

SMOOTHING EFFECT FOR SCHRÖDINGER BOUNDARY VALUE PROBLEMS

N. BURQ

ABSTRACT. We show the necessity of the non trapping condition for the plain smoothing effect ($H^{1/2}$) for Schrödinger equation with Dirichlet boundary conditions in exterior problems. We also give a class of trapped obstacles (Ikawa's example) for which we can prove a weak ($H^{1/2-\varepsilon}$) smoothing effect

RÉSUMÉ. On démontre que l'hypothèse de non capture est nécessaire pour l'effet régularisant ($H^{1/2}$) pour l'équation de Schrödinger avec conditions aux limites de Dirichlet à l'extérieur d'un domaine de \mathbb{R}^d . On donne aussi une classe d'obstacles captifs (l'exemple d'Ikawa) pour lesquels on démontre un effet régularisant affaibli ($H^{1/2-\varepsilon}$).

1. INTRODUCTION

Consider $u = e^{it\Delta}u_0$ solution of the Schrödinger equation

$$(1.1) \quad \begin{cases} (i\partial_t + \Delta)u = 0 \text{ in } \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d). \end{cases}$$

It is well known that $u \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^d))$ satisfies the following smoothing effect (for any $s > 1/2$ if $d \geq 3$)

$$(1.2) \quad \|u\|_{L^2(\mathbb{R}_t; \dot{H}_s^{1/2}(\mathbb{R}^d))} \leq C \|u_0\|_{L^2},$$

where

$$(1.3) \quad \dot{H}_s^{1/2} = \{u \in \mathcal{D}'(\mathbb{R}^d); \langle x \rangle^{-s} \Delta^{1/4} u \in L^2(\mathbb{R}^d)\}.$$

This result, which can be proved by explicit calculations, has been extended to more complicated operators, satisfying a non trapping assumption (see the results of Constantin - Saut [9], Ben-Artzi - Devinatz [2], Ben-Artzi - Klainerman [1], Doi [12, 11], and Kato - Yajima [19]). It has been recently extended to the case of boundary value problems by Gérard, Tzvetkov and the author [7].

On the other hand, in [10] Doi has proved that, for Schrödinger operators in \mathbb{R}^d , the non trapping assumption is necessary for the $H^{1/2}$ smoothing effect.

In this paper we extend this latter result to the case of boundary value problems. Our first result reads as follows:

Theorem 1. *Consider an arbitrary smooth domain with boundary $\Omega \subset \mathbb{R}^d$, with no infinite order contact with its boundary (see the precise definition in Section 3), and P a second order self adjoint operator on $L^2(\Omega)$, with domain $D \subset H_0^1(\Omega)$ and such that the boundary is non characteristic. Denote by $\varphi_s : {}^bT^*\Omega \setminus \{0\} \rightarrow {}^bT^*\Omega \setminus \{0\}$ the bicharacteristic flow of the*

operator P (given by the integral curves of the Hamiltonian vector field of the principal symbol of P reflecting on the boundary according to the law of geometric optics—see Section 3—) defined on the boundary cotangent bundle. Let $A \in \Psi^{(1/2)}$ be a classical tangential pseudodifferential operator of order $1/2$. Suppose that $(z_0, \zeta_0) \in {}^bT^*\Omega \setminus \{0\}$ satisfy the trapping assumption:

$$(1.4) \quad \int_{-\infty}^0 |\sigma_{1/2}(A)(\varphi_s(x_0, \zeta_0))|^2 ds = +\infty,$$

where $\sigma_{1/2}(A)$ is the principal symbol of the operator A . Then for any $t_0 > 0$ the map

$$(1.5) \quad u_0 \in C_0^\infty \subset L^2(\Omega) \mapsto Ae^{itP}u_0 \in L^2([0, t_0]; L^2(M))$$

is not bounded (even for data with fixed compact support).

Remark 1.1. The assumption (1.4) can be essentially fulfilled in two distinct cases

- (1) If A is compactly supported (in the z variable), then (1.4) means that the bicharacteristic starting from (z_0, ζ_0) spends an infinite time in the support of A , which corresponds to a “trapped trajectory”
- (2) If A is not compactly supported, a typical example is (in the case $P = -\Delta$) $A(z, D_z) = a(|z|)|D_z|^{1/2}$, then (1.4) might correspond to a lack of decay of $a(x)$ at infinity: suppose that the trajectory starting from (z_0, ζ_0) is not trapped; hence it leaves any compact set and for $\pm s \rightarrow +\infty$, $(z(s), \zeta(s)) \sim (s\zeta_\pm, \zeta_\pm)$ and (1.4) is equivalent to $|a|^2 \notin L^1(\mathbb{R})$ (and we recover the usual assumption required for proving the smoothing effect, see [11]).

Remark 1.2. We could have added lower order terms to P and supposed that the Cauchy problem is well posed in L^2 (in case of first order terms). The condition (1.4) has in this case to be modified.

Remark 1.3. In [11, 10], Doi proves this result in the case of a manifold without boundary and gives some variants of this result for operators of higher order, and with weights in times. The proof we present below is essentially self contained in this case and it can also handle these variants modulo slight modifications. The proof in presence of a boundary is much more technical.

Remark 1.4. For $P = -\Delta_g$ the x -projection of the integral curves of H_p are the geodesics for the metric g .

Remark 1.5. The smoothness assumption can be relaxed to C^2 coefficients and C^3 domains (and even to C^1 coefficients, but the assumption (1.4) is then more complicated since the Hamiltonian flow is no more well defined). We also can prove Theorem 1 for systems (see Remark 3.1).

Having Theorem 1 in mind, a natural question is whether a weakened version of (1.2) might hold for some trapping geometries. In the case of a stable (elliptic) trapped trajectory, the existence of quasi-modes well localized along this trajectory shows that no such result may hold (see Remark 4.2). However in the case of hyperbolic trapped trajectories, we do obtain such a weak smoothing effect:

Theorem 2. Consider $\Theta = \cup_{i=1}^N \Theta_i \subset \mathbb{R}^d$ a finite union of strictly convex obstacles satisfying the assumptions of Section 4. Denote by $\Omega = \Theta^c$ its complement. Then for any $\varepsilon > 0$ and $\chi \in C_0^\infty(\mathbb{R}^d)$ there exists $C > 0$ such that for any $u_0 \in L^2(\Omega)$,

$$(1.6) \quad \|\chi e^{it\Delta_D} u_0\|_{L^2(\mathbb{R}_t; H^{1/2-\varepsilon}(\Omega))} \leq C \|u_0\|_{L^2(\Omega)}$$

Remark 1.6. This result was proved in [7] with no ε loss under the non trapping assumption: “any geodesic of the metric g reflecting on the boundary according to the laws of geometric optics goes to the infinity” which is clearly no fulfilled here.

To prove Theorem 1 we will follow the same kind of strategy as in Doi’s paper [11, 10]. However we will replace in his argument the use of Egorov’s theorem by the use of the theorem of propagation of Wigner measures, which has three advantages: first it simplifies the rest of the proof, second it allows to relax assumptions (on the regularity of the coefficients) and finally the proof holds also for a (system of) boundary value problem (whereas Egorov’s Theorem is not true in these cases)

To prove Theorem 2 we reduce, following [7], the estimate (1.6) to obtaining estimates for the outgoing resolvent of Δ_D , $(-\Delta - (z \pm i0))^{-1}$. Then we show that these estimates can be deduced from a combination of other estimates proved by M. Ikawa [16, 17, 18] and some form of the maximum principle.

The article is written as follows: in Section 2 we recall the definition of Wigner measures which will be used in the sequel and we prove Theorem 1 in the simpler case where $\Theta = \emptyset$. In Section 3 we give the necessary modifications required to handle the general case $\Theta \neq \emptyset$. In Section 4 we prove Theorem 2. Finally we have stated at the end of Section 4 an application of our smoothing result to the global existence of non linear Schrödinger equations.

Acknowledgements : I would like to thank C. Zuily for discussions about the results in this article and the referees whose observations lead to substantial improvements in the exposition. This work was completed during a stay at the Department of Mathematics of the University of California, Berkeley, partially funded by a N.A.T.O. fellowship and the Miller Institute for Basic Research in Sciences. I thank these institutions.

2. PROOF OF THEOREM 1: THE CASE OF EMPTY BOUNDARY

2.1. Wigner measures. In this section we recall the definition of Wigner measures (or semi-classical measures) introduced by Gérard – Leichtnam [14] and Lions – Paul [22] (see also the survey by Gérard – Markowich – Mauser – Poupaud [15]). We work in the context of functions of $1+d$ variables $((t, z))$ in $L_{loc}^2(\mathbb{R}_t; L^2(\mathbb{R}_z^d)) = \mathcal{L}^2$ and we have adapted the definitions in [14, 22] to fit our purpose.

Definition 2.1. We will say that a sequence of functions $(f_n) \in \mathcal{L}^2$ is bounded in \mathcal{L}^2 if for any $\varphi \in C_0^\infty(\mathbb{R}_t)$, the sequence (φf_n) is bounded in L^2 .

