# LOWER BOUNDS FOR SHAPE RESONANCES WIDTHS OF LONG RANGE SCHRÖDINGER OPERATORS

*by*

Nicolas Burq

***Abstract.*** — In this article we give general lower bounds of shape resonance widths in the semi-classical limit for Schrödinger operators in the exterior of smooth compact obstacles with Dirichlet or Neuman boundary conditions and with long range dilation analytic potentials. These lower bounds are exponentially small with respect to the Planck constant.

***Résumé.*** — On donne dans cet article des bornes inférieures universelles, en limite semiclassique, de la taille des résonances de forme associées aux l'opérateurs de Schrödinger à l'exterieur d'obstacles avec des conditions au bord de Dirichlet ou de Neumann et des potentiels à longue portée analytiquement dilatables. Ces bornes inférieures sont exponentiellement petites par rapport à la constante de Planck.

## 1. Introduction, hypotheses

The purpose of this article is the study of the localization of resonances for the semi-classical Schrödinger operator near positive energy levels. The shape resonances are the metastable states of a system whose evolution is described by a Hamiltonian, $H$, depending upon the Planck constant $h$. An important feature of the resonances is their lifetime (the reverse of their width) which describes the time needed by a wave packet associated to the resonance and initially localized in a compact region of the space, to develop a non zero probality density in the outside region. In this paper we shall study the semi-classical approximation when the planck constant tends to 0. Most results about the shape resonances are concerned about the *existence* of resonances and *upper* bounds for their width (see the works by J.M. Combes, P. Duclos, M. Klein and R. Seiler [**CDKS87**], P.D. Hislop and I.M. Sigal [**HS89**] and by B. Helffer and J. Sjöstrand [**HS86**]) However, in the case of trapped energy levels, there are very few results about *non existence* of resonances or more generally about *lower* bounds for the width of shape resonances. To our knowledge, the only known results about such matters were particular cases: for a Schrödinger operator in space dimension 1 equal to $-\partial_x^2 + V(x)$ with a potential compactly supported E.M. Harrel [**Har82**] gave lower bounds. In space dimension $d \geq 1$, B. Helffer and J. Sjöstrand [**HS86**] for analytic dilation potential with a unique non degenerate local minimum and energy near the bottom of the well gave a complete asymptotic expansion of the resonances. Later on C.Fernandez and R.Lavine [**FL90**] extended the results by E.Harrel to the case of space dimension $d \geq 1$ and potential with compact support. Finally the author [**Bur98**] has given such lower bounds for the Helmoltz operator (null potential) outside any obstacle (whose complementary is connected), with Dirichlet conditions on the boundary. These results have been recently extended to superexponentially decaying perturbations of the Laplace operator ($\mathcal{O}(e^{-|x|^{2+\varepsilon}})$ by G. Vodev [**Vod00**], using the same kind of methods. In fact, the methods developped in [**Bur98**] could apply to compactly supported perturbations of the Laplace operator (in the semi-classical framework).

The result we obtain in this article is the following:

***Theorem 1***. — *Consider a long range ($\mathcal{O}\left(1/|x|^{\beta}\right), \beta > 0$) self adjoint analytic dilation perturbation of the semi-classical Laplace operator $-h^2\Delta$ (satisfying the assumptions [1..4] of section 2). Then for any $0 < E_m < E_M < +\infty$, there exists $C, d, h_0 > 0$ such that for any $0 < h < h_0$, the operator $P(h)$ has no resonance in the set*

$$\{z \in \mathbb{C};\ \operatorname{dist}(z, [E_m, E_M]) \leq Ce^{-d/h} \tag{1.1}$$

***Remark 1.1***. — *It is very likely that our method could deal with unbounded energy level (e.g. to control the constants with respect to $E_M$). However, the question of the null energy is very different and probably deserves different methods.*

The result of theorem 1 means that near any strictly positive energy $E$, the resonances can not accumulate to the real axis faster than exponentially. In fact, we shall obtain a more precise result: the resonances are defined using complex scaling as the eigenvalues of certain non self adjoint operator. We shall give an (exponentially large) estimation of the norm of the resolvent of this operator on any compact $K \subset \mathbb{R}_+^*$.

A natural question would be to determine the optimal constant $d$ near a fixed level of energy (e.g. for $K$ a small neighbourhood of a point $E$) in terms of the geometry of the problem. In the one dimensional case or if the potential and the metric are rotation invariant, it is possible to recover, following our method, some classical results and to express $C$ in terms of the Agmon distance. However, in the general case, the study of the Helmoltz resonator and results by A. Martinez on the tunneling effect for the double well [**Mar88**], section 7, make us think that the optimal constant is more likely to be a microlocal quantity than a local one.

The starting point for the proof of this theorem is the same as in [**Bur98**]: use Carleman estimates to deal with the region where one has no information (any compact region). Then in [**Bur98**] it was possible to conclude using that the operator was equal to $-h^2\Delta$ outside of a compact set, by making computations on the unperturbed outgoing resolvent of the Laplace operator. Here we shall use the fact that the operator $P$ tends to $-h^2\Delta$ when $r$ tends to $+\infty$ in order to go back from Carleman estimates to Mourre type estimates (positive commutators estimates) and then perform analytic dilation to conclude. These two steps rely heavily on explicit and quite technical computations. We were unfortunately unable to give a more conceptual proof of the result.

***Remark 1.2.*** — *It is likelly that the result might also hold without the assumption that the perturbation tends to zero at infinity, provided that one assumes the existence of certain escape functions. For the sake of simplicity we did not want to get into such refinments.*

***Remark 1.3.*** — *The* dilation analyticity *assumption is necessary to define the resonances. However, dropping this assumption and replacing it by a symbol-like decay for the perturbation, the methods developped in this paper (and the limit absorption principle) would lead to a weighted estimate for the outgoing resolvent. To keep this paper a reasonable length, we leave this for a forthcoming article.*

The paper is organized as follows: in part 2 we give some basic facts about complex scaling. These results are due to J. Sjöstrand and M. Zworski. We also state a result (theorem 3) about the resolvent of the dilated operators which implies theorem 1. In part 3 we recall some results on Carleman estimates for semi-classical second order boundary value problems. In the context of the Laplace operator, these results are due to G. Lebeau and L. Robbiano [**LR95b, LR95a**]. Here we state the results for Helmoltz-like operators and for the convenience of the reader we outline the proofs. In the most calculatory part of the paper (part 4) we construct the phase functions that we shall use in part 6 and 7 to prove the main technical result of the article (propositions 6.4 and 7.4. In part 5 we show how theorem 3 and 4 can be deduced from these technical propositions (and from Carleman estimates). In section 8, we give a byproduct of this work concerning the decay of the local energy for solutions to the wave equation with asymptotically flat metric (announced below as theorem 2). Finally ,in an appendix we give an outline of some technical results used to simplify the form of our operators. Almost all the constants that appear in the paper are fixed in section 4 by the construction of these phase functions. This is the reason why we had to write this section at this point of the paper. However, for a first lecture, we strongly recommend the reader to postpone the analysis of the proofs of the results in section 4.

To finish this introduction let us try to give a flavour of some consequences of our results for solutions of wave equation that we shall developp in section 8.

Consider $\Theta \subset \mathbb{R}^d$ a (possibly empty) smooth obstacle whose complementary, $\Omega = \Theta^c$, is connected. Suppose that the boundary of $\Theta$, $\partial\Theta$ is the disjoint union of two components, $\Gamma_D$ and $\Gamma_N$. Note $g$ a smooth (positive definite) metric $g = (g_{i,j})_{i,j=1,\dots d}$ in $\Omega$, satisfying the *smoothness, analyticity at infinity, ellipticity and asymptotic flatness* assumptions of section 2. Consider for $(u_0, u_1) \in C_0^\infty(\Omega)$, $u(t)$ the solution of the wave equation:

$$\left(\partial_t^2 - \frac{1}{\sqrt{\det g}}\partial_{x_i}\sqrt{\det g}\, g^{i,j}(x)\,\partial_{x_j}\right) u = 0 \tag{1.2}$$

$$u\mid_{\Gamma_D} = 0 \qquad \frac{\partial u}{\partial n}\mid_{\Gamma_N} + b(x)\,u\mid_{\Gamma_N} = 0 \tag{1.3}$$

$$u\mid_{t=0} = u_0 \qquad \partial_t u\mid_{t=0} = u_1 \tag{1.4}$$

with $\partial_n$ the normal derivative associated to the metric $g$ and $(g^{i,j}) = (g_{i,j})^{-1}$ and $b(x) \in \mathbb{R}$ (to have a self adjoint operator).

***Theorem 2.*** — *For any $k \in \mathbb{R}^+$ and any $R > 0$ there exists a constant $C > 0$ such that for any initial data, $(u_0, u_1) \in C_0^\infty(\Omega)^2$ with support in the ball $\Omega \cap B(0,R)$ and any $t > 1$ we have*

$$\|u(t)\|_{H^1(\Omega\cap B(0,R))} + \|\partial_t u(t)\|_{L^2(\Omega\cap B(0,R))} \le \frac{C}{(\log t)^k}\left[\|u_0\|_{H^{1+k}} + \|u_1\|_{H^k}\right] \tag{1.5}$$

*There exists $R_0 > 0$ such that for any $R > R_0$ there exists a constant $C > 0$ such that for any initial data, $(u_0, u_1) \in H^1 \times L^2$ with support in the ring $\Omega \cap (B(0,R) \setminus B(0,R_0))$ we have*

$$\begin{aligned}\int_0^{+\infty} \|u(t)\|^2_{H^1(\Omega\cap(B(0,R)\setminus B(0,R_0)))} + \|\partial_t u(t)\|^2_{L^2(\Omega\cap(B(0,R)\setminus B(0,R_0)))} \\ \le C\left[\|u_0\|^2_{H^1} + \|u_1\|^2_{L^2}\right]\end{aligned} \tag{1.6}$$

## 2. A review of complex scaling in the semi-classical case

**2.1. Assumptions.** — Let us first precise the framework of this article:

Let $\Theta \subset \mathbb{R}^d$ be a (possibly empty) smooth obstacle whose complementary, $\Omega = \Theta^c$, is connected. Suppose that the boundary of $\Theta$, $\partial\Theta$ is the disjoint union of two components, $\Gamma_D$ and $\Gamma_N$. Consider a semi-classical *self adjoint* operator defined on $L^2(\Omega)$:

$$Q = \sum_{i,j=1}^d a_{i,j}(x) hD_{x_i} hD_{x_j} + \sum_{i=1}^d (b_i(x,h))\, hD_{x_i} + V(x,h), \qquad D_x = \frac{\partial_x}{i} \tag{2.1}$$

with domain

$$\mathcal{D} = \{u \in H^2(\Omega)\,;\, u\mid_{\Gamma_D} = 0 \text{ and } \frac{\partial u}{\partial n}\mid_{\Gamma_N} + b(x)\,u\mid_{\Gamma_N} = 0\} \tag{2.2}$$

where $\partial_n$ is the normal to the boundary with respect to the metric $g = (a_{i,j})^{-1}$. Observe that $\Gamma_D$ or $\Gamma_N$ or the both of them can be empty and that the self adjointness of the operator $Q$ imposes compatibility conditions for the Neumann part of the boundary condition.

Following J. Sjöstrand [**Sjö97**], we make the following assumptions about the operator $Q$

1. **(Smoothness.)** The functions $a_{i,j}, b_i, V$ and their derivatives are (uniformly with respect to $1 > h > 0$) bounded for the $C^\infty\left(\mathbb{R}^d\right)$ topology (in any compact set)
2. **(Analyticity at infinity.)** There exists $\theta_0, \varepsilon > 0$ and $R_0 > 0$ such that the functions $a_{i,j}, a_i, b_i, V$ and $W$ are analytic in x in the domain

$$\Lambda_{\theta_0,R_0,d_0} = \Big\{ r\omega; \omega \in \mathbb{C},\ \operatorname{dist}\left(\omega, \mathbb{S}^{d-1}\right) < d_0, \quad r \in \mathbb{C},\ |r| > R_0,\ \arg(r) \in [-\theta_0, \theta_0] \Big\} \tag{2.3}$$

3. **(Ellipticity.)** The operator $Q$ is supposed to be elliptic

$$\exists C > 0; \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^d \sum_{i,j=1}^{d} a_{i,j}(x)\,\xi_i\xi_j \geq C|\xi|^2 \tag{2.4}$$

4. **(Asymptotic flatness.)** The operator $Q$ is close to $-h^2\Delta$ at infinity:

$$\begin{gathered} \exists \beta > 0, M > 0;\ \forall x \in \Lambda_{\theta_0,R_0,d_0},\ \forall 0 < h < 1, \\ |a_{i,j}(x) - \delta_{i,j}| \leq \frac{M}{|x|^\beta}, \qquad |b_i(x,h)| + |V|(x,h) \leq \frac{M}{|x|^\beta} \end{gathered} \tag{2.5}$$

***Remark 2.1.*** — *According to Cauchy formula,* (2.5) *imply (in $\Lambda_{\theta_0/2,R_0+1,d_0/2}$)*

$$|\partial_x^\alpha a_{i,j}(x)| + |\partial_x^\alpha b_i(x)| + |\partial_x^\alpha V|(x) \leq \frac{M}{|x|^{|\alpha|+\beta}} \tag{2.6}$$

We shall note

$$\begin{aligned} Q_0(x, D_x) &= \sum_{i,j=1}^{d} a_{i,j}(x)\, hD_{x_i} dD_{x_j} \\ q(x,\xi,h) &= \sum_{i,j=1}^{d} a_{i,j}(x)\,\xi_i\xi_j + \sum_{i=1}^{d} b_i(x,h)\,\xi^i \end{aligned} \tag{2.7}$$

It was shown by J. Sjöstrand [**Sjö97**] that with these hypotheses one can define the resonances of the semi classical operator $Q$ in any sector $S_{\theta_0} = \{z \in \mathbb{C}; z = e^{-i\theta}r,\ r \geq 0, \theta \in [0,\theta_0]\}$ and that these resonances form a discrete set. However, before performing the analytic dilation, we shall make some reductions concerning the form of the operator $Q$.

**2.2. Reductions for the operator $Q$.** — The proof of the following result is given in appendix A:

***Proposition 2.2.*** — *There exists $R_0 > 0$, $\theta_1 > 0$ and a holomorphic change of coordinates:*

$$\chi : R_0 + ]0, +\infty[e^{i]-\theta_0,\theta_0[} \times B(\mathbb{S}^{d-1}, \varepsilon) \to \Omega \subset \mathbb{C}^d \tag{2.8}$$

*(where $B(\mathbb{S}^{d-1}, \varepsilon)$ is an $\varepsilon$-neighborhood of $\mathbb{S}^{d-1}$ in the complexified $d-1$ dimensionnal sphere) satisfying the following:*

1. *$\chi$ induces a diffeomorphism from $B(0,R_0)^c$ to $\Omega\cap\mathbb{R}^d$*
2. *Seen as a map from $\mathbb{C}^d$ to $\mathbb{C}^d$, $\chi$ is asymptotically equal to the identity: Note $\chi_r$ and $\chi_\sigma$ the components of $\chi$ in radial coordinates. Then*

$$\|\chi_\sigma(r,\sigma)-\sigma\|\leq\frac{\epsilon}{|r|^\beta}\tag{2.9}$$

*and*

$$\|\frac{\chi_r(r,\sigma)-r}{r}\|\leq\frac{\epsilon}{|r|^\beta}\tag{2.10}$$

*where $\epsilon$ can be taken small if $R_0$ is large.*

*and such that in this new coordinate system the operator $Q$ still has the form (2.1), satisfies the analyticity and asymptotic flatness (with possibly different constants $\theta_0$, $d_0$ and $M$) assumption and furthermore its second order part is equal to (in polar coordinates, $(r,\sigma)$)*

$$-(h\partial_r)^2-h^2P_2(r,\sigma,\frac{h\nabla_\sigma}{r})\tag{2.11}$$

*with $P_2$ a quadratic form. In this coordinate system, the operator $Q$ is self adjoint for the measure $\mu=f(r,\sigma)drdL$, image of the Lebesgue measure on $\mathbb{R}^d$. ($dL$ is the Lebesgue measure on $\mathbb{S}^{d-1}$ and $f\sim r^{d-1}$)*

***Remark 2.3***. — *This result is only a global (and holomorphic) version at infinity of a well known result saying that the terms $\partial_r\partial_\sigma$ can be eliminated by the choice of a geodesic coordinate system. The holomorphic version is straightforward by solving the geodesic equations in the complex space; and the globality at infinity follows from the fact that the operator is asymptotically flat.*

We'll also obtain in appendix A the following result:

***Proposition 2.4***. — *There exists a holomorphic function $k=f^{1/2}e^{-\hat{f}/h}$ with $\hat{f}\mid_{\mathbb{R}^d}\in i\mathbb{R}$ such that the conjugate operator $P=k^{-1}Qk$ is self adjoint for the measure $drdL$ and has the following form (for $|x|\geq R_0$):*

$$P=-h^2\partial_r^2-h^2\frac{\Delta_\sigma}{r^2}+q_2(r,\sigma,\frac{h\nabla_\sigma}{r})+q_1(r,\sigma,\frac{h\nabla_\sigma}{r},h)+V(r,\sigma,h)\tag{2.12}$$

*where $\Delta_\sigma$ is (in the $\sigma$ coordinate system) the Laplace operator on the sphere $\mathbb{S}^{d-1}$ and $q_2$, $q_1$ and $V$ are self adjoint operators (of orders 2, 1 and 0 respectively) such that if*

$$q_2(r,\sigma,\frac{h\nabla_\sigma}{r})=\sum_{i,j}q_2^{i,j}(r,\sigma)\frac{h\partial_{\sigma_i}}{r}\frac{h\partial_{\sigma_i}}{r}\tag{2.13}$$

$$q_1(r,\sigma,\frac{h\nabla_\sigma}{r},h)=\sum_k q_1^k(r,\sigma,h)\frac{h\partial_{\sigma_i}}{r}\tag{2.14}$$

*then the coefficients* $q_2^{i,j}$, $q_1^k$ *and* $V$ *are holomorphic in* $\Lambda_{\theta'_0,R'_0,d'_0}$ *and satisfy there the following estimates (uniformly with respect to* $0<h<1$*)*

$$\left|q_2^{i,j}(r,\sigma)\right|+\left|q_1^k(r,\sigma,h)\right|+|V(r,\sigma,h)|\leq\frac{M}{|r|^{\beta}}. \tag{2.15}$$

***Remark 2.5.*** — *In the first part of this proposition, the conjugation by* $f^{1/2}$ *makes the operator self adjoint with respect to the metric* $drdL$ *and the conjugation by* $e^{\hat{f}/h}$ *does eliminate the first order term in* $\partial_r$. *The fact that* $\hat{f}$ *is purely imaginary is a simple consequence of the fact that this first order term is self adjoint (hence the coefficient of* $\partial_r$ *is purely imaginary). The second part of the proposition is simply the traduction that the operator is (taking the conjugation into account) close to the operator*

$$P_0=-h^2\partial_r^2-h^2\frac{\Delta_{\mathbb{S}^{d-1}}}{r^2} \tag{2.16}$$

***Remark 2.6.*** — *Since the functions* $q_2^{i,j}$, $q_1^k$ *and* $q_0$ *are holomorphic in conic domains, the Cauchy formula imply (after disminishing the angle of these domains), that we have for any* $\alpha\in\mathbb{N}^d$,

$$\begin{aligned}\left|\partial_r^{\alpha_1}\partial_\sigma^{\alpha_2}\left(q_2^{i,j}\right)(r,\sigma)\right|&\leq\frac{M}{|r|^{\beta+|\alpha_1|}},\\ \left|\partial_r^{\alpha_1}\partial_\sigma^{\alpha_2}\left(q_1^{k}\right)(r,\sigma,h)\right|&\leq\frac{M}{|r|^{\beta+|\alpha_1|}},\\ \left|\partial_r^{\alpha_1}\partial_\sigma^{\alpha_2}\left(V\right)(r,\sigma,h)\right|&\leq\frac{M}{|r|^{\beta+|\alpha_1|}}\end{aligned} \tag{2.17}$$

Here, we work in a (finite number of) local coordinate system on the sphere, but for the sake of simplicity, we shall do as if this coordinate system was global.

***Remark 2.7.*** — *In the sequel we'll always work in the coordinate system above and we'll always perform the conjugation of proposition 2.4. In fact in the set* $\Omega\cap B(0,R_0)$ *where this special coordinate system does not exist, the analysis we use to treat what happens (Carleman estimates) is invariant by change of coordinates. So, for our purpose, the fact that this coordinate system does not exist everywhere will be transparent.*

**2.3. Analytic dilation.** — We now recall the definition of resonances by Sjöstrand [**Sjö97**] in this framework. We follow very closely [**Sjö97**].

For given $\varepsilon_0>0$, $B>>1$, we can construct a smooth function $f_\theta(t):[0,\pi]\times[0,+\infty[\to\mathbb{C}$, injective for any $\theta$ with the following properties:

(i) $f_\theta(t)=t$ for $0\leq t\leq B$
(ii) $0\leq\arg(f_\theta(t))\leq\theta,\partial_t f_\theta\neq 0$
(iii) $\arg(f_\theta(t))\leq\arg(\partial_t f_\theta(t))\leq\arg(f_\theta(t))+\varepsilon_0$
(iv) $f_\theta(t)=e^{i\theta}t$ for $t\geq T_0$ where $T_0$ depends only on $\varepsilon_0$ and $B$.
(v) $\arg(f_\theta(t))$ is an increasing function of $\theta$ and $t$.
(vi) $\arg(f_\theta(t))>0$ for $t>B$.

