Control and stabilization from geodesic domains

Nicolas Burq[1]

Université Paris-Sud, Université Paris-Saclay,
Laboratoire de Mathématiques d'Orsay, UMR 8628 du CNRS

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Control of waves

Consider the wave equation on a Riemanian manifold M_g, $a \in L^\infty(M), a \geq 0,\ T > 0$

$$(\partial_t^2 - \Delta)u = f \times 1_{(0,T)} \times a(x), \quad (u\mid_{t=0}, \partial_t u\mid_{t=0}) = (u_0, u_1)$$

Given $(u_0, u_1) \in \mathcal{H}^1 = H^1(M) \times L^2(M)$ initial data and $(v_0, v_1) \in \mathcal{H}^1$ target data in energy space, can we choose f in suitable space such that

$$(u\mid_{t=T}, \partial_t u\mid_{t=T}) = (v_0, v_1)?$$

Natural space for f is $L^2((0,T) \times M)$. If answer yes: exact controlability

Stabilization for waves

$$(\partial_t^2 - \Delta + a(x)\partial_t)u = 0,$$
$$(u\,|_{t=0}, \partial_t u\,|_{t=0}) = (u_0, u_1) \in H^1 \times L^2 = \mathcal{H}^1$$

The natural energy is decaying ($a \geq 0$)

$$E(u)(t) = \int_M |\nabla_x u|^2 + |\partial_t u|^2 dx, \frac{d}{dt}E(t) = \int_M -a(x)|\partial_t u|^2 dx$$

Question: speed of decay of $E(u)(t)$?

- The energy of all solutions tend to 0 iff there exists no non trivial stationary equilibrium, i.e.
  $-\Delta e = \lambda^2 e, a \times e = 0 \Rightarrow e = 0.$
- Semi-group property: If there exists a uniform rate $f(t)$,

$$\forall (u_0, u_1) \in \mathcal{H}^1, E(u)(t) \leq f(t)E(u)(0), \lim_{t \to +\infty} f(t) = 0,$$

then can choose $f(t) = Ce^{-ct}$ (uniform) stabilization.

Observation and HUM duality imply equivalence

- There exists a rate $f(t)$ such that $\lim_{t\to+\infty} f(t) = 0$ and

$$\forall (u_0, u_1) \in H^1(M) \times L^2(M), E(u)(t) \leq f(t)E(u)(0).$$

(and then can choose $f(t) = Ce^{-ct}$)

- $\exists T > 0, c > 0; \forall (u_0, u_1) \in H^1(M) \times L^2(M)$, if u is the solution to the damped wave equation, then

$$E(u)(0) \leq C \int_0^T \int_M 2a(x)|\partial_t u|^2 dxdt.$$

- $\exists T > 0, c > 0; \forall (u_0, u_1) \in H^1(M) \times L^2(M)$, if u is the solution to the undamped wave equation then

$$E(u)(0) \leq C \int_0^T \int_M 2a(x)|\partial_t u|^2 dxdt.$$

- There exists $T > 0$ such that The wave equation is exactly controlable in time T (and we can take the time given by observation)

The geometric control assumption for waves

$(a \in C^0(M), T)$ controls geometrically (M, g) if every geodesic starting from any point $x_0 \in M$ in any direction ξ_0, $\gamma_{(x_0,\xi_0)}(s)$, encounters $\{a > 0\}$ in time smaller that T

Theorem (Rauch-Taylor, Bardos-Lebeau-Rauch 88', N.B- P.G.)

