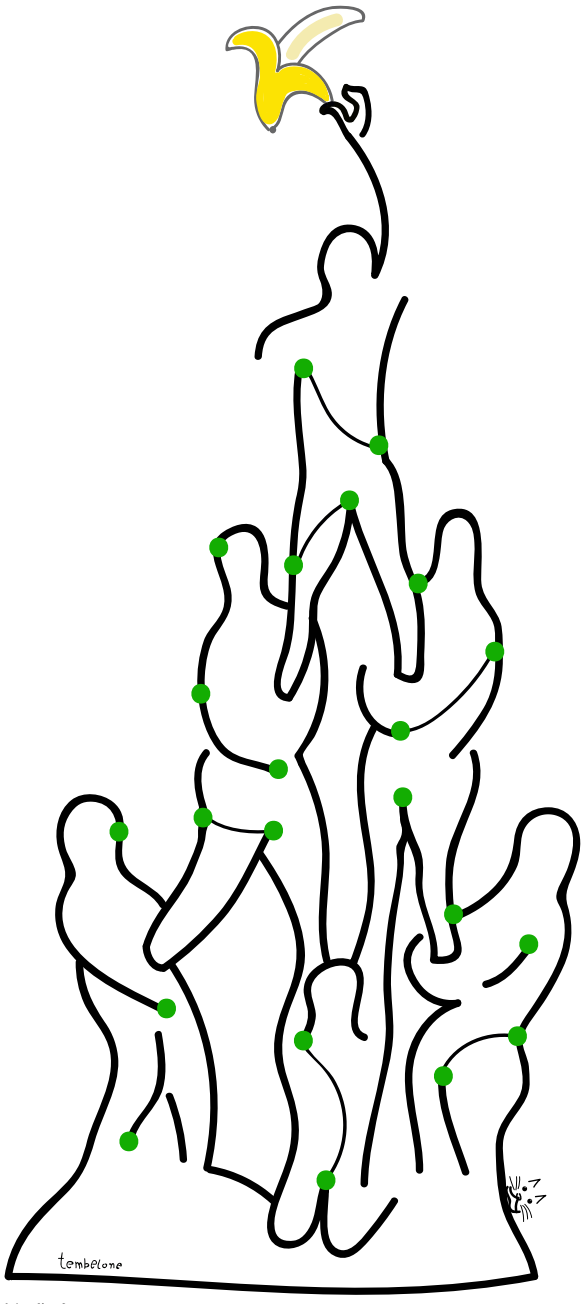
Peeling random planar maps

Nicolas Curien



Introduction

What is a random *discrete* planar geometry?

When thinking of planar lattices, the reader may immediately come up with "Euclidean" planar graphs such as $\mathbb{Z}^2$, the triangular grid, or a hyperbolic lattice. Those examples are absolutely not random. To make them more amusing, one could "perturb" them using a (supercritical) percolation or by considering the Poisson-Voronoi tessellation in the Euclidean or hyperbolic plane (and their numerous variants). However, those random graphs have a large scale structure which stay deterministic and in some sense is close to the *a priori* continuous geometry we started with to construct those graphs (i.e. Euclidean or hyperbolic geometry). If one wants to see what a *really random* planar geometry is, one should build is without using any reference to a pre-existing geometry.

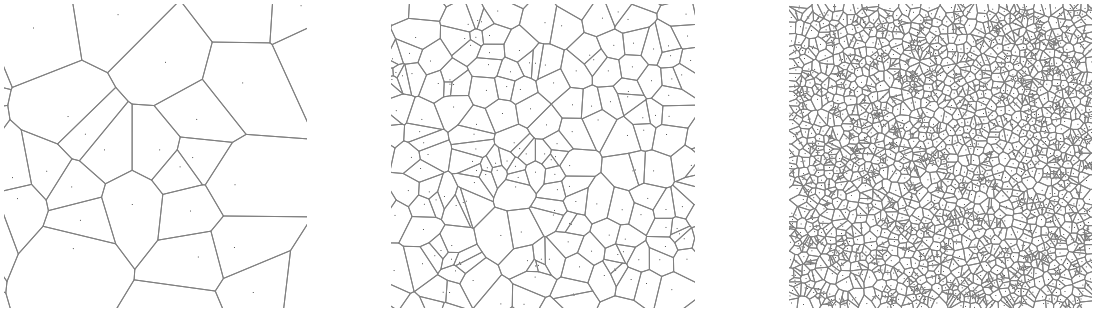


Figure 1: The geometry of random Voronoï diagram in $\mathbb{R}^2$ becomes *deterministic* at large scale and reveals the *a priori* Euclidean plane it is constructed from.