For technical reasons, we'll suppose that the function $f_\theta(t)$ satisfies the following additional assumptions

(vii) $f_\theta(t) = te^{i\kappa_\theta(t)}$ with $\kappa_\theta$ a real, increasing function such that

$$|\kappa'_\theta|^2 + |\kappa''_\theta|^2 + |\kappa'''_\theta|^2 + |\kappa^{(4)}_\theta|^2 \leq C_\theta \kappa_\theta \tag{2.18}$$

The analytic dilation will be performed for $\theta \in [0, \theta_0]$. We will also suppose that the angle $\theta_0 > 0$ is taken so small that

$$-\mathrm{Im}\left(\frac{e^{-i\kappa_{\theta_0}(t)}}{(1+it\kappa'_{\theta_0}(t))^2}\right) \geq \frac{3}{4}(\kappa_{\theta_0} + r\kappa'_{\theta_0})(t), \qquad \mathrm{Im}(e^{i\kappa_{\theta_0}(t)}) \geq \frac{3}{4}\kappa_{\theta_0}(t) \tag{2.19}$$

Consider the map

$$\kappa_{\theta_0} : \mathbb{R} \ni x = t\omega \mapsto f_{\theta_0}(t)\,\omega \in \mathbb{C}, t = |x|. \qquad (\omega \in \mathbb{S}^{d-1}) \tag{2.20}$$

Then the image of $\kappa_{\theta_0}$, $\Gamma_{\theta_0}$, is a totally real manifold which coincides with $\mathbb{R}^d$ along $B(0, B)$.

Define

$$L^2_{\theta_0} = L^2(\Gamma_{\theta_0} \setminus \Theta) \tag{2.21}$$

$$\mathcal{H}_{0,\theta_0} = \{u \in H^1(\Gamma_{\theta_0})\,;\ u\,|_{\Gamma_D} = 0\} \tag{2.22}$$

$$\mathcal{D}_{\theta_0} = \{u \in H^2(\Gamma_{\theta_0})\,; u\,|_{\Gamma_D} = 0, \left(\frac{\partial u}{\partial n} + b(x)u\right)|_{\Gamma_N} = 0 \tag{2.23}$$

where $H^2(\Gamma_{\theta_0})$ and $H^1(\Gamma_{\theta_0})$ are equipped with their natural semi-classical norms:

$$\begin{aligned} \|u\|^2_{\mathcal{H}_{0,\theta_0}} &= \|u\|^2_{L^2} + \|h\nabla u\|^2_{L^2} \\ \|u\|^2_{\mathcal{D}_{\theta_0}} &= \|u\|^2_{L^2} + \|h\nabla u\|^2_{L^2} + \sum_{i,j} \|hD_i hD_j u\|^2_{L^2} \end{aligned} \tag{2.24}$$

Lets consider the operator $P_{\theta_0}(h)$ acting on $L^2(\Gamma_{\theta_0} \setminus \Theta)$ with domain $\mathcal{D}_{\theta_0}$ which inside $B(0, B)$ coincide with $P(h)$ and outside is (in the polar coordinates coming from $\kappa_{\theta_0}$) equal to

$$P_{\theta_0}(h) = P\left(f_{\theta_0}(t)\,\omega, f'_{\theta_0}(t)\,D_t, D_\omega\right) \tag{2.25}$$

The following lemma is shown in [**Sjö97**] (lemma 5.1 and 5.2)

***Lemma 2.8***. — *If $z \in \mathbb{C} \setminus \{0\}$, $\arg(z) \neq -2\theta_0$, then the operator $P_{\theta_0} - z$ from $\mathcal{D}_{\theta_0}$ to $L^2_{\theta_0}$ is a Fredholm operator of index 0. Furthermore, if $\arg(z) < -2\theta_1 \leq -2\theta_2$ then*

$$dim\ Ker(P_{\theta_1} - z) = dim\ Ker(P_{\theta_2} - z) \tag{2.26}$$

This lemma and the analytic Fredholm theory show that the spectrum of $P_\theta$ in $\mathbb{C} \setminus e^{-2i\theta}[0, +\infty[$ is discrete and included in $]-\infty, 0] \cup e^{-2i[0,\theta[}[0, +\infty[$, the part in $]-\infty, 0]$ beeing the eigenvalues of the operator $P(h)$.

***Definition 2.9***. — *We say that $z \in e^{-2i[0,\theta[}[0, +\infty[$ is a resonance for the operator $P(h)$ if and only if $z \in \sigma(P_\theta)$ for some (and hence all) $\theta > -\arg(z)/2$.*

Since the operator $P_\theta$ is a Fredholm operator of index 0 it is equivalent to define the resonances as the eigenvalues of the operator $P_\theta(h)$. The theorem 1 is now a straightforward consequence of the following result:

***Theorem 3***. — *For any compact set, $K \subset \mathbb{R}_+^*$ there exists $B$ and $\theta_0 > 0$, $C, C', h_0 > 0$ such that for any $0 < h < h_0$ and any $e \in K$, the operator $P_{\theta_0}(h) - e\mathrm{Id}$ is invertible on $L^2(\Gamma_{\theta_0})$ and one has*

$$\| (P_{\theta_0}(h) - e\mathrm{Id})^{-1} \|_{L^2(\Gamma_{\theta_0})} \leq Ce^{C'/h} \tag{2.27}$$

***Remark 2.10***. — *We are going to prove theorem 3 for some particular choices of $B$ and of the function $f_\theta$. However it is a consequence of the works by J. Sjöstrand and M. Zworski* [**SZ91**] *(see the proof of lemma 3.4) that theorem 3 (i.e. the estimate* (2.27) *for a particular choice of $B$ and $\theta_0$ implies that the estimate* (2.27) *is still valid (with a different constant $C$)* for any choice *of $B' > B$ and $0 < \theta' < \theta_0$, satisfying assumptions (i) to (vi). Even more, we could perform the analytic dilation in the initial coordinate system in $\mathbb{R}^d$ (instead of doing it in the coordinate system of appendix A) and estimate* (2.27) *would still be valid.*

As a by-product of the proof of theorem 3 we obtain the following:

***Theorem 4***. — *There exists $R_1$ such that for any $R_2 > 0$ and any compact set, $K \subset \mathbb{R}_+^*$ there exists $C, h_0 > 0$ such that for any $0 < h < h_0$ and any $e \in K$ one has*

$$\|1_{R_1 \leq |x| \leq R_2} (P(h) - e\mathrm{Id})^{-1} 1_{R_1 \leq |x| \leq R_2}\|_{L^2} \leq \frac{C}{h} \tag{2.28}$$

*where $(P(h) - e\mathrm{Id})^{-1}$ denotes the outgoing resolvent of the operator $P$.*

***Remark 2.11***. — *The $C/h$ bound in estimate* (2.28) *is the natural bound obtained in non trapping cases for the resolvent truncated in balls. Here we see that if the cuttoff is performed in rings, then the natural estimate is always valid. This result is a weak (but very general) version of a recent result obtained by Martinez in a particular case showing that in the case of a potential with a unique non degenerate minimum, even if there are resonances exponentially close to the real axis, at these points the residue of the scattering matrix is exponentially small; these two exponentials compensating exactly each other.*

***Remark 2.12***. — *By a simple scaling argument, we can suppose that the compact set, $K$ is equal to* $[1, 2]$.

## 3. A review of Carleman estimates

In this section we give some Carleman estimates up to the boundary for the Schrödinger operator. Such kind of estimates were originaly proved by Hormander in the interior, then extended by G. Lebeau and L. Robbiano for the Laplace operator with Dirichlet

([**LR95a**]) and Neumann ([**LR95b**]) conditions. The proof of these estimates for the Schrödinger operator is essentially the same as the proof by Lebeau and Robbiano.

Let $\Omega$ be a smooth, bounded domain of $\mathbb{R}^d$. Let $e \in [1,2]$, $P^e = P(h) - e$ be the Schrödinger operator with semi-classical principal symbol $p^e(x,\xi,h)$.

$$p^e(x,\xi,h) = \sum_{i,j} a_{i,j}(x)\xi_i\xi_j + \sum_i b_i(x,h)\xi_i + V(x,h) - e \tag{3.1}$$

Consider a function $\varphi \in C^\infty(\mathbb{R}^d)$ which satisfies the hypoellipticity assumptions (uniformly with respect to $0 < h < h_0$):

$$\exists c > 0; \forall x \in \overline{\Omega}\ \forall \xi \in \mathbb{R}^d,\ p^e_\varphi(x,\xi) = 0 \Rightarrow \{\mathcal{R}e\, p^e_\varphi, \mathrm{Im}\, p^e_\varphi\} \geq c \tag{3.2}$$

where $\{f,g\} = \sum_i \partial_{\xi_i} f \partial_{x_i} g - \partial_{x_i} f \partial_{\xi_i} g$ is the Poisson bracket of the two functions $f$ and $g$. We shall also suppose that

$$\frac{\partial\varphi}{\partial n} \mid_{\partial\Omega} \neq 0 \text{ and } \nabla\varphi \neq 0 \text{ in } \overline{\Omega}. \tag{3.3}$$

The operator $P^e_\varphi$ is defined by $P^e_\varphi = e^{\varphi/h} P^e e^{-\varphi/h}$. His principal symbol is

$$p^e_\varphi = p^e(x, \xi + i\varphi'_x, h). \tag{3.4}$$

If the function $f$ satisfies the boundary conditions (2.2), then the conjugate function $g = e^{\varphi/h} u$ satisfies the following:

$$g \mid_{\Gamma_D} = 0, \qquad (h\frac{\partial g}{i\partial n} + ihbg + i\nabla\varphi \cdot \partial_n @ g) \mid_{\Gamma_N} = 0 \tag{3.5}$$

that we'll note (with $k = -(i\nabla\varphi \cdot \partial_n + ihb)$,

$$g \mid_{\Gamma_D} = 0, \qquad h\frac{\partial g}{i\partial n} \mid_{\Gamma_N} = kg \mid_{\Gamma_N}. \tag{3.6}$$

***Proposition 3.1.*** — *Let $\varphi$ satisfy conditions (3.2) and (3.3). There exists $C > 0$, $h_1 > 0$, such that for any $0 < h < h_1$ and any $g \in C^\infty(\overline{\Omega})$*

$$\begin{aligned}\int_\Omega |P^e_\varphi(g)|^2 + h\int_{\partial\Omega} \left(|g \mid_{\partial\Omega}|^2 + |h\partial_{x'} g \mid_{\partial\Omega}|^2 + |h\partial_{x_n} g \mid_{\partial\Omega}|^2\right) \\ \geq Ch\int_\Omega \left(|g|^2 + |h\nabla g|^2\right).\end{aligned} \tag{3.7}$$

***Proposition 3.2.*** — *Let $\varphi$ satisfy conditions (3.2) and (3.3). Note $\Gamma = \Gamma_D \cup \Gamma_N \subset \partial\Omega$ a union of connected components of the boundary $\partial\Omega$. Suppose that $\frac{\partial\varphi}{\partial n} > 0$ on $\Gamma$ ($\frac{\partial}{\partial n}$ is the incoming normal unit on the boundary), then there exists $c > 0$, $h_1 > 0$, such that for any $0 < h < h_1$ and any $g \in C^\infty(\overline{\Omega})$ such that $g \mid_{\Gamma_D} = 0$, and $(hD_n f \mid_{\Gamma_N} = kf \mid_{\Gamma_N}$,*

$$\begin{aligned}\int_\Omega |P^e_\varphi(g)|^2 + h\int_{\partial\Omega\setminus\Gamma} \left(|g \mid_{\partial\Omega}|^2 + |h\partial_{x'} g \mid_{\partial\Omega}|^2 + |h\partial_{x_n} g \mid_{\partial\Omega}|^2\right) \\ \geq Ch\int_\Omega \left(|g|^2 + |h\nabla g|^2\right).\end{aligned} \tag{3.8}$$

The analogues of these propositions are proved by G. Lebeau and L. Robbiano [**LR95a, LR95b**], for the Laplace operator. We shall give here an outline of the proof in our particular case.

***Remark 3.3.*** — *Estimations such as (3.7) and (3.8) are stated globally. However, by means of partitions of unity, it is clearly possible to reduce the proof to* local *proofs (provided that the phase $\varphi$ is* globally *defined). They also depend clearly only on the (semi classical) principal part of the operator. Hence the conjugation by any function $h$-independant is of no effect.*

In the interior such estimations are classical. Hence, for the proof of propositions 3.1 and 3.2, we can assume that the function $g$ is supported by a small neighbourhood of the boundary (remark that in any case the result near the boundary implies the result in the interior). Taking a geodesic coordinate system for the second order elliptic operator $P_0$, component, we can assume also that $\Omega = \{(x', x_d) \in \mathbb{R}^d;\ x_d > 0\},\ g \in C_0^\infty(K)$, the set of functions $C^\infty$ up to the boundary of $\mathbb{R}^d_+$, supported by $K = \{x \in \mathbb{R}^d; |x| \leq D\}$ and that $P - e$ is equal to

$$P - e = h^2 D^2_{x_d} + R(x_d, x', h\frac{Dx'}{i}) + 2ahD_{x_d}, \tag{3.9}$$

with $R$ a tangential differential operator of order 2. Conjugating the operator $P$ by a suitable function, we also can assume that the operator considered is self adjoint with respect to the measure $dx_d dx'$. In the sequel, we shall dropp the $e$ exponent and suppose that $e$ is fixed.

Since the operator $P$ is self adjoint, $a(x,h) \in \mathbb{R}$, we can then perform a conjugation with $e^{ig(x_d,x',h)}$, $\frac{dg}{dx_d} = a(x_d, x', h)$ the $h\partial_{x_d}$ component, leading to an operator

$$P - e = h^2 D^2_{x_d} + R(x_d, x', h\frac{Dx'}{i}) \tag{3.10}$$

with a possibly different $R$. Remark that we conjugate the operator by an $h$-dependent function; however, $g$ is uniformly bounded with respect to $h$ as well as any derivative and everything works as if $g$ was $h$-independant.

We recall now how G. Lebeau et L. Robbiano proceed to prove estimations (3.7) and (3.8) for operators of the form (3.10). The technical details are in [**LR95a**], part 3, pages 347 to 351 and [**LR95b**] part 5.

We shall denote by $S^j$ the space of $C^\infty$ tangential symbols $a(x, \xi', h)$ defined in the boundary of $K$ et $\xi' \in \mathbb{R}^{d-1}$, such that for $0 < h < h_0$

$$\forall \alpha, \beta,\ \exists C\ \forall x, \xi', h,\ |\partial_x^\alpha \partial_{\xi'}^\beta a| \leq (1 + |\xi'|)^{j-|\beta|} \tag{3.11}$$

Let $S^j_{\mathrm{cl}} \subset S^j$ be the subspace of the symbols of the form $a(x,\xi,h) = a_0(x,\xi) + hb(x,\xi,h)$ with $b \in S^{j-1}$. Let $\mathcal{E}^j$ (resp. $\mathcal{E}^j_{\mathrm{cl}}$) be the space of tangential pseudo-differential operators, $A = Op(a)\,;\ a \in S^j$ (resp. $a \in S^j_{\mathrm{cl}}$) defined by

$$Op(a)\, u(x_d, x') = (2\pi h)^{1-d} \int e^{i(x'-y')\xi'/h} a(x_d, x', \xi', h)\, u(x_d, y')\, dy' d\xi'. \tag{3.12}$$

For $A \in \mathcal{E}^j_{\mathrm{cl}}$ we shall note $\sigma(a) = a_0$ the principal symbol of $A$ et $\mathcal{D}^j \subset \mathcal{E}$ the subspace of the differential operators.

Let $\widetilde{Q}_1 = \frac{P_\varphi - P_\varphi^*}{2i} = \mathrm{Im} P_\varphi$ and $\widetilde{Q}_2 = \frac{P_\varphi + P_\varphi^*}{2} = \mathrm{Re} P_\varphi$,with principal symbols $2q_1 + 2\varphi'_{x_d}\xi_d = \mathrm{Im} p_\varphi$ and $q_2 + 2\xi_d^2 = \mathrm{Re} p_\varphi$. Consider, for $f, g \in C_0^\infty(K)$

$$((f,g))_0 = \int_{\mathbb{R}^d_+} f\overline{g}dx, \qquad (f,g)_0 = \int_{\mathbb{R}^{d-1}} f\mid_{x_d=0} \overline{g}\mid_{x_d=0} dx'$$

$$((f,g))_1 = \int_{\mathbb{R}^d_+} h\nabla f\overline{\nabla g} + f\overline{g}dx, \tag{3.13}$$

$$(f,g)_1 = \int_{\mathbb{R}^{d-1}} h\nabla f\mid_{x_d=0} \overline{\nabla g}\mid_{x_d=0} + f\mid_{x_d=0} \overline{g}\mid_{x_d=0} dx'$$

and $\|f\|_0$, $|f|_0$, $\|f\|_1$, $|f|_1$, the associated norms.

$$\begin{aligned} \|P_\varphi f\|_0^2 &= \|\widetilde{Q}_2 f\|_0^2 + \|\widetilde{Q}_1 f\|_0^2 + i\big((\widetilde{Q}_1 f, \widetilde{Q}_2 f)\big) - i\big((\widetilde{Q}_2 f, \widetilde{Q}_1 f)\big)_0 \\ &= \|\widetilde{Q}_2 f\|_0^2 + \|\widetilde{Q}_1 f\|_0^2 + i\big(\big(\left[\widetilde{Q}_2, \widetilde{Q}_1\right] f, f\big)\big) + h\beta(f), \end{aligned} \tag{3.14}$$

with ($D_x = \frac{h\partial_x}{i}$)

$$\begin{aligned} \beta(f) &= \left(\widetilde{Q}_1 f, D_{x_d} f\right)_0 + \left(\left(D_{x_d}\widetilde{Q}_1 - 2\varphi'_{x_d}\widetilde{Q}_2 f\right), f\right)_0 \\ &= \left(2\varphi'_{x_d} D_{x_d} f, D_{x_d} f\right)_0 + (A_1 f, D_{x_d} f)_0 + (A'_1 D_{x_d} f, f)_0 + (A_2 f, f)_0 \end{aligned} \tag{3.15}$$

where $A_i$ belong to $\mathcal{D}^i$.

The principal symbol of the operator $i\left[\widetilde{Q}_2, \widetilde{Q}_1\right]$ is $h\{\mathrm{Re} p_\varphi, \mathrm{Im} p_\varphi\}$ and Gårding inequality gives from (3.4) (see [**LR95b**], paragraphe 3, lemme 2) that

$$\|P_\varphi f\|_0^2 - \mathrm{Re} h\beta(f) \geq ch\left(\|f\|_0^2 + \|h\nabla f\|_0^2\right) + h^2 \mathrm{Re}\left(D_{x_d} f + G_1 f, G_0 f\right)_0, \tag{3.16}$$

with $G_0 \in \mathcal{E}^0$ and $G_1 \in \mathcal{D}^1$, which implies 3.1. Noticing that with the hypotheses of proposition 3.2, near $\Gamma_D$, the only term appearing in (3.15) is $\left(2\varphi'_{x_d} D_{x_d} f, D_{x_d} f\right)_0 \geq 0$, we also obtain proposition 3.2 near $\Gamma_D$.

We now restrict ourselves to the case $\Gamma_N$. Note

$$\mathcal{E}^+ = \{(x,\xi') \in K \times \mathbb{R}^d; q_2(x,\xi') + \frac{q_1(x,\xi')^2}{\left(\frac{\partial\varphi}{\partial n}\right)^2} > 0\} \tag{3.17}$$

$$\mathcal{Z} = \{(x,\xi') \in K \times \mathbb{R}^d; q_2(x,\xi') + \frac{q_1(x,\xi')^2}{\left(\frac{\partial\varphi}{\partial n}\right)^2} = 0\} \tag{3.18}$$

$$\mathcal{E}^- = \{(x,\xi') \in K \times \mathbb{R}^d; q_2(x,\xi') + \frac{q_1(x,\xi')^2}{\left(\frac{\partial\varphi}{\partial n}\right)^2} > 0\} \tag{3.19}$$

Following [**LR95b**], we shall use the technique of Calderón projectors (and elliptic calculus) in $\mathcal{E}^+$ and Carleman inequalities in $\mathcal{Z}$ and $\mathcal{E}^-$.

The following result is proved in [**LR95b**] (lemme 5.3).

***Lemma 3.4.*** — *For $(x,\xi') \in \mathcal{E}^+$, $p_\varphi$ has two roots, $r_\pm$ with $Imr_+ > 0$ and $Imr_- < 0$; for $(x,\xi') \in \mathcal{Z}$, one of the two roots is real and for $(x,\xi') \in \mathcal{E}^-$, the imaginary parts of the two roots has the same sign as $\partial_d\varphi$.*

The following two lemma (lemme 5.4 and 5.5 of [**LR95b**]) are the keys to the proof of proposition 3.2 near $\Gamma_N$.

***Lemma 3.5* (elliptic estimates).** — *Let $\chi \in S^0$, with compact support in $x$. Assume that there exists $\delta > 0$ such that on the support of $\chi$ one has*

$$q_2\left(x,\xi'\right) + \frac{q_1\left(x,\xi'\right)^2}{\left(\frac{\partial\varphi}{\partial x_d}\right)^2} \geq \delta\left(1+|\xi'|^2\right) \tag{3.20}$$

*and $\partial_d\varphi > 0$ on $\{x_n = 0\}$. Then there exists $h_1, C > 0$ such that $\forall 0 < h < h_0$ and all $f \in C_0^\infty(K)$*

$$\|P_\varphi f\|_0^2 + h|(hD_d - k)\,Op(\chi)\,f|_1^2 + h^2\|f\|_1^2 \geq C\left(\|Op(\chi)\,f\|_1^2 + h|Op(\chi)\,f\,|_{x_d=0}\,|_1^2\right). \tag{3.21}$$

***Lemma 3.6* (Carleman estimates).** — *Let $\varphi$ satisfy conditions (3.2) and (3.3). Let $\chi \in S^0$, with compact support in $x$. Assume that on the support of $\chi$, $|\xi'|$ is bounded and*

$$\left(\partial_n\varphi\right)^2 - q_2 > 0 \tag{3.22}$$

*Then there exists $h_1, C > 0$ such that $\forall 0 < h < h_0$ and all $f \in C_0^\infty(K)$*

$$\begin{aligned}\|P_\varphi f\|_0^2 &+ h|(hD_d - k)\,Op(\chi)\,f\,|_{x_d=0}\,|_0^2 + h^2\|f\|_1^2 \\ &+ h|f\,|_{x_d=0}\,|_0|(hD_d - k)\,Op(\chi)\,f\,|_{x_d=0}\,|_0 + h^2|f\,|_{x_d=0}\,|_0^2 \geq \\ &\qquad C\left(h\|Op(\chi)\,f\|_1^2 + h|Op(\chi)\,f\,|_{x_d=0}\,|_1\right).\end{aligned} \tag{3.23}$$

Now proposition 3.2 near $\Gamma_N$ is a simple consequence of these two lemma. Indeed, it is enough to construct $\chi_1$ satisfying the hypotheses of lemma 3.5 and $\chi_2$ satisfying the hypotheses of lemma 3.6 such that $\chi_1 + \chi_2 \geq 1$ and to sum the inequalities obtained (it is easy to see that the error terms can be absorbed). For $|\xi'|$ large enough, (3.20) is satisfied and so the construction of $\chi_{1,2}$ is straightforward.