Definition 2.2. We will say that an operator A is bounded on \mathcal{L}^2 if there exists $\varphi \in C_0^\infty(\mathbb{R}_t)$ such that for any $f \in \mathcal{L}^2$,

$$\|Af\|_{L_{t,z}^2} \leq C \|\varphi f\|_{L_{t,z}^2}.$$

Denote by $(x, \xi) = (t, z, \tau, \zeta)$ a point in $T^*\mathbb{R}^{d+1}$; and consider for $a(x, \xi) \in C_0^\infty(\mathbb{R}^{2d+2})$ and $\varphi \in C_0^\infty(\mathbb{R}_t)$ equal to 1 near the support of a , the operator $\text{Op}_\varphi(a)(x, hD_x)$ defined on \mathcal{L}^2 by

$$(2.1) \quad \begin{aligned} \text{Op}_\varphi(a)(x, hD_x)f &= \text{Op}_\varphi(a)(t, z, hD_t, hD_z)f \\ &= \frac{1}{(2\pi)^{d+1}} \int e^{i(t\cdot\tau+z\cdot\zeta)} a(t, z, h\tau, h\zeta) \widehat{\varphi(t)} f(\tau, \zeta) d\tau d\zeta. \end{aligned}$$

The operator $\text{Op}(a)_\varphi(t, z, hD_t, hD_z)$ is (uniformly with respect to $0 < h < 1$) bounded on \mathcal{L}^2 and we have the following weak form of the Gårding inequality

Proposition 2.3. *For any $a \in C_0^\infty(\mathbb{R}^{2d+2})$ and any sequence (f_n) bounded in \mathcal{L}^2 and $(h_n) \in]0, 1[$; $\lim_{n \rightarrow +\infty} h_n = 0$,*

$$(2.2) \quad a(x, \xi) \geq 0 \Rightarrow \liminf_{n \rightarrow +\infty} \text{Re} (\text{Op}(a)_\varphi(x, h_n D_x) f_n, f_n)_{L^2(\mathbb{R}^{d+1})} \geq 0.$$

To prove this result consider for $\varepsilon > 0$, $\psi \in C_0^\infty(\mathbb{R}^{2d+2})$ equal to 1 near the (t, z, τ, ζ) projection of the support of a and $b = \varphi(t)\sqrt{\varepsilon + a}\psi(t, z, \tau, \zeta) \in C_0^\infty(\mathbb{R}^{2d+2})$. Then the symbolic calculus shows

$$(2.3) \quad \begin{aligned} 0 &\leq (\text{Op}(b)_\varphi^* \text{Op}(b)_\varphi f_n, f_n) \\ &= (\text{Op}(a)_\varphi(x, h_n D_x) f_n, f_n)_{L^2} + \varepsilon (\varphi(t)\psi^2(x, h_n D_x)\varphi(t) f_n, f_n)_{L^2} + \mathcal{O}(h_n), \end{aligned}$$

hence, taking the \liminf and using that $\liminf(\alpha_n + \beta_n) \leq \liminf(\alpha_n) + \limsup(\beta_n)$, we get

$$(2.4) \quad \liminf_{n \rightarrow +\infty} \text{Re} (\text{Op}(a)_\varphi(x, h_n D_x) f_n, f_n)_{L^2(\mathbb{R}^d)} + \varepsilon \limsup_{n \rightarrow +\infty} \| \psi(x, h_n D_x)\varphi(t) f_n \|^2 \geq 0.$$

When $\varepsilon > 0$ tends to 0 we obtain Proposition 2.3.

By the symbolic calculus, the operator $\text{Op}(a)_\varphi$ is modulo operators bounded on \mathcal{L} by $\mathcal{O}(h^\infty)$, independent of the choice of the function φ . For conciseness, we will drop in the sequel the index φ . As in [14] (see also [3]) we can prove:

Proposition 2.4. *Consider a sequence (f_n) bounded in \mathcal{L}^2 . There exist a subsequence (n_k) and a positive Radon measure on \mathbb{R}^{2d+2} , μ , such that for any $a \in C_0^\infty(\mathbb{R}^{2d+2})$*

$$(2.5) \quad \lim_{k \rightarrow +\infty} (\text{Op}(a)(x, h_{n_k} D_x) f_{n_k}, f_{n_k})_{L^2} = \langle \mu, a(x, \xi) \rangle.$$

The idea for extracting such a sequence is to fix a and consider the bounded sequence $(L(a)_n) = (\text{Op}(a)(x, h_n D_x) f_n, f_n)_{L^2}$. By compactness we can extract a subsequence which converges. Iterating this process for a sequence (a_j) dense in C_0^∞ , we obtain, by diagonal extraction, a sequence (f_{n_k}) such that the limit exists for any a_j . By (2.2) the limit defines a positive functional on a dense subset of C_0^∞ (hence this limit is continuous for the C^0 topology). It is consequently a Radon measure and the limit (2.5) exists for any $a \in C_0^\infty$. For the sake of conciseness we shall denote again by (f_n) the extracted subsequences.

The measure μ represents at points (x_0, ξ_0) the oscillations of the sequence (f_n) at point x_0 and scale ξ_0/h_n . The oscillations at frequencies smaller than h_n^{-1} are concentrated in $\{\xi_0 = 0\}$ whereas the oscillations at higher ($\gg h_n^{-1}$) frequencies are lost.

2.2. Invariance of the Wigner measure.

2.2.1. *Elliptic regularity.* Suppose that the sequence (f_n) is solution of the equation

$$(2.6) \quad (ih_n \partial_t + h_n^2 P) f_n = \mathcal{O}(h_n)_{\mathcal{L}^2}$$

Take $a \in C_0^\infty$ and consider first

$$(2.7) \quad (\text{Op}(a)(x, h_n D_x)(ih_n \partial_t + h_n^2 P) f_n, f_n)_{L^2} = o(1).$$

Taking into account that the operator $\text{Op}(a)(x, h_n D_x)(ih_n \partial_t + h_n^2 P)$ is equal to $\text{Op}(a \times (-\tau + p(z, \zeta)))(x, h D_x)$ modulo an operator bounded by $\mathcal{O}(h_n)$ on \mathcal{L}^2 and passing to the limit in (2.7) we obtain

$$(2.8) \quad \langle \mu, a(x, \xi)(-\tau + p(z, \zeta)) \rangle = 0$$

from which we deduce:

Proposition 2.5. *The measure μ is supported in the semi-classical characteristic set of the operator:*

$$(2.9) \quad \text{Char}(ih_n \partial_t + h_n^2 P) = \{(x, \xi) = (t, z, \tau, \zeta); \tau = p(z, \zeta)\}$$

Remark 2.6. *Suppose that the sequence (f_n) is solution of the equation (2.6). Then for any $a \in C_0^\infty(\mathbb{R}^{2d})$, the function*

$$(2.10) \quad t \mapsto (\text{Op}(a)(z, h_n D_z) f_n |_{t, f_n |_{t}})_{L^2(\mathbb{R}^d)}(t)$$

is, according to (2.6), locally uniformly equicontinuous. Hence using Ascoli's theorem, it is possible to extract a subsequence (f_{n_k}) (independent of t) such that there exist a family of positive measures μ_t continuous with respect to t and such that for any t and any $a \in C_0^\infty(\mathbb{R}^{2d})$ we have

$$(2.11) \quad \lim_{n \rightarrow +\infty} (\text{Op}(a)(z, h_n D_z) f_n, f_n)_{L^2(\mathbb{R}^d)}(t) = \langle \mu_t, a \rangle.$$

Of course, from μ_t one can recover the measure μ (assuming that the extracted sequences are the same):

$$(2.12) \quad \mu = dt \otimes \delta_{\tau=p(z, \zeta)} \otimes \mu_t.$$

2.2.2. *Propagation of the Wigner measure.* Suppose now that

$$(2.13) \quad (ih_n \partial_t + h_n^2 P) f_n = o(h_n)_{\mathcal{L}^2}.$$

Consider the bracket ($P^* = P$)

$$(2.14) \quad \begin{aligned} & h_n^{-1} ([\text{Op}(a)(x, h_n D_x), ih_n \partial_t + h_n^2 P] f_n, f_n)_{L^2} \\ &= h_n^{-1} ((ih_n \partial_t + h_n^2 P) \text{Op}(a)(x, h_n D_x) f_n, f_n)_{L^2} + o(1), \\ &= o(1). \end{aligned}$$

