Non-linear Schrödinger boundary value problems

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The non linear Schrödinger equation in a smooth domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions:

$$NLS \begin{cases} i\partial_t u + \Delta u = F(u, \overline{u}), \\ F(z, \overline{z}) = \dfrac{\partial V}{\partial \overline{z}}(x), \\ V = V(|z|) \text{ (gauge invariance)} \\ u \mid_{t=0} = u_0, \\ u \mid_{\partial\Omega} = 0. \end{cases}$$

Question: How does the presence (and the geometry) of the boundary influence the large time behaviour of solutions of (NLS) (if these exist)?

Motivations:

1. Some models of Bose-Einstein condensates.
2. Gross Pitaevskii equation (models for the superfluid Helium)= exterior of several balls.
3. Some models in fiber optics.
4. Natural mathematical question.

Conservation laws

1. Energy conservation

$$E(u)(t) = \int_\Omega \|\nabla u\|^2 + V(|u|) = E(u)(t=0) \tag{1}$$

2. Charge conservation

$$\|u\|_{L^2(\Omega)}(t) = \|u_0\|_{L^2(\Omega)} \tag{2}$$

two cases for the long time dynamic

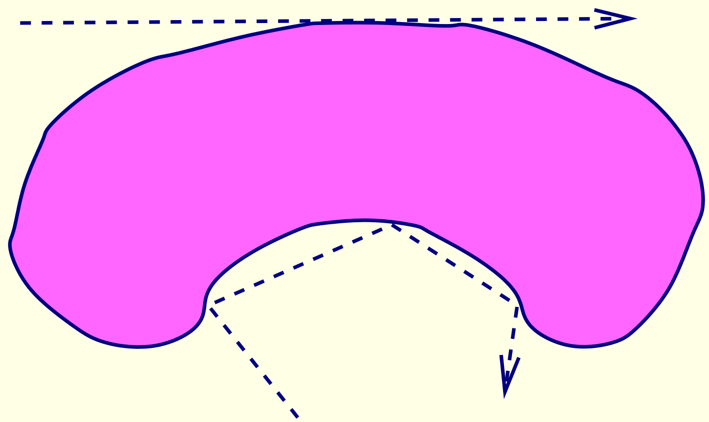
1. Defocusing case: The energy controls the $\int_\Omega \|\nabla u\|^2$ norm