Lets give a sketch of the proof of lemma 3.5 and 3.6:

Set $u = \mathrm{Op}(\chi_1)\,f$, $f_1 = P_\varphi f$ and $f_0 = hD_d f - kf$ solutions of

$$\begin{cases} P_\varphi u = \mathrm{Op}(\chi_1)\,f_1 + [P_\varphi, \mathrm{Op}(\chi_1)]\,f = u_1, \ |u_1| \leq C\left(|f_1| + h\|f\|_1\right) \\ \left(hD_d u - \mathrm{Op}(k)\,u\right)|_{x_d=0} = u_0, \ |u_0|_1 \leq C\left(|f_0|_1 + h|f\,|_{x_d=0}\,|_1\right) \\ u_0 = \mathrm{Op}(\chi_1)\,f_0 + [hD_d - \mathrm{Op}(k)\,, \mathrm{Op}(\chi_1)]\,f\,|_{x_d=0} \end{cases} \tag{3.24}$$

Let $\underline{u}$ be the 0 extension of $u$ in $x_d < 0$ and $\gamma_0(u) = u\mid_{x_d=0}$, $\gamma_1(u) = -hi\partial_n u\mid_{x_d=0}$. Taking into account that $r_+ + r_- = -2i\varphi'_d$, we obtain by the jump formula,

$$P_\varphi \underline{u} = \underline{f}_1 - h^2\gamma_0(u)\otimes\delta'_{x_d=0} + \frac{h}{i}\left[\gamma_1(u) - \operatorname{Op}(r_+ + r_-)\gamma_0(u)\right]\otimes\delta_{x_d=0} \tag{3.25}$$

Using a parametrix of the operator $P_\varphi$ (since $u = \operatorname{Op}(\chi)f$ is supported in the elliptic region) we can solve equation (3.24) and taking again the traces on $x_d = 0$, we obtain by standard elliptic semi-classical calculous ( [**LR95b**], equation (3.48)):

$$\gamma_0(u) = \operatorname{Op}(a)\gamma_0(u) + \operatorname{Op}(b)\gamma_1(u) + v \tag{3.26}$$

with $a \in S^0$, $\sigma(\operatorname{Op}(a)) = \frac{\tilde\chi_1 r_-}{r_+ + r_-}\mid_{x_d=0}$, $\tilde\chi_1$ equal to one near $\operatorname{supp}(\chi_1)$, $b \in S^{-1}$, $\sigma(\operatorname{Op}(b)) = \frac{\tilde\chi_1}{r_+ + r_-}\mid_{x_d=0}$ and $v$ satisfies

$$|v|_1 \le Ch^{-1/2}\left[\|f_1\|_0 + h\|f\|_1 + h^2|hD_d f\mid_{x_d=0}|_0\right] \tag{3.27}$$

Now $\gamma_0(u)$ satisfies the equation

$$\left[\operatorname{Op}(1-a) - \operatorname{Op}(b)\operatorname{Op}(k)\right]\gamma_0(u) = v + u_0 \tag{3.28}$$

with

$$\sigma\left(\operatorname{Op}(1-a) - \operatorname{Op}(b)\operatorname{Op}(k)\right)\mid_{\operatorname{supp}(\chi_1)} = \frac{r_+ - k}{r_+ + r_-} \tag{3.29}$$

but, according to lemma 3.4, we have

$$r_+ - k = r_+ + i\partial_d\varphi \neq 0, \tag{3.30}$$

so the operator $(\operatorname{Op}(1-a) - \operatorname{Op}(b)\operatorname{Op}(k))$ is elliptic, which gives the estimation

$$|\gamma_0(u)|_1 \le C\left[|u_0|_1 + |v|_1 + O(h^\infty\|f\|_1)\right] \tag{3.31}$$

and also (since $\gamma_1(u) = u_0 + k\gamma_0(u)$)

$$|\gamma_1(u)|_1 \le C\left[|u_0|_1 + |v|_1 + O(h^\infty\|f\|_1)\right] \tag{3.32}$$

Using again the parametrix of the operator $P_\varphi$, equations (3.24), (3.29) and (3.32) we obtain the estimate (3.21)).

To prove lemma 3.6, we apply estimate (3.16) to the function $u = \operatorname{Op}(\chi_2)f$, satisfying (with $f_1 = P_\varphi f$ and $f_0 = hD_d f - \operatorname{Op}(k)f$)

$$\begin{cases} P_\varphi u = \operatorname{Op}(\chi_2)f_1 + [P_\varphi, \operatorname{Op}(\chi_2)]f = u_2,\ |u_2| \le C(|f_1| + h\|f\|_1) \\ (hD_d u - \operatorname{Op}(k)u)\mid_{x_d=0} = u_3,\ |u_3|_0 \le C(|f_0|_0 + h\|f_{x_d=0}\|_0) \\ u_3 = \operatorname{Op}(\chi_2)f_0 + [hD_d - \operatorname{Op}(k), \operatorname{Op}(\chi_2)]f\mid_{x_d=0} \end{cases} \tag{3.33}$$

Now,

$$\beta(u) = (Bu, u) + R(f, f_0) \tag{3.34}$$

with

$$\begin{aligned} \sigma(B) &= 2\partial_n\varphi|k|^2 - 2\partial_n\varphi q_2 \\ &= 2\partial\varphi_n(|\partial_n\varphi|^2 - q_2) \end{aligned} \tag{3.35}$$

and

$$|R(f,f_0)| \leq C\left[|f|_{x_d=0}|_0|f_0|_0 + |f_0|_0 + h|f|_{x_d=0}|_0^2\right] \tag{3.36}$$

Since $u = \mathrm{Op}(\chi_2)f$ and on the support of $\chi_2$, one has $\sigma(B) \geq c > 0$, by Gårding inequality, we obtain

$$(Bu,u) \geq \frac{c}{2}|\mathrm{Op}(\chi_2)f|_{x_d=0}|_1^2 \tag{3.37}$$

and we get estimate(3.21)

***Remark 3.7.*** — *If the function $g$ is supported in the exterior of a (fixed) neighbourhood of some part of the boundary of $\Omega$, $\Gamma_0$, then proposition 3.2 is still true even if we no longer suppose that $\frac{\partial\varphi}{\partial n} \neq 0$ on $\Gamma_0$ since the proof is local and the result is empty (hence true) near $\Gamma_0$.*

***Remark 3.8.*** — *Suppose that the phase $\varphi$ can be choosen independently with respect to the parameter $e \in [1,2]$. Then it is a consequence of the proof we present that the constants $h_1$ and $C$ can be choosen independantly with respect to $e \in [1,2]$ (in fact, from the proofs, these constants can be chosen* locally *independantly and a compacity argument gives the global independance).*

## 4. Construction of phase functions

This section is devoted to the construction of the phase functions we shall use later to conjugate the operator $P$. For $|x|$ large, we work in the coordinate system of section 2.2. In the sequel, except if explicitaly precised otherwise, every $L^2$ norm considered will be the $L^2$ norm for which the operator $Q$ is self adjoint.

**4.1. Construction of phases in any bounded domain.** — Fix $R > 0$. In this part we shall construct two phases $\varphi_1$ et $\varphi_2$ satisfying the hypoellipticity conditions of Hörmander (3.2) in $\Omega \cap B(0,R)$, except on a finite number of balls such that on the one hand, where one phase does not satisfy the hypotheses, the other one does and is strictly bigger than the first one, and on the other hand, far from the l'obstacle, the functions $\varphi_1$ et $\varphi_2$ are equal and radial. This construction is essentially done in [**Bur98**], the only difference is that we supposed that the operator $P - e$ was equal to $h^2\Delta - 1$ outside a compact set, whereas here we suppose only that the difference between $P$ and $-h^2\Delta$ tends to 0.

Take $R > 0$ such that $\Theta \subset B(0,R)$. The following result is proved in[**Bur98**], proposition 3.1.

***Proposition 4.1.*** — *For $R > 0$ such that $\Theta \subset B(0,R)$, there exists two functions $\psi_{1,2} \in C^\infty\left(\overline{\Theta^c \cap B(0,R)}\right)$ such that $\partial_n\psi_i|_{\partial\Theta} > 0$ ($\partial_n$ is the interior normal to $\Theta^c$), with only non degenerate critical points and such that on $C(0,R)$, the circle centered in $O$ with radius $R$, $\nabla\psi_i \cdot x > 0$ and where $\nabla\psi_i = 0$ (a finite number of points),* *one has $\nabla\psi_{i+1} \neq 0$ and $\psi_{i+1} > \psi_i$ ($\psi_3 = \psi_1$). Finally the functions $\psi_i$ are radial and equal near $C(0,R)$.*

***Remark 4.2.*** — *We deduce that there exists a finite number of points, $x_{i,j} \in \Theta^c; i = 1,2; j = 1, \cdots, N_i$, a constant $\varepsilon > 0$ such that $B(x_{i,j}, 2\varepsilon) \subset \Theta^c \cap B(0,R)$, $B(x_{1,j}, 2\varepsilon) \cap B(x_{2,j}, 2\varepsilon) = \emptyset$ and on $B(x_{i,j}, 2\varepsilon)$, $\psi_i < \psi_{i+1}$ ($\psi_3 = \psi_1$).*

*Note $\Omega_i = \Theta^c \cap B(0,R) \cap (\cup_j B(x_{i,j}, \varepsilon))^c$.*

We are looking for $\varphi_i$ satisfying (3.2) on $\Omega_i$, of the form $e^{\beta\psi_i}$, $\beta >> 1$. It is enough to check that for $\psi \in C^\infty\left(\overline{\Omega}\right)$ such that $|\nabla\psi| \geq c > 0$ then for $\beta > 0$ large enough, $\{\mathrm{Re} p_\varphi, \mathrm{Im} p_\varphi\} > c' > 0$ if $p_\varphi(x,\xi) = 0$.

We have $\varphi' = \beta e^{\beta\psi}\psi'$, $\varphi'' = e^{\beta\psi}\left(\beta^2\psi'\ {}^t\psi' + \beta\psi''\right)$ and if $p_\varphi = 0$ then ${}^t\xi A\xi + a(x)\cdot\xi = {}^t\varphi A\varphi + e + a(x)\cdot\varphi'$ hence $|\xi| \leq C\beta e^{\beta\psi}$. The semi-classical principal symbol of the operator $P$ has the following form:

$$\begin{aligned} p(x,\xi) &= {}^t\xi A(x)\xi + b(x,h)\cdot\xi + c(x,h), \\ p_\varphi(x,\xi) &= {}^t\xi A(x)\xi - {}^t\varphi' A(x)\varphi' + b(x,h)\cdot\xi + c(x,h) + 2ib(x,h)\cdot\varphi' + 2iA(x)\varphi' \end{aligned} \tag{4.1}$$

with $A = (a_{i,j}(x))$ uniformly definite positive, $b$ and $c$ real. A direct computation gives if $p_\varphi(x,\xi) = 0$,

$$\{\mathrm{Re} p_\varphi, \mathrm{Im} p_\varphi\} = 4e^{3\beta\psi}\left(\beta^4\ \left({}^t\psi' A\psi'\right)^2 + \mathcal{O}\left(\beta^3\right)\right), \tag{4.2}$$

which implies the result since $\psi' \neq 0$ et $A >> cId$, $c > 0$.

***Remark 4.3.*** — *In the computation above, the choice of $\beta$ (hence the choice of function $\varphi_{1;2}$) is clearly independant of the choice of $e \in [1,2]$.*

**4.2. Stabilisation of the phases outside of a ball.** — We are going to define now the phases on $B(0,R)^c \cap B(0,R_1)$ ($R_1$ to be fixed) so that near the set $\{|x| = R_1\}$ the phases are small. This part is essentially the same as in [**Bur98**] except for minor technical modifications to take into account that the perturbation of the metric and the potential are not compactly supported.

At the level of the symbols, according to the reduction in section 2.2, if $R$ is taken large enough, we can suppose that the operator $P$ is as in (2.12) for $|x| \geq R$, and that we have, for $r \geq R$

$$\begin{gathered} |q_2(r,\sigma,\frac{\eta}{r})| \leq \frac{1}{8}\left\|\frac{\eta}{r}\right\|^2, \qquad |r\partial_r q_2(r,\sigma,\frac{\eta}{r})| \leq \frac{1}{8}\left\|\frac{\eta}{r}\right\|^2 \\ |q_1(r,\sigma,\frac{\eta}{r},h)| \leq \frac{1}{8}\left\|\frac{\eta}{r}\right\|, \qquad |r\partial_r q_1(r,\sigma,\frac{\eta}{r},h)| \leq \frac{1}{8}\left\|\frac{\eta}{r}\right\| \\ |V(r,\sigma,h)| \leq \frac{1}{8}, \qquad |r\partial_r V| \leq \frac{1}{8} \end{gathered} \tag{4.3}$$

At the level of the operators, we shall also make the following assumptions on the choice of $R$: we suppose that for any $r \geq R$ and any function $w$ on the sphere $\mathbb{S}^{d-1}$

$$\begin{aligned}(4.4)\quad & \left|\int_{\mathbb{S}^{d-1}} 2q_2(r,\sigma,\frac{h\partial_\sigma}{r})w\overline{w}\right| + \left|\int_{\mathbb{S}^{d-1}} q_1(r,\sigma,\frac{h\partial_\sigma}{r})w\overline{w}\right| \\ & + \left|\int_{\mathbb{S}^{d-1}} r\partial_r q_2(r,\sigma,\frac{h\partial_\sigma}{r})w\overline{w}\right| + \left|\int_{\mathbb{S}^{d-1}} r\partial_r q_1(r,\sigma,\frac{h\partial_\sigma}{r})w\overline{w}\right| \\ & \leq \frac{1}{8}\left(\int_{\mathbb{S}^{d-1}} |\frac{h\nabla_\sigma}{r}w|^2 + |w|^2\right)\end{aligned}$$

and for any $r \geq R$, $z = re^{i\theta}$, $\theta \leq \theta_0$,

$$(4.5)\quad \begin{aligned}\int_{\mathbb{S}^{d-1}} \left|\left(q_1(z,\sigma,\frac{h\partial_\sigma}{r},h) - q_1(r,\sigma,\frac{h\partial_\sigma}{r},h)\right)w\right|^2 \leq \frac{\theta}{16}\left(\int_{\mathbb{S}^{d-1}} |\frac{h\nabla_\sigma}{r}w|^2 + |w|^2\right) \\ \int_{\mathbb{S}^{d-1}} \left|\left(q_2(z,\sigma,\frac{h\partial_\sigma}{r},h) - q_2(r,\sigma,\frac{h\partial_\sigma}{r},h)\right)w\right|^2 \leq \frac{\theta}{16}\left(\int_{\mathbb{S}^{d-1}} |\frac{h\nabla_\sigma}{r}w|^2 + |w|^2\right)\end{aligned}$$

***Remark 4.4.*** — *Of course,* (4.4) *is fulfilled if $R$ is large enough because the coefficients of $q_2$ and their derivatives with respect to $\sigma$ are bounded by $M/r^\beta$, whereas the derivatives with respect to $r$ are bounded by $M/r^{\beta+1}$; and the same remark holds for $q_1$. To check* (4.5), *remark that if $q_2(r,\sigma,\frac{h\partial_\sigma}{r}) = \sum q_2^{i,j}(r,\sigma)h\partial_{\sigma_i}h\partial_{\sigma_j}$, then*

$$(4.6)\quad \sum q_2^{i,j}(z,\sigma) - \sum q_2^{i,j}(r,\sigma) = (z-r)\int_0^1 \partial_r q_2^{i,j}(z+s(z-r),\sigma)ds$$

*and from the bound on the $r$-derivatives, we deduce*

$$(4.7)\quad |\partial_2^\alpha \partial_1 q_2^{i,j}(z+s(z-r),\sigma)| \leq \frac{M_\alpha}{r^{\beta+1}}$$

*which implies*

$$(4.8)\quad \int_{\mathbb{S}^{d-1}} \left|\left(q_2(z,\sigma,\frac{h\partial_\sigma}{r},h) - q_2(r,\sigma,\frac{h\partial_\sigma}{r},h)\right)w\right|^2 \leq \frac{M|z-r|}{|r|^{\beta+1}}\left(\int_{\mathbb{S}^{d-1}} |\frac{h\nabla_\sigma}{r}w|^2 + |w|^2\right)$$

*The same argument gives also the result for the $q_1$ term.*

Since near the set $\{|x| = R\}$ the phases $\varphi_1$ and $\varphi_2$ are equal, we will take the same extension for the two phases and dropp the index $i$. We are looking for radial extensions for $r > R$. The function $p(r,\theta,\rho,\eta)$ has the form:

$$(4.9)\quad p(r,\sigma,\varrho,\eta) = \varrho^2 + \frac{\|\eta\|^2}{r^2} + q_2\left(r,\sigma,\frac{\eta}{r}\right) + q_1\left(r,\sigma,\frac{\eta}{r},h\right) + V(r,\sigma,h) - e$$

and

$$(4.10)\quad p_\varphi(r,\sigma,\varrho,\eta) = p(r,\sigma,\varrho,\eta) - (\varphi'(r))^2 + 2i\varrho\varphi'(r).$$

If $p_\varphi = 0$ (and $\varphi'(r) \neq 0$), we have

$$\varrho = 0 \quad \text{and} \quad \frac{\|\eta\|^2}{r^2} + q_2\left(r,\sigma,\frac{\eta}{r}\right) + q_1\left(r,\sigma,\frac{\eta}{r}\right) + V(r,\sigma,h) = (\varphi'(r))^2 + e \tag{4.11}$$

which gives from (4.3)

$$\left(\frac{4}{5}(\varphi')^2 + \frac{1}{2}\right) \leq \left\|\frac{\eta}{r}\right\|^2 \leq \left(\frac{4}{3}(\varphi')^2 + 3\right) \tag{4.12}$$

and on the set $\{p_\varphi = 0\}$, we have, using that $q_2$ is a quadratic form and $q_1$ a linear form,

$$\begin{aligned}
\{\mathrm{Re}p_\varphi, \mathrm{Im}p_\varphi\} &= 4(\varphi')^2\varphi'' + 2\left(-\partial_1 q_1\left(r,\sigma,\frac{\eta}{r}\right) - \partial_1 q_2\left(r,\sigma,\frac{\eta}{r}\right) - \partial_r V\right)\varphi' \\
&\quad + \frac{\varphi'}{r}\left(4\frac{\|\eta\|^2}{r^2} + 4q_2\left(r,\sigma,\frac{\eta}{r}\right) + 2q_1\left(r,\sigma,\frac{\eta}{r}\right)\right) \\
&= \frac{\varphi'}{r}\left(4\frac{\|\eta\|^2}{r^2} + 4q_2 + 2q_1 - 2r\partial_1 q_2 - 2r\partial_1 q_1 - 2\partial_r V\right) \\
&= \frac{\varphi'}{r}\left(4\left(\frac{\|\eta\|^2}{r^2} + q_2 + q_1 + V + r\varphi'\varphi''\right)\right) \\
&\quad - 2\frac{\varphi'}{r}\left(r\partial_1 q_2 + r\partial_1 q_1 + r\partial_r V + 2V + q_1\right)
\end{aligned} \tag{4.13}$$

From (4.3)and (4.12)

$$\begin{aligned}
&|r\partial_1 q_2 + r\partial_1 q_1 + r\partial_r V + 2V + q_1| \\
&\quad \leq \left(\frac{1}{8}\|\eta/r\|^2 + \frac{1}{4}\|\eta/r\| + \frac{3}{8}\right) \leq \left(\frac{1}{4}\|\eta/r\|^2 + \frac{1}{2}\right) \leq \frac{(\varphi')^2}{3} + \frac{5}{4}
\end{aligned} \tag{4.14}$$

and from (4.11) and (4.13) we obtain (on the set $p_\varphi = 0$)

$$\begin{aligned}
\{\mathrm{Re}p_\varphi, \mathrm{Im}p_\varphi\} &\geq \frac{\varphi'}{r}\left(4(\varphi')^2 + 4e - \frac{1}{3}(\varphi')^2 - \frac{5}{4} + 4r\varphi'\varphi''\right) \\
&\geq \varphi'\left(\frac{\frac{11}{3}(\varphi')^2 + \frac{11}{4}}{r} + 4\varphi'\varphi''\right)
\end{aligned} \tag{4.15}$$

Our problem is reduced to the obtention of a phase function defined for $r \geq R$ and satisfying $\varphi' > 0$ and

$$4\varphi'\varphi'' + \frac{11}{4r} \geq 0, \quad \varphi'(R) \text{ given} \tag{4.16}$$

In fact, we'll construct $\varphi$ such that

$$20\varphi'\varphi'' + \frac{1}{r} \geq 0, \tag{4.17}$$

which implies (4.16).

