \le C_0 h^{-d}(a_h - b_h) \tag{4.10}$$

By decomposing $L^2(M)$ as a direct sum of spaces

$$L^2(M) = \oplus_k E_{h_k}$$

where E_{h_k} correspond to the choice $b_h - a_h = Mh$, we can prove the analogs of Theorems 1, 2. The only essential difference is that of course the elements of E_{h_k} are no more *exact* eigenfunctions, but rather *approximate* eigenfunctions:

$$-\mathbf{\Delta} e = h_k^{-2} e + \mathcal{O}(h_k^{-1})_{L^2(M)}$$

On Zoll manifolds (where all geodesics are periodic, the clustering of the eigenvalues (see [**1**]) allows to improve to

$$u \in E_{h_k} \Rightarrow -\mathbf{\Delta} u = h_k^{-2} u + O(1)_{L^2(M)}$$

5. Probability calculus on spheres

PROPOSITION 5.1. *Let us denote by p_N the uniform probability measure on the unit sphere $S(N)$ of $\mathbb{R}^N$. We have (see* [**5**, Appendix A]*)*

$$p_N(|x_1| > \cos(\theta)) = \mathbf{1}_{t\in[0,1[} D_N \int_0^\theta \sin^{N-2}(\varphi) d\varphi \tag{5.1}$$

with $D_N = \frac{c_1 c_{N-1}}{c_N}$ where c_k is the $k-1$-dimensional volume of the sphere $S(k)$.

Notice that

$$1 = D_N \int_0^{\pi/2} \sin^{N-2}(\varphi) d\varphi = D_N \int_0^1 (1-t^2)^{\frac{N-3}{2}} \tag{5.2}$$
$$= \frac{D_N}{2} \int_0^1 (1-u)^{\frac{N-3}{2}} u^{-\frac{1}{2}} du = \frac{D_N}{2} \beta(\frac{N-1}{2}, \frac{1}{2})$$

where β is the beta function of Weierstrass,

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

which implies

$$D_N = 2\frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})}.$$

Finally let us recall the standard concentration of measure property (see e.g. Ledoux [**12**])

PROPOSITION 5.2. *Let us consider a Lipshitz function F on the sphere $\mathbb{S}^d = S(d+1)$ (endowed with its geodesi distance and with the uniform probability measure, μ_d). Let us define its median value $\mathcal{M}(F)$ by*

$$\mu(F \geq \mathcal{M}(F)) \geq \frac{1}{2}, \qquad \mu(F \leq \mathcal{M}(F)) \geq \frac{1}{2}.$$

Then for any $r > 0$, we have

$$\mu(|F - \mathcal{M}(F)| > r) \leq 2e^{-(d-1)\frac{r^2}{2\|F\|^2_{Lips}}} \tag{5.3}$$

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