The approach we will take in these lecture notes is to consider *random maps*, which are discrete surfaces obtained by gluing of polygons. This can be seen as a discretization procedure to construct natural probability measures on the space of all possible geometries of a given topological type[1]. Our probability measures on the space of maps will be of product form, i.e. similar to random walk measures on discrete paths or to Bienaymé–Galton–Watson measures on the set of discrete trees. The prototype example of such a measure is the uniform measure on all triangulations (i.e. gluing of n triangles which form a two-dimensional manifold) of the sphere: This space is finite and one can wonder about the typical geometric properties (genus, number of vertices, diameter,...) of a triangulation T_n with n triangles sampled uniformly at random.

[1]This line of research is motivated in part by the theory of quantum gravity where, in an effort to unify quantum mechanics and gravitation using Feynman's path-integral approach, physicists consider space-time itself as a random geometric object (at least at small scales). We refer the curious reader to the book [10] for background on this point of view which we will unfortunately not develop in these lecture notes. Of course, the interest lies in the physical dimension 3 (or even $3+1$) but so far very little rigorous progress have been made there whereas the specific case of dimension 2 has witnessed much success over the last decades (mostly due to the conformal invariance paradigm).

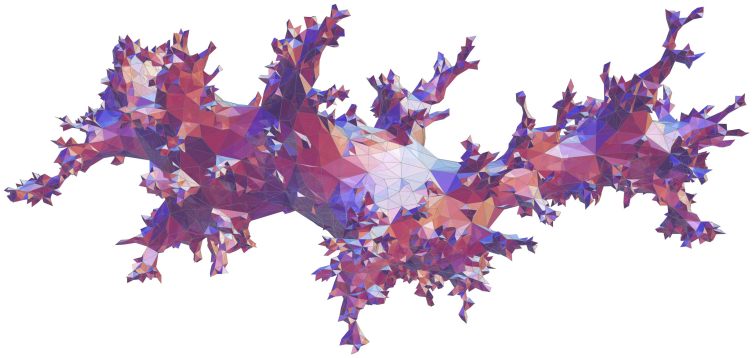


Figure 2: A random uniform triangulation with 10000 triangles embedded in $\mathbb{R}^3$ using the GraphPlot function of mathematica.

Spatial Markov Property. The underlying idea we use to study the geometry of random surfaces is to manage to encode the two-dimensional randomness into one-dimensional random processes which are much easier to control. This idea has successfully been applied in the theory of random trees using their encodings so-called depth first, breath first, Łukasiewicz paths and variations thereof, see e.g. [138], and is also at work in two-dimensional Liouville Quantum Gravity, where quantum surfaces are described using their exploration by variants of SLE processes [106]. The implementation of this idea needs two "ingredients" :

an *exploration method* and a *spatial Markov property.*

In the context of random planar maps, the spatial Markov property says that after a region of the map has been explored, the law of the remaining part only depends on the perimeter of the discovered region (at a high level, this comes from the product form of our measures). It is the two-dimensional analog of the standard and useful Markov property of random processes. This circle of ideas was first conceived in the physics literature (without a precise justification): Watabiki [190] introduced the so-called "peeling process", which is a growth process discovering the random lattice step by step and used it to derive the so-called "two-point" function of 2D quantum gravity. A rigorous version of the peeling process and its Markovian properties was later given by Angel [13] and recently generalized by [53]. The flexibility of the peeling process is that *different exploration methods* can yield the understanding of *different type of geometric properties.*

These lecture notes (do not) contain

The goal of these lecture notes is to present in a unified fashion the recent applications of the peeling process for random planar maps that have been proved over the last decade. In order to avoid technicalities, we stick to the simpler case of

bipartite planar maps

although there is little doubt that the machinery can be extended to the general case and in particular to the case of random triangulations (and for the Uniform Infinite Planar Triangulations of Angel & Schramm). In particular, **we shall not discuss** in details the scaling limit theory of random planar maps and the construction of the **Brownian Sphere** by Le Gall [140] and Miermont [159]. These breakthroughs were made possible thanks to bijective constructions of maps from labeled trees (e.g. Schaeffer construction, or Bouttier–Di Francesco–Guitter construction) and their continuous counterparts based on the Brownian snake. Similarly, the reader may find a bitter disappointment that **we shall not touch upon** the theory of **Liouville Quantum gravity**: thanks to several breakthrough works of Miller and Sheffield, a deep relation is now understood between random planar maps and Liouville Quantum Gravity (based on the $2d$ Gaussian Free Field), and this link is key in many recent activities around scaling limits of random planar maps by Gwynne, Holden, Miller, Sun... We refer the interested reader to the Saint-Flour lectures [158, 160] or to [142, 116] for details.