Taking into account that the operator

$$(2.15) \quad h_n^{-1} [\text{Op}(a)(x, D_x), ih_n \partial_t + h_n^2 P]$$

is equal to

$$(2.16) \quad \frac{1}{i} \{a, -\tau + p(z, \zeta)\}(x, h_n D_x) + \mathcal{O}(h_n)_{\mathcal{L}(\mathcal{L}^2)},$$

where the Poisson bracket of a and q , $\{a, q\}$, is defined by

$$(2.17) \quad \{a, q\} = \nabla_{\tau, \zeta} a \cdot \nabla_{t, z} q - \nabla_{t, z} a \nabla_{\tau, \zeta} q,$$

we can pass to the limit in (2.14) and obtain:

$$(2.18) \quad \langle \mu, \{a, -\tau + p(z, \zeta)\} \rangle = 0,$$

or equivalently (with $H_{\tau-p(z, \zeta)}$ the Hamiltonian vector field of $\tau - p$)

$$(2.19) \quad H_{\tau-p(z, \zeta)}(\mu) = (\partial_t - H_p)\mu = 0.$$

Gathering Proposition 2.5 and (2.19), we have proved

Proposition 2.7. *The measure μ is invariant along the integral curves of the vector field $H_{\tau-p}$ drawn on the surface $\{\tau = p(z, \zeta)\}$. Equivalently, if we denote by φ_s the Hamiltonian flow of the function $p(z, \zeta)$ on $T^*\mathbb{R}^2$ and if μ_t is as in (2.12), we have the equality for any $s \in \mathbb{R}$*

$$(2.20) \quad \mu_s = \varphi_s^*(\mu_0) \Leftrightarrow \langle \mu_s, a \circ \varphi_s \rangle = \langle \mu_0, a \rangle.$$

2.3. Proof of Theorem 1 in the case $\Omega = \mathbb{R}^d$. Take (z_0, ζ_0) satisfying the assumption (1.4) and consider $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\int |\varphi|^2 = 1$ and

$$(2.21) \quad u_{0,n} = n^{d/4} \varphi(n^{1/2}(z - z_0)) e^{in(z-z_0) \cdot \zeta_0}.$$

Denote by $v_n = e^{itP} u_{0,n}$ the corresponding solution of the Schrödinger equation. To prove Theorem 1, we are going to show:

$$(2.22) \quad \forall \varepsilon > 0 \quad \lim_{n \rightarrow +\infty} \|A(z, D_z)v_n\|_{L^2([0, \varepsilon] \times \mathbb{R}^d)} = +\infty$$

if $A \in S^{1/2}(\mathbb{R}^{2d})$ satisfies the assumptions of Theorem 1.

For this we compute, with $h_n = 1/n$ and $\Psi \in C_0^\infty(\mathbb{R}^{2d})$, $0 \leq \Psi \leq 1$ equal to 1 near 0 and $\alpha > 0$ fixed,

$$(2.23) \quad \begin{aligned} \|A(z, D_z)v_n\|_{L^2([0, \varepsilon] \times \mathbb{R}^d)}^2 &= \int_0^\varepsilon (A^*(z, D_z)A(z, D_z)v_n, v_n)_{L^2(\mathbb{R}_z^d)} dt \\ &\geq \int_0^\varepsilon (\Psi(\alpha z, \alpha h_n D_z)A^*(z, D_z)A(z, D_z)\Psi(\alpha z, \alpha h_n D_z)v_n, v_n)_{L^2(\mathbb{R}_z^d)} dt \\ &\quad - C \\ &\geq \int_0^\varepsilon (h_n^{-1}b^*(z, h_n D_z)b(z, h_n D_z)v_n, v_n)_{L^2(\mathbb{R}_z^d)} dt - C, \end{aligned}$$

with $b(z, \zeta) = \sigma_{1/2}(A)(z, \zeta)\Psi(\alpha z, \alpha \zeta)$.

But, for any T , if n is large enough

$$(2.24) \quad \int_0^\varepsilon (h_n^{-1}b^*(z, h_n D_z)b(z, h_n D_z)v_n, v_n)_{L^2(\mathbb{R}_z^d)} dt \geq \int_0^{h_n T} (h_n^{-1}b^*(z, h_n D_z)b(z, h_n D_z)v_n, v_n)_{L^2(\mathbb{R}_z^d)} dt.$$

Denote by $u_n(s, z) = v(h_n s, z)$, the solution of the semi-classical Schrödinger equation

$$(2.25) \quad (ih_n \partial_s + h_n^2 P)u_n = 0$$

we obtain for any $T > 0$

$$(2.26) \quad \int_0^\varepsilon (h_n^{-1} b^*(z, h_n D_z) b(z, h_n D_z) v_n, v_n)_{L^2(\mathbb{R}_z^d)} dt \geq \int_0^T (b^*(z, h_n D_z) b(z, h_n D_z) u_n, u_n)_{L^2(\mathbb{R}_z^d)} ds.$$

According to (2.21), the Wigner measure, μ_0 , of the sequence $(u_n |_{t=0})$ is equal to

$$\delta_{(z, \zeta)=(z_0, \zeta_0)}.$$

From Proposition 2.7 and (2.12), we deduce that the Wigner measure, μ_s , of $(u_n |_{t=s})$ is equal to $\delta_{(z, \zeta)=\varphi_{-s}(z_0, \zeta_0)}$, where φ_s is the flow of H_p .

Hence (for T fixed)

$$(2.27) \quad \lim_{n \rightarrow +\infty} \int_0^T (b^*(z, h_n D_z) b(z, h_n D_z) u_n, u_n)_{L^2(\mathbb{R}^d)} dt \\ = \int_0^T \langle \mu_s, b \rangle ds \\ = \int_0^T |b|^2(\varphi_{-s}(z_0, \zeta_0)) ds \\ = \int_0^T |\sigma_{1/2}(A)(\varphi_{-s}(z_0, \zeta_0))|^2 |\Psi(\alpha \varphi_{-s}(z_0, \zeta_0))|^2 ds.$$

From (2.23), (2.24), (2.26) and (2.27) we deduce (if α is chosen small enough) that for any $T > 0$ and with a fixed constant C independent of T :

$$(2.28) \quad \liminf_{n \rightarrow +\infty} \|A(z, D_z) v_n\|_{L^2([0, \varepsilon] \times \mathbb{R}^d)}^2 \geq \int_0^T |\sigma_{1/2}(A)(\varphi_{-s}(z_0, \zeta_0))|^2 ds - C$$

Letting T tend to the infinity (and using the assumption (1.4)), we obtain (2.22).

3. PROOF OF THEOREM 1 FOR A DIRICHLET PROBLEM

In this section we are going to give the outline of the proof of Theorem 1 in the general case. In fact the proof is essentially the same as in the previous section. The differences are that we have to define Wigner measures for sequences bounded in $L_{\text{loc}}^2(\mathbb{R}t; L^2(\Omega))$ and prove the elliptic (Proposition 2.5) and propagation (Proposition 2.7) results for these measures. Then we will construct a sequence of initial data whose Wigner measure is $\delta_{(z_0, \zeta_0)}$ where (z_0, ζ_0) satisfies the assumption (1.4) and the sequence of solutions of the Schrödinger equation with these initial data will prove the result. Fortunately, all these constructions have already been done (see the works by Gérard – Leichtnam [14], Miller [26, 25], Burq – Lebeau [8] and Burq [5]) in some slightly different settings. All that we have to do is to adapt these constructions to our framework and to glue the pieces together.

For the sake of completeness, we are going to give an outline of the constructions. However, we insist on the fact that in this section, most of the material is taken from the works cited above.

Remark 3.1. *For simplicity, we have restricted the study to the case of a scalar equation; however, following [8], it would not be much more difficult to prove the result for systems.*

3.1. Geometry. Denote by $M = \mathbb{R}_t \times \Omega$, $x = (t, z) \in M$ and by bTM the bundle of rank $d + 1$ whose sections are the vector fields tangent to ∂M , ${}^bT^*M$ the dual bundle (Melrose's compressed cotangent bundle) and $j : T^*M \rightarrow {}^bT^*M$ the canonical map. In any coordinate system where $M = \{x = (x_n > 0, x')\}$, the bundle bTM is generated by the fields $\frac{\partial}{\partial x'}$, $x_n \frac{\partial}{\partial x_n}$ and j is defined by

$$(3.1) \quad j(x_n, x', \xi_n, \xi') = (x_n, x', v = x_n \xi_n, \xi').$$

Denote by $\text{Car}\tilde{P}$ the semi-classical characteristic manifold of $\tilde{P} = ih\partial_t + h^2P$ and Z its projection

$$(3.2) \quad \text{Car}\tilde{P} = \left\{ (x, \xi) = (t, z, \tau, \zeta) \in T^*\mathbb{R}^d \mid_{\overline{M}}; p(x, \xi) = \tau \right\}, \quad Z = j(\text{Car}\tilde{P}).$$

The set Z is a locally compact metric space.

Consider, near a point $x_0 \in \partial M$ a geodesic system of coordinates for which $x_0 = (0, 0)$, $M = \{(x_n, x') \in \mathbb{R}^+ \times \mathbb{R}^d\}$ and the operator \tilde{P} has the form (near x_0)

$$(3.3) \quad \tilde{P} = -h^2 D_{x_n}^2 + R(x_n, x', hD_{x'}) + hQ(x, hD_x),$$

with R a second order tangential operator and Q a first order operator.