2. Focusing case otherwise

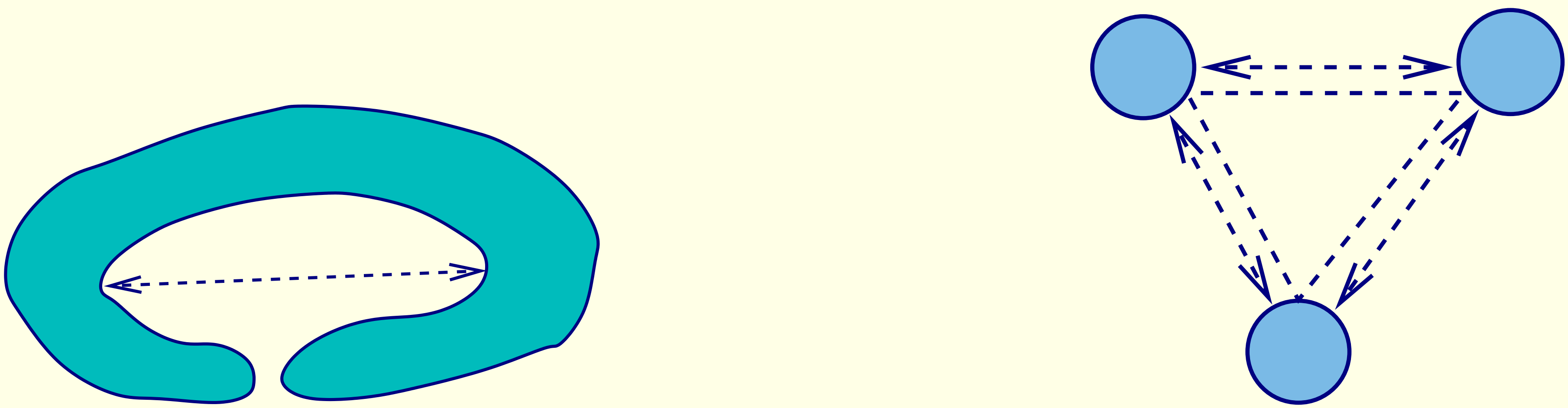
Two partial answers

1. The case of a disk: $\Omega = B(0,1)$

2. The case of non trapping exterior domains



A non trapping obstacle



Two trapping obstacles

A. The Schrödinger equation in a disk: instability features

Theorem 1 (Subcritical instability). *Consider (NLS) in a disk* $D = B(0,1)$ *with a (focusing) cubic non linearity*

$$F(u, \overline{u}) = -|u|^2 u \tag{3}$$

Then for any $t > 0$ *and any* $s \in [0, 1/3[$, *the flow map*

$$u_0 \in H^s(D) \mapsto u(t) \in H^s(D) \tag{4}$$

is, on any ball of H^s, *NOT uniformly continuous*

Remarks

1. If $\Omega = \mathbb{R}^2$: stability for cubic (NLS) in H^s for any $s \geq 0$
2. If $\Omega = \mathbb{T}^2$: stability for cubic (NLS) in H^s for any $s > 0$ (Bourgain)
3. If $\Omega = \mathbb{R}$: instability for cubic (NLS) in H^s for any $s < 0$ (Kenig, Ponce and Vega for focusing case and Christ, Colliander, Tao for defocusing)
4. If $\Omega = \mathbb{S}^2$: instability for cubic (NLS) in H^s for any $s \in [0, 1/4[$ (Burq, Gerard, Tzvetkov)
5. Same result for

$$|u|^2 u \to \langle u \rangle^\alpha u, \qquad 0 < \alpha \leq 2 \tag{5}$$

6. Higher dimensions: For example if $d = 5$ we obtain unstability in energy space H_0^1 on the ball for non linearities $F = (1 + |u|^2)^\alpha u, \qquad 0 < \alpha < \alpha_0$

B. The Schrödinger equation in a non trapping domain: global well posedness

Theorem 2 (Well posedness in H^1).
$\Omega \subset \mathbb{R}^d$ *non trapping*

$$F(u,\overline{u}) = |u|^\alpha u, \alpha < \frac{2}{d-2} \tag{6}$$

(defocusing case) Then for any $u_0 \in H_0^1(\Omega)$*, the Cauchy problem (NLS) admits a unique global (stable) solution* $u \in C(\mathbb{R}_t; H_0^1(\Omega))$

If $d = 3$ *and* $\alpha = 2$ *(cubic nonlinearity), small initial data results.*

Rk: $\Omega = \mathbb{R}^d$: $\alpha_c = \frac{4}{d-2}$

Theorem 3 (Well posedness in L^2).
$\Omega \subset \mathbb{R}^d$ *non trapping*

$$F(u, \overline{u}) = -|u|^\alpha u, \alpha < \frac{2}{d} \tag{7}$$

Then for any $u_0 \in L^2(\Omega)$, *the Cauchy problem (NLS) admits a unique global (stable) solution* $u \in C(\mathbb{R}_t; L^2(\Omega))$

Rk: $\Omega = \mathbb{R}^d$: $\alpha_c = \frac{4}{d}$

Ideas for the instability result:

Initial data concentrating on the boundary:

$$\begin{aligned} u_{n,0} &= \kappa \varphi_n, \\ \varphi_n(r,\theta) &= n^{\frac{2}{3}-s} e^{in\theta} J_n(z_{n,1} r) \end{aligned} \tag{8}$$

where J_n is the n-th Bessel function and $z_{n,1}$ its first positive zero.

$$\begin{aligned} -\Delta \varphi_n &= z_{n,1}^2 \varphi_n \\ \|\varphi_n\|_{L^p} \sim n^{-\frac{2}{3p}-\frac{1}{3}}&, \|\varphi_n\|_{H^s} \sim 1 \end{aligned} \tag{9}$$