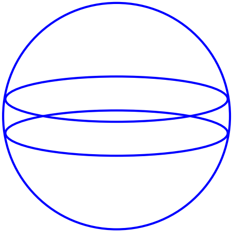
$a \in C^0(M)$ *geometric control is equivalent to observability (and hence control and stabilization) for wave equations.* $a \in L^\infty(M)$ *Strong Geometric Control is sufficient for observability which implies Weak Geometric Control.*

$$\begin{gathered}\exists T, c > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T), \exists \delta > 0;\\ a \geq c \text{ a.e. on } B(\gamma_{\rho_0}(s), \delta).\end{gathered} \tag{SGCC}$$

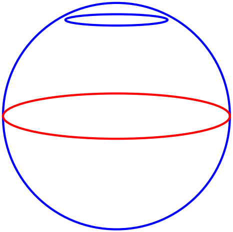
$$\exists T > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T); \gamma_{\rho_0}(s) \in \operatorname{supp}(a) \tag{WGCC}$$

supp(a) is the support (in the distributional sense) of a,

The geometric control assumption



Yes



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Some examples on tori

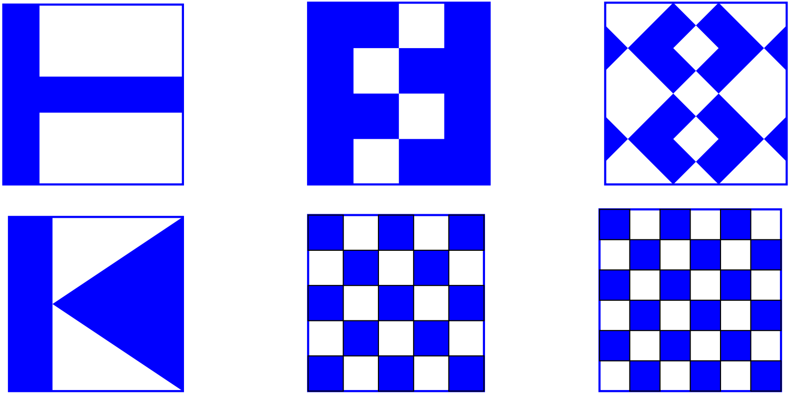
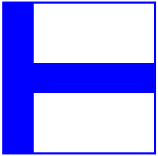


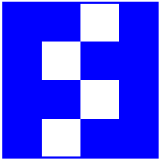
Figure: Checkerboards: the damping a is equal to 1 in the blue region, 0 elsewhere. The geodesics are (periodized) straight lines. The first example satisfies (SGCC) while all others satisfy (WGCC) but not (SGCC)

Stabilization for wave equations: the result

Theorem (Does Stabilization holds? $a = 1$ in blue region 0 otherwise)



YES 80'
(Taylor-Rauch)



YES
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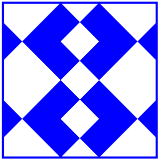

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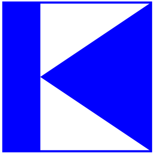

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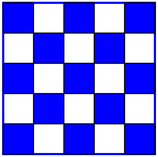

YES
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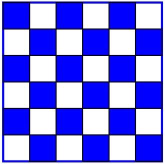


NO
(NB-PG 16)

Another geometric condition

When the manifold is a two dimensional torus and the damping a is a linear combination of characteristic functions of rectangles, i.e. there exists N rectangles (or polygons), $R_j, j = 1, \dots N$ (disjoint and non necessarily vertical), and $0 < a_j, j = 1, \dots, N$ such that

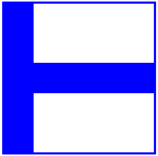
$$a(x) = \sum_{j=1}^{N} a_j 1_{x \in R_j}, \tag{1}$$

(piecewise smooth domains, no infinite contact with geodesics= much easier)

Theorem (NB–P. Gérard 15-17)

Stabilization holds for the waves on $\mathbb{T}^2$ iff there exists $T > 0$ such that all geodesics (straight lines) of length T either encounters the interior of one of the rectangles or follows for some time one of the sides of a rectangle R_{j_1} on the left and for some time one of the sides of another (possibly the same) rectangle R_{j_2} on the right.