We recall now the usual decomposition of $T^*\partial M$ (in this coordinate system). Denote by $r(x_n, x', \xi')$ the semi-classical principal symbol of R and $r_0 = r \mid_{x_n=0}$. Then $T^*\partial M$ is the disjoint union of $\mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$ with

$$(3.4) \quad \mathcal{E} = \{r_0 < 0\}, \mathcal{G} = \{r_0 = 0\}, \mathcal{H} = \{r_0 > 0\}.$$

Remark that j gives a natural identification between $Z \mid_{\partial M}$ and $\mathcal{H} \cup \mathcal{G} \subset T^*\partial M$. In \mathcal{G} we distinguish between the *diffractive* points $\mathcal{G}^{2,+} = \{r_0 = 0, r_1 = \partial_{x_n} r \mid_{x_n=0} > 0\}$ and the *gliding* points $\mathcal{G}^- = \{r_0 = 0, r_1 = \partial_{x_n} r \mid_{x_n=0} \leq 0\}$. We will make the assumption (Ω has no infinite order contact with its tangents) that for any $\varrho_0 \in T^*\partial M$, there exists $N \in \mathbb{N}$ such that

$$H_{r_0}^N(r_1) \neq 0$$

The definition of the generalized bicharacteristic flow, φ_s associated to the operator P is essentially the definition given in [24]:

Definition 3.2. *A generalized bicharacteristic curve $\gamma(s)$ is a continuous curve from an interval $I \subset \mathbb{R}$ to Z such that*

- (1) *if $s_0 \in I$ and $\gamma(s_0) \in T^*M$ then close to s_0 , γ is an integral curve of the Hamiltonian vector field $H_{\tilde{p}}$*
- (2) *If $s_0 \in I$ and $\gamma(s_0) \in \mathcal{H} \cup \mathcal{G}^{2,+}$ then there exists $\varepsilon > 0$ such that for $0 < |s - s_0| < \varepsilon$, $x_n(\gamma(s)) > 0$*
- (3) *If $s_0 \in I$ and $\gamma(s_0) \in \mathcal{G}^-$ then for any function $f \in C^\infty(T^*\mathbb{R}^{d+1} \mid_{\overline{M}})$ satisfying the symmetry condition*

$$(3.5) \quad \forall \varrho_0 \in Z, \forall \widehat{\varrho}_0, \tilde{\varrho}_0 \in j^{-1}(\varrho_0) \cap \text{Car}(\tilde{P}), f(\widehat{\varrho}_0) = f(\tilde{\varrho}_0)$$

then

$$\frac{d}{ds} f(j(\gamma(s)) \mid_{s=s_0} = H_{\tilde{p}} \mid_{j^{-1}(\gamma(s_0))} f(j^{-1}(\gamma(s_0)))$$

It is proved in [24] that under the assumption of no infinite order contact, through every point $\varrho_o \in {}^bT^*M \setminus \{0\}$ there exists a unique generalized bicharacteristic (which is furthermore a limit of bicharacteristics having only hyperbolic contacts with the boundary). This defines the flow Φ . Finally remark that since $\tilde{p} = p - \tau$ We have consequently a natural flow, φ on $\text{Char}P \subset {}^bT^*\Omega$ (the generalized flow of $p(z, \zeta)$) given by

$$(3.6) \quad \Phi_s(t, \tau, z, \zeta) = (t - s, \tau, \varphi_s(z, \zeta))$$

3.2. Wigner measures. Consider functions $a = a_i + a_\partial$ with $a_i \in C_0^\infty(T^*M)$, and $a_\partial \in C_0^\infty(\mathbb{R}^{2d-1})$. Such symbols are quantized in the following way: take $\varphi_i \in C_0^\infty(M)$ (resp $\varphi_\partial \in C_0^\infty(\mathbb{R}^d)$) equal to 1 near the x -projection of $\text{supp}(a_i)$ (resp the x -projection of $\text{supp}(a_\partial)$) and define

$$(3.7) \quad \text{Op}_{\varphi_i, \varphi_\partial}(a)(x, hD_x)f = \frac{1}{(2\pi h)^d} \int e^{i(x-y)\cdot\xi/h} a_i(x, \xi) \varphi_i(y) f(y) dy d\xi \\ + \frac{1}{(2\pi h)^{d-1}} \int e^{i(x'-y')\cdot\xi'/h} a_\partial(x_n, x', \xi) \varphi_\partial(x_n, y') f(x_n, y') dy' d\xi'.$$

Remark that according to the symbolic semi-classical calculus, the operator $\text{Op}_{\varphi_i, \varphi_\partial}(a)$ does not depend on the choice of functions $\varphi_i, \varphi_\partial$, modulo operators on \mathcal{L}^2 of norms bounded by $O(h^\infty)$. As in the previous section, we shall in the sequel drop the index $\varphi_i, \varphi_\partial$

Denote by \mathcal{A} the space of the operators which are a finite sum of operators obtained as above in suitable coordinate systems near the boundary and for $A \in \mathcal{A}$, by $a = \sigma(A)$ the semiclassical symbol of the operator A . For such functions a we can define $\kappa(a) \in C^0(Z)$ by

$$(3.8) \quad \kappa(a)(\rho) = a(j^{-1}(\rho))$$

(the value is independent of the choice of $j^{-1}(\rho)$ since the operator is tangential).

The set

$$(3.9) \quad \{\kappa(a), a = \sigma(A), A \in \mathcal{A}\}$$

is a locally dense subset of $C_c^0(Z)$.

3.3. Elliptic regularity. Consider a sequence (f_k) bounded in $\mathcal{L}^2 = L_{\text{loc}}^2(\mathbb{R}_t; L^2(\Omega))$, solution of the equation (with $\lim_{k \rightarrow +\infty} h_k = 0$)

$$(3.10) \quad \begin{cases} (ih_k \partial_t + h_k^2 P) f_k = o(h_k)_{L_{\text{loc}}^2(\mathbb{R}_t; L^2(\Omega))} \\ u|_{\partial\Omega} = 0. \end{cases}$$

The same argument as in section 2.2.1 shows:

Proposition 3.3. *If a_i is equal to 0 near $\text{Car}\tilde{P}$ then*

$$(3.11) \quad \lim_{k \rightarrow +\infty} (\text{Op}(a_i)(x, h_k D_x) f_k, f_k)_{L^2} = 0,$$

and the analysis of the boundary value problem shows:

Proposition 3.4. *If a_∂ is equal to 0 near Z (i.e. a_i is supported in the elliptic region) then*

$$(3.12) \quad \lim_{k \rightarrow +\infty} (\text{Op}(a_\partial)(x_n, x', h_k D_{x'}) f_k, f_k)_{L^2} = 0.$$

3.4. Definition of the measure. The analog of proposition 2.4 is:

Proposition 3.5. *There exists a subsequence (k_p) and a Radon positive measure μ on Z such that*

$$(3.13) \quad \forall Q \in \mathcal{A} \quad \lim_{p \rightarrow \infty} (Q f_{k_p}, f_{k_p})_{L^2} = \langle \mu, \kappa(\sigma(Q)) \rangle.$$

The proof of this result in the interior of Ω is the same as in Section 2 and near a boundary point, it relies on the Gårding inequality for tangential operators (see G. Lebeau [21] for a proof in the classical context and [14, 3] for the semi-classical construction). As before, we denote again by (f_k) the extracted sequence.

Proposition 3.6 (First properties of the measure μ).

$$(3.14) \quad \mu(\mathcal{H}) = 0,$$

$$(3.15) \quad \limsup_{k \rightarrow +\infty} |(\text{Op}(a)h_k D_{x_n} f_k, f_k)_{L^2}| \leq C \sup_{\varrho \in \text{supp}(a)} |r|^{1/2} |a|.$$

The relation (3.14) is a simple consequence of the micro-local analysis of the boundary problem near a point $\varrho_0 \in \mathcal{H}$, for which a parametrix for the solution can be written in terms of a semi-classical Fourier integral operator, by geometric optics methods. To prove (3.15) compute (with $\varphi \in C_0^\infty$ equal to 1 near the t -projection of the support of a)

$$(3.16) \quad \begin{aligned} \left| (\text{Op}(a)h_k D_{x_n} f_k, f_k)_{L^2} \right| &\leq \| \text{Op}(a)h_k D_{x_n} f_k \|_{L^2} \| \varphi(t) f_k \|_{L^2} \\ &\leq (h_k D_{x_n} \text{Op}(a)^* \text{Op}(a)h_k D_{x_n} f_k, f_k)_{L^2}^{1/2} \| \varphi(t) f_k \|_{L^2} \\ &\leq (\text{Op}(a)^* \text{Op}(a)h_k^2 D_{x_n}^2 f_k, f_k)_{L^2}^{1/2} \| \varphi(t) f_k \|_{L^2} + o(1) \\ &\leq (\text{Op}(a)^* \text{Op}(a)(R - \tilde{P})f_k, f_k)_{L^2}^{1/2} \| \varphi(t) f_k \|_{L^2} + o(1) \\ &\leq (\text{Op}(a)^* \text{Op}(a)Rf_k, f_k)_{L^2}^{1/2} \| \varphi(t) f_k \|_{L^2} + o(1) \end{aligned}$$

and we obtain

$$(3.17) \quad \limsup_{k \rightarrow +\infty} \left| (\text{Op}(a)h_k D_{x_n} f_k, f_k)_{L^2} \right| \leq C |\langle \mu, a^2 r \rangle|^{1/2} \leq C \sup_{\varrho \in \text{supp}(a)} |a| |r|^{1/2}.$$