φ_n concentrates on the caracteristic manifold

$$\varrho^2 + \frac{\eta^2}{r^2} = \varrho^2 + \frac{n^2}{r^2} = z_{n,1}^2 \sim n^2 \Rightarrow \varrho = o(n), r \sim 1. \tag{10}$$

Ansatz for the solution of (NLS): The solution of (NLS) with $u_{n,0}$ as initial data satisfies:

$$u_n(t,\cdot) = \begin{cases} \kappa e^{-itz_{n,1}^2} e^{it(\kappa^2\omega_n)} \varphi_n + \mathcal{O}(n^{-4s})_{H^s} \text{ if } 0 < s < 1/3 \\ \kappa e^{-itz_{n,1}^2} e^{it(\kappa^2\omega_n + \kappa^4\theta_n)} \varphi_n + o(1)_{H^s} \text{ if } s = 0 \end{cases}$$

where

$$\omega_n = \frac{\|\varphi_n\|_{L^4}^4}{\|\varphi_n\|_{L^2}^2} \sim n^{2/3-2s} \to_{n\to+\infty} +\infty$$

And θ_n has a limit when $n \to +\infty$.

Instability: Take κ_1 and κ_2 arbitrarily close from each other, due to the phase shift (and $\omega_n \to +\infty$), the solutions $u_{n,1}$ and $u_{n,2}$ are not close from each other at $t > 0$

The Ansatz:

Eigenfunctions of $-\Delta$:

$$\varphi_{p,k}(r,\theta) = e^{ip\theta} J_n(z_{n,k} r) \tag{11}$$

$z_{n,k}$ is the k-th zero of the Bessel function J_n.

Eigenvalues are $z_{p,k}^2$

Gauge invariance:

$$u_n(t,r,\theta+\varphi) = u_n(t,r,\theta) e^{in\varphi} \tag{12}$$

Spann $u_n(t,r,\theta)$ on the L^2 eigenfunctions basis $\varphi_{p,k}$. due to gauge invariance non null components only for $p = n$Hence frequencies involved are $z_{n,k}$ (greater than $z_{n,1}$)

Conservation laws:

$$|u_{n,1}(t)|^2 \|\varphi_n\|_{L^2}^2 + \|q(t)\|_{L^2}^2 = \|\varphi_n\|_{L^2}^2 \tag{13}$$

$$\begin{aligned} |u_{n,1}(t)|^2 \|\nabla\varphi_n\|_{L^2}^2 + \|\nabla q(t)\|_{L^2}^2 - \frac{\kappa^2}{2}\|u_{n,1}(t)\varphi_n + q\|_{L^4}^4 \\ = \|\nabla\varphi_n\|_{L^2}^2 - \frac{\kappa^2}{2}\|\varphi_n\|_{L^4}^4 \end{aligned} \tag{14}$$

Write

$$\begin{aligned} u_n(t,r,\theta) &= \alpha(t)\varphi_{n,1} + \sum_{k\geq 2}\alpha_k(t)\varphi_{n,k} \\ &= \alpha(t)\varphi_{n,1} + q(t) \end{aligned} \tag{15}$$

The use of the conservation laws and the separation of eigenvalues $z_{n,1} - z_{n,2}$ gives a good control on $\|q\|_{H^s}$ which plugged into the differential equation satisfied by α proves the ansatz

Ideas for the non trappingness result

Theorem 4 (Smoothing effect).

- $\Theta \neq \emptyset \subset \mathbb{R}^d$ *non trapping and smooth* (C^3),
- $\Omega = \Theta^c$, $\chi \in C_0^\infty(\mathbb{R}^d)$,

Then, for any $s \in \mathbb{R}$:

$$(i\partial_t + \Delta_D)u = 0, \qquad u\,|_{\partial\Omega} = 0, \qquad u\,|_{t=0} = 0$$

$$\Rightarrow \|\chi(x)u\|_{L^2(\mathbb{R}_t; H_D^{\frac{1}{2}}(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}$$

$$(i\partial_t + \Delta_D)v = \chi(x)f(x,t), \qquad u\,|_{\partial\Omega} = 0, \qquad u\,|_{t<<0} = 0$$

$$\Rightarrow \|\chi(x)v(t,x)\|_{L^2(\mathbb{R}_t; H_D^1(\Omega))} \leq C\|f\|_{L^2(\mathbb{R}_t; L^2(\Omega))}$$