Stabilization for wave equations: the result



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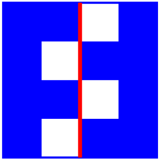

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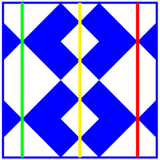

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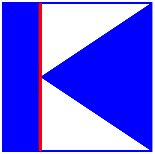

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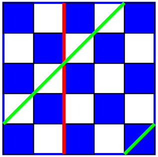

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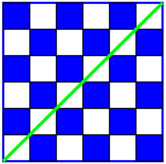

NO

A geometric control condition for *control*

When the manifold is a two dimensional surface and a is a linear combination of characteristic functions of *geodesic polygons*, *i.e. there exists N polygons, $R_j, j = 1, \ldots N$ (disjoint and non necessarily vertical), and $0 < a_j, j = 1, \ldots, N$ such that*

$$a(x) = \sum_{j=1}^{N} a_j 1_{x \in R_j}, \tag{2}$$

Theorem (NB 17)

Let $T > 0$. Then exact controlability holds for the waves on M^2 if and only if a generalized geometric condition (defined in terms of an ODE on a sphere bundle over S^*M) is satisfied. Roughly speaking it says that all geodesics of length T either encounters the interior of one of the polygons or follows for some time one of the sides of a polygon R_{j_1} and there exists $s > 0$ such all neighbour geodesics spend an amount of time of order at least s in the interior of one of the polygons

The result on the sphere and the torus

Theorem (NB 17)

Let $T > 0$. Then exact controlability holds for the waves on M^2 if and only if There exists $\alpha > 0$ such that for almost every $(x_0, \xi_0) \in S^*M$,

$$\int_0^T a(x(s, x_0, \xi_0))ds \geq \alpha > 0.$$

here $(x(s, x_0, \xi_0), \xi(s, x_0, \xi_0)$ is the bicharacteristic starting from (x_0, ξ_0) at $s = 0$).

Control for wave equations: the result on the sphere

On spheres, geodesics are great circles and the generalized geometric condition reduces to checking

–The geodesic enters the *interior of the control region* $\omega = \{a(x) > 0\}$ *or*

– The geodesic follows (some) sides of (some) polygons and we check the following algorithm. Write the whole oriented geodesic circle

$$\Gamma = \Gamma^u \cup \Gamma^d \cup \Gamma^0,$$

(parts of Γ *which encounter the side of a polygon on the upper (lower) hemisphere–or not). Then any choice of oriented diameter* D *separates any piece of geodesic* $\gamma(0, T)$ *into three (possibly empty) pieces*

$$\gamma^l \cup \gamma^r \cup \gamma^c,$$

corresponding to the part on the left (right) of the diameter or on the diameter. Then we assume that we never have

$$\left(\gamma^l \subset \Gamma^u \text{ and } \gamma^r \subset \Gamma^d\right).$$

Contradiction argument

Want to prove observation estimates for half wave solutions with spectrally localized initial data (h small enough)

$$(\partial_t^2 - \Delta)u = 0, u_0 = 1_{a < -h^2\Delta < b} u_0, u_1 = 1_{a < -h^2\Delta < b} u_1 \qquad a < 1 < b$$

$$\|u_0\|_{L^2}^2 \leq C \int_0^T \int_\omega |u|^2(x,t) dxdt \ \omega = \{a > 0\}$$

Assume false then there exists sequences

$$a_n, b_n \to 1, u_n \in L^2, 1_{a_n < -h_n^2 \Delta < b_n} u_n = u_n,$$

such that

$$\|u_n\|_{L^2} = 1, \int_0^T \int_\omega |u_n|^2(x,t) dxdt = o(1)$$

First microlocalization

Scales: $t, X \sim 1, \tau, \Xi \sim h^{-1}$. Consider operators

$$a(t, X, hD_t, hD_X), a \in C_0^\infty(T^*M).$$

If $a \geq 0$ then (Gårding) $a(t, X, hD_t, hD_X) \geq -Ch$.