3.5. Invariance of the measure. Consider now a sequence (f_k) bounded in $L_{\text{loc}}^2(\mathbb{R}_t; L^2(\Omega))$, solution of the equation (with $\lim_{k \rightarrow +\infty} h_k = 0$)

$$(3.18) \quad \begin{cases} (ih_k \partial_t + h_k^2 P) f_k = o(h_k)_{L_{\text{loc}}^2(\mathbb{R}_t; L^2(\Omega))} \\ u|_{\partial\Omega} = 0 \end{cases}$$

Proposition 3.7. *Consider $q \in C^\infty(T^*\mathbb{R}^{d+1} \setminus \overline{M})$ satisfying the symmetry condition (3.5). In general $\{\tilde{p}, q\} = -2\xi_n \partial_{x_n} q + \{r, q\}$ is not a function defined on Z (because of the ξ_n dependence). To obtain a function on Z , we take the convention*

$$(3.19) \quad \{\tilde{p}, q\} \stackrel{\text{def}}{=} -2\xi_n \partial_{x_n} q 1_{\varrho \notin \mathcal{H}} + \{r, q\}.$$

This function is μ -integrable and, thanks to (3.14), μ -almost everywhere continuous.

Then, with this convention, the measure μ satisfies

$$(3.20) \quad \langle \mu, \{\tilde{p}, q\} \rangle = 0$$

The proof of Proposition 3.7 is simply integration by parts (and some careful study of the terms arising). We give it below:

Since in M , the equation (3.20) is simply (2.18), we restrict the study to the case where q is supported near a point $\varrho_0 \in T^*\partial M$. Suppose first only that $q \in C^\infty(T^*\mathbb{R}^{d+1} |_{\overline{M}})$. From Malgrange preparation theorem, there exist functions $q_0(x_n, x', \xi'), q_1(x_n, x', \xi') \in C^\infty$ such that

$$(3.21) \quad q|_{\text{Car}\tilde{P}} = q_0|_{\text{Car}\tilde{P}} + \xi_n q_1|_{\text{Car}\tilde{P}}.$$

Let $Q = \text{Op}(q_0) + \text{Op}(q_1)hD_{x_n}$ and compute $(P^* = P)$

$$(3.22) \quad h_k^{-1} \left((\tilde{P}Q - Q\tilde{P})f_k, f_k \right)_{L^2}.$$

Two integrations by part, (3.3) and the boundary condition $f_k|_{x_n=0} = 0$ show that

$$(3.23) \quad \begin{aligned} h_k^{-1} \left([\tilde{P}, Q]f_k, f_k \right)_{L^2} &= h_k^{-1} \left(\tilde{P}Qf_k, f_k \right)_{L^2} + o(1) \\ &= -i \left(Q_1|_{x_n=0} h_k D_{x_n} f_k|_{x_n=0}, h_k D_{x_n} f_k|_{x_n=0} \right)_{L^2(\mathbb{R}_{x'}^{d-1})}. \end{aligned}$$

On the other hand $[\tilde{P}, Q]$ can be written under the form

$$\frac{i}{h_k} [\tilde{P}, Q] = A_0 + A_1 h_k D_{x_n} + A_2 \tilde{P} + h A_3$$

where A_0, A_1 and A_2 are tangential operators, A_3 is differential of order at most 1 in D_{x_n} , and on $\text{Car}\tilde{P}$ we have

$$(3.24) \quad a_0 + a_1 \xi_n = \{\tilde{p}, q\}.$$

From (3.14) we deduce that μ -almost everywhere

$$(3.25) \quad a_0 + a_1 \xi_n \mathbf{1}_{x_n > 0} = \{\tilde{p}, q\}.$$

Consequently

$$(3.26) \quad \begin{aligned} h_k^{-1} \left([\tilde{P}, Q]f_k, f_k \right)_{L^2} &= h_k^{-1} \left(\tilde{P}Qf_k, f_k \right)_{L^2} + o(1) \\ &= \left(-i(A_0 + A_1 h_k D_{x_n} + A_2 \tilde{P} + o(1))f_k, f_k \right)_{L^2} + o(1). \end{aligned}$$

Passing to the limit in (3.26) we obtain

$$(3.27) \quad \lim_{k \rightarrow +\infty} \left((A_0 + A_1 h_k D_{x_n} + A_2(\tilde{P})f_k, f_k \right)_{L^2} = \langle \mu, a_0 \rangle + \lim_{k \rightarrow +\infty} (A_1 h_k D_{x_n} f_k, f_k)_{L^2}.$$

Take $\varepsilon > 0$ and $\varphi \in C_0^\infty(]-1, 1[)$ equal to 1 near 0. Decompose

$$(3.28) \quad \begin{aligned} A_1 &= \left(1 - \varphi\left(\frac{x_n}{\varepsilon}\right)\right)A_1 + \text{Op}\left(\varphi\left(\frac{x_n}{\varepsilon}\right)\varphi\left(\frac{r(x_n, x', \xi')}{\varepsilon}\right)\right)A_1 \\ &\quad + \text{Op}\left(\varphi\left(\frac{x_n}{\varepsilon}\right)\left(1 - \varphi\left(\frac{r(x_n, x', \xi')}{\varepsilon}\right)\right)\right)A_1. \end{aligned}$$

The first term in the right hand side of (3.28) is supported in the interior of Ω ; its contribution to the limit in (3.27) is equal to

$$(3.29) \quad \langle \mu, (1 - \varphi(\frac{x_n}{\varepsilon}))a_1 \xi_n \rangle$$

The contribution of the second term is, according to (3.15), smaller than

$$(3.30) \quad C \sup_{\varrho \in \text{supp}(\varphi(\frac{x_n}{\varepsilon})(\varphi(\frac{r(x_n, x', \xi')}{\varepsilon})))} |r|^{1/2} |a_1| \leq C\varepsilon^{1/2}$$

and the contribution of the last term is smaller than

$$(3.31) \quad \|h_k D_{x_n} f_k\| \left\| A_1^* \text{Op} \left(\varphi(\frac{x_n}{\varepsilon})(1 - \varphi(\frac{r(x_n, x', \xi')}{\varepsilon})) \right)^* f_k \right\| \\ \leq C \langle \mu, |a_1|^2 \varphi^2(\frac{x_n}{\varepsilon})(1 - \varphi(\frac{r(x_n, x', \xi')}{\varepsilon}))^2 \rangle^{1/2} + o(1)$$

Passing to the limit $\varepsilon \rightarrow 0$ we obtain that the contribution of the first term is equal to

$$(3.32) \quad \langle \mu, a_1 \xi_n 1_{x_n > 0} \rangle,$$

the contribution of the second term is (according to (3.30)) equal to 0 and the contribution of the last term is, according to (3.14), smaller than

$$(3.33) \quad \langle \mu, a_1^2 1_{x_n=0} 1_{r \neq 0} \rangle = \langle \mu, a_1^2 1_{\varrho \in \mathcal{H}} \rangle = 0.$$

Finally we have proved

$$(3.34) \quad \lim_{n \rightarrow +\infty} -i (Q_1 |_{x_n=0} h_k D_{x_n} f_k |_{x_n=0}, h_k D_{x_n} f_k |_{x_n=0})_{L^2(\mathbb{R}_{x'}^{d-1})} = -i \langle \mu, \{\tilde{p}, q\} \rangle.$$

But, if q satisfies the symmetry condition (3.5), the function $q |_{x_n=0}$ is independent of ξ_n on $\text{Car}P$. Hence $q_1 |_{x_n=0} = 0$ on \mathcal{H} and consequently on $\overline{H} = \mathcal{H} \cup \mathcal{G}$; and the left hand side in (3.34) tends to 0, which proves Proposition 3.7.