Proof: TT^* argument, Fourier transform + standard scattering estimates

$$T = \chi e^{it\Delta_D} : L^2(\Omega) \to L^2_t; H_D^{1/2} \tag{16}$$

$$\Leftrightarrow T^* : f \in L^2_t; H_D^{-1/2} \mapsto \int_{-\infty}^{+\infty} e^{-is\Delta_D} \chi(\cdot) \chi f(s, \cdot) \in L^2(\Omega)$$

$$\Leftrightarrow TT^* : f \in L^2_t; H_D^{-1/2}$$

$$\mapsto \int_{-\infty}^{+\infty} \chi(x) e^{i(t-s)\Delta_D} \chi(\cdot) f(s, \cdot) \in L^2_t; H_D^{1/2} \tag{17}$$

but

$$
\begin{aligned}
TT^*f &= \int_{-\infty}^{+\infty} \chi(x) e^{i(t-s)\Delta_D} \chi(\cdot) f(s,\cdot) \\
&= \int_{-\infty}^{t} + \int_{t}^{+\infty}
\end{aligned}
\tag{18}
$$

Prove the result for $\int_{-\infty}^{t}$ $\Leftrightarrow$ non-homogeneous result

u solution of

$$i\partial_t u + \Delta_D u = \chi f, u \text{ and } f\,|_{t<<0} = 0 \tag{19}$$

Fourier transform (w. r. to t) are analytic in $\Im m z < 0$

$$(-(\tau - i\varepsilon) + \Delta_D)\widehat{u}(\tau, \cdot) = \chi(x)\widehat{f}(\tau.\cdot)$$

$$\Rightarrow \chi(x)\widehat{u}(\tau - i\varepsilon, x) = \chi(x)(-(\tau - i\varepsilon) + \Delta_D)^{-1}(\chi(\cdot)\widehat{f}(\tau.\cdot)) \tag{20}$$

Take $\varepsilon \to 0$, the estimate follows from

1. The Fourier transform is an isometry on $L^2(\mathbb{R}_t; H)$ if H is an Hilbert space

2. The resolvent $\chi(x)(-(\tau - i\varepsilon) + \Delta_D)^{-1}\chi(\cdot)$ is uniformly bounded from $H_D^{-1/2}(\Omega)$ to $H_D^{1/2}(\Omega)$

Proposition 5. *The cut-off resolvent satisfies (uniformly with respect to* $\varepsilon > 0$ *and* $\tau \in \mathbb{R}$*)*

$$\|\chi(x)(-(\tau \pm i\varepsilon) + \Delta_D)^{-1}\chi(\cdot)\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{C}{\sqrt{1+|\tau|}} \quad (21)$$

$$\Rightarrow \|\chi(x)\left(-(\tau \pm i\varepsilon) + \Delta_D\right)^{-1}\chi(\cdot)\|_{H_D^{-1/2}\to H_D^{1/2}} \leq C \quad (22)$$

1. Hight frequencies: non-trapping assumption,
    a) Lax Phillips, Morawetz: multiplier methods,
    b) Melrose, Sjöstrand: propagation of singularities,
    c) Vasy-Zworski: semi-classicalMourre estimates,
    d) Burq: semi-classical measures (low regularity)
2. Low frequencies: $\Theta \neq \emptyset$ if $d = 2$:, Vainberg, Morawetz for $d = 2$, Burq for $d \geq 3$: no non-trapping assumption

Remark: For Neumann boundary conditions, Low frequency estimates are open. The proof above gives Local in time smoothing effect and the results on global wellposedness are true in this case also

Strichartz estimates

Proposition 6. *For any* $T > 0$, $\chi \in C_0^\infty(\mathbb{R}^d)$, $s \in [0, 1]$