Proposition

There exists a subsequence (now we drop all sub-indexes) and a positive measure μ (on continuous functions on T^*M) such that

$$\lim_{n \to +\infty} \Big(a(t, X, h_n D_t, h_n D_X) u_n, u_n \Big)_{L^2_{t,X}} = \langle \mu, a \rangle.$$

$$supp\ (\mu) \subset \{(t, \tau, X, \Xi); 1 = \tau^2 = \|\Xi\|^2_{g(x)} = p(X, \Xi)$$

$$\mu(T^*(0, Y) \times M) = T, \qquad \partial_t \mu = H_p \mu, \qquad \mu \mid_{(0,T) \times \omega} = 0.$$

As a consequence, μ is supported on bicharacteristics which do not encounter ω but hence graze $\partial\omega$ on left or right

Second microlocalization

Understand at finer scales how the mass can concentrate on the geodesic from left or right. Work in a geodesic coordinate system (x, y) where

$$-\Delta = -\partial_x^2 - \partial_y^2(1 + x^2\kappa(y) + O(x^3)),$$

where the geodesic is given by $\{x = 0\}$ (and the bicharacteristic by $\{(x = 0, \xi = 0)\}$) and $\kappa(y)$ is the gauss curvature of the surface at point y.

Scales: 2 different regimes

- Transversal HF

$$t, y \sim 1, \tau = \eta = 1 + o(1), h^{1/2} \ll \|x, \xi\| = o(1)$$

- Transversal LF

$$t, y \sim 1, \tau = \eta = 1 + o(1), \quad \|(x, \xi)\| \leq Ch^{1/2}$$

Describe concentration at these scales and conclude contradiction.

2-pseudodifferential operators

– Symbols: functions $a(t, y, z, \tau, \eta, \zeta) \in S$ the class of smooth compactly supported in the (y, η) variables and polyhomogeneous of degree 0 near infinity in the (z, ζ) variables

$$|\partial_{t,y,\tau,\eta}^{\alpha} \partial_z^{\gamma} \partial_{\zeta}^{\delta} a| \leq C(1 + |z| + |\zeta|)^{-(\gamma+\delta)}.$$

– Operators : $\chi \in C_0^{\infty}(\mathbb{R}^2)$ equal to 1 near 0,

$$\mathrm{Op}_h(a) = a(t, y, hD_t, hD_y, h^{-1/2}x, h^{1/2}D_x)$$

$$\mathrm{Op}_{h,\epsilon}(a) = \ \mathrm{Op}(a \times \chi(\epsilon z, \epsilon\zeta)),$$

$$\mathrm{Op}_h^{\epsilon}(a) = \ \mathrm{Op}(a \times (1 - \chi)(\epsilon z, \epsilon\zeta)),$$

– Bad pseudodifferential calculus for $\mathrm{Op}(a)$ and $\mathrm{Op}_{\epsilon}(a)$
L^2 boundedness but no symbolic calculus , no Gårding
– Good pseudodifferential calculus $\mathrm{Op}^{\epsilon}(a)$ (gain ϵ^2)
symbolic calculus, and Gårding
Approach inspired from works by Fermanian, Nier and Anantharaman-Macia for Schrödinger on tori. Here $S_{\frac{1}{2},\frac{1}{2}}$ calculus, Nier $S_{1,1}$, A–M, $S_{0,0}$ calculus

2-microlocal measure: transversal HF (for tori $\epsilon \to h^\epsilon$)

$$t, y \sim 1, \tau = \eta = 1 + o(1), h^{1/2} \ll \|x, \xi\| = o(1),$$

If $a \geq 0$ then (Gårding) $\mathrm{Op}^\epsilon(a) \geq -C\epsilon$.

Proposition

There exists a subsequence and a positive measure ν^+ (on continuous functions on $T^*N \times \mathbb{R}^2$ homogeneous of degree 0 at infinity in (z, ζ)) such that

$$\lim_{\epsilon \to 0} \lim_{n \to +\infty} \left(Op^\epsilon_{h_n}(a) u_n, u_n \right)_{L^2_{t,y,x}} = \langle \nu^+, a \rangle.$$

$$supp\ (\nu^+) \subset \{(t, \tau, y, z, \eta, \zeta); 1 = \tau = \eta\}$$

The projection of ν on the (t, y, τ, η)variables is bounded by the previous (1)-microlocal measure and additional propagation holds

$$\left(\partial_t - \partial_y - \zeta \partial_z + z\kappa(y)\partial_\zeta\right)\nu^+ = 0$$