Organization of the book:

- **Background.** After a detour through random maps with unrestricted genus (Chapter I), we remind the combinatorial background on planar maps in Chapter II. Chapter III is the analytic core of the book, it contains all the miraculously explicit formulas for counting (bipartite) planar maps we shall use. These are expressed in terms of weight sequences and Boltzmann measures we will use to define random maps.

- **Peeling process.** The second part of the book carefully defines what a peeling exploration is and derives the law of these processes (Chapter IV) under the Boltzmann measures. These laws are tightly connected to that of a random walk which we use to classify the weight sequences in Chapter V and derive scaling limits of the peeling exploration in Chapter X.

- **Infinite random maps.** Those explorations are used in the third part of the book to construct infinite random maps (the half-plane and plane versions in Chapters VI and VII and their hyperbolic analogs in Chapter VIII) of which the famous Uniform Infinite Planar Triangulation and Quadrangulation are examples. Maps with simple boundary are treated in Chapter IX and are only used later to study Bernoulli site percolation in Chapter XI. This chapter can be skipped during the first reading.

- **Applications.** We then use peeling explorations to study properties of our random maps:

  - *Percolations.* In Chapter XI we study the (Bernoulli) percolations on (half-plane) Boltzmann maps and by designing appropriate peeling algorithms we compute explicitly the percolation thresholds in full generality for bond, site or face percolation. Chapter XII contains more details on the study of bond percolation on infinite plane maps and their hyperbolic analogs.

– *Volume growth.* Chapter XIII studies the volume growth of our random maps with respect to several distances: primal distance, dual distance and First-passage percolation distance. It heavily uses (conditioned versions of) the stable Lévy process popping-up in the scaling limit of the perimeter and volume in a peeling exploration (see Chapter X). Those results are put in perspective of the "scaling limit theory" of random maps which we briefly sketch in Chapter XIV.

– The remaining two chapters XV and XVI concentrate on the behavior of the *simple random walk* on our infinite maps. After recalling the beautiful application of the Circle Packing theory to deduce recurrence of certain limit of finite random maps, we establish a clear distinction between the Boltzmann infinite maps and their hyperbolic analogs by studying the Liouville property. Finally, Chapter XVI establishes an anomalous subdiffusive behavior of the random walk on infinite planar Boltzmann maps.

Much of the content of this book is due to other authors: credits and pointers are given in the bibliographical notes included as the end of each chapter. The lecture notes also contain several open problems (to the author's knowledge in 2019 with a few updates in 2022) scattered along the pages. We end this introduction by opening a mathematical umbrella:

In these pages, by **graph**, we mean a locally finite multi-graph, i.e. where loops and multiple edges are allowed but where the vertex degrees are all finite. Unless explicitly mentioned, our graphs are connected.

Acknowledgments. The cover picture is due to Tembelone ©. These lecture notes were designed for the 49th Saint-Flour summer school given in 2019, although a first version of these were written for the "Cours Peccot" which I gave at College de France in May 2016. But in the meantime, Covid made its apparition and delayed the publication... Many thanks go to the patient readers, Thomas Budzinski, Jean-François Le Gall, Johannes Wiesel, who found typos/inaccuracies in previous versions of these notes. A special thank for Sébastien Martineau and Rémi Peyre for uncountably many judicious remarks and LaTeX lessons, and to Emmanuel Kammerer and Tanguy Lions for catching important mathematical inaccuracies during a careful reading. I am indebted to my co-authors without whom those notes would not exist.

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Part I:
(Planar) Maps

In this part, we introduce the concept of maps which will be our discrete surfaces. After a quick glance at the behavior of random maps with no constraint on the genus, we dive into the theory of *planar* maps. In particular, we present the "miraculous" enumeration of bipartite planar maps obtained by Tutte and its consequences for the model of random Boltzmann planar map.