The solution, $k$ of the equation $k'k + \frac{1}{24r} = 0$ with $k(R) = \varphi'(R)$ is

$$k = \sqrt{(\varphi'(R))^2 - \log\left(\frac{r}{R}\right)/12}. \tag{4.18}$$

Take $R_1 = Re^{11\varphi'^2(R)}$, $\kappa > 0$ small and define $\widetilde{\varphi}'$ on $\Omega$ by

$$\widetilde{\varphi}'(r) = \begin{cases} \varphi'(r) & \text{if } r < R \\ k(r) & \text{if } R \leq r < R_1 e^{-12\kappa^2} \\ \kappa & \text{if } R_1 e^{-12\kappa^2}. \end{cases} \tag{4.19}$$

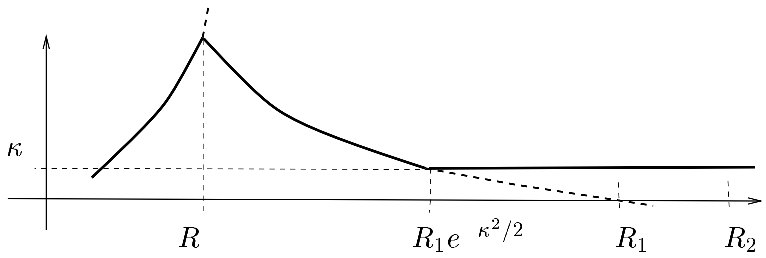


Figure 1: The function $\widetilde{\varphi}'$

***Remark 4.5***. — *Remark that the constant $R_1$ does depend only on the value of $\varphi'(R)$, i.e. only on the geometry of the problem for $|x| \leq R$ (i.e. on the nature of the perturbation of the Laplace operator in a* fixed *compact set).*

Let $H \in C^\infty(\mathbb{R})$ equal to 0 for $r < R - \alpha/2$ to 1 for $r > R - \alpha/3$, $0 \leq H \leq 1$, $H' \geq 0$ and $h \in C_0^\infty(]0,1[)$, positive, of integral equal to 1. Note

$$h_\eta(r) = \frac{1}{\eta} h\left(\frac{r}{\eta}\right) \tag{4.20}$$

and

$$\varphi'_\eta = (1 - H)\widetilde{\varphi}' + h_\eta * \left(H\widetilde{\varphi}'\right) \tag{4.21}$$

The function $\varphi'_\eta(r)$ converges uniformly to $\widetilde{\varphi}'(r)$ $(\eta \to 0^+$ and

$$\tfrac{d}{dr}\left(\varphi'_\eta\right)(r) = (-H')\widetilde{\varphi}' + h_\eta * (H'\widetilde{\varphi}') + (1 - H)\widetilde{\varphi}'' + h_\eta * (H\widetilde{\varphi}'') \tag{4.22}$$

but for $r > R$, $\widetilde{\varphi}'' \geq \frac{-1}{r\widetilde{\varphi}'}$ and for $R - \alpha/3 \leq r \leq R$, since the function $\varphi$ has been built such that $\varphi'' >> 1$, $\widetilde{\varphi}'' \geq \frac{-1}{r\widetilde{\varphi}'}$. We obtain (taking $\alpha > 0$ small enough)

$$\tfrac{d}{dr}\left(\varphi'_\eta\right)(r) \geq (-H')\widetilde{\varphi}' + h_\eta * (H'\widetilde{\varphi}') + (1 - H)\tfrac{-1}{r\widetilde{\varphi}'} + h_\eta * \left(H \times \tfrac{-1}{r\widetilde{\varphi}'}\right). \tag{4.23}$$

The right hand side converges uniformly to $\frac{-1}{r\widetilde{\varphi}'}$, hence there exists $c > 0$ such that if $\eta > 0$ is small enough, $\varphi''_\eta \varphi'_\eta + \frac{1}{22r} \geq c > 0$, for $r \in [R - \alpha, R_2]$.

Now, this construction allows us to construct phases $\varphi_{1,2}$ satisfying the hypoellipticity assumptions of Hörmander (3.2), in $\Omega \cap B(0, R_2)$, with $\varphi_{1,2}$ radial and $\varphi' = \kappa$ arbitrarily small (independently of $R_2$) near $\{|x| = R_2\}$.

**4.3. Construction of the phases near infinity.** — In the preceding sections we have constructed phases satisfying the hypoellipticity assumptions of Hörmander and such that $\varphi' = \kappa > 0$ for $r \geq R_1$, the constant $R_1$ being independant of the choice of $\kappa > 0$. In this section we arrive at the most delicate point of the construction of our phases: we are going to construct a family (depending on the parameter $h > 0$) of phases in $\{|x| \geq R_1\}$ equal to the phases constructed above near $\{|x| = R_1\}$ and such that $\varphi'(r, h) = 8h/r$ for $r$ very large. The difficulty in this construction is that we want to get some positivity in the bracket $[\mathrm{Re}P_\varphi, \mathrm{Im}P_\varphi]$. Of course, contrarily to the previous constructions which depends only on the principal symbol of the operator $P$, this construction does depend on the subprincipal symbol. Consequently, the fact that this construction is performed in the coordinate system of section 2 such that the operator satisfies (2.12) is important.

The most natural candidate for this construction would have been to take $\varphi'(r) = k1_{r \leq R_1}$ where $k$ is defined by (4.18). A first remark is that we have to work with smooth phase functions. Note $R_2 = R_1 + 4$ and consider the function $\Psi'(r) = e^{1/(r-R_2)}1_{r \leq R_2} \in C^\infty(\mathbb{R})$. Since near $R_2$, $\Psi'$ and $\Psi''$ are small, $\Psi$ satisfies

$$22\Psi\Psi' + \frac{1}{r} \geq 0 \tag{4.24}$$

for $|r - R_2| \leq \varepsilon, \varepsilon > 0$. Now suppose that $\kappa > 0$ has been chosen smaller than $e^{-1/\varepsilon}$. Then by smoothing the continuous function equal to $\varphi$ for $r \leq R_2 + \log(\kappa)$ and equal to $\Psi$ for $r \geq R_2 + \log(\kappa)$ we get

***Lemma 4.6.*** — *There exists a function $\hat{\varphi}$ equal to $\varphi$ for $r \leq R_2 - 2\varepsilon$ and such that*

- *i) $21\hat{\varphi}'\hat{\varphi}'' + \frac{1}{r} \geq 0$ for $R \leq R_2$*
- *ii) $\hat{\varphi}'(r) = e^{1/(r-R_2)}1_{r \leq R_2}$ near $r = R_2$ and for $r \geq R_2$.*

***Remark 4.7.*** — *Since for $r \geq R_2$, the function $\hat{\varphi}$ is constant ($\hat{\varphi}' = 0$), it does no more satisfies the hypoellipticity assumptions and we have to modify it for $r > R_2$ (and also for $r$ close to $R_2$).*

Consider a non negative increasing function $\chi_1 \in C^\infty$ equal to 0 for $r \leq 1$, to 1 for $r \geq 10$ and such that $0 \leq \chi_1' \leq 1/8$. We will also need to suppose that near $\{r = 1\}$, we have

$$\chi_1(r) = e^{-\frac{1}{r-1}} 1 \,|_{r>1} \tag{4.25}$$

Note

$$H(r, h) = \frac{h}{h^{1/2} + \chi_1(r)} \tag{4.26}$$

A direct calculus gives

***Lemma 4.8.*** — *There exists $C > 0$ such that $\forall r \in \mathbb{R}$, $\forall 0 < h < 1$,*

$$|H(r, h)| \leq Ch^{1/2}, \qquad |H'(r, h)| \leq C, \qquad |H''(r, h)| \leq Ch^{-1/2}, \tag{4.27}$$

Take a non negative function $\chi_2 \in C_0^\infty(]-3,12[)$, equal to 1 on $[-1,10]$ such that

$$|\chi_2'| \leq 1, \qquad |\chi_2''| \leq 2 \tag{4.28}$$

***Lemma 4.9.*** — *There exists a function $g \geq 0$ such that*

- *i) $g$ is equal to 0 for $r \leq R_2 + 1$*
- *ii) $g' > 0$ for $r > R_2 + 1$*
- *iii) $g'' \geq 2$ and $g' \geq 2$ for $r \in [R_2 + 10, R_2 + 12]$*
- *iv) $g'' \geq 0$ for $R_2 + 1 \leq r \leq R_2 + 12$*
- *v) $\exists R_3 > R_2$; $\forall r \geq R_3$, $8/r = g'(r)$*
- *vi) $\forall r \geq R_2 + 12$, $(g')^2 + 7hg'' \geq 0$ and $g' \geq \frac{7}{r}$*

Indeed, the only serious restrictions on $g$ are imposed by conditions v) and vi), but the solution to $(k')^2 + 7k'' = 0$ is $k = 7/r$ and $(8/r)^2 + 7(8/r)' = 8/r^2 > 0$ which shows that condition v) is compatible with vi). From now on, we shall note $\varphi$ the function equal to $\hat{\varphi}$ for $r \leq R_1$ and such that

$$\varphi'(r) = \hat{\varphi}'(r) + \chi_2(r - R_2)H(r - R_2, h) + hg'(r) \tag{4.29}$$

***Lemma 4.10.*** — *For any $\varepsilon > 0$ there exists $h_0 > 0$ such that for any $0 < h < h_0$ and any $r \geq R_1$*

$$h|\varphi'''| \leq \varepsilon\frac{\varphi'}{r}, \qquad h|\varphi''|^2 \leq \varepsilon\frac{\varphi'}{r} \tag{4.30}$$

The proof of this lemma depends on the position of the point $r$:

- i) For $r \geq R_3$ $\varphi'(r) = \frac{8h}{r}$ and the result follows by a simple computation
- ii) For $R_2 + 10 \leq r \leq R_3$, according to lemma 4.9 iii) and vi), we have

$$\varphi' = H\chi_2 + hg' \geq hg' \geq h\inf(2, \frac{7}{R_3}) \tag{4.31}$$

  $|\varphi''| + |\varphi'''| \leq Ch$ and the result follows by compacity.
- iii) For $R_2 + 1 \leq r \leq R_2 + 10$,

$$\begin{aligned}\varphi' &= \frac{h}{h^{1/2} + \chi_1} + hg'\\ \varphi'' &= -\frac{h}{(h^{1/2} + \chi_1)^2}\chi_1' + hg''\\ \varphi''' &= 2\frac{h}{(h^{1/2} + \chi_1)^3}(\chi_1')^2 - \frac{h}{(h^{1/2} + \chi_1)^2}\chi_1'' + hg'''\end{aligned} \tag{4.32}$$

  Close to $r = R_2 + 1$ we have

$$\chi_1 = e^{-\frac{1}{r-(R_2+1)}} 1\,|_{r \geq R_2+1} \tag{4.33}$$

  Choose $3 < \alpha < 4$.

1. For $R_2 + 1 - \frac{\alpha}{\log h} \leq r$, we have $\chi_1 \geq h^{1/\alpha}$. From (4.32) we obtain since $\alpha > 3$,

$$h|\varphi'''| \leq Ch^{2-3/\alpha} + Ch^{2-2/\alpha} + Ch^2 = o(h) \tag{4.34}$$

   Since $\varphi' \geq ch$ (with $c > 0$) we obtain the first inequality of lemma 4.10 (for $R_2 + 1 - \frac{\alpha}{\log h} \leq r$). The second inequality is obtained similarly.
2. For $R_2 + 1 \leq r \leq R_2 + 1 - \frac{\alpha}{\log h}$. Choose $\eta > 1$ such that $\alpha\eta < 4$. There exists $\varepsilon > 0$ and $C > 0$ such that for $0 < x < \varepsilon$ we have

$$\frac{1}{x^2}e^{-\frac{1}{x}} \leq Ce^{-\frac{1}{\eta x}}, \qquad \frac{1}{x^3}e^{-\frac{1}{x}} \leq Ce^{-\frac{1}{\eta x}}, \qquad \frac{1}{x^4}e^{-\frac{1}{x}} \leq Ce^{-\frac{1}{\eta x}} \tag{4.35}$$

   This gives for $R_2 + 1 \leq r \leq R_2 + 1 - \frac{\alpha}{\log h}$, according to (4.33),

$$\begin{aligned} |\chi_1'| &\leq Ce^{\frac{\log h}{\eta\alpha}} = h^{\frac{1}{\eta\alpha}} \\ |\chi_1''| &\leq Ce^{\frac{\log h}{\eta\alpha}} = h^{\frac{1}{\eta\alpha}} \end{aligned} \tag{4.36}$$

   From (4.32) we obtain since $\alpha\eta < 4$,

$$h|\varphi'''| \leq Ch^{2-3/2+2/(\eta\alpha)} + Ch^{2-1+1/(\eta\alpha)} + Ch^2 = o(h) \tag{4.37}$$

and we obtain the first inequality. The second one is obtained similarly.

– iv) For $R_2 - 1 \leq r \leq R_2 + 1$,

$$\varphi' = \hat{\varphi}' + h^{1/2} \geq h^{1/2} \qquad \varphi'' = \hat{\varphi}'' \qquad \varphi''' = \hat{\varphi}''' \tag{4.38}$$

and the result is obvious

– v) For $\leq r \leq R_2 - 1$, $\varphi' = \hat{\varphi}' + \chi_2 h^{1/2} \geq c > 0$, $|\varphi''| \leq C$, $|\varphi'''| \leq C$.

***Proposition 4.11***. — *There exists $h_0 > 0$ such that for any $0 < h < h_0$ and any $r \geq R_1$, we have*

$$\varphi'(\frac{1}{r} + 20\varphi'\varphi'') \geq 0 \tag{4.39}$$

According to lemma 4.10 it is enough to prove

$$\varphi'(\frac{1}{r} + \frac{41}{2}\varphi'\varphi'') \geq 0 \tag{4.40}$$

The proof of this result depends again on the position of the point $r$:

– i) For $r \geq R_3$, $\varphi' = hg' = \frac{8h}{r}$ and a simple computation gives the result.

– ii) For $R_2 \leq r \leq R_3$, we have

$$|\varphi'| \leq Ch^{1/2}, \quad |\varphi''| \leq C, \tag{4.41}$$

and the result follows

– iii) For $R_2 - 3 \leq r \leq R_2$, $\varphi' = \hat{\varphi}' + h^{1/2}\chi_2(r - R_2)$ which implies

$$\begin{aligned} \varphi' &= \hat{\varphi}' + \mathcal{O}(h^{1/2}) \\ \varphi'' &= \hat{\varphi}'' + \mathcal{O}(h^{1/2}) \end{aligned} \tag{4.42}$$

and we obtain from lemma 4.6, i),

$$\varphi'(\frac{1}{r}+\frac{41}{2}\varphi'\varphi'')\geq\varphi'(\frac{1}{40r}-Ch^{1/2}) \tag{4.43}$$

which, by a compacity argument implies proposition (4.40).

– iv) For $r\leq R_2-3$, $\varphi=\hat{\varphi}'$ and the result follows from lemma 4.6 i).

***Proposition 4.12***. — *There exists $h_0>0$ such that for any $0<h<h_0$, we have*

$$(\varphi')^2-2h\frac{\varphi'}{r}+4h\varphi''-h^2|\varphi'''|\geq 0 \tag{4.44}$$

The proof of this proposition depends on the position of $r$.

– i) For $R_3\leq r$, $\varphi'=hg'=\frac{8h}{r}$ and a simple computation gives the result.

– ii)For $R_2+12\leq r\leq R_3$, $\varphi'=hg'$ and from lemma 4.9 we have

$$\frac{h}{r}\leq\frac{\varphi'}{7},\qquad ,4\varphi''\geq\frac{-4}{7}(\varphi')^2 \tag{4.45}$$

and from lemma 4.10 (with $\varepsilon<1$), we obtain

$$\begin{aligned}(\varphi')^2-2h\frac{\varphi'}{r}+4h\varphi''-h^2|\varphi'''|&\geq h^2(g')^2(1-\frac{2}{7}-\frac{4}{7})-h^3|g'''|\\&\geq h^2\frac{(g')^2}{7}-h^3|g'''|\geq h^2(g')^2(\frac{1}{7}-\frac{\varepsilon}{r})\end{aligned} \tag{4.46}$$

– iii) For $R_2+10\leq r\leq R_2+12$, $\varphi'=hg'+h\chi_2\geq 2h$ from lemma 4.9, iii); and since from lemma 4.8, $g''\geq 2$ and $|\chi_2'|\leq 1$, we have $\varphi''\geq 0$ and

$$(\varphi')^2-2h\frac{\varphi'}{r}+4h\varphi''-h^2|\varphi'''|\geq(\varphi')^2-2h\frac{\varphi'}{r}-Ch^3\geq 2h\left(2h-\frac{2h}{10}\right)-Ch^3. \tag{4.47}$$

– iv) For $R_2+1\leq r\leq R_2+10$, $\varphi'=H+hg'\geq H\geq h-Ch^{3/2}$ and since $R_2\geq 12$,

$$\varphi'-\frac{2h}{r}\geq\frac{2}{3}\varphi' \tag{4.48}$$

Since $g'\geq 0$, $g''\geq 0$

$$(\varphi')^2-2h\frac{\varphi'}{r}+4h\varphi''-h^2|\varphi'''|\geq\frac{2}{3}H^2+4hH'-h^2|H''|-Ch^3 \tag{4.49}$$

But

$$hH'=-H^2\chi_1',\qquad h^2H''=2H^3(\chi_1')^2-hH^2\chi_1'' \tag{4.50}$$

and since $|\chi_1'|\leq 1/8$, we obtain

$$\begin{aligned}(\varphi')^2-2h\frac{\varphi'}{r}+4h\varphi''-h^2|\varphi'''|&\geq H^2(\frac{2}{3}-\frac{1}{2}-H|\chi_1'|^2-h|\chi_1''|)-Ch^3\\&\geq H^2(\frac{1}{6}-Ch^{1/2})-Ch^3\geq\frac{h^2}{6}-Ch^{5/2}\end{aligned} \tag{4.51}$$

– v) For $R_2 \leq r \leq R_2 + 1$

$$\varphi' = h^{1/2} + hg' \tag{4.52}$$

and

$$(\varphi')^2 - 2h\frac{\varphi'}{r} + 4h\varphi'' - \frac{h^2}{2}|\varphi'''| = h + \mathcal{O}(h^{3/2}). \tag{4.53}$$

– vi) For $r \leq R_2$, $\varphi' = \hat{\varphi}' + \chi_2 h^{1/2}$. Since $\hat{\varphi} = 0$ for $r \geq R_2$ there exists $1 > \alpha > 0$ such that for $R_2 - \alpha \leq r \leq R_2$,

$$\frac{\hat{\varphi}'}{r} \leq \frac{1}{8} \qquad \text{and } |\hat{\varphi}''| \leq \frac{1}{8} \tag{4.54}$$

Then (for $R_2 - \alpha \leq r \leq R_2$), since $\chi_2 = 1$, $\varphi' \geq h^{1/2}$,

$$(\varphi')^2 - 2h\frac{\varphi'}{r} + 4h\varphi'' - \frac{h^2}{2}|\varphi'''| \geq h - \frac{h}{4} - \frac{h}{4} - Ch^{3/2} \tag{4.55}$$

For $r \leq R_2 - \alpha$, $\hat{\varphi}' \geq c > 0$ and

$$(\varphi')^2 - 2h\frac{\varphi'}{r} + 4h\varphi'' - \frac{h^2}{2}|\varphi'''| \geq c^2 - Ch^{1/2}. \tag{4.56}$$

***Proposition 4.13***. — *There exists $C > 0$ and $h_0 > 0$ such that for any $0 < h < h_0$ and any $r \geq R_1$,*

$$|\varphi''| \leq C, \qquad |\varphi''| \leq Ch^{-1/2}\varphi', \qquad |\varphi'''| \leq Ch^{-3/2}\varphi' \tag{4.57}$$

The proof of the first inequality is straightforward. For the others we distinguish three cases:

– i) For $r \leq R_2$, we have $\varphi' \geq h^{1/2}$ and

$$|\varphi''| \leq C, \qquad |\varphi'''| \leq C \tag{4.58}$$

– ii) For $R_2 \leq r \leq R_3$, we have $\varphi' \geq H \geq h - Ch^{3/2}$ and

$$|\varphi''| \leq C, \qquad |\varphi'''| \leq Ch^{-1/2} \tag{4.59}$$

– iii) For $r \geq R_3$ we have $\varphi' = hg' = \frac{8h}{r}$ and the result follows.

## 5. Localization

Consider the phase functions, $\varphi_1$ and $\varphi_2$ constructed in section 4. Perform the analytic dilation as in section 2.3 for the operator $P$, with $B \geq R_3$ ($R_3$ is defined in lemma 4.9). Note $A = R_1$ ($R_1$ is defined in section 4.2). Consider the operator $P_{\theta_0}$ obtained. Take $\chi_1 \in C_0^\infty(\mathbb{R}^d)$ equal to 1 for $|x| \leq A$ and with support in $\{x; |x| \leq A+1\}$. From the (global) Carleman estimates of proposition 3.2, we obtain

***Proposition 5.1.*** — *There exists $c, C > 0$, $h_0 > 0$, $\forall 0 < h < h_0$, $e \in [1,2]$, $\forall u \in (\mathcal{D}_{\theta_0})$*

(5.1)
$$\int_\Omega (e^{2\varphi_1/h}+e^{2\varphi_2/h})|\chi_1(P-e)u|^2 drdL \geq ch\int_\Omega (e^{2\varphi_1/h}+e^{2\varphi_2/h})(|\chi_1 u|^2+|h\nabla\chi u|^2)drdL - Ch^2\int_{A\leq|x|\leq A+1} e^{2\varphi/h}\left(|h\nabla u|^2+|u|^2\right)drdL$$

Indeed, to prove proposition 5.1 remark that

$$\begin{aligned}
&\int_\Omega (e^{2\varphi_1/h}+e^{2\varphi_2/h})|\chi_1(P-e)u|^2 \\
&\geq \frac{1}{2}\int_\Omega (e^{2\varphi_1/h}+e^{2\varphi_2/h})|(P-e)\chi_1 u|^2 - 2\int_\Omega (e^{2\varphi_1/h}+e^{2\varphi_2/h})|[P,\chi_1]u|^2 \\
&\geq \frac{1}{2}\int_\Omega (e^{2\varphi_1/h}+e^{2\varphi_2/h})|(P-e)\chi_1 u|^2 - Ch^2\int_{A\leq r\leq A+1} e^{2\varphi/h}(|h\nabla u|^2+|u|^2)
\end{aligned} \tag{5.2}$$

Consider cut-off $\Psi_1, \Psi_2$ supported respectively in $\Omega_1\cup\{R\leq|x|\leq A+1\}$ and $\Omega_2\cup\{R\leq|x|\leq A+1\}$ (the sets $\Omega_i$ are defined in remark 4.2) and such that $\Psi_1^2+\Psi_2^2 = 1$ on the support of $\chi_1$. Apply proposition 3.2 to function $\Psi_i\chi_1 u$ and phases $\varphi_i$. and $\varphi_2$. Remark that due to the (finite number of) critical points of the functions $\varphi_i$, some error terms due to cut-off appear. However, when one sums the two inequalities obtained on $\Omega_{1;2}\cup\{R\leq|x|\leq A+1\}$ with $\varphi_1$ and $\varphi_2$, since on these small balls around the critical points of $\varphi_i$, $\varphi_{i+1}$ has no critical point and $\varphi_{i+1} > \varphi_i$ ($\varphi_3 = \varphi_1$), the error terms due to cut-off $\Psi_i$ to avoid these balls can be absorbed. See [**Bur98**, section 3.2] for a detailled analysis.