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \qquad p \geq 2 \tag{23}$$

$$\|e^{it\Delta_D} u_0\|_{L^p([0,T];L^q(\Omega))} \leq C \|u_0\|_{H_D^{\frac{1}{p}}(\Omega)} \tag{24}$$

Similarly we have nonhomogeneous Strichartz estimates with losses of derivatives

1. If $\Omega = \mathbb{R}^d$ and $\Delta = \Delta_0$: Strichartz estimates with no loss (Ginibre, Velo, Kato, Yajima, Cazenave Weissler)

2. If $\Omega = \mathbb{R}^d$ and $\Delta = \Delta_g$, non trapping compactly supported perturbation of the metric: Strichartz estimates with no loss (Staffilani, Tataru)

3. If $\Omega = M$ compact riemanian manifold: loss of $\frac{1}{p}$ derivatives (Burq, Gerard, Tzvetkov, + can be deduced from the results in Staffilani, Tataru)

4. Here Strichartz estimates $L_t^p; L_x^q$ with loss of $\frac{1}{p}$ derivatives

Two last results look the same but are completly different:

Gain in Strichartz with respect to Sobolev embedding:

$$H^1(\mathbb{R}^d) \mapsto L^{\frac{2d}{d-2}}(R^d) \tag{25}$$

$$\|e^{it\Delta_0}u_0\|_{L^2(\mathbb{R}_t;L^{\frac{2d}{d-2}}(R^d))} \le C\|u_0\|_{L^2(\mathbb{R}^d)} \tag{26}$$

$$\begin{aligned} \text{Gain } &= 1 \text{ derivative} \\ &= 1/2 \text{ for the smoothing effect} \\ &\quad + 1/2 \text{ semi-classical Strichartz} \end{aligned} \tag{27}$$

Strichartz estimate: proof

1. Close to the obstacle: Sobolev embedding from the smoothing effect
2. Far from the obstacle: Strichartz for $e^{it\Delta_0}$ + smoothing effect gives usual Strichartz (with no loss (Staffilani, Tataru))

Close to the obstacle ($\chi = 1$ close to Θ)

$$\begin{aligned} \|\chi e^{it\Delta_D} u_0\|_{L^2([0,T]);H^1_D)} &\leq \|u_0\|_{H^1/2_D} \\ \|\chi e^{it\Delta_D} u_0\|_{L^\infty([0,T]);L^2)} &\leq \|u_0\|_{L^2} \end{aligned} \tag{28}$$

Interpolate

$$\|\chi e^{it\Delta_D} u_0\|_{L^p([0,T];H_D^{\frac{2}{p}})} \leq \|u_0\|_{H_D^{\frac{1}{p}}} \tag{29}$$

Sobolev embedding:

$$H_D^{\frac{1}{p}} \mapsto L^q(\Omega), \frac{2}{p} + \frac{d}{q} = \frac{d}{2} \tag{30}$$

Far from the obstacle (Staffilani Tataru)

$$(i\partial_t + \Delta)(1-\chi)u = [\Delta, \chi]u = \widetilde{\chi} f \text{ bounded in } L^2_t; H^{-1/2}_D$$

$$(1-\chi)u = e^{it\Delta_0}(1-\chi)u_0 + \int_0^t e^{i(t-s)\Delta_0}\widetilde{\chi} f$$

First term= OK. Imagine second term =

$$\int_0^T e^{i(t-s)\Delta_0}\widetilde{\chi} f = e^{it\Delta_0}\int_0^T e^{-is\Delta_0}\widetilde{\chi} f \tag{31}$$

Dual of smoothing effect (for Δ_0):

$$\int_0^T e^{-is\Delta_0}\widetilde{\chi} : L^2_t; H^{-1/2}_D \mapsto L^2 \tag{32}$$

Usual Strichartz:

$$e^{it\Delta_0} : L^2(\Omega) \mapsto L^p_t; L^q(\mathbb{R}^d) \tag{33}$$

replace $\int_0^t$ by $\int_0^T$:

Theorem 7 (Christ Kiselev).