Proposition 3.8 (see [6] and [26]). *We have*

$$(3.35) \quad \mu(\mathcal{G}^{2,+}) = 0$$

Consider a point $\varrho_0 \in \mathcal{G}^{2,+}$. Apply (3.34) to a family of functions $q = \xi_n \times q_\varepsilon$ with

$$(3.36) \quad q_\varepsilon = \varphi(\frac{x_n}{\varepsilon^{1/3}}) \varphi(\frac{r(x_n, x', \xi')}{\varepsilon}) a(x_n, x', \xi').$$

Then we get

$$(3.37) \quad \lim_{n \rightarrow +\infty} \left(\varphi(\frac{r(0, x', h_k D_{x'})}{\varepsilon}) a(0, x', h_k D_{x'}) h_k D_{x_n} f_k |_{x_n=0}, h_k D_{x_n} f_k |_{x_n=0} \right)_{L^2(\mathbb{R}_{x'}^{d-1})} \\ = \langle \mu, -2\xi_n^2 \partial_{x_n} (\varphi(\frac{x_n}{\varepsilon^{1/3}}) \varphi(\frac{r(x_n, x', \xi')}{\varepsilon}) a(x_n, x', \xi')) \rangle \\ - \langle \mu, \partial_{x_n} r \varphi(\frac{x_n}{\varepsilon^{1/3}}) \varphi(\frac{r(x_n, x', \xi')}{\varepsilon}) a(x_n, x', \xi') + \xi_n \{r, q_\varepsilon\}' \rangle,$$

where $\{r, q_\varepsilon\}'$ is the Poisson bracket with respect to the x', ξ' variables. On the support of the measure μ , $\xi_n^2 = r(x_n, x', \xi')$. Hence we can apply the dominated convergence theorem and obtain that the right hand side in (3.37) tends to

$$(3.38) \quad \langle \mu, -\partial_{x_n} r(0, x', \xi') a(x_n, x', \xi') 1_{x_n=0} 1_{r=0} \rangle = \langle \mu, -\partial_{x_n} r(0, x', \xi') a(x_n, x', \xi') 1_{\rho \in \mathcal{G}} \rangle$$

According to the assumption $\varrho_0 \in \mathcal{G}^{2,+}$, $\partial_{x_n} r > 0$ at the point ϱ_0 . If the support of a is chosen small enough so that $\partial_{x_n} r > 0$ on this support, then the right hand side in (3.38) is non positive. On the other hand by Gårding inequality the limit on the left hand side is non negative. Both sides are then equal to 0. This implies Proposition 3.8.

It is now possible to prove as in [8, Théorème 1] (see also [5]), by measure theory methods, that the invariance of the measure μ along the generalized bicharacteristic flow is equivalent to Propositions 3.7 and 3.8 (in fact the proof of this result is presented in [8, Section 3.3] for classical measures, in the more general context of systems, but the proof for semi-classical measures is the same word by word).

3.6. Proof of Theorem 1. All that remains to do to complete the proof of Theorem 1 in the case of a Dirichlet boundary value problem is to construct a sequence of initial data $(u_{0,n})$ and a sequence $h_n; \lim_{n \rightarrow +\infty} h_n = 0$, such that the sequence of solutions of the semi-classical Schrödinger equations admits

$$(3.39) \quad dt \otimes \delta_{\tau=p(z_0, \zeta_0)} \otimes \delta_{(z, \zeta)=(\varphi_{-t}(z_0, \zeta_0))}$$

as Wigner measure.

In the case where the bicharacteristic starting from $(t_0 = 0, \tau_0 = p(z_0, \zeta_0), z_0, \zeta_0)$ has an interior point $(t_1, \tau = \tau_0, z_1 \in \Omega, \zeta_1)$, we perform the construction as in the previous section, since by finite speed of propagation (modulo $\mathcal{O}(h^\infty)$), the boundary is not seen, (3.39) is satisfied close to $(t_1, \tau = \tau_0, z_1, \zeta_1)$. Using the propagation result, we deduce that (3.39) is satisfied everywhere.

In the case where the bicharacteristic starting from $(t_0 = 0, \tau = \tau_0, z_0, \zeta_0)$ has no interior point, we know that it can be approximated by bicharacteristics γ_k which have an interior point (see [23, 24]). For these bicharacteristics, we can construct sequences of initial data $(u_{n,k})$ associated to $(h_{n,k}); \lim_{n \rightarrow +\infty} h_{n,k} = 0$. Taking $(u_{n_k, k})$ with n_k large enough, as initial data matches our aim.

The rest of the proof of the estimate (2.22) in the case of a boundary value problem is now the same as in Section 2.

4. SMOOTHING EFFECT

In this section we prove a weaker smoothing effect for a class of trapping obstacles.

4.1. Assumptions. Consider $\Theta \subset \mathbb{R}^d$ a compact smooth obstacle whose complement, $\Omega = \Theta^c$ is connected. Let Δ_D be the Laplace operator acting on $L^2(\Omega)$, with domain $D = H^2(\Omega) \cap H_0^1(\Omega)$. Denote, for $u_0 \in L^2(\Omega)$, by $e^{-it\Delta_D} u_0 = u$ the solution of the Schrödinger equation with Dirichlet boundary conditions:

$$(4.1) \quad \begin{cases} (i\partial_t - \Delta)u = 0 \text{ on } \mathbb{R}_t \times \Omega, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0. \end{cases}$$

We suppose that $\Theta = \cup_{i=1}^N \Theta_i \subset \mathbb{R}^d$ is the union of a finite number of strictly convex obstacles, Θ_i satisfying:

- For any $1 \leq i, j, k \leq N$, $i \neq j$, $j \neq k$, $k \neq i$, one has

$$(4.2) \quad \text{Convex Hull}(\Theta_i \cup \Theta_j) \cap \Theta_k = \emptyset.$$

- Denote by κ the infimum of the principal curvatures of the boundaries of the obstacles Θ_i , and L the infimum of the distances between two obstacles. Then if $N > 2$ we assume that $\kappa L > N$ (no assumption if $N = 2$).

Remark 4.1. *If there are only two obstacles, then the assumptions are automatically fulfilled. The first assumption is essentially technical, whereas the second one is an assumption about the strong hyperbolicity of the dynamical system given by the billiard flow.*

In this case, since there are trapped trajectories (for example any line minimizing the distance between two obstacles is trapped), we have shown in Section 3 that the plain smoothing effect $H^{1/2}$ does not hold. However, the result below (a more precise version of Theorem 2) shows that the smoothing effect with a logarithmic loss still holds.

Theorem 3. *Under the assumptions above, for any $\chi \in C_0^\infty(\mathbb{R}^d)$ there exists $C > 0$ such that the solution of*

$$(4.3) \quad \begin{cases} (i\partial_t - \Delta)u = 0 \text{ on } \mathbb{R}_t \times \Omega, \\ u|_{t=0} = u_0, \\ u|_{\partial\Omega} = 0 \end{cases}$$

and the solution of

$$(4.4) \quad \begin{cases} (i\partial_t - \Delta)v = \chi f; \chi f \text{ compactly supported in time,} \\ v|_{t < 0} = 0, \\ v|_{\partial\Omega} = 0 \end{cases}$$

satisfy:

$$(4.5) \quad \begin{aligned} \|\chi u\|_{L^2(\mathbb{R}_t; H_D^{1/2,-}(\Omega))} &\leq C \|u_0\|_{L^2(\Omega)}, \\ \|\chi v\|_{L^2(\mathbb{R}_t; H_D^{1/2,-}(\Omega))} &\leq C \|\chi f\|_{L^2(\mathbb{R}_t; H_D^{-1/2,+}(\Omega))}, \end{aligned}$$

where $H_D^{1/2,-} = D((Id - \Delta_D)^{1/4} \log^{-1/2}(2Id - \Delta_D))$ and $H^{-1/2,+} = (H_D^{1/2,-})'$. In particular

$$\forall \varepsilon > 0, H^{-1/2+\varepsilon}(\Omega) \subset H_D^{-1/2,+} \subset H^{-1/2}(\Omega), \quad H^{1/2}(\Omega) \subset H_D^{1/2,-} \subset H^{1/2-\varepsilon}(\Omega)$$

with continuous injections.

Remark 4.2. *In the case where there exist an elliptic (stable) periodic trajectory, it is possible to construct quasi modes with compact support, i.e. functions $(e_n)_{n \in \mathbb{N}}$ with compact supports associated to a particular sequence $(\lambda_n) \rightarrow +\infty$ and satisfying*

$$(4.6) \quad \begin{aligned} -\Delta e_n &= \lambda_n e_n + r_n, \\ \|r_n\|_{H^N} &\leq C_{N,M} \lambda_n^{-M}, \quad \forall N, M \in \mathbb{N}. \end{aligned}$$

From this we deduce easily that the sequence of solutions of the Schrödinger equation with initial data (e_n) is, for any $\varepsilon > 0$, $s > 0$, not bounded in $L^1([0, \varepsilon]; H_{loc}^s)$; which implies that no smoothing effect at all is true any more. Under the assumptions of Theorem 3, the periodic trajectories are hyperbolic (unstable), which forbids the construction of such well localized quasi-modes.