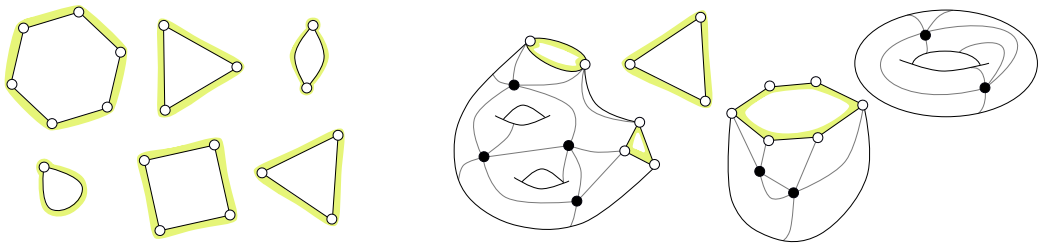


Figure 3: Gluing polygons to form a discrete surface.

Chapter I : Discrete random surfaces in high genus

In this chapter, we introduce the concept of *map*, which will be our definition of discrete surfaces in these lecture notes, and describe a few basic properties. We will be rather imprecise and refer the reader to standard textbooks for a more rigorous treatment. We then investigate the behavior of random maps (with no restriction on the genus) using an exploration process and will see that those random surfaces typically have a very high genus (almost the highest possible) and that their degree distribution follows the Poisson–Dirichlet paradigm. This chapter can be taken as an appetizer to the exploration of random *planar* maps, which is the core of these lecture notes.

1.1 What is a map? – different points of view

1.1.1 Gluing of polygons and a first exploration

The continuous surfaces (2-dimensional oriented connected compact manifolds without boundaries) are classified up to homeomorphism by a single integer parameter $g \in \{0, 1, 2, ...\}$, the genus, counting the number of handles of the space. Heuristically speaking, *maps* are discretizations of those surfaces: Imagine that you are given a finite set of polygons whose sides are numbered from 1 to $2n$, where $n \geq 1$. Given a matching (an involution without fixed points) of $\{1, 2, \dots, 2n\}$ we can create a surface by gluing the sides of the polygons according to the matching. This results in a 2-manifold[1] (which may be disconnected) on which the images of the glued edges form an embedded graph. When the gluing remains connected, we get a map:

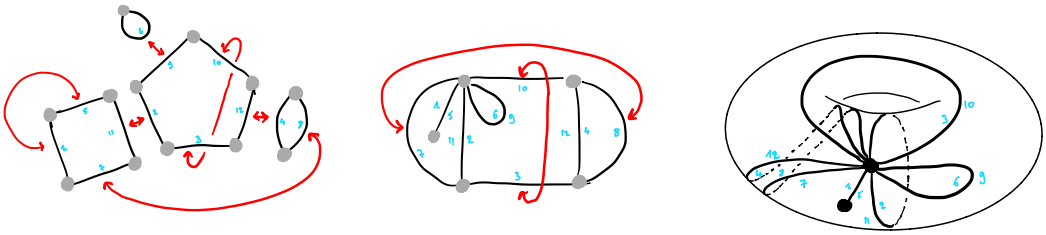


Figure 1.1: Example of a gluing of polygons by matching their edges pairwise. The labeling of the edges of the polygons is in blue.

[1] This seemingly trivial statement becomes wrong in higher dimension: for example in dimension 3 there are gluings of tetrahedra which do not form a 3-manifold, see [73] for a recent work on this subject.

Definition 1.1 (Maps as gluing of polygons). *A labeled map with n edges is a connected gluing of labeled polygons whose total perimeter is $2n$ obtained by pairing their edges two by two.*

In the gluing operation one always identifies two sides of two (possibly equal) polygons to form an **edge** of the map. A convenient way to represent this is to imagine that the sides of the polygons are all oriented clockwise around each polygon, and that during the gluing operation one always identifies two sides with opposite orientation[2]: we will say that the resulting edge of the map is divided into **two half-edges or oriented edges** and these are labeled by $\{1, 2, \dots, 2n\}$. It is easy to see then that there is a finite number of labeled maps with n edges.