Consider now $\chi_2 \in C_0^\infty(\Gamma_{\theta_0})$ and $\chi_3 \in C^\infty(\Gamma_{\theta_0})$ such that $1 = \chi_1+\chi_2+\chi_3$, and

$$\chi_2 = \begin{cases} 0 \text{ for } |x|\notin[A,B+1] \\ 1 \text{ for } |x|\in[A+1/2,B+1/2]\end{cases} \quad , \qquad \chi_3 = \begin{cases} 0 \text{ for } |x|\leq B+1/2 \\ 1 \text{ for } |x|\geq B+1\end{cases} \tag{5.3}$$

The rest of the article will be devoted to the proof of the following result (remark that on the support of $\chi_2$, $\varphi_1 = \varphi_2$). *We recall that the $L^2$ norms we consider are the natural norms for which the operator $P$ is self adjoint. For $|x|\geq A$, the corresponding integration measure is simply the measure $drdL$.*

***Proposition 5.2.*** — *There exists $c, C > 0$, $h_0 > 0$, $\forall 0 < h < h_0$, $e \in [1,2]$, $\forall u \in (\mathcal{D}_{\theta_0})$, with $v = e^{\varphi/h}\chi_2 u$,*

$$\begin{aligned}(5.4)\quad \int e^{2\varphi/h}|\chi_2(P_\theta - e)u|^2 - e^{2\varphi(B,h)/h} Im\left(e^{i\kappa_{\theta_0}}(P_\theta - e)u, u\right)_{L^2} \geq \\ c\left(\|(\varphi')^{1/2}h\partial_r v\|_{L^2}^2 + ch\|(\frac{\varphi'}{r})^{1/2}h\frac{\nabla\sigma}{r}v\|_{L^2}^2 + h\|(\frac{\varphi'}{r})^{1/2}v\|_{L^2}^2\right) \\ + ce^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(|h\nabla u|^2 + |u|^2) - Ch^2\int_{A\leq|x|\leq A+1} e^{2\varphi/h}|h\nabla u|^2 + |u|^2\end{aligned}$$

***Remark 5.3.*** — *The last but one term in the estimate (5.4) is an elliptic estimate and corresponds to the fact that for $r > B$ (e.g. after the analytic dilation) the operator $P_{\theta_0} - e$ is elliptic (in the semi-classical meaning). Taking into account that for $A \leq r \leq B$ the phase $\varphi'$ varies from $\varphi' \geq c > 0$ for $r$ near $A$ to $\varphi' > Ch$ for $r$ near $B$. The first terms in the right hand side of (5.4) correspond to subelliptic estimate with loose of $1/2$ derivatives near $A$ (Carleman type estimate) and $1$ derivative near $B$ (Mourre type estimate). Finally the last term is an error term due to cut-off.*

Admitting this proposition for a while, we are in position to prove theorem 3. Consider $u, f \in L^2(\Gamma_{\theta_0})$ solutions to

$$(5.5)\qquad (P_\theta - e)u = f$$

Note $\tilde{\varphi}$ a function equal to $\max(\varphi_1, \varphi_2)$ for $r \leq B + 2$, constant for $r \geq B + 3$ and such that for $r \geq B$,

$$(5.6)\qquad |\varphi(r,h) - \varphi_{1;2}(B,h))| \leq Ch$$

Summing (5.1) and (5.4) we get

(5.7)

$$\begin{aligned}&4\int_{\Gamma_\theta} e^{2\tilde{\varphi}/h}|(P_{\theta_0} - e)u|^2 - e^{2\varphi(B,h)/h}\mathrm{Im}\left(e^{i\kappa_{\theta_0}}(P_{\theta_0} - e)u, u\right)_{L^2(\Gamma_{\theta_0})} \\ &\geq ch\int_{\Gamma_{\theta_0}} e^{2\tilde{\varphi}/h}(|\chi_1 u|^2 + |h\nabla\chi_1 u|^2) \\ &+ c\left(\|(\tilde{\varphi}')^{1/2}e^{\tilde{\varphi}/h}h\partial_r\chi_2 u\|_{L^2}^2 + h\|(\frac{\tilde{\varphi}'}{r})^{1/2}e^{\tilde{\varphi}/h}h\frac{\nabla\sigma}{r}\chi_2 u\|_{L^2}^2 + h\|(\frac{\tilde{\varphi}'}{r})^{1/2}e^{\tilde{\varphi}/h}\chi_2 u\|_{L^2}^2\right) \\ &+ ce^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(|h\nabla u|^2 + |u|^2) - Ch^2\int_{A\leq r\leq A+1} e^{2\tilde{\varphi}/h}(|h\nabla u|^2 + |u|^2)\end{aligned}$$

But taking into account that for $A \leq r \leq A + 1$, we have $\varphi' \geq \alpha > 0$, we easilly see that the last term can be absorbed by the four first ones (provided that $h > 0$ is small enough) and we obtain

(5.8)

$$4\int_{\Gamma_\theta} e^{2\tilde\varphi/h}|(P_{\theta_0}-e)u|^2 - e^{2\tilde\varphi(B,h)/h}\mathrm{Im}\left(e^{i\kappa_{\theta_0}}(P_{\theta_0}-e)u,u\right)_{L^2}$$
$$\geq ch\int e^{2\tilde\varphi/h}(|\chi_1 u|^2+|h\nabla\chi_1 u|^2)+ce^{2\tilde\varphi(B,h)/h}\int\kappa_{\theta_0}(|h\nabla u|^2+|u|^2)$$
$$+c\left(\|(\tilde\varphi')^{1/2}e^{\tilde\varphi/h}h\partial_r\chi_2 u\|^2_{L^2}+h\|(\frac{\tilde\varphi'}{r})^{1/2}e^{\tilde\varphi/h}h\frac{\nabla\sigma}{r}\chi_2 u\|^2_{L^2}+h\|(\frac{\tilde\varphi'}{r})^{1/2}e^{\tilde\varphi/h}\chi_2 u\|^2_{L^2}\right)$$

or

$$\|e^{\tilde\varphi/h}(P_{\theta_0}-e)u\|^2_{L^2}+e^{-\tilde\varphi(B,h)/h}\|e^{-\tilde\varphi/h}(P_{\theta_0}-e)\|_{L^2}\|e^{\tilde\varphi/h}u\|_{L^2} \geq ch^2(\|e^{\tilde\varphi/h}u\|^2_{L^2}+\|e^{\tilde\varphi/h}h\nabla u\|^2_{L^2}) \tag{5.9}$$

which gives, since $0\leq\tilde\varphi\leq Cte$,

$$\|e^{\tilde\varphi/h}u\|^2_{L^2}\leq\frac{C'}{h^2}\left(e^{2\tilde\varphi(B,h)/h}\|e^{-\tilde\varphi/h}(P_{\theta_0}-e)u\|^2_{L^2}+\|e^{\tilde\varphi/h}(P_{\theta_0}-e)u\|^2_{L^2}\right) \tag{5.10}$$

and we obtain theorem 3.

Suppose now that $(P_{\theta_0}-e)u$ is supported in the set $R_2+12\leq r$, where $\varphi(r,h)=\varphi(B,h)+\mathcal{O}(h)$. Then, since

$$\left|e^{2\varphi(B,h)/h}\mathrm{Im}\left(e^{i\kappa_{\theta_0}}(P_{\theta_0}-e)u,u\right)_{L^2(\Gamma_{\theta_0})}\right|\leq\|e^{\tilde\varphi/h}(P_{\theta_0}-e)u\|_{L^2}\|e^{\tilde\varphi/h}u\|_{L^2} \tag{5.11}$$

from (5.10) we obtain

$$\|e^{\tilde\varphi/h}u\|^2_{L^2}\leq\frac{C'}{h^2}\|e^{\tilde\varphi/h}(P_{\theta_0}-e)u\|^2_{L^2} \tag{5.12}$$

and we obtain theorem 4: indeed, since for $R_2+12\leq r\leq B$, we have $\varphi(r,h)=\varphi(B)+\mathcal{O}(h)$, (5.12) implies the following

$$\|1_{R_2+12\leq r\leq B}(P_{\theta_0}-e)^{-1}1_{R_2+12\leq r\leq B}f\|^2_{L^2}\leq\frac{C}{h^2}\|f\|^2_{L^2} \tag{5.13}$$

Since

$$1_{R_2+12\leq r\leq B}(P_{\theta_0}-e)^{-1}1_{R_2+12\leq r\leq B}=1_{R_2+12\leq r\leq B}(P-e)^{-1}1_{R_2+12\leq r\leq B} \tag{5.14}$$

where $(P-e)^{-1}$ is the outgoing resolvent, we obtain theorem 3 (for the ring $\{R_2+12\leq r\leq B\}$); and to conclude it is enough to remark that in the constructions above, $B$ can be taken arbitrarily large (in which case the constant $C$ depends on $B$).

## 6. The case of a potential

In this section we give the proof of proposition 5.2 for the simpler case where the operator we consider has the following form:

$$Q=-h^2\Delta+V(x,h) \tag{6.1}$$

in the exterior of a compact set.

In this case the reductions of appendix A are straightforward: Take $R+1 \leq R_1 = A \leq R_3 \leq B$ and suppose that $R$ is large enough so that for $r \geq R$ the operator $P$ is as in (6.1). Consider $\varrho$ a smooth function equal to 1 for $r \leq A-1$ and to $r^{(d-1)/2}$ for $r \geq A$. The operator $\varrho P \varrho^{-1}$ has the following form:

$$\begin{aligned}(6.2)\quad P = -h^2(\partial_r^2 + \Delta_\sigma) + h\left(2\varrho^{-1}\varrho' - \frac{d-1}{r}\right) h\partial_r \\ + h^2\left(\frac{d-1}{r}\varrho^{-1}\varrho' - 2\varrho^{-2}(\varrho')^2 + \varrho^{-1}\varrho''\right) u\end{aligned}$$

hence for $r \geq A$,

$$P = \varrho Q \varrho^{-1} = -h^2(\partial_r^2 + \frac{\Delta_\sigma}{r^2}) + \tilde{V}(x,h) \tag{6.3}$$

where

$$\tilde{V}(x,h) = V(x,h) + h^2\frac{(d-1)(d-3)}{4r^2} \tag{6.4}$$

For the sake of concisness, we shall still note $\tilde{V}$, $V$ (since for our purpose $\tilde{V}$ satisfies exactly the same assumptions as $V$). With this notation the operator $P$ is self adjoint for the measure $\mu = \varrho^{-2}r^{d-1}drdL$ where $dL$ is the Lebesgue measure on the $d-1$ sphere. We'll note

$$(u,v)_{L^2} \tag{6.5}$$

The $L^2$ scalar product for this measure.

We fix $\theta_0$ and note $P_{\theta_0}$ the operator obtained from $P$ by the analytic dilation process for $r \geq B$ as in section 2. Consider 2 functions $u \in \mathcal{D}_{\theta_0}$ and $f \in L^2_{\theta_0}$ with supports in the set $\{A < r\}$ and such that (for a number $e \in [1,2]$),

$$(P_{\theta_0} - e)u = f \tag{6.6}$$

### 6.1. Dissipation estimates. —

***Proposition 6.1.*** — *For any $\delta > 0$ there exists $h_0$ such that for any $0 < h < h_0$ and $u \in \mathcal{D}_{\theta_0}$*

$$\begin{aligned}(6.7)\quad & -Im\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0} - e)u, u\right)_{L^2} \\ & \geq \frac{1}{2}\int(\kappa_{\theta_0}(r) + r\kappa_{\theta_0}(r)')|h\partial_r u|^2 + \kappa_{\theta_0}(r)(|\frac{h\nabla_\sigma}{r}u|^2 + |u|^2)drdL - h^2\varepsilon(\delta)\int_{B\leq r\leq B+\delta}|u|^2\end{aligned}$$

*with*

$$\lim_{\delta\to 0}\varepsilon(\delta) = 0 \tag{6.8}$$

Take $\Psi \in C^\infty(\mathbb{R}^d)$ equal to 0 for $|x| \leq A$ and equal to 1 for $|x| \geq A+1$. Remark that since the operator $P$ is self adjoint,

$$\begin{aligned}\text{(6.9)}\quad &\operatorname{Im}\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0}-e)u,u\right)_{L^2}\\ &=\operatorname{Im}\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0}-e)\Psi u,\Psi u\right)_{L^2}+\operatorname{Im}\left((P-e)(1-\Psi)u,\Psi u\right)_{L^2}\\ &\quad+\operatorname{Im}\left((P-e)(1-\Psi)u,(1-\Psi)u\right)_{L^2}+\operatorname{Im}\left((P-e)\Psi u,(1-\Psi)u\right)_{L^2}\\ &=\operatorname{Im}\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0}-e)\Psi u,\Psi u\right)_{L^2}\end{aligned}$$

According to this remark, to compute $\operatorname{Im}\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0}-e)u,u\right)_{L^2}$, we can replace $u$ by $\Psi u$. In the following we still note $u$, but suppose that $u$ is supported in the set $\{|x| \geq A\}$. We have

$$\text{(6.10)}\qquad P_{\theta_0} = -\left(\frac{e^{-i\kappa_{\theta_0}(r)}}{1+ir\kappa_{\theta_0'}(r)}h\partial r\right)^2 - h^2 e^{-2i\kappa_{\theta_0}(r)}\frac{h^2\Delta_\sigma}{r^2} + V(re^{i\kappa_{\theta_0}(r)},\sigma,h)$$

and from (2.19),

$$\begin{aligned}\text{(6.11)}\quad &-\operatorname{Im}\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0}-e)u,u\right)_{L^2}\\ &=\int \operatorname{Im}\left(-\frac{e^{-i\kappa_{\theta_0}(r)}}{(1+ir\kappa'_{\theta_0}(r))^2}\right)|h\partial_r u|^2+\operatorname{Im}\left(-e^{-i\kappa_{\theta_0}(r)}\right)|h\frac{\nabla_\sigma}{r}u|^2 drdL\\ &+\int Im\left(e^{i\kappa_{\theta_0}(r)}(e-V(re^{i\kappa_{\theta_0}(r)},\sigma,h))\right)|u|^2drdL - h\operatorname{Im}(\int g(r)h\partial_r u\overline{u}drdL)\\ &\geq \int \frac{3}{4}(\kappa_{\theta_0}(r)+r\kappa'_{\theta_0}(r))|h\partial_r|^2+\kappa_{\theta_0}(r)(\frac{3}{4}|h\frac{\nabla_\sigma}{r}u|^2+\frac{1}{2}|u|^2)drdL-h\operatorname{Im}(g(r)h\partial_r u\overline{u})drdL\end{aligned}$$

with

$$\text{(6.12)}\qquad g(r) = \partial_r\left(\frac{1}{1+ir\kappa'_{\theta_0}(r)}\right)\frac{e^{-i\kappa_{\theta_0}(r)}}{1+ir\kappa'_{\theta_0}(r)}.$$

We have used here that

$$\text{(6.13)}\qquad V(re^{i\kappa_{\theta_0}(r)}) = V(r) + r(e^{i\kappa_{\theta_0}(r)}-1)\partial_r V(z)$$

for some $z \in ]r, re^{i\kappa_{\theta_0}(r)}[$ and since $V(r)$ is real we obtain from (4.3),

$$\text{(6.14)}\qquad |\operatorname{Im}(e^{i\kappa_{\theta_0}(r)}V(re^{i\kappa_{\theta_0}(r)}))| \leq |\kappa_{\theta_0}(r)||V(r)| + |\kappa_{\theta_0}(r)||r\partial_r V(z)| \leq \frac{|\kappa_{\theta_0}(r)|}{4}.$$

from which we deduce (using (2.19))

$$\text{(6.15)}\qquad |\operatorname{Im}(e^{i\kappa_{\theta_0}(r)}(e-V(re^{i\kappa_{\theta_0}(r)})))| \geq \frac{1}{2}|\kappa_{\theta_0}(r)|.$$

An integration by parts shows that the last term in (6.11) is smaller (in absolute value) than

$$\frac{h}{2}\int |Reg||h\partial_r u||u| + \frac{h^2}{2}\int |\partial_r g||u|^2 \tag{6.16}$$

but

$$g(r) = \frac{-i(r\kappa''_{\theta_0}(r) + \kappa'_{\theta_0}(r))e^{-i\kappa_{\theta_0}(r)}}{(1+ir\kappa'_{\theta_0}(r))^2} \tag{6.17}$$

and from (2.18)

$$\begin{aligned} |\mathrm{Re} g| &\le C(|\kappa'_{\theta_0}(r)| + |\kappa''_{\theta_0}(r)|)(|\kappa_{\theta_0}(r)| + |\kappa'_{\theta_0}(r)|) \le C|\kappa_{\theta_0}(r)|, \\ |\partial_r g| &\le C\left(|\kappa'''_{\theta_0}(r)| + |\kappa''_{\theta_0}(r)| + |\kappa_{\theta_0}(r)'|\right) \end{aligned} \tag{6.18}$$

The contribution of the first term in (6.16) is smaller than

$$Ch\int \kappa_{\theta_0}(r)\left(|h\partial_r u|^2 + |u|^2\right). \tag{6.19}$$

To study the second term we write

$$h^2\int |\partial_r g||u|^2 = h^2\int_{B\le r\le B+\delta} |\partial_r g||u|^2 + h^2\int_{B+\delta\le r} |\partial_r g||u|^2 \tag{6.20}$$

The second term in the right hand side of (6.20) is smaller than

$$C(\delta)h^2\int \kappa_{\theta_0}(r)|u|^2 \tag{6.21}$$

and since for $r\ge\delta$, $\kappa_{\theta_0}(r)\ge\kappa_{\theta_0}(r)(\delta)>0$, it can be (for fixed $\delta>0$) absorbed by the right hand side of (6.11); whereas the first term in the righthand side of (6.20) is smaller than

$$\varepsilon(\delta)h^2\int_{B\le r\le B+\delta}|u|^2, \quad \text{with} \quad \varepsilon(\delta) = \sup_{B\le r\le B+\delta}|\partial_r g| \tag{6.22}$$

and (from (6.18))

$$\lim_{\delta\to 0}\varepsilon(\delta) = 0. \tag{6.23}$$

From (6.11), (6.19), (6.20), (6.21) and (6.22), we obtain proposition 6.1.

**6.2. Commutators estimates.** — In this section we prove positive commutator type estimates for a conjugate of the operator $P$.

Consider the function $\varphi(r,h)$ constructed in section 4. Note

$$\begin{aligned} P_\varphi &= e^{\varphi/h}Pe^{-\varphi/h} \\ &= -h^2\partial_r^2 - \frac{h^2\Delta_\sigma}{r^2} + V - (\varphi')^2 + i\left(\varphi' h\frac{\partial_r}{i} + h\frac{\partial_r}{i}\varphi'\right) \\ &= \mathrm{Re}P_\varphi + i\mathrm{Im}P_\varphi \end{aligned} \tag{6.24}$$

Remember that $\chi_2 \in C_0^\infty(\mathbb{R}^d)$ with support in $\{A \leq r \leq B+1\}$ and equal to 1 in $\{A+1/2 \leq r \leq B+1/2\}$.