Theorem 3 is deduced from the following estimate of the cut-off resolvent:

Proposition 4.3. *Suppose that the obstacle Θ satisfies the assumptions in Theorem 3 above. Then the resolvent of the operator Δ_D , $(-\Delta_D - \lambda)^{-1}$ (which is analytic in $\mathbb{C} \setminus \mathbb{R}^+$) satisfies:*

$$(4.7) \quad \forall \chi \in C_0^\infty(\mathbb{R}^2), \exists C > 0; \forall \lambda \in \mathbb{R}, 0 < \varepsilon \ll 1, \\ \|\chi(-\Delta_D - (\lambda \pm i\varepsilon))^{-1}\chi\|_{L^2 \rightarrow L^2} \leq \frac{C \log(2 + |\lambda|)}{1 + \sqrt{|\lambda|}}.$$

We are going to prove this estimate for $\lambda \gg 1$. The proof for $|\lambda| \ll 1$ can be found in [4, Annexe B.2]; whereas the result for $c \leq |\lambda| \leq C$ follows from the Rellich uniqueness Theorem (see [20] or [4, Annexe B.1]) and the result for $\lambda < -\varepsilon$ is clear because in this case the operator is semi-classically elliptic.

Let us perform a change of variables $\lambda = \tau^2$ and consider $\chi(-\Delta_D - (\tau^2))^{-1}\chi$ which is holomorphic in $\{\text{Im}\tau < 0\}$ and satisfies there (according to the standard estimate for self adjoint operators),

$$(4.8) \quad \|(-\Delta_D - (\tau^2))^{-1}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \frac{1}{|\tau| |\text{Im}\tau|}.$$

M. Ikawa proved in [16, 17] and more precisely in [18, Theorem 2.1] (see also the work by C. Gérard [13] where such an estimate is implicit) that under the assumptions above, the following estimate on the cut-off resolvent holds:

Theorem 4 (Ikawa, [18, Theorem 2.1]). *The cut-off resolvent: $\chi(-\Delta_D - (\lambda \pm i\varepsilon))^{-1}\chi$ admits a holomorphic continuation in a strip of the upper half plane*

$$(4.9) \quad \{\tau \in \mathbb{C}; |\tau| > 1, \text{Im}\tau \leq \alpha\}, \alpha > 0$$

and satisfies there (for a large N):

$$(4.10) \quad \|\chi(-\Delta_D - (\tau^2))^{-1}\chi\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C|\tau|^N$$

Remark 4.4. *In [18, Theorem 2.1] the proof is done with the additional assumption that the dimension of space is equal to 3 (which is the relevant dimension the author had in mind for applications to the wave equation). However the proof could be equally performed in any space dimension $d \geq 2$ (see [13] in the case $N = 2$, $d \geq 2$).*

Using (4.8) and (4.10) (and writing $\tau = h^{-1}z$, $z \sim 1$, $h \rightarrow 0$), one easily sees, with

$$(4.11) \quad f(h, z) = (\chi(-h^2\Delta_D - z^2)^{-1}\chi u, v)_{L^2(\Omega)}; u, v \in L^2(\Omega)$$

that (4.7) for large $|\lambda|$ follows from Theorem 4 and the following semi-classical maximum principle (a variant of Phrögmen Lindelöf principle) adapted from the work by Tang–Zworski [28]:

Lemma 4.5. *Suppose that $f(h, z)$ is a family of holomorphic functions defined for $0 < h < 1$ in a neighbourhood of*

$$(4.12) \quad \Omega(h) = [1/2, 3/2] \times i[h\alpha, -h\alpha]$$

such that

$$(4.13) \quad \begin{aligned} |f(h, z)| &\leq Ch^{-M} \text{ on } \Omega(h), \\ |f(h, z)| &\leq \frac{1}{|\text{Im}z|} \text{ on } \Omega(h) \cap \{\text{Im}z < 0\}. \end{aligned}$$

Then there exists $h_0 > 0$, $C > 0$, such that for any $0 < h < h_0$

$$(4.14) \quad |f(h, z)| \leq C \frac{\log(h^{-1})}{h} \text{ on } [4/5, 6/5]$$

To prove this lemma, first consider the function

$$(4.15) \quad \varphi(z, h) = (\pi h)^{-1/2} \int e^{-\frac{(x-z)^2}{h}} \Psi(x) dx$$

where $\Psi \in C_0^\infty([2/3, 4/3])$ is non negative and equal to 1 in $[3/4, 5/4]$. Then the function $\varphi(z, h)$ satisfies:

- (1) $\varphi(z, h)$ is holomorphic in $\Omega(h)$,
- (2) $|\varphi(z, h)| \leq C$ in $\Omega(h)$,
- (3) $|\varphi(z, h)| \geq c > 0$ in $[4/5, 6/5]$,
- (4) $|\varphi(z, h)| \leq Ce^{-c/h}$ on $\Omega(h) \cap \{|\text{Re}z - 1| \geq 1/2\}$.

Then apply the maximum principle to the function $g(z, h) = e^{-iN \log(h)z/h} \varphi(z, h) f(z, h)$ on the domain

$$(4.16) \quad \tilde{\Omega}(h) = [1/2, 3/2] \times i[h\alpha, -h/\log(h^{-1})].$$

Using the bounds (4.13) on f and the properties of φ above, we can estimate g by

$$(4.17) \quad \begin{aligned} |g(z, h)| &\leq Ch^{N\alpha-M} \text{ on } \partial\tilde{\Omega}(h) \cap \{\text{Im}z = h\alpha\}, \\ |g(z, h)| &\leq C_N e^{-c/h} \text{ on } \partial\tilde{\Omega}(h) \cap \{\text{Re}z \in [1/2, 3/2]\}, \\ |g(z, h)| &\leq C_N \frac{\log(h^{-1})}{h} \text{ on } \partial\tilde{\Omega}(h) \cap \{\text{Im}z = \frac{-h}{\log(h^{-1})}\}. \end{aligned}$$

Taking N large enough and applying the maximum principle we get

$$(4.18) \quad |g(z, h)| \leq C' \frac{\log(h^{-1})}{h} \text{ on } \tilde{\Omega}(h)$$

which implies

$$(4.19) \quad |f(z, h)| \leq C' \frac{\log(h^{-1})}{h} \text{ on } [4/5, 6/5],$$

and ends the proof of Lemma 4.5.

We deduce from (4.7):

$$(4.20) \quad \|\chi(-\Delta_D - (\lambda \pm i\varepsilon))^{-1} \chi\|_{H^{-1/2,+} \rightarrow H^{1/2,-}} \leq C.$$

Indeed, for bounded λ , integrations by parts show that we can in fact replace $H^{1/2,-}$ by $H_0^1(\Omega)$ and $H^{-1/2,+}$ by $H^{-1}(\Omega)$ and for large λ we decompose, with $\Psi \in C_0^\infty([1/2, 2])$ equal to 1 close to 1,

$$(4.21) \quad u = (P - \lambda)^{-1} \chi f = \Psi\left(\frac{-\Delta_D}{\lambda}\right)u + (1 - \Psi\left(\frac{-\Delta_D}{\lambda}\right))u.$$

We get by the functional calculus of self adjoint operators

$$(4.22) \quad \|(1 - \Psi\left(\frac{-\Delta_D}{\lambda}\right))u\|_{H_0^1(\Omega)} \leq C\|(1 - \Psi\left(\frac{-\Delta_D}{\lambda}\right))\chi f\|_{H^{-1}(\Omega)}.$$

On the other hand, the function $v = \Psi\left(\frac{-\Delta_D}{\lambda}\right)u$ satisfies

$$(4.23) \quad (P - \lambda)v = \Psi\left(\frac{-\Delta_D}{\lambda}\right)\chi f.$$

If $\tilde{\chi} \in C_0^\infty(\mathbb{R}^d)$ is equal to one on the support of χ , we have modulo negligible terms

$$(4.24) \quad \tilde{\chi}\Psi\left(\frac{-\Delta_D}{\lambda}\right)\chi = \Psi\left(\frac{-\Delta_D}{\lambda}\right)\chi$$

because

$$\tilde{\chi}\Psi\left(\frac{-\Delta_D}{\lambda}\right)\chi - \Psi\left(\frac{-\Delta_D}{\lambda}\right)\chi = [\tilde{\chi}, \Psi\left(\frac{-\Delta_D}{\lambda}\right)]\chi$$

and on the support of $\nabla\tilde{\chi}$, the operator P is a differential operator and consequently $\Psi\left(\frac{-\Delta_D}{\lambda}\right)$ is a pseudodifferential operator on this set (see for example Sect.4 of [27]).

According to (4.23), (4.24) and Lemma 4.5, we get

$$(4.25) \quad \left(\frac{\sqrt{|\lambda|}}{\log(2 + |\lambda|)}\right)^{1/2} \|\chi\Psi\left(\frac{-\Delta_D}{\lambda}\right)u\|_{L^2} \leq C \left(\frac{\log(2 + |\lambda|)}{\sqrt{|\lambda|}}\right)^{1/2} \|\Psi\left(\frac{-\Delta_D}{\lambda}\right)\chi f\|_{L^2}$$

to replace the weights in λ above by the $H^{\pm 1/2, \mp}$ -norms (i.e. to replace the weights in λ by weights in $-\Delta_D$), it is enough to check that modulo negligible terms, if $\tilde{\Psi}$ is a fonction equal to 1 on the support of Ψ ,

$$(4.26) \quad \tilde{\Psi}\left(\frac{-\Delta_D}{\lambda}\right)\chi\Psi\left(\frac{-\Delta_D}{\lambda}\right) = \chi\Psi\left(\frac{-\Delta_D}{\lambda}\right)$$

which follows from the same arguments as above.