Instead of considering labeled maps we will usually consider **rooted maps** which are obtained by declaring that two labeled maps equivalent if they can be obtained from one another by relabeling the half-edges $2, 3, \dots, 2n$ (the label 1 must coincide). The root edge of a map $\mathfrak{m}$ is the distinguished oriented edge labeled 1. It is easy to see by connectedness that a rooted map corresponds exactly to an equivalence class of $(2n-1)!$ labeled maps. Hence, the combinatorial nature of labeled or rooted maps is exactly the same and we shall usually drop the adjective *labeled* or *rooted* in what follows and simply speak of maps.

If $\mathfrak{m}$ is map, we denote respectively

$$\mathsf{Edges}(\mathfrak{m}), \mathsf{Vertices}(\mathfrak{m}) \text{ and } \mathsf{Faces}(\mathfrak{m})$$

the set of its edges, vertices (the endpoints of edges) and faces (the polygons we started with). The precise definition of these objects does not really matter, what will be important are the incidence relations between those elements. The **degree** $\deg(f)$ of a face f (we sometimes also say the **perimeter** of f) is the number of half-edges incident to this face, that is the degree of its underlying polygon (in particular if an edge lies completely "inside" a face then it is counted twice in the degree). Similarly the degree $\deg(x)$ of a vertex x is the number of half-edges incident to x (in particular loops attached to x are counted twice). The incidence relations between vertices and edges naturally yield a finite graph which we denote by $\mathsf{Graph}(\mathfrak{m})$ whose vertices are $\mathsf{Vertices}(\mathfrak{m})$ and whose edges are $\mathsf{Edges}(\mathfrak{m})$. It is a *multi-graph* since multiple edges and loops are allowed. Beware, the graph structure of $\mathfrak{m}$ carries a weaker structure than that of a map, see Figure 2.2 and Section 1.1.2

Genus

It is clear that if we perform the topological gluing which a map $\mathfrak{m}$ encodes then we end up with an orientable 2-manifold whose genus is well-defined and which we denote by

$$\mathsf{Genus}(\mathfrak{m}) \in \{0, 1, 2, 3, \dots\}.$$

[2]If we allow identification of edges with the same orientation, we fall into the realm of *non-orientable* surfaces, which escapes from the scope of these lectures. Another heuristic way to formulate this is to imagine that the polygons have two sides, a black side and a white side, and we only glue sides with the same colors when identifying two edges.

Perhaps the most well-known theorem about maps is Euler's relation which relates the numbers of edges, faces and vertices to the genus of the map.

Theorem 1.2 (*Euler*)

For any map $\mathfrak{m}$*, we have*

$$\#\mathsf{Vertices}(\mathfrak{m}) + \#\mathsf{Faces}(\mathfrak{m}) - \#\mathsf{Edges}(\mathfrak{m}) \quad = \quad 2\,(1 - \mathsf{Genus}(\mathfrak{m}))\,. \tag{1.1}$$



In particular, a map with n edges cannot have a genus larger than $n/2$. We will prove Euler's formula through an exploration process of the map which gives the flavor of what we will be doing in these lecture notes. We fix a configuration $\mathcal{P}$ of (oriented) polygons with total perimeter $2n$ whose sides have been labeled from 1 to $2n$ and let ω be a pairing of its edges. We denote by $\mathbf{M}_{\mathcal{P},\omega}$ the resulting discrete surface obtained after the gluing which may be disconnected. We will construct step by step the discrete surface $\mathbf{M}_{\mathcal{P},\omega}$ by matching the edges 2 by 2. More precisely, we will create a sequence

$$S_0 \to S_1 \to \cdots \to S_n = \mathbf{M}_{\mathcal{P},\omega}$$

of "combinatorial surfaces" where S_0 is made of the set of labeled polygons whose perimeters are specified by $\mathcal{P}$ and where we move on from S_i to S_{i+1} by identifying two edges of the pairing ω. More specifically, S_i will be a union of labeled maps with distinguished faces called the holes (they are in yellow in the figures below). The holes are made of the edges which are not yet paired, they form simple faces which cannot share any vertex, we call them the **active boundary** of the surface and denote it by $\partial^* S_i$. We write $|\partial^* S_i|$ for half of the total perimeter of the active boundary so that we have