***Proposition 6.2.*** — *There exists $h_0 > 0$ such that for any $0 < h < h_0$, $e \in [1,2]$ and any $u \in \mathcal{D}_{\kappa_{\theta_0}(r)}$, if $v = \chi_2 u e^{\varphi/h}$*

$$2\|(P_\varphi - e)v\|_{L^2}^2 \geq \|\varphi' h\partial_r v\|_{L^2}^2 + h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + \frac{h}{2}\left(\frac{\varphi'}{r}v, v\right)_{L^2} \tag{6.25}$$

Compute

(6.26)
$$\begin{aligned} I = \|(P_\varphi - e)v\|_{L^2}^2 &= \|\mathrm{Re}(P_\varphi - e)v\|_{L^2}^2 + \|\mathrm{Im}(P_\varphi - e)v\|_{L^2}^2 + i\left([\mathrm{Re}P_\varphi, \mathrm{Im}P_\varphi]v, v\right)_{L^2} \\ &= (A) + (B) + (C) \end{aligned}$$

$$(B) = \|2\varphi' h\partial_r v + h\varphi'' v\|_{L^2}^2 \geq 2\|\varphi' h\partial_r v\|_{L^2}^2 - h^2\|\varphi'' v\|_{L^2}^2 \tag{6.27}$$

$$\begin{aligned} (C) = &\left((-4h^2\partial_r\varphi'' h\partial_r - 4h\frac{\varphi'}{r}\frac{h^2\Delta_\sigma}{r^2} + 4h(\varphi')^2\varphi'' v, v\right)_{L^2} \\ &+ \left((-h^3\partial_r\varphi''' + h^3\varphi'''\partial_r - 2h\varphi'\partial_r V)v, v\right)_{L^2} \\ = &(1) + (2) + (3) + (4) + (5) + (6) \end{aligned} \tag{6.28}$$

$$(1) = 4h\left(\varphi'' h\partial_r v, h\partial_r v\right)_{L^2} \tag{6.29}$$

$$(2) = 4h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} \tag{6.30}$$

$$(3) = 4h\left((\varphi')^2\varphi'' v, v\right)_{L^2} \tag{6.31}$$

$$|(4) + (5) + (6)| \leq h^2\left(\left(|\varphi'''| h\partial_r v, h\partial_r v\right)_{L^2} + \left(|\varphi'''| v, v\right)_{L^2}\right) + 2h\left(\varphi'|\partial_r V| v, v\right)_{L^2} \tag{6.32}$$

We need the following lemma:

***Lemma 6.3.*** — *There exists $h_0 > 0$ such that for any $0 < h < h_0$*

$$\begin{aligned} \left(\frac{\varphi'}{r}v, v\right)_{L^2} \leq 2&\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} \\ &+ 2\left(\frac{\varphi'}{r}h\partial_r v, h\partial_r v\right)_{L^2} + 3\|\left(\frac{\varphi'}{r}\right)^{1/2}(P_\varphi - e)v\|_{L^2}^2 \end{aligned} \tag{6.33}$$

Proof of Lemma 6.3: Compute

$$
\begin{aligned}
&\mathrm{Re}\left((P_\varphi - e)v, \frac{\varphi'}{r}v\right)_{L^2}\\
&= \left(\frac{\varphi'}{r}h\partial_r v, h\partial_r v\right)_{L^2} + \left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2}\\
&- \left(\frac{\varphi'}{r}(e+(\varphi')^2 - V)v, v\right)_{L^2} + h\mathrm{Re}\left(\left(\frac{\varphi'}{r}\right)' h\partial_r v, v\right)_{L^2}\\
&- h\left(\varphi'\left(\frac{\varphi'}{r}\right)' v, v\right)_{L^2}\\
&= \left(\frac{\varphi'}{r}h\partial_r v, h\partial_r v\right)_{L^2} + \left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2}\\
&- \left(\left[\frac{\varphi'}{r}(e+(\varphi')^2 - V) + h\varphi'\left(\frac{\varphi'}{r}\right)' + \frac{h^2}{2}\left(\frac{\varphi'}{r}\right)''\right] v, v\right)_{L^2}
\end{aligned}
\tag{6.34}
$$

From proposition 4.13, we obtain

$$
\begin{aligned}
\mathrm{Re}\left((P_\varphi - e)v, \frac{\varphi'}{r}v\right)_{L^2} &= \left(\frac{\varphi'}{r}h\partial_r v, h\partial_r v\right)_{L^2} + \left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2}\\
&- \left(\frac{\varphi'}{r}\left(e+(\varphi')^2 - V + \mathcal{O}(h^{1/2})\right)v, v\right)_{L^2}
\end{aligned}
\tag{6.35}
$$

but

$$
\begin{aligned}
\left|\mathrm{Re}\left((P_\varphi - e)v, \frac{\varphi'}{r}v\right)_{L^2}\right| &\leq \|\left(\frac{\varphi'}{r}\right)^{1/2}(P_\varphi - e)v\|_{L^2} \times \|\left(\frac{\varphi'}{r}\right)^{1/2} v\|_{L^2}\\
&\leq \frac{3}{2}\|\left(\frac{\varphi'}{r}\right)^{1/2}(P_\varphi - e)v\|^2_{L^2} + \frac{1}{6}\|\left(\frac{\varphi'}{r}\right)^{1/2} v\|^2_{L^2}
\end{aligned}
\tag{6.36}
$$

From (4.3) and (6.36) we obtain lemma 6.3.

We come back to the proof of proposition 6.2. From (6.27), (6.30) and lemma 6.3, we get

$$
\begin{aligned}
&(B) + (2) + 3h\left\|\left(\frac{\varphi'}{r}\right)^{1/2}(P_\varphi - e)v\right\|^2_{L^2}\\
&\geq \left(\left[2(\varphi')^2 - 2h\frac{\varphi'}{r}\right] h\partial_r v, h\partial_r v\right)_{L^2} + h\left(\frac{\varphi'}{r}(4-2)\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2}\\
&\qquad + h\left(\left(\frac{\varphi'}{r} - h(\varphi'')^2\right)v, v\right)_{L^2}
\end{aligned}
\tag{6.37}
$$

and we obtain from (6.26), (6.29), (6.31), (6.32),

$$
\begin{aligned}
(6.38)\quad I + 3h\Big\| \left(\frac{\varphi'}{r}\right)^{1/2} (P_\varphi - e)v\Big\|^2_{L^2} &\geq 2h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} \\
&+ \left(\left[2(\varphi')^2 - 2h\frac{\varphi'}{r} + 4h\varphi'' - h^2|\varphi'''|\right] h\partial_r v, h\partial_r v\right)_{L^2} \\
&+ h\left(\left[\frac{\varphi'}{r} + 4(\varphi')^2\varphi'' - h(\varphi'')^2 - h|\varphi'''| - 2\varphi'|\partial_r V|\right] v, v\right)_{L^2}
\end{aligned}
$$

From lemma 4.10, (4.3), proposition 4.11 and proposition 4.12 we obtain

$$
\begin{aligned}
(6.39)\quad & I + 3h\Big\| \left(\frac{\varphi'}{r}\right)^{1/2} (P_\varphi - e)v\Big\|^2_{L^2} \\
&\geq \left((\varphi')^2 h\partial_r v, h\partial_r v\right)_{L^2} + 2h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + \frac{h}{2}\left(\frac{\varphi'}{r}v, v\right)_{L^2}
\end{aligned}
$$

proving proposition 6.2.

**6.3. The dissipation-commutator estimate.** — In this section we show how propositions 6.1 and 6.2 can be mixed together to give an estimate on solution to the analitically dilated problem.

***Proposition 6.4* (Proposition 5.2).** — *There exists $C > 0$ and $h_0 > 0$ such that for any $0 < h < h_0$ and any $u \in \mathcal{D}_{\theta_0}$, with $v = \chi_2 e^{\varphi/h} u$*

$$
\begin{aligned}
(6.40)\quad & C\|\chi_2 e^{\varphi/h}(P_{\theta_0} - e)u\|^2_{L^2} - Ce^{2\varphi(B,h)/h} Im\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0} - e)u, u\right)_{L^2} \\
&\geq \left((\varphi')^2 h\partial_r v, h\partial_r v\right)_{L^2} + h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + h\left(\frac{\varphi'}{r}v, v\right)_{L^2} \\
&+ e^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(r)\left(|h\nabla u|^2 + |u|^2\right) - Ch^2\int_{A\leq r\leq A+1} e^{2\varphi/h}(|u|^2 + |h\nabla u|^2)
\end{aligned}
$$

From proposition 6.1

$$
\begin{aligned}
(6.41)\quad & -\operatorname{Im}\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0} - e)u, u\right)_{L^2} \\
&\qquad \geq \frac{1}{2}\int \kappa_{\theta_0}(r)(|h\nabla u|^2 + |u|^2) - \varepsilon(\delta)h^2\int_{B\leq r\leq B+\delta}|u|^2
\end{aligned}
$$

From proposition 6.2

$$
(6.42)\qquad 2\|(P_\varphi - e)v\|^2 \geq \|\varphi' h\partial_r v\|^2_{L^2} + h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + \frac{h}{2}\left(\frac{\varphi'}{r}v, v\right)_{L^2}
$$

But

$$(6.43)\quad \begin{aligned}(P_\varphi - e)v &= e^{\varphi/h}(P_{\theta_0} - e)\chi_2 u + e^{\varphi/h}(P - P_{\theta_0})\chi_2 u\\ &= e^{\varphi/h}\chi_2(P_{\theta_0} - e)u + e^{\varphi/h}[P_{\theta_0}, \chi_2]u + e^{\varphi/h}(P - P_{\theta_0})\chi_2 u\\ &= (1) + (2) + (3)\end{aligned}$$

Since for $r \geq B$, we have $\varphi(r,h) = \varphi(B,h) + \mathcal{O}(h)$, we obtain

$$(6.44)\quad \begin{aligned}\|(2)\|^2_{L^2} \leq Ch^2 \int_{A\leq r\leq A+1/2} & e^{2\varphi/h}(|h\nabla u|^2 + |u|^2)drdL\\ &+ Ch^2 e^{2\varphi(B,h)/h}\int_{B+1/2\leq r\leq B+1}(|h\nabla u|^2 + |u|^2)drdL\end{aligned}$$

Since the coefficients of the operator $(P - P_{\theta_0})$ are equal to 0 for $r \leq B$, it is easy to see from (6.10) that

$$(6.45)\quad \|(3)\|^2_{L^2} \leq Ce^{2\varphi(B,h)/h}\int (|\kappa_{\theta_0}(r)| + |\kappa_{\theta_0}(r)'| + |\kappa_{\theta_0}(r)''|)^2 \sum_{|\alpha|\leq 2} |(h\partial_x)^\alpha \chi_2 u|^2$$

***Lemma 6.5***. — *There exists $C > 0$ and $h_0 > 0$ such that for any $0 < h < h_0$, $u \in \mathcal{D}_{\theta_0}$,*

$$(6.46)\quad \|(3)\|^2_{L^2} \leq Ce^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(r)\sum_{|\alpha|\leq 1}|(h\partial_x)^\alpha u|^2 + \kappa_{\theta_0}(r)|\chi_2(P_{\theta_0} - e)u|^2$$

Proof of lemma 6.5

$$(6.47)\quad P_{\theta_0}(\chi_2\kappa_{\theta_0}(r)u) = \chi_2\kappa_{\theta_0}(r)(P_{\theta_0} - e)u + [P_{\theta_0}, \chi_2\kappa_{\theta_0}(r)]u + e\chi_2\kappa_{\theta_0}(r)u$$

By standard semi-classical elliptic estimates for the operator $P_{\theta_0}$, we obtain

$$(6.48)\quad \begin{aligned}\sum_{|\alpha|\leq 2}\|(h\partial_x)^\alpha\chi_2\kappa_{\theta_0}(r)u\|^2_{L^2} \leq C\big(&\|\chi_2\kappa_{\theta_0}(r)(P_{\theta_0} - e)u\|^2_{L^2}\\ &+ \|e\chi_2\kappa_{\theta_0}(r)u\|^2_{L^2} + \|[P_{\theta_0}, \chi_2\kappa_{\theta_0}(r)]u\|^2_{L^2}\big)\end{aligned}$$

Hence

$$(6.49)\quad \begin{aligned}\sum_{|\alpha|=2}\|\kappa_{\theta_0}(r)(h\partial_x)^\alpha\chi_2 u\|^2_{L^2} \leq C\Big(&\|\chi_2\kappa_{\theta_0}(r)(P_{\theta_0} - e)u\|^2_{L^2}\\ &+ \sum_{|\alpha|\leq 1}\||(\kappa_{\theta_0}(r)| + |\kappa_{\theta_0}(r)'| + |\kappa_{\theta_0}(r)''|)(h\partial_x)^\alpha u\|^2_{L^2}\Big)\end{aligned}$$

By the same way we get similar estimates for

$$(6.50)\quad \sum_{|\alpha|=2}\|\kappa'_{\theta_0}(r)(h\partial_x)^\alpha\chi_2 u\|^2_{L^2},\ \text{and}\ \sum_{|\alpha|=2}\|\kappa''_{\theta_0}(r)(h\partial_x)^\alpha\chi_2 u\|^2_{L^2},$$

which gives

$$(6.51)\quad \sum_{|\alpha|=2} \|(|\kappa_{\theta_0}(r)| + |\kappa_{\theta_0}(r)'| + |\kappa_{\theta_0}(r)''|)(h\partial_x)^\alpha \chi_2 u\|_{L^2}^2$$
$$\leq C[\|\chi_2(|\kappa_{\theta_0}(r)| + |\kappa_{\theta_0}(r)'| + |\kappa_{\theta_0}(r)''|)(P_{\theta_0} - e)u\|_{L^2}^2$$
$$+ \sum_{|\alpha|\leq 1} \|(\kappa_{\theta_0}(r)| + |\kappa_{\theta_0}(r)'| + |\kappa_{\theta_0}(r)''| + |\kappa_{\theta_0}(r)'''| + |\kappa_{\theta_0}(r)^{(4)}|)(h\partial_x)^\alpha u\|_{L^2}^2$$

According to (2.18) we obtain lemma 6.5.

We come back to the proof of proposition 6.4. Take $N \in \mathbb{N}$ to be fixed later. From (6.41), (6.43) and (6.42),

$$(6.52)\quad I = 2\|e^{\varphi/h}\chi_2(P_{\theta_0} - e)u\|^2 - Ne^{2\varphi(B,h)/h}\mathrm{Im}\left(e^{i\kappa_{\theta_0}(r)}(P_{\theta_0} - e)u, u\right)_{L^2}$$
$$\geq \|(P_\varphi - e)v\|^2 - 4[\|(2)\|^2 + \|(3)\|^2]$$
$$+ Ne^{2\varphi(B,h)/h}\left[\frac{1}{2}\int \kappa_{\theta_0}(r)(|h\nabla u|^2 + |u|^2) - \varepsilon(\delta)h^2 \int_{B\leq r\leq B+\delta} |u|^2\right]$$

From (6.42), (6.44)and lemma 6.5 we obtain

$$I + Ce^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(r)|(\chi_2 P_{\theta_0} - e)u|^2$$
$$\geq \frac{1}{2}\|\varphi' h\partial_r v\|_{L^2}^2 + \frac{h}{2}\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + \frac{h}{4}\left(\frac{\varphi'}{r}v, v\right)_{L^2}$$
$$+ Ne^{2\varphi(B,h)/h}\left[\frac{1}{2}\int \kappa_{\theta_0}(r)(|h\nabla u|^2 + |u|^2)\right]$$
$$(6.53)\quad - Ch^2\int_{A\leq r\leq A+1/2} e^{2\varphi/h}(|u|^2 + |h\nabla u|^2)$$
$$- N\varepsilon(\delta)h^2 e^{2\varphi(B)/h}\int_{B\leq r\leq B+\delta} |u|^2$$
$$- Ch^2 e^{2\varphi(B,h)/h}\int_{B+1/2\leq r\leq B} (|u|^2 + |h\nabla u|^2)$$
$$- Ce^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(r)(|u|^2 + |h\nabla u|^2)$$

To get proposition 6.4, we have to eliminate in (6.53) the three last terms (terms (6), (7) and (8). First fix $N \geq 2(C+1)$ Then

$$(6.54)\quad (4) + (8) = Ne^{2\varphi(B,h)/h}\left[\frac{1}{2}\int \kappa_{\theta_0}(r)(|h\nabla u|^2 + |u|^2)\right]$$
$$- Ce^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(r)(|u|^2 + |h\nabla u|^2)$$
$$\geq e^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(r)(|u|^2 + |h\nabla u|^2)$$

Then choose $0 < \delta < 1/2$ such that for $0 < h < h_0$

$$N\varepsilon(\delta)e^{2\varphi(B)/h}h^2 \leq \frac{1}{8}\inf_{r\in[B,B+1/2]} h\frac{\varphi'}{r}e^{2\varphi(r)/h} \tag{6.55}$$

$$\Leftrightarrow \varepsilon(\delta) \leq \frac{1}{8N}\inf_{r\in[B,B+1/2]} e^{2\frac{(\varphi(r,h)-\varphi(B,h))}{h}}\frac{g'(r)}{r} \tag{6.56}$$

with $\varphi' = hg' = \frac{8h}{r}$ near $B$ ($g$ is as in lemma 4.9). Such a choice is clearly possible since for $r \geq B$ we have $\varphi(r,h) = \varphi(B,h) + 8h\log(\frac{r}{B})$ and $\lim_{\delta\to 0}\varepsilon(\delta) = 0$.

Since $v = e^{\varphi/h}u$ for $B \leq r \leq B+1/2$, from (6.55) we obtain

$$\begin{aligned}(3) + (6) = \frac{h}{4}\left(\frac{\varphi'}{r}v, v\right)_{L^2} - N\varepsilon(\delta)h^2e^{2\varphi(B,h)/h}\int_{B\leq r\leq B+\delta}|u|^2 \\ \geq \frac{h}{8}\left(\frac{\varphi'}{r}v, v\right)_{L^2}\end{aligned} \tag{6.57}$$

Finally, since for $B \leq r \leq B+1/2$, we have $\kappa_{\theta_0}(r) \geq c > 0$, we obtain, using (6.54)

$$\begin{aligned}(4) + (8) + (7) \geq= e^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(r)(|u|^2 + |h\nabla u|^2) + (7) \\ \geq (1 - \frac{C}{c}h^2)e^{2\varphi(B,h)/h}\int \kappa_{\theta_0}(r)(|u|^2 + |h\nabla u|^2)\end{aligned} \tag{6.58}$$

With these choices of $N, \delta$ we obtain proposition 6.4.

## 7. The general case

In this section we come back to the general case. We work here in the coordinate system and after the conjugations of section 2.2. Since all the computations will be performed on functions supported in the set $\{|x| \geq A\}$, the use of these reductions is licit. As in section 6, take $R + 1 \leq A \leq R_3 \leq< B$ and suppose that $R$ is large enough so that for $r \geq R$ the operator $P$ satisfies (2.12), (2.15) and (2.13). We note as in section 6 $P_{\theta_0}$ the operator obtained from $P$ by the analytic dilation process presented in section 2.3. The point in this section is to prove the analogues of propositions 6.1 and 6.2. Unfortunatly, we can not obtain these propositions (in the general case) by a simple perturbation argument from the results obtained in the potential case, but more likelly from a perturbation of their proofs.

### 7.1. Dissipation estimates. —

***Proposition 7.1.*** — *For any $\delta > 0$ there exists $h_0$ such that for any $0 < h < h_0$ and $u \in \mathcal{D}_{\theta_0}$*

$$\begin{aligned}&-Im\left(e^{i\kappa_{\theta_0}}(P_{\theta_0} - e)u, u\right)_{L^2} \\ &\geq \frac{1}{2}\int(\kappa_{\theta_0} + r\kappa'_{\theta_0})|h\partial_r u|^2 + \kappa_{\theta_0}(|\frac{h\nabla_\sigma}{r}u|^2 + |u|^2)drdL - h^2\varepsilon(\delta)\int_{B\leq r\leq B+\delta}|u|^2\end{aligned} \tag{7.1}$$

*with*

$$\lim_{\delta\to 0}\varepsilon(\delta)=0 \tag{7.2}$$

As in section 6 to prove this result, since the operator $P$ is self adjoint,we can suppose that the function $u$ is supported in the set $\{|x|\geq A\}$. We have

$$\begin{aligned} P_{\theta_0} = &-\left(\frac{e^{-i\kappa_{\theta_0}}}{1+ir\kappa'_{\theta_0}}h\partial r\right)^2 - h^2e^{-2i\kappa_{\theta_0}}\frac{\Delta_\sigma}{r^2} \\ &+h^2e^{-2i\kappa_{\theta_0}}q_2(re^{i\kappa_{\theta_0}},\sigma,\frac{h\partial_{\sigma_i}}{r})+e^{-i\kappa_{\theta_0}}q_1(re^{i\kappa_{\theta_0}},\sigma,\frac{h\partial_{\sigma_i}}{r},h)+V(re^{i\kappa_{\theta_0}},\sigma,h)\end{aligned}\tag{7.3}$$

and

$$\begin{aligned} &-\operatorname{Im}\left(e^{i\kappa_{\theta_0}}(P_{\theta_0}-e)u,u\right)_{L^2} \\ &= \int \operatorname{Im}\left(-\frac{e^{-i\kappa_{\theta_0}}}{(1+ir\kappa'_{\theta_0})^2}\right)|h\partial_r u|^2 + \operatorname{Im}\int -e^{-i\kappa_{\theta_0}}\frac{\|\nabla_\sigma u\|^2}{r^2} \\ &+\operatorname{Im}\int e^{-i\kappa_{\theta_0}}q_2(r,\sigma,\frac{h\partial_{\sigma_i}}{r})u\overline{u}drdL + \int Im\left(e^{i\kappa_{\theta_0}}(e-V(re^{i\kappa_{\theta_0}},\sigma,h))\right)|u|^2drdL \\ &-h\operatorname{Im}\left(\int g(r)h\partial_r u\overline{u}drdL\right)+\operatorname{Im}\int q_1(re^{i\kappa_{\theta_0}},\sigma,\frac{h\partial_{\sigma_i}}{r})u\overline{u}drdL \\ &+\operatorname{Im}\int e^{-i\kappa_{\theta_0}}\left(q_2(re^{i\kappa_{\theta_0}},\sigma,\frac{h\partial_{\sigma_i}}{r})-q_2(r,\sigma,\frac{h\partial_{\sigma_i}}{r})\right)u\overline{u}drdL \\ &= (1)+(2)+(3)+(4)+(5)+(6)+(7)\end{aligned}\tag{7.4}$$

with (from (2.19))

$$(1)\geq\frac{3}{4}\int(\kappa_{\theta_0}+r\kappa'_{\theta_0})|h\partial_r u|^2, \qquad (2)\geq\frac{3}{4}\int\kappa_{\theta_0}\frac{\|\nabla_\sigma u\|^2}{r^2}\tag{7.5}$$

Since $q_2$ is self adjoint, we have

$$(3)=\int\operatorname{Im}(e^{-i\kappa_{\theta_0}})q_2(r,\sigma,\frac{h\partial_{\sigma_i}}{r})u\overline{u}\tag{7.6}$$

and from (4.3) and (4.4)

$$|(3)|\leq\frac{1}{16}\int\kappa_{\theta_0}(|h\frac{\nabla_\sigma}{r}u|^2+|u|^2)\tag{7.7}$$

As in section 6, we obtain

$$(4)\geq\frac{3}{4}\int\kappa_{\theta_0}|u|^2\tag{7.8}$$

and

$$|(5)|\leq C(\delta)h\int\kappa_{\theta_0}(|h\partial_r u|^2+|u|^2)+\varepsilon(\delta)h^2\int_{B\leq r\leq B+\delta}|u|^2\tag{7.9}$$

Remark that since the operator $q_1(r,\sigma,\frac{h\partial_{\sigma_i}}{r})$ is self adjoint, we have $(6)=\mathrm{Im}(6')$ with

$$(6')=\int\left(q_1(re^{i\kappa_{\theta_0}},\sigma,\frac{h\partial_{\sigma_i}}{r})-q_1(r,\sigma,\frac{h\partial_{\sigma_i}}{r})\right)u\overline{u} \tag{7.10}$$

From (4.5) we deduce

$$|(6')|\leq\frac{1}{16}\left(\int\kappa_{\theta_0}|h\frac{\nabla_\sigma}{r}u|^2+\int\kappa_{\theta_0}|u|^2\right) \tag{7.11}$$

We also have from (4.5)

$$\begin{aligned}|(7)|&=|\int e^{-i\kappa_{\theta_0}}\left(q_2(re^{i\kappa_{\theta_0}},\sigma,\frac{h\partial_{\sigma_i}}{r})-q_2(r,\sigma,\frac{h\partial_{\sigma_i}}{r})\right)u\overline{u}|\\&\leq\frac{1}{16}\left(\int\kappa_{\theta_0}|h\frac{\nabla_\sigma}{r}u|^2+\int\kappa_{\theta_0}|u|^2\right)\end{aligned} \tag{7.12}$$

From (7.5), (7.7), (7.8), (7.9), (7.11)and (7.12) we obtain

$$\begin{aligned}&-\mathrm{Im}\left(e^{i\kappa_{\theta_0}}(P_{\theta_0}-e)u,u\right)_{L^2}\\&\geq\int\left(\frac{3}{4}-C(\delta)h\right)(\kappa_{\theta_0}+r\kappa'_{\theta_0})|h\partial_r u|^2+\left(\frac{3}{4}-\frac{3}{16}\right)\kappa_{\theta_0}(|\frac{h\nabla_\sigma}{r}u|^2drdL\\&\qquad+\left(\frac{3}{4}-\frac{3}{16}-C(\delta)h\right)|u|^2)drdL-h^2\varepsilon(\delta)\int_{B\leq r\leq B+\delta}|u|^2drdL\end{aligned} \tag{7.13}$$

For $h_0$ small enough (depending on $\delta>0$), we obtain proposition 7.1

**7.2. Commutator estimates.** — Consider the function $\varphi(r,h)$ constructed in section 4. Note

(7.14)

$$\begin{aligned}P_\varphi&=e^{\varphi/h}Pe^{-\varphi/h}\\&=-h^2\partial_r^2-h^2\frac{\Delta_\sigma}{r^2}+q_2(r,\sigma,h\frac{\partial_\sigma}{r})+q_1(r,\sigma,h\frac{\partial_\sigma}{r},h)+V-(\varphi')^2+i(\varphi'h\frac{\partial_r}{i}+h\frac{\partial_r}{i}\varphi')\\&=\mathrm{Re}P_\varphi+i\mathrm{Im}P_\varphi\end{aligned}$$

Consider $\chi_2\in C_0^\infty(\mathbb{R}^d)$ with support in $\{A\leq r\leq B+1\}$ and equal to 1 in $\{A+1/2\leq r\leq B+1/2\}$.