Following [7], Theorem 3 is now a consequence of (4.20). For the sake of completeness and since the argument is short, we recall it: firstly remark that by TT^* argument it suffices to study the second (inhomogeneous) case. Indeed denote by $T = \chi e^{-it\Delta_D}$. The continuity of T from L^2 to $L^2(\mathbb{R}_t; H^{1/2,-})$ is equivalent to the continuity of the adjoint operator

$$(4.27) \quad T^* f = \int_{\mathbb{R}} e^{is\Delta_D} \chi f(s) ds$$

from $L^2(\mathbb{R}_t; H^{-1/2,+})$ to L^2 , which in turns is equivalent to the continuity of the operator TT^* from $L^2(\mathbb{R}_t; H^{-1/2,+})$ to $L^2(\mathbb{R}_t; H^{1/2,-})$. But

$$(4.28) \quad \begin{aligned} TT^* f(t) &= \int_{\mathbb{R}} \chi e^{i(s-t)\Delta_D} \chi f(s) ds \\ &= \int_{s < t} \chi e^{i(s-t)\Delta_D} \chi f(s) ds + \int_{t < s} \chi e^{i(s-t)\Delta_D} \chi f(s) ds \end{aligned}$$

and (by time inversion), it clearly suffices to prove the continuity of any one of the terms in the right hand side, which is the second (inhomogeneous) part of Theorem 3.

Consider now (v, f) solution of (4.4). By translation invariance we can suppose that f (and hence v) is supported in $\{t > 0\}$. The Fourier transforms of v and f are (according to the support property) holomorphic in the set $\{\text{Im} z < 0\}$ and satisfy there, according to (4.4)

$$(4.29) \quad (-z + \Delta) \hat{u}(z, \cdot) = \chi \hat{f}(z, \cdot).$$

Taking $z = \tau - i\varepsilon$, $\tau \in \mathbb{R}$ and having ε tend to 0, using (4.20), we get

$$(4.30) \quad \|\chi \hat{u}\|_{L^2(\mathbb{R}_\tau; H^{1/2,-})} \leq C \|\chi \hat{f}\|_{L^2(\mathbb{R}_\tau; H^{-1/2,+})}$$

and since the Fourier transform is an isometry on $L^2(\mathbb{R}; H)$ if H is a Hilbert space, we get (4.5).

Finally, as in [7], we can deduce from Theorem 3:

Theorem 5 (Global existence for 2-d defocusing NLS). *Consider $\Theta \subset \mathbb{R}^2$ an obstacle which is the union of N strictly convex obstacles satisfying the assumptions above. Denote by $\Omega = \Theta^c$ its complement and let P be a polynomial with real coefficients. For every $u_0 \in H_0^1(\Omega)$, there exists a unique maximal solution $u \in C(I, H_0^1(\Omega))$ of the equation*

$$(4.31) \quad i\partial_t u + \Delta u = P'(|u|^2)u, \quad u(0, x) = u_0(x).$$

Moreover we have:

- (i) *If $\|u_0\|_{H_0^1(\Omega)}$ is bounded from above, the length of $I \cap \mathbb{R}_\pm$ is bounded from below by a positive constant.*
- (ii) *For any finite p , $u \in L_{loc}^p(I, L^\infty(\Omega))$.*
- (iii) *If $P(r) \rightarrow +\infty$ as $r \rightarrow +\infty$, $I = \mathbb{R}$.*
- (iv) *If $u_0 \in H_D^s(\Omega)$ for some $s > 1$, $u \in C(I, H_D^s(\Omega))$. In particular if $u_0 \in C_0^\infty(\Omega)$, $u \in C^\infty(I \times \Omega)$.*

REFERENCES

- [1] M. Ben Artzi and S. Klainerman. Decay and regularity for the schrödinger equation. *Journal d'Analyse Mathématique*, 58:25–37, 1992.
- [2] M. Ben-Artzi and A. Devinatz. Regularity and decay of solutions to the Stark evolution equation. *J. Funct. Anal.*, 154(2):501–512, 1998.
- [3] N. Burq. Mesures semi-classiques et mesures de défaut. *Séminaire Bourbaki*, Mars 1997.
- [4] N. Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Mathematica*, 180:1–29, 1998.
- [5] N. Burq. Semi-classical estimates for the resolvent in non trapping geometries. *Int. Math. Res. Notices*, 5:221–241, 2002.
- [6] N. Burq and P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. *Comptes Rendus de L'Académie des Sciences*, pages 749–752, 1997. t.325, Série I.

- [7] N. Burq, P. Gerard, and N. Tzvetkov. On non linear schrödinger equations in exterior domains. *Preprint*, 2002.
- [8] N. Burq and G. Lebeau. Mesures de défaut de compacité, application au système de Lamé. *Ann. Sci. École Norm. Sup. (4)*, 34(6):817–870, 2001.
- [9] P. Constantin and J.C. Saut. Local smoothing properties of Schrödinger equations. *Indiana Univ. Math. J.*, 38(3):791–810, 1989.
- [10] S. Doi. Smoothing effect for schrödinger evolution equation and global behaviour of geodesic flow. *Math. Ann.*, 318:355–389, 2000.
- [11] S.I. Doi. Remarks on the Cauchy problem for Schrödinger-type equations. *Comm. Partial Differential Equations*, 21(1-2):163–178, 1996.
- [12] S.I. Doi. Smoothing effects of Schrödinger evolution groups on Riemannian manifolds. *Duke Math. J.*, 82(3):679–706, 1996.
- [13] C. Gérard. Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes. *Supplément au Bulletin de la Société Mathématique de France*, 116, 1988.
- [14] P. Gérard and E. Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Mathematical Journal*, 71:559–607, 1993.
- [15] P. Gérard, P. Markowich, N. J. Mauser, and F. Poupaud. Homogenization limits and Wigner transforms. *Comm. Pure Appl. Math.*, 50(4):323–379, 1997.
- [16] M. Ikawa. Decay of solution of the wave equation in the exterior of two convex bodies. *Osaka Journal of Mathematics*, 19:459–509, 1982.
- [17] M. Ikawa. On the poles of the scattering matrix for two convex obstacles. *Journal of Mathematics of the Kyoto University*, 23:127–194, 1983.
- [18] M. Ikawa. Decay of solution of the wave equation in the exterior of several convex bodies. *Annales de l'Institut Fourier*, 38(2):113–146, 1988.
- [19] Tosio Kato and Kenji Yajima. Some examples of smooth operators and the associated smoothing effect. *Rev. Math. Phys.*, 1(4):481–496, 1989.
- [20] P. D. Lax and R. S. Phillips. *Scattering theory*. Number 26 in Pure and Applied Mathematics. Academic Press, 2 edition, 1989.
- [21] G. Lebeau. Equation des ondes amorties. In A. Boutet de Monvel and V. Marchenko, editors, *Algebraic and Geometric Methods in Mathematical Physics*, pages 73–109. Kluwer Academic, The Netherlands, 1996.
- [22] P.L. Lions and T. Paul. Sur les mesures de Wigner. *Rev. Mat. Iberoamericana*, 9(3):553–618, 1993.
- [23] R.B. Melrose and J. Sjöstrand. Singularities of boundary value problems I. *Communications in Pure Applied Mathematics*, 31:593–617, 1978.
- [24] R.B. Melrose and J. Sjöstrand. Singularities of boundary value problems II. *Communications in Pure Applied Mathematics*, 35:129–168, 1982.
- [25] L. Miller. Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary. *J. Math. Pures Appl., IX. Ser.*, 79(3):227–269, 2000.
- [26] L. Miller. Refraction d'ondes semi-classiques par des interfaces franches. (refraction of semiclassical waves by sharp interfaces). *C. R. Acad. Sci., Paris, Ser. I, Math.*, 325(4):371–376, 1997.
- [27] J. Sjöstrand. A trace formula and review of some estimates for resonances. In *Microlocal Analysis and Spectral Theory*, volume 490 of *NATO ASI series C*, pages 377–437. Kluwer, 1997.
- [28] S.H. Tang and M. Zworski. Resonance expansions of scattered waves. *Comm. Pure Appl. Math.*, 53(10):1305–1334, 2000.

UNIVERSITÉ PARIS SUD, MATHÉMATIQUES, BÂT 425, 91405 ORSAY CEDEX
E-mail address: Nicolas.burq@math.u-psud.fr
URL: <http://www.math.u-psud.fr/~burq>