$$|\partial^* S_i| = n - i.$$

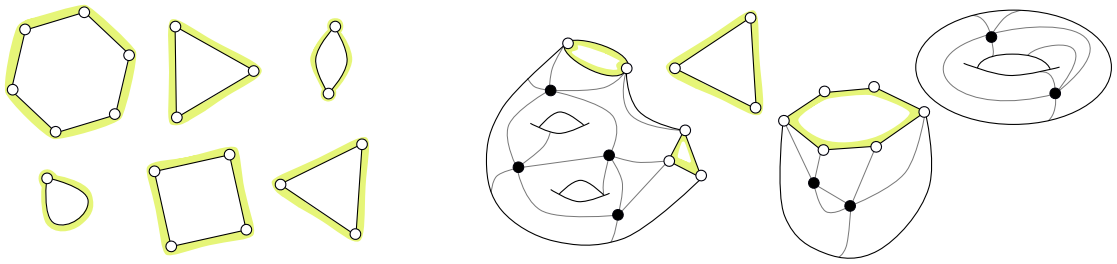


Figure 1.2: Starting configuration (on the left) and a typical state of the exploration (on the right). Here and later the labeling of the oriented edges does not appear for the sake of visibility. The final vertices of the graph are black dots whereas "temporary" vertices are in white. Notice on the right-hand side that S_i contains a closed surface without boundary: if this happens the final surface $S_n = \mathbf{M}_{\mathcal{P}}$ is disconnected.

To go from S_i to S_{i+1} we select an edge on $\partial^* S_i$ which we call *the edge to peel* (in red in the figures below) and identify it with its partner edge in ω (in green in the figures below) also belonging to $\partial^* S_i$. We now describe the possible outcomes of the peeling of one edge. The reader should keep in mind that our surfaces are always labeled and oriented and that when identifying two edges we glue them in a way compatible with the orientation.

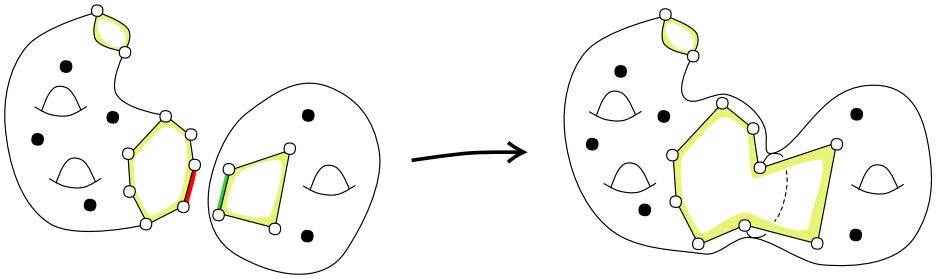


Figure 1.3: If we identify two edges of different components, the holes and the components merge (their genus and final vertices add). In particular, the perimeter of the resulting hole is the sum of the perimeters of the former holes minus 2.

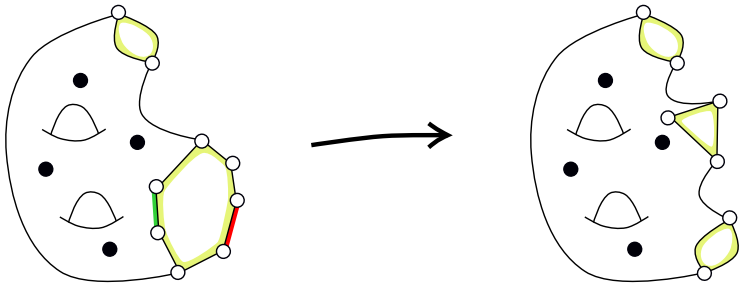


Figure 1.4: If we identify two edges on the same hole then the hole splits into two holes of perimeters p_1' and p_1'' so that $p_1' + p_1'' + 2$ is the perimeter of the initial hole.

The reader may wonder when, during this process, the vertices of $\mathbf{M}_{\mathcal{P},\omega}$ are created. Actually, vertices should be seen above as holes of perimeters 0, they can thus appear as a specialization of the above scenarii as shown in Figure 1.6.

In each of the above scenarii, it is easy to check that the variation of the quantity

$$\chi(S_i) = \sum_{C_j \subset S_i} \#\mathsf{Vertices}(C_j) + \#\mathsf{Faces}(C_j) - \#\mathsf{Edges}(C_j) - 2 + 2\mathsf{Genus}(C_j)$$

is zero, where the sum runs over the connected components C_j of S_i (of course, we count the holes of each components as faces). Notice in particular that $\sum_j \#\mathsf{Edges}(C_j) = 2n - i$. We have thus