$$T \;:\; L^p(\mathbb{R}; B_1) \to L^q(\mathbb{R}; B_2) \tag{34}$$

locally integrable kernel $K(t,s)$ *with values bounded operators from* B_1 *to* B_2 *Banach spaces. Suppose that* $p < q$. *Then*

$$\widetilde{T}f(t) = \int_{s<t} K(t,s)f(s)ds$$

is bounded from $L^p(\mathbb{R}; B_1)$ *to* $L^q(\mathbb{R}; B_2)$ *and*

$$\|\widetilde{T}\|_{L^p(\mathbb{R};B_1)\to L^q(\mathbb{R};B_2)} \le (1 - 2^{-(p^{-1}-q^{-1})})^{-1}\|T\|_{L^p(\mathbb{R};B_1)\to L^q(\mathbb{R};B_2)}. \tag{35}$$

Well posedness in H_0^1

1. $d = 2$: Fixed point in
   $X_T = L^\infty(]0,T[; H_0^1) \cap L^p(]0,T[; W^{1-1/p,q})$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $p > \alpha$

2. $d = 3$: Fixed point in
   $X_T = L^\infty(]0,T[; H_0^1) \cap L^p(]0,T[; W^{1,\frac{18}{7}})$

3. $d \geq 4$: "Fixed point" in
   $X_T = L^\infty(]0,T[; H_0^1) \cap L^p(]0,T[; W^{1,q})$, $\frac{1}{p} + \frac{d}{q} = \frac{d}{2}$, $p > 2$, p close to 2

Remark: for $d \geq 5$, $\alpha_c = \frac{2}{d-2} < 1$ and stability unknown.

Well posedness in L^2

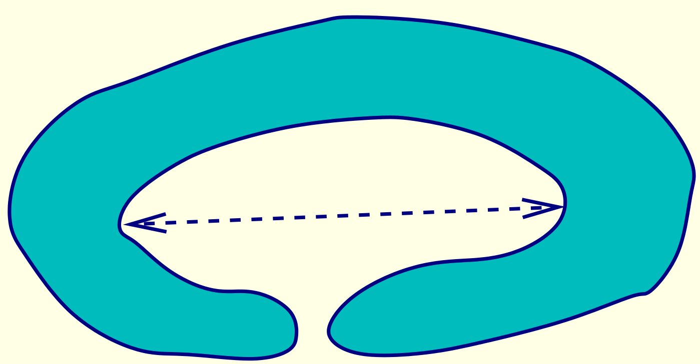
Fixed point in $X_T = L^\infty(]0,T[;L^2) \cap L^p(]0,T[;L^q)$,
$\frac{1}{p} + \frac{d}{q} = \frac{d}{2}$, $p > 2$, p close to 2

Comments

1. Any gain on the semi-classical Strichartz will improve the results (the gains arising from Smoothing effect and from semi-classical Strichartz cumulate)

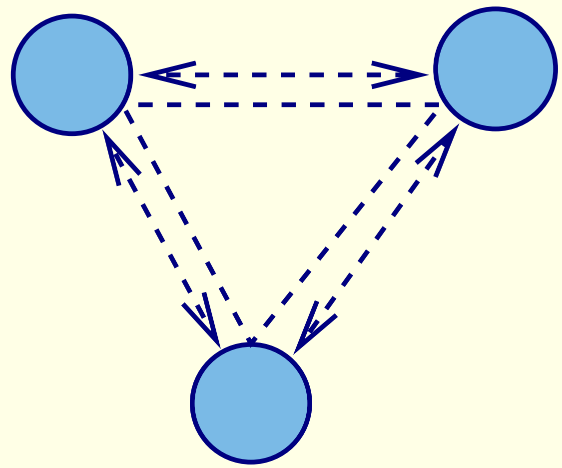
2. The non trapping assumption is known to be necessary for the $H^{1/2}$ smoothing effect (Doï if $\partial\Omega = \emptyset$, Burq for boundary value problems)

3. If there exists an elliptic trapped trajectory then the existence of quasimodes shows that no smoothing at all is true



No smoothing effect

4. In some cases of hyperbolic trapped trajectories (not too many convex obstacles sufficiently distant) The method gives a $H^{1/2}$ smoothing effect with a logarithmic loss



Smoothing effect with a logarithmic loss