***Proposition 7.2.*** — *There exists $h_0>0$ such that for any $0<h<h_0$, $e\in[1,2]$ and any $u\in\mathcal{D}_\theta$, if $v=\chi_2u$*

$$2\|(P_\varphi-e)v\|^2_{L^2}\geq\|\varphi'h\partial_r v\|^2_{L^2}+h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v,\frac{h\nabla_\sigma}{r}v\right)_{L^2}+\frac{h}{2}\left(\frac{\varphi'}{r}v,v\right)_{L^2} \tag{7.15}$$

To prove this result, we follow closely the proof of proposition 6.2. Compute

(7.16)

$$\begin{aligned}I=\|(P_\varphi-e)v\|^2_{L^2}&=\|\mathrm{Re}(P_\varphi-e)v\|^2_{L^2}+\|\mathrm{Im}(P_\varphi-e)v\|^2_{L^2}+i\left([\mathrm{Re}P_\varphi,\mathrm{Im}P_\varphi]v,v\right)_{L^2}\\&=(A)+(B)+(C)\end{aligned}$$

As in (6.27),

$$(B) \geq 2\|\varphi' h\partial_r v\|^2_{L^2} - h^2\|\varphi'' v\|^2_{L^2} \tag{7.17}$$

(7.18)

$$(C) = \left(\left(-4h^2\partial_r\varphi'' h\partial_r - h^3\partial_r\varphi''' + \varphi''' h^3\partial_r - 4h\frac{\varphi'}{r^3}h^2\Delta_\sigma + 4h(\varphi')^2\varphi''\right)v, v\right)_{L^2}$$
$$-\left(\left(2h\varphi'\frac{d}{dr}(q_2(r,\sigma,h,h\frac{\partial_\sigma}{r})) + 2h\varphi'\frac{d}{dr}(q_1(r,\sigma,h,h\frac{\partial_\sigma}{r}))\right)v, v\right)_{L^2} + 2h(\varphi'\partial_r V v, v)_{L^2}$$

Taking into account that $q_2$ is quadratic with respect to $\partial_\sigma/r$ and that $q_1$ is linear we obtain

$$\begin{aligned}
(C) &= \left(4\varphi'' h\partial_r v, h\partial_r v\right)_{L^2} + 4h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + 4h\left((\varphi')^2\varphi'' v, v\right)_{L^2}\\
&\quad + \left((-h^3\partial_r\varphi''' + h^3\varphi'''\partial_r)v, v\right)_{L^2} - 2h\left(\varphi'(\partial_r q_2 - 2\frac{q_2}{r})v, v\right)_{L^2}\\
&\quad - 2h\left(\varphi'(\partial_r q_1 - \frac{q_1}{r})v, v\right) - 2h\left(\varphi'\partial_r V v, v\right)_{L^2}\\
&= (1) + (2) + (3) + (4) + (5) + (6) + (7)
\end{aligned} \tag{7.19}$$

The terms (1), (2) and (3) are the same as in (6.28) and we have as in (6.29), (6.31) and (6.32)

$$(1) = 4h\left(\varphi'' h\partial_r v, h\partial_r v\right)_{L^2} \tag{7.20}$$

$$(2) = 4h\left(\frac{\varphi'}{r}h\frac{\nabla_\sigma}{r}v, h\frac{\nabla_\sigma}{r}v\right)_{L^2} \tag{7.21}$$

$$(3) = h\left((\varphi')^2\varphi'' v, v\right)_{L^2} \tag{7.22}$$

$$|(4) + (7)| \leq h^2\left(\left(|\varphi'''|h\partial_r v, h\partial_r v\right)_{L^2} + \left(|\varphi'''|v, v\right)_{L^2}\right) + 2h\left(\varphi'|\partial_r V|v, v\right)_{L^2} \tag{7.23}$$

From (4.4) we have

$$|(5) + (6)|) \leq \frac{h}{8}\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + \frac{h}{8}\left(\frac{\varphi'}{r}v, v\right)_{L^2} \tag{7.24}$$

We need the following lemma:

***Lemma 7.3.*** — *There exists $h_0 > 0$ such that for any $0 < h < h_0$*

$$\left(\frac{\varphi'}{r}v, v\right)_{L^2} \leq 2\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + 2\left(\frac{\varphi'}{r}h\partial_r v, h\partial_r v\right)_{L^2} + 3\|\left(\frac{\varphi'}{r}\right)^{1/2}(P_\varphi - e)v\|^2_{L^2} \tag{7.25}$$

Proof of Lemma 7.3: The same calculus as in the proof of lemma 6.3 gives

$$
\begin{aligned}
&\mathrm{Re}\left((P_\varphi - e)v, \frac{\varphi'}{r}v\right)_{L^2} \\
&= \left(\frac{\varphi'}{r}h\partial_r v, h\partial_r v\right)_{L^2} + \left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} \\
&+ \left(\frac{\varphi'}{r}q_2(r,\sigma,\frac{h\nabla_\sigma}{r}v, v\right)_{L^2} + \left(\frac{\varphi'}{r}q_1(r,\sigma,\frac{h\partial_\sigma}{r})v, v\right)_{L^2} \\
&- \left(\left[\frac{\varphi'}{r}(e+(\varphi')^2 - V) - h\varphi'\left(\frac{\varphi'}{r}\right)' - \frac{h^2}{2}\left(\frac{\varphi'}{r}\right)''\right]v, v\right)_{L^2}
\end{aligned}
\tag{7.26}
$$

From (4.4) we have

$$
\begin{aligned}
&\left|\left(\frac{\varphi'}{r}q_2(r,\sigma,\frac{h\partial_\sigma}{r})v, v\right)_{L^2}\right| + \left|\left(\frac{\varphi'}{r}q_1(r,\sigma,\frac{h\partial_\sigma}{r})v, v\right)_{L^2}\right| \\
&\qquad \leq \frac{1}{8}\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + \frac{1}{8}\left(\frac{\varphi'}{r}v, v\right)_{L^2}
\end{aligned}
\tag{7.27}
$$

From (4.3), proposition 4.13 and(7.27) we obtain

$$
\begin{aligned}
&(\frac{7}{8} - Ch^{1/2})\left(\frac{\varphi'}{r}v, v\right)_{L^2} \leq \left(\frac{\varphi'}{r}\left(e + (\varphi')^2 - V - h\varphi'\left(\frac{\varphi'}{r}\right)' - \frac{h^2}{2}\left(\frac{(\varphi')}{r}\right)''\right)v, v\right)_{L^2} \\
&\leq |\mathrm{Re}\left((P_\varphi - e)v, \frac{\varphi'}{r}v\right)_{L^2}| + \left(\frac{\varphi'}{r}h\partial_r v, h\partial_r v\right)_{L^2} \\
&\qquad + \frac{9}{8}\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + \frac{1}{8}\left(\frac{\varphi'}{r}v, v\right)_{L^2}
\end{aligned}
\tag{7.28}
$$

but

$$
\begin{aligned}
\left|\mathrm{Re}\left((P_\varphi - e)v, \frac{\varphi'}{r}v\right)_{L^2}\right| &\leq \|\left(\frac{\varphi'}{r}\right)^{1/2}(P_\varphi - e)v\|_{L^2} \times \|\left(\frac{\varphi'}{r}\right)^{1/2} v\|_{L^2} \\
&\leq \frac{3}{2}\|\left(\frac{\varphi'}{r}\right)^{1/2}(P_\varphi - e)v\|_{L^2}^2 + \frac{1}{6}\|\left(\frac{\varphi'}{r}\right)^{1/2} v\|_{L^2}^2
\end{aligned}
\tag{7.29}
$$

From (7.28) and (7.29) we obtain lemma 7.3.

We come back to the proof of proposition 7.2. From (7.17), (7.24) and lemma 7.3, we get

$$
\begin{aligned}
&(B) + (2) + 3h\left\|\left(\frac{\varphi'}{r}\right)^{1/2}(P_\varphi - e)v\right\|_{L^2}^2 \\
&\geq \left((2(\varphi')^2 - 2h\frac{\varphi'}{r})h\partial_r v, h\partial_r v\right)_{L^2} + h\left(\frac{\varphi'}{r}(4-2)\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} \\
&\qquad + h\left(\left(\frac{\varphi'}{r} - h(\varphi'')^2\right)v, v\right)_{L^2}
\end{aligned}
\tag{7.30}
$$

and we obtain from (7.20), (7.22), (7.23), (7.24) and (4.3)

(7.31)

$$I + 3h\left\| \left(\frac{\varphi'}{r}\right)^{1/2} (P_\varphi - e)v\right\|_{L^2}^2 \geq \left((2(\varphi')^2 - 2h\frac{\varphi'}{r} + 4h\varphi'' - h^2|\varphi'''|)h\partial_r v, h\partial_r v\right)_{L^2}$$
$$+ h\left(\frac{\varphi'}{r}(4-2-1/8)\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2}$$
$$+ h\left(\left(\frac{\varphi'}{r}(1-1/8) + (\varphi')^2\varphi'' - h(\varphi'')^2 - h|\varphi'''| - 2\varphi'|\partial_r V|\right)v, v\right)_{L^2}$$

From lemma 4.10, proposition 4.11 and proposition 4.12 we obtain

$$(7.32)\quad I + 3h\left\| \left(\frac{\varphi'}{r}\right)^{1/2} (P_\varphi - e)v\right\|_{L^2}^2$$
$$\geq \left((\varphi')^2 h\partial_r v, h\partial_r v\right)_{L^2} + \frac{15h}{8}\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + \frac{h}{2}\left(\frac{\varphi'}{r}v, v\right)_{L^2}$$

proving proposition 7.2.

**7.3. The dissipation-commutator estimate.** — As in section 6.3, propositions 7.1 and 7.2 can be mixed together to give an estimate on solution to the analitically dilated problem.

***Proposition 7.4* (Proposition 5.2).** — *There exists $C > 0$ and $h_0 > 0$ such that for any $0 < h < h_0$ and any $u \in \mathcal{D}_{\theta_0}$, with $v = \chi_2 e^{\varphi/h} u$*

$$(7.33)\quad C\|\chi_2 e^{\varphi/h}(P_{\theta_0} - e)u\|_{L^2}^2 - Ce^{2\varphi(B)/h} Im\left(e^{i\kappa_{\theta_0}}(P_{\theta_0} - e)u, u\right)_{L^2}$$
$$\geq \left((\varphi')^2 h\partial_r v, h\partial_r v\right)_{L^2} + h\left(\frac{\varphi'}{r}\frac{h\nabla_\sigma}{r}v, \frac{h\nabla_\sigma}{r}v\right)_{L^2} + h\left(\frac{\varphi'}{r}v, v\right)_{L^2}$$
$$+ e^{2\varphi(B)/h}\int \kappa_{\theta_0}\left(|h\nabla u|^2 + |u|^2\right) drdL - Ch^2\int_{A\leq r\leq A+1} e^{2\varphi/h}(|u|^2 + |h\nabla u|^2) drdL$$

## 8. Decay for the local energy of solution of the wave equation

In this section we show how theorem 2 can be deduced from theorem 3 for the estimate (1.5) and from theorem 4 for the estimate (1.6).

Note $P_0 = \Delta_g$ the Laplace operator associated to the metric $g$. Note $P$ the operator $-h^2 P_0$ and $P_{\theta_0}$, the operator obtained from $P$ by analytic dilation (for suitable $\theta_0$ and $B >> 1$).

From theorem 3 (with $e = 1$), we obtain

$$(8.1)\qquad \|(P_{\theta_0} - e)^{-1}\|_{L^2(\Gamma_{\theta_0})} \leq Ce^{C'/h}, \qquad e = 1$$

By a perturbation argument, estimate (8.1) is stil valid (with a different $C$) for $|\mathrm{Im} e| \leq \frac{C^{-1}}{2}e^{-C'/h}$. Remark that if $\chi \in C_0^\infty(\mathbb{R}^d)$ is supported in the set $\{|x| \leq B\}$, (the analytic dilation has been performed for $r \geq B$), then $\chi(P_{\theta_0} - e)^{-1}\chi$ is the truncated outgoing resolvent of the operator $P$. This remark and (8.1) gives

$$\|\chi(P-e)^{-1}\chi\|_{L^2(\Gamma_{\theta_0})} \leq Ce^{C'/h}, \qquad |\mathrm{Im} e| \leq \frac{C^{-1}}{2}e^{-C'/h} \tag{8.2}$$

Coming back by scaling to the resolvent for the operator $P_0$, we obtain the following estimate for the outgoing resolvent of the operator $P_0$, $R(\lambda) = (P_0 - \lambda^2)^{-1}$ (with different $C$, $C'$).

***Proposition 8.1***. — *There exists $\lambda_0 > 0$, $C, C'$ such that the outgoing resolvent of the operator $P_0$ is holomorphic in the set*

$$\{\lambda \in \mathbb{C} \ ; \ Im\lambda < \frac{1}{C}e^{-C'|Re\lambda|}\} \cap \{|\lambda| > \lambda_0\}. \tag{8.3}$$

*and satisfies there the estimate:*

$$\|\chi R(\lambda)\,\chi\|_{\mathcal{L}(L^2;L^2)} \leq Ce^{C'|Re\lambda|}. \tag{8.4}$$

The same analysis as in the appendix B of [**Bur98**] shows that $\chi R(\lambda)\,\chi$ has no pole for $\lambda$ real, $0 < |\lambda| \leq \lambda_0$ (and hence, by compacity in a neighborhood of this set except close to 0) and that it is bounded close to 0 in any angular sector $\Lambda$, disjoint from the imaginary axis:

$$\Lambda = \{\lambda \in \mathbb{C}^*; \ \arg(\lambda) \in [-\theta_1, \theta_1\}; \quad 0 < \theta_1 < \pi/2, \quad |\lambda| < \varepsilon << 1\} \tag{8.5}$$

with these estimates we are in position to apply [**Bur98**, Theorem3] which gives (1.5).

For the proof of (1.6), the same analysis (and scaling) as above implies that if $\chi \in C_0^\infty(\mathbb{R}^d)$, is supported in the set $\{R_1 \leq |x| \leq R_2\}$ and equal to 1 in the set $\{R_1 + 1/2 \leq |x| \leq R_2 - 1/2\}$ then the outgoing resolvent $R(\lambda) = (P_1 - \lambda^2)^{-1}$ satisfies the following estimate

$$\|\chi(P_0 - \lambda^2)^{-1}\chi\|_{\mathcal{L}(L^2;L^2)} \leq \frac{C}{|\lambda|} \tag{8.6}$$

Take $\tilde{\chi} \in C^\infty_{0(\mathbb{R}^d)}$ supported in the set where $\chi$ is equal to 1 and equal to 1 in the set $\{R_1 + 2/3 \leq |x| \leq R_2 - 2/3\}$.

Remark that (8.6) implies ,

$$\|\tilde{\chi}(P_0 - \lambda^2)^{-1}\tilde{\chi}\|_{\mathcal{L}(L^2;H^1)} \leq C \tag{8.7}$$

We recall the classical equality (see [**Bur98**, (0.2)])

$$(P_0 - \lambda^2)^{-1} = -\int_0^{+\infty} e^{it\lambda}\frac{\sin(\sqrt{-P_1})}{\sqrt{-P_1}} \tag{8.8}$$

(As compared to [**Bur98**, (0.2)], the minus sign before the integral appears since here we consider $P_0 - \lambda^2$ instead of $-P_0 + \lambda^2$ and the minus sign in the exponent because according to our conventions the resonances here are the conjugate of the resonances in [**Bur98**]).

From (8.8) we obtain

$$\frac{\sin(t\sqrt{-P_1})}{\sqrt{-P_1}}1_{t>0} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-it\lambda}(P_1-\lambda^2)^{-1} = \mathcal{F}^{-1}((P_1-\lambda^2)^{-1}) \tag{8.9}$$

Suppose now that $f \in L^2$ is supported in the set $\{|x| \in [R_1+2, R_2-2]\}$. By finite speed of propagation (and if $R_1$ is large enough so that the speed is close to 1) $\frac{\sin(\sqrt{-P_1})}{\sqrt{-P_1}}f$ is for $t \in [0,1]$ supported in the set where $\chi = 1$. Consider $\Psi \in C^\infty(\mathbb{R})$ an increasing function equal to 0 for $t \leq 0$ and to 1 for $t \geq 1$. Note

$$u(t) = \left(\Psi(t)\frac{\sin(t\sqrt{-P_1})}{\sqrt{-P_1}}f\right) \tag{8.10}$$

From the equation

$$\left(\partial_t^2 - P_1\right)(\Psi(t)u(t,x)) = \Psi''(t)u(t,x) + 2\Psi'(t)\partial_t u(t,x) = g(t,x) \tag{8.11}$$

and Duhamel formula we obtain

$$\Psi(t)u(t,x) = \left(\frac{\sin(s\sqrt{-P_1})}{\sqrt{-P_1}} \star_s g(s)\right)(t) \tag{8.12}$$

Remark that in (8.12) $g(s)$ is bounded in $L^\infty(\mathbb{R}_s; L^2(\Omega))$ by $\|f\|_{L^2}$ and supported in $[0,1]$, hence is bounded in $L^2(\mathbb{R}_s; L^2(\Omega))$by $\|f\|_{L^2}$. Remark also that

$$\tilde{\chi}\Psi(t)u(t,x) = \chi\left(\frac{\sin(s\sqrt{-P_1})}{\sqrt{-P_1}}\tilde{\chi} \star_s g(s)\right)(t) \tag{8.13}$$

indeed, $g(s)$ is, by the finite speed of propagation argument we mentionned above, supported (in the $x$ variable) in the region where $\tilde{\chi}$ is equal to 1. By taking Fourier transforms in (8.13) we obtain

$$\mathcal{F}_{r\mapsto\lambda}\left(\chi\Psi(t)u(t,x)\right)(\lambda) = -\chi(P_1-(-\lambda)^2)^{-1}\chi\mathcal{F}_{s\mapsto\lambda}(g(s))(\lambda) \tag{8.14}$$

From (8.7) we obtain that the right hand side in (8.14) is bounded in $L^2(\mathbb{R}_\lambda; H^1(\Omega))$ by $\|f\|_{L^2}$. By Parseval identity, we obtain that

$$\|\chi\Psi(t)u(t,x)\|_{L^2(\mathbb{R}_t;H^1)} \leq C\|f\|_{L^2} \tag{8.15}$$

which is exactly (1.6) for initial data $(u_0,u_1) = (0,f)$. The proof for initial data of the form $(u_0,u_1) = (g,0)$ is similar.