Proof of propagation

Key remark

$$-h^2\Delta = -h^2\partial_y^2(1 - x^2\kappa(y) - h^2\partial_x^2 + O(x^3))$$
$$= \mathrm{Op}(\eta^2(1 + hz^2\kappa(y) + h\zeta^2 + O(hz^2x)) \quad (3)$$

Compute

$$\frac{i}{2h}\Big[(h^2\partial_t^2 - h^2\Delta, \mathrm{Op}^{\epsilon}_{h_n}(a)\Big] =$$
$$\mathrm{Op}^{\epsilon}_{h_n}(-\tau\partial_t(a) + \eta\partial_y(a) + \zeta\partial_z(a) - \eta^2\kappa(y)z\partial_\zeta(a) + O(\epsilon^2) + O(x)$$

implies

$$0 = \lim_{\epsilon\to 0}\lim_{n\to+\infty}\frac{i}{2h}\Big(\Big[(h^2\partial_t^2 - h^2\Delta, \mathrm{Op}^{\epsilon}_{h_n}(a)\Big]u_n, u_n\Big)_{L^2_{t,y,x}}$$
$$= \langle(\partial_t - \partial_y - \zeta\partial_z + \kappa(y)z\partial_\zeta)\nu, a\rangle$$

Conclusion in the transversal HF regime

- The measure ν^+ is invariant by the flow defined by previous equation
- It is supported on geodesics grazing $\partial\omega$
- By contradiction assumption

$$\int_{(0,T)}\int_{\omega}|u|^2(t,x)dxdt = o(1),$$

  We deduce that if near points $(t,y,x=0)$ such that $(y,x=0)\in\partial\omega^r$ then ν_+ is supported in $\{z\leq 0\}$
- The geometric hypothesis implies that $\nu_+\equiv 0$

2-microlocal measure: transversal LF

We are looking at $(\chi_\epsilon = \chi(\epsilon \cdot))$

$$\lim_{\epsilon \to 0} \lim_{n \to +\infty} \Big((a \times \chi_\epsilon)(t, y, h_n D_t, h_n D_y, h_n^{-1/2} x, h_n^{1/2} D_x) u_n, u_n \Big)_{L^2}$$

change variables in x, $z = h^{-1/2} x$, $v_n(z) = h^{1/4} u_n(h^{1/2} z)$ We are now looking at

$$\lim_{\epsilon \to 0} \lim_{n \to +\infty} \Big((a \times \chi_\epsilon)(t, y, h_n D_t, h_n D_y, z, D_z) v_n, v_n \Big)_{L^2}$$

Due to the presence of the cut off χ_ϵ for any fixed ϵ, the operators

$$(a \times \chi_\epsilon)(t, y, h_n D_t, h_n D_y, z, D_z)$$

are semi-classical operators in the (t, y) variable with values compact operators on $L^2_x = H$
The sequence v_n is bounded in $L^2_{loc}(\mathbb{R}^2_{t,y}; H)$.

2-microlocal measure: transversal LF

Proposition (P. Gérard 90')

There exists a subsequence and a positive measure ν^- on continuous functions on $T^*\mathbb{R}^2$ with values trace class operators

$$\lim_{n\to+\infty}\Big((a\times\chi_\epsilon)(t,y,h_nD_t,h_nD_y,z,D_z)v_n,v_n\Big)_{L^2}$$
$$=Tr\langle\nu^-,(a\chi_\epsilon)(t,y,\tau,\eta,z,D_z)\rangle.$$

Radon-Nikodym: $\nu^- = A(t,y,\tau,\eta)d\rho$, *where* A *is trace class*

Proposition (Saut Scheurer?)

Consider the following classical one dimensional harmonic oscillator

$$(i\partial_s - \frac{\partial_z^2}{2} + \frac{\kappa(-s)z^2}{2})u = 0.$$

Assume that u *vanishes on* $(\alpha,\beta)_s\times\mathbb{R}_z^+$. *Then* $u\equiv 0$.

Theorem

As soon as Γ^+ *or* Γ^- *is non empty, the measure* ν^- *is identically* 0.

The harmonic oscillator

$$t, y \sim 1, \tau = \eta = 1 + o(1), |z| + |D_z| \le \epsilon^{-1},$$

$$(ih_n\partial_t + \sqrt{-h_n^2\Delta})u_n = 0,$$

$$-h^2\Delta = -h^2\partial_y^2(1 + h\kappa(y)z^2) - h\partial_z^2 + O_\epsilon(h^{3/2})$$

$$\sqrt{-h^2\Delta} = -ih\partial_y\sqrt{(1 + h\kappa(y)z^2) - h\frac{\partial_z^2}{-h^2\partial_y^2}} + O_\epsilon(h^{3/2})$$

$$= -ih\partial_y + h(-\frac{\partial_z^2}{2} + \frac{\kappa(y)z^2}{2}) + o_\epsilon(h) \quad (4)$$

$$(ih_n(\partial_t - \partial_y) + h_n(-\frac{\partial_z^2}{2} + \frac{\kappa(y)z^2}{2})v_n = o_\epsilon(h).$$

$$s = \frac{t-y}{2}, r = \frac{t+y}{2} \Rightarrow (i\partial_s + H_{r-s})w_n = o_\epsilon(h)$$

$$\Leftrightarrow w_n = S_r(s,0)(w_n\,|_{s=0}) + o(1). \quad (5)$$

An elementary argument in infinite dimension

Conjugate (micro-locally) with the inverse of the evolution to replace the equation $(i\partial_s + H_{r-s})w_n = 0$ by $i\partial_s \tilde{w}_n = 0$.

$$\tilde{w}_n = S_r(0,s)w_n = S_r(s,0)^* w_n,$$

Let $\tilde{\nu}^-$ be the measure of the new sequence $\tilde{w}_n$. Then

$$\tilde{\nu}^- = S_r(0,s)\nu S_r(0,s)^*$$

is independent of the variable s. It writes

$$\tilde{\nu}^- = A(r)d\lambda_r \otimes ds \otimes \delta_{\sigma=0} \otimes \delta_{\rho=2},$$

Where $A(r)$ is a family of hermician trace class operators and $d\lambda_r$ a non negative measure. Let $e_n(r) \in H$ a Hilbert Basis diagonalizing $A(r)$ (eigenvalues λ_n). We get

$$\tilde{\nu}^- = \sum_n \lambda_n \langle \cdot, e_{n,r}\rangle_H e_{n,r} d\lambda_r \otimes ds \otimes \delta_{\sigma=0} \otimes \delta_{\rho=2},$$

$$\nu^- = \sum_n \lambda_n \langle \cdot, S(s,0)e_{n,r}\rangle_H S(s,0)e_{n,r} d\lambda_r \otimes ds \otimes \delta_{\sigma=0} \otimes \delta_{\rho=2},$$

Conclusion in the transversal LF regime

$$\nu^- = \sum \lambda_n \langle \cdot, S(s,0)e_{n,r}\rangle_H S(s,0)e_{n,r} d\lambda_r \otimes ds \otimes \delta_{\sigma=0} \otimes \delta_{\rho=2},$$

If $\zeta(s,r)$ is supported in the region where the bicharacteristic grazes a part of Γ^r, then for any $\epsilon > 0$,

$$\lim_{n\to+\infty} \int_{s,r} \chi(s,r) \int_{z\in(0,\delta h^{-1/2})} |w_n|^2 dzdrds = 0,$$

We deduce that for any $\epsilon > 0$,

$$\langle \nu^-, \zeta(s,r) 1_{x>0} \chi(\epsilon(z, D_z))\rangle = 0$$

$$\Rightarrow 0 = \int_{r,s} \zeta(r,s) \mathrm{Tr}(\sum_n \lambda_n \langle \cdot, S(s,0)e_{n,r}\rangle_H 1_{z>0} S(s,0)e_{n,r} d\lambda_r ds,$$

$$= \int_{r,s} \zeta(r,s) \sum_n \lambda_n \int_{z>0} |S(s,0)e_{n,r}|^2 dsd\lambda_r \quad (6)$$

(we used that $S(s,o)e_n$ is a Hilbert basis of H). Hence implies $\forall n, \lambda_n = 0$ and $\nu^- = 0$.