## Appendix A
### Existence of a geodesic coordinate system at infinity

In this section we give an outline of the proof of proposition 2.2

Note $Z = \{z = (z_1,\dots,z_d) \in \mathbb{C}^d; \sum z_i^2 = 1\}$ the complexified $d-1$ sphere and $\mathbb{S}_\varepsilon^{d-1}$ the intersection of $Z$ with an $\varepsilon$ neighborhood of $\mathbb{R}^d$ (in $\mathbb{C}^d$). The purpose of this part is the following result:

***Proposition A.1.*** — *$\exists \varepsilon, \theta_1 > 0$, $R_0 >> 1$ and*

$$\chi : x = (r\sigma), (r,\sigma) \in R_0 + ]0, +\infty[\times e^{i]-\theta_1,\theta_1[} \times \mathbb{S}^{d-1}_{\varepsilon} \mapsto \chi(x) \in \Omega \subset \mathbb{C}^d \tag{A.1}$$

*a holomorphic diffeomorphism such that*

1.

$$\chi(]R_0, +\infty[\times \mathbb{S}^{d-1}_{\mathbb{R}^d}) = (]R_0, +\infty[\times \mathbb{S}^{d-1}_{\mathbb{R}^d}) \tag{A.2}$$

2. *Denoting $\chi_r$ and $\chi_\sigma$ the radial coordinates, we have*

$$\left| \frac{\chi_r(r,\sigma)}{r} - 1 \right| \leq \frac{C}{|r|^\beta} \tag{A.3}$$

*and*

$$|\chi_\sigma(r,\sigma) - \sigma| \leq \frac{C}{|r|^\beta} \tag{A.4}$$

3. *The principal symbol of the operator $\chi_*(Q)$ is equal to*

$$\varrho^2 + q_2(r,\sigma,\frac{\eta}{r}) \tag{A.5}$$

***Remark A.2.*** — *Remark that* (A.3) *and* (A.4) *imply that*

$$\frac{\|\chi(r,\sigma) - r\sigma\|}{|r|} \leq \frac{C}{|r|^\beta} \tag{A.6}$$

*which means that the change of coordinate is asymptotically the identity. Proposition A.1 is nothing but a global result of a well known result: the existence of geodesic coordinate systems which allows, modulo the choice of a good coordinate system, to suppress the $\partial_r\partial_\sigma$ terms. Here two ingredients come into account: the global existence due to the global assumption on the* asymptotic flatness *and the analyticity property due to the* analyticity at infinity *assumption.*

Since most of the work is standard estimates on solutions of differential equations, we will not give all the details of the (fastidious) proof of this result. We'll emphasize on the global existence of the geodesic coordinate system.

Note

$$q(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \tag{A.7}$$

the classical principal symbol of the operator $Q$. We work in polar coordinates and still note $q(r,\sigma,\varrho,\eta)$ the function $q$ in the polar (and their dual) coordinates. We recall that this function is a second order polynomial in the $(\rho, \eta/r)$ variables with coefficients holomorphic with respect to $(r,\sigma)$ in set of the form (with $d > 0, R_0 > 0$)

$$\Lambda_{\theta_0,R_0,d_0} = \{(r,\sigma) \in \mathbb{C}^d;\ \text{dist}\left(\sigma, \mathbb{S}^{d-1}\right) < d_0, r \in e^{i]-\theta_0,\theta_0[}]R_0, +\infty[\} \tag{A.8}$$

Consider the following system of differential equations:

$$\frac{d}{ds}r = \frac{1}{2}\partial_\rho q,\ \frac{d}{ds}\sigma = \frac{1}{2}\partial_\eta q,\ \frac{d}{ds}\rho = -\frac{1}{2}\partial_r q,\ \frac{d}{ds}\eta = -\frac{1}{2}\partial_\sigma q, \tag{A.9}$$

with initial data

$$r\mid_{s=0} = r_0,\ \sigma\mid_{s=0} = \sigma_0 \text{ close to } \mathbb{S}^{d-1}_{\mathbb{R}^d},\ \rho_{s=0} = \varrho_0,\ \eta_{s=0} = \eta_0 \tag{A.10}$$

For any $\varepsilon > 0$ and $R_0 > 0$, note

$$D(\varepsilon, R_0) = \{(r,\sigma,\varrho,\eta) \in \mathbb{C}^{2d}; |r - R_0| < \varepsilon,\ \mathrm{dist}(\sigma, \mathbb{S}^{d-1}_{\mathbb{R}^d}) < \varepsilon, |\varrho - 1| < \varepsilon, \\ p(r,\sigma,\varrho,\eta) = 1\} \tag{A.11}$$

***Proposition A.3.*** — *$\exists\theta_1 > 0$, $\varepsilon > 0$, $R_1$, $\forall R_0 > R_1$, $\forall(r,\sigma,\varrho,\eta) \in D(\varepsilon, R_0)$ the system* (A.9), (A.10) *has a unique solution defined for $s \in e^{i[-\theta_1,\theta_1]}\mathbb{R}^+$ which is holomorphic with respect to $(s, r_0, \sigma_0, \varrho_0, \eta_0)$*

Fix $\varepsilon, R_0 > 0$. Then by standart results about differential equations, the system (A.9), (A.10) has a unique solution for $s$ in a neighborhood of 0 in $\mathbb{C}$, for initial data in $D(\varepsilon, R_0)$. We recall that, according to the *asymptotic flatness* assumption, we have $\forall x \in \Lambda_{\theta_0,R_0,d_0}$,

$$|q(r,\sigma,\rho,\eta) - (\varrho^2 + \frac{|\eta|^2}{r^2})| \le \frac{M}{|r|^\beta}(\varrho^2 + \frac{|\eta|^2}{r^2}) \tag{A.12}$$

Furthermore, by Cauchy formula, (and possibly reducing $\theta_0$), we have

$$|\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\partial_4^{\alpha_4}(p(r,\sigma,\rho,\frac{\eta}{r}) - (\varrho^2 + \frac{|\eta|^2}{r^2}))| \le \frac{M_\alpha}{|r|^{\beta+\alpha_1}}(\varrho^2 + \frac{|\eta|^2}{r^2}) \tag{A.13}$$

Note $I \subset \mathbb{R}^+$ the following set: $s_0 \in I$ if and only if for any $(r,\sigma,\varrho,\eta) \in D(\varepsilon, R_0)$ the system (A.9), (A.10) has a unique solution defined for $s \in [0, s_0]e^{i[-\theta_0/2,\theta_0/2]}$ and such that $\forall s \in [0, s_0]e^{i[-\theta_0/2,\theta_0/2]}$

$$\begin{gathered} |\dot r - 1|(s) \le \frac{1}{8} + \frac{M}{|r|^\beta(s)}, \qquad |\dot\varrho|(s) \le \frac{M}{|r|^{\beta+1(s)}} \\ |\dot\sigma(s)| \le \frac{M}{|r|^{\beta+1}(s)}, \qquad |\dot\eta(s)| \le \frac{M}{|r|^\beta(s)} \end{gathered} \tag{A.14}$$

***Lemma A.4.*** — *$\exists\varepsilon_0 > 0, R_1 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $R_0 > R_1$ then $I = \mathbb{R}^+$.*

To prove lemma A.4, we prove that $I$ is non empty, closed and open in $\mathbb{R}^+$. $I$ is clearly closed in $\mathbb{R}^+$, non empty since it contains a neighborhood of 0 in $\mathbb{R}^+$. To show that it is open, we first prove

***Lemma A.5.*** — *$\exists\varepsilon_0 > 0, R_1 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $R_0 > R_1$, then if $s_0 \in I$, for any $s \in [0, s_0]e^{i[-\theta_0/2,\theta_0/2]}$, the point $(r,\sigma,\varrho,\eta)(s)$ satisfies*

$$(r,\sigma) \in \Lambda_{2\theta_0/3,R_1,d_0/2}, \qquad |\varrho - 1| \le \frac{1}{16}, \qquad |\frac{\eta}{r}| \le \frac{4}{3}, \qquad |r| \ge R_0 + \frac{|s|}{2} \tag{A.15}$$

Proof: Fix $s$ and consider for $t \in [0, |s|]$,

$$t \mapsto (r_t, \sigma_t, \varrho_t, \eta_t) = (r, \sigma, \varrho, \eta)(\frac{s}{|s|}t) \tag{A.16}$$

We have from (A.14)

$$|\frac{d}{dt} r_t - \frac{s}{|s|}| \leq \frac{1}{8} + \frac{M}{|r_t|^\beta} \tag{A.17}$$

If $R_0$ is taken large, then the right hand side in (A.17) is for $t = 0$ smaller than $1/4$. Hence, $|r_t|$ is increasing for small $t$ and hence for all $t$ and $|\frac{d}{dt} r_t - \frac{s}{|s|}| \leq \frac{1}{4}$; which implies

$$\frac{3}{4} \leq \frac{|r|}{|s| + R_0} \leq \frac{5}{4} \tag{A.18}$$

Using this inequality and (A.14), we obtain

$$|\sigma - \sigma_0| \leq \frac{C}{R_0^\beta}, \quad |\eta - \eta_0| \leq C(R_0 + |s|)^{1-\beta}, \quad |\varrho - \varrho_0| \leq \frac{C}{R_0^\beta} \tag{A.19}$$

Hence

$$\left|\frac{\eta}{r}\right| \leq \frac{C}{(R_0 + |s|)^\beta} \tag{A.20}$$

Comming back to (A.9), we obtain since $\varrho^2 + \left|\frac{\eta}{r}\right|^2$ is bounded from (A.12)

$$|\dot{r} - \varrho| \leq \frac{C}{|r|^\beta} \leq \frac{C}{(R_0 + |s|)^\beta} \tag{A.21}$$

and from (A.19) and (A.21)

$$|\dot{r} - 1| \leq \frac{C}{R_0^\beta} + \varepsilon \tag{A.22}$$

Now, if $\varepsilon$ is small enough and $R_0$ large enough, we obtain from (A.22) that $r$ stays in the set $e^{i[-2\theta_0/3, 2\theta_0/3]}\mathbb{R}^+$ (because $s \in e^{i[-\theta_0/2, \theta_0/2]}\mathbb{R}^+$) and from (A.19) that $\sigma$ stays close to $\mathbb{S}^{d-1}$. The rest of the estimate (A.15) is now a simple consequence of (A.19), (A.20), (A.18) and of the inequality $|\varrho_0 - 1| \leq \varepsilon$.

We come back to the proof of lemma A.4. Suppose that $s_0 \in I$. Consider $z_0 = s_0 e^{i\theta}$, $\theta \in [-\theta_0/2, \theta_0/2]$. According to lemma A.5, we have

$$(r, \sigma)(z_0) \in \Lambda_{\frac{2\theta_0}{3}, R_0, \frac{d_0}{2}}, \qquad |r|(z_0) \geq R_0 + \frac{s_0}{2} > R_0, \qquad |\varrho(z_0)| \leq \frac{17}{16} \tag{A.23}$$

We can solve the equations (A.9),with initial data $(r, \sigma, \varrho, \eta)(z_0)$ at $z_0$ for $z$ close to $z_0$, with a solution satisfying

$$\begin{aligned} |\dot{r} - \varrho| &\leq \frac{M}{|r|^\beta}, \qquad |\dot{\sigma}| \leq \frac{M}{|r|^{1+\beta}} \\ |\dot{\varrho}| &\leq \frac{M}{|r|^{1+\beta}}, \qquad |\dot{\eta}| \leq \frac{M}{|r|^\beta} \end{aligned} \tag{A.24}$$

as long as the solution $(r, \sigma)(z)$ stays in $\Lambda_{\theta_0, R_0, d_0}$ (and at least for $|z - z_0|$ small). Since $|\varrho - 1|(z_0) \leq \frac{1}{16}$, for $|z - z_0|$ small enough, we have $|\varrho - 1|(z_0) \leq \frac{1}{8}$ and we deduce that the solution of (A.9) with initial data $(r,\sigma,\varrho,\eta)(z_0)$ at $z_0$ exists and satisfies (A.14) for $|z-z_0|$ small enough. This shows that the solution can be prolonged near $z_0$, satisfying (A.14). Using that $\{s_0\}\times e^{i[-\theta_0/2,\theta_0/2]}$ is compact, we obtain that the set $I$ is open in $\mathbb{R}^+$. This concludes the proof of lemma A.4.

Using the equation satisfied by the differential of the application $(r_0,\sigma_0,\rho_0,\eta_0)\mapsto(r,\sigma,\rho,\eta)\,(s)$, we get (using (A.12), (A.13), (A.18) and (A.20))

$$\begin{aligned}
\left|\frac{d}{ds}dr\right| &\le \frac{C}{(R_0+|s|)^{1+\beta}}|dr| + \frac{C}{(R_0+|s|)^{\beta}}\,|d\sigma| \\
&\quad + C\,|d\rho| + \frac{C}{(R_0+|s|)^{1+\beta}}|d\eta| \\
\left|\frac{d}{ds}d\sigma\right| &\le \frac{C}{(R_0+|s|)^{2+\beta}}\,|dr| + \frac{C}{(R_0+|s|)^{1+\beta}}\,|d\sigma| \\
&\quad + \frac{C}{(R_0+|s|)^{1+\beta}}\,|d\rho| + \frac{C}{(R_0+|s|)^{2+\beta}}|d\eta| \\
\left|\frac{d}{ds}d\rho\right| &\le \frac{C}{(R_0+|s|)^{2+\beta}}|dr| + \frac{C}{(R_0+|s|)^{1+\beta}}\,|d\sigma| \\
&\quad + \frac{C}{(R_0+|s|)^{1+\beta}}\,|d\rho| + \frac{C}{(R_0+|s|)^{2+\beta}}\,|d\eta| \\
\left|\frac{d}{ds}d\eta\right| &\le \frac{C}{(R_0+|s|)^{1+\beta}}|dr| + \frac{C}{(R_0+|s|)^{\beta}}\,|d\sigma| \\
&\quad + \frac{C}{(R_0+|s|)^{\beta}}\,|d\rho| + \frac{C}{(R_0+|s|)^{1+\beta}}\,|d\eta|
\end{aligned} \tag{A.25}$$

differentiating the equation

$$q(r,\sigma,\varrho,\eta) = q(R_0,\sigma_0,\varrho_0,\eta_0) \tag{A.26}$$

and using that $\varrho$ is close to 1, we obtain

$$|d\varrho| \le \frac{C}{(R_0+|s|)^{1+\beta}}|dr| + \frac{C}{(R_0+|s|)^{\beta}}|d\sigma| + \frac{C}{(R_0+|s|)^{1+\beta}}|d\eta| \tag{A.27}$$

which allows to eliminate the $|d\varrho|$ terms in the right hand side of (A.25). Note

$$f(s) = \frac{|dr-\mathrm{Id}|^2}{(R_0+|s|)^{2-\beta}} + \frac{|d\eta-\mathrm{Id}|^2}{(R_0+|s|)^{2-\beta}} + |d\sigma-\mathrm{Id}|^2 + |d\rho-\mathrm{Id}|^2 \tag{A.28}$$

From (A.25) and(A.27) we obtain

$$\left|\frac{d}{ds}f\right| \le \frac{C}{(R_0+|s|)^{1-\beta/2}} + \frac{C}{(R_0+|s|)^{1-\beta/2}} \tag{A.29}$$

and we obtain by Gronwall inequality

$$f(s) \le \frac{C}{(R_0+|s|)^{1-\beta/2}} \tag{A.30}$$

For any $\sigma_0 \in \mathbb{S}^{d-1}_\varepsilon$ note $\varrho_0$ the unique solution (defined by the implicit function theorem) close to 1 of the equation $q(R_0, \sigma_0, \varrho_0, \eta_0) = 1$ which exists (and is holomorphic) if $\varepsilon > 0$ is small and $R_0$ large enough. Using (A.12), (A.13), it is easy to see that

$$|\varrho - 1| \leq \frac{C}{(R_0)^\beta}, \qquad |\partial_{\sigma_0} \varrho_0| \leq \frac{C}{(R_0)^\beta} \tag{A.31}$$

Now consider the following map:

$$\begin{aligned}\chi_0 : (t, \sigma_0) \in R_0 + ]0, +\infty[ \times e^{i[-\theta_0/2\beta, \theta_0/2\beta]} \times (\mathbb{S}^{d-1} + B_{\mathbb{C}^d}(0, \varepsilon)) \\ \mapsto \left( r^{\beta/2}((t - R_0^{\beta/2})^{2/\beta}), \sigma((t - R_0^{\beta/2})^{2/\beta}) \right)\end{aligned} \tag{A.32}$$

where $(r, \sigma, \varrho, \eta)(s)$ is the solution of (A.9) with initial data $(R_0, \sigma_0, \varrho_0(\sigma_0), \eta_0 = 0)$.

The differential of the map $\chi_0$ is

$$d\chi_0 = \begin{pmatrix} r^{\beta/2-1} \times (t - R_0^{\beta/2})^{2/\beta - 1} \partial_s r & r^{\beta/2-1} (\partial_{\sigma_0} r + \partial_{\sigma_0} \varrho_0 \partial_{\varrho_0} r) \\ \frac{2}{\beta(t - R_0^{\beta/2})^{2/\beta - 1}} \partial_s \sigma & \partial_{\sigma_0} \sigma + \partial_{\sigma_0} \varrho_0 \partial_{\varrho_0} \sigma \end{pmatrix} \tag{A.33}$$

Hence, from (A.30) and(A.31), we have

$$|d\chi_0 - \mathrm{Id}| \leq \frac{C}{(R_0)^{\beta/2}} \tag{A.34}$$

On the other hand, from (A.18), (A.20) and the fact that $q = 1$, we deduce

$$|\rho - 1| \leq \frac{C}{(R_0 + |s|)^\beta} \tag{A.35}$$

From (A.25) we deduce

$$|\dot{r} - 1| \leq \frac{C}{(R_0 + |s|)^\beta} \tag{A.36}$$

Note $g(s) = r^{\beta/2}(s^{2/\beta}) - (s - R_0^{\beta/2}$. From (A.36) and Gronwall inequality we deduce

$$|g|(s) \leq \frac{C}{\left(R_0 + |s|^{2/\beta}\right)^{\beta/2}} \tag{A.37}$$

This is equivalent to the fact that the $r$-component of $\chi_0$ is close to the identity:

$$|\chi_{0,r}(t, \sigma) - t| \leq \frac{C}{(R_0 + |s|)^{\beta/2}} \tag{A.38}$$

From (A.14) we have that the $\sigma$ component of $\chi_0$ is also close to the identity.

Now, the map $\chi_0$ and its derivative are close to the identity (if $R_0$ is taken large), so $\chi_0$ is a *global* diffeomorphism onto its image. Coming back to the map

$$\tilde{\chi} : R_0 + ]0, +\infty[ e^{i[-\theta_0/2\beta, \theta_0/2\beta]} \times (\mathbb{S}^{d-1}_{\mathbb{R}^d} + B_{\mathbb{C}^d}(0, \varepsilon)) \mapsto (r(s - R_0), \sigma(s - R_0)) \subset \mathbb{C}^{d+1} \tag{A.39}$$

we obtain that it is an holomorphic diffeomorphism onto its image.

Remark also that from (A.14), $\lim_{|s|\to+\infty}\sigma(s)$ exists. Indeed, we have

$$|\sigma(s,\sigma_0)-\sigma(s',\sigma_0)| \leq |\int_s^{\frac{|s'|}{|s|}s}\dot\sigma| + |\int_{\frac{|s'|}{|s|}s}^{s'}\dot\sigma| \leq \frac{M}{|s|^\beta}+\frac{M}{|s'|^\beta} \tag{A.40}$$

It is again possible to show that the map $\Psi : \sigma_0 \mapsto \lim_{s\to+\infty}$ is an holomorphic diffeomorphism from $\mathbb{S}^{d-1}+B_{\mathbb{C}^d}(0,\varepsilon)$ onto a neighborhood of $\mathbb{S}^{d-1}$, $V$ in $\mathbb{C}^d$. Finally, the change of variables we are interested in, is the following (for $\varepsilon$ small enough):

$$\begin{aligned}\chi : (r,\sigma) \in R_0+]0,+\infty[\times e^{i[-\theta_0/2,\theta_0/2]}\times\mathbb{S}_\varepsilon^{d-1}\\ \mapsto \chi_0(r,\Psi^{-1}(\sigma))_r\times\chi^{-1}(r,\sigma)_\sigma\end{aligned} \tag{A.41}$$

It is now easy to check that $\chi$ satisfies the desired asumptions.

**A.1. Reductions for the operator $P$.** — The operator $P$ (seen in the new coordinate system) is self adjoint for the measure $\mu$ obtained from $dx$ by the change of variables. Hence $\mu$ has the following form ($dL$ is the Lebesgue measure on the $d-1$-dimensionnal sphere)

$$\mu = f(r,\sigma)drdL \tag{A.42}$$

with $f(r,\sigma)$ a holomorphic function in a conic neighborhood of $B(0,R_0)^c$ satisfying there

$$|f(r,\sigma)-r^{d-1}| \leq C|r|^{d-1-\beta/2} \tag{A.43}$$

Note $P_1 = f^{1/2}Pf^{-1/2}$. It is easy to see that the operator $P_1$ is self adjoint for the measure $drdL$ and satisfies all the other assumptions [1,. . . , 4] of page 4 we have made about $P$. Notice that (since $P_1^*=P_1$), we can suppose that the first order term in $h\partial_r$ is equal to

$$q_1(r,\sigma,h\partial_r) = a_1(r,\sigma,h)h\partial_r + h\partial_r a_1(r,\sigma,h) \tag{A.44}$$

with $a_1(r,\sigma)\in i\mathbb{R}$; and that the 0 order term (the potential) is real.

Note

$$\hat f(r,\sigma) = \int_{z=R_0}^r a_1(r,\sigma)dz \tag{A.45}$$

Then $\hat f$ is purely imaginary and the operator $P_2 = e^{\hat f/h}P_1e^{-\hat f/h}$ is self adjoint for the measure $drdL$ and has the following form

$$P_2 = -h^2\partial_r^2 + q_2(r,\sigma,\frac{h\partial_\sigma}{r}) + \tilde q_1(r,\sigma,\frac{h\nabla_\sigma}{r},h) + V(r,\sigma,h) \tag{A.46}$$

with different $\tilde q_1$ and $V$, but still satisfying the assumptions [1,. . . , 4] of page 4 we have made about $P$.

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Nicolas Burq, Université Paris Sud, Mathématiques, Bât 425, 91405, Orsay Cedex
*E-mail :* `Nicolas.burq@math.u-psud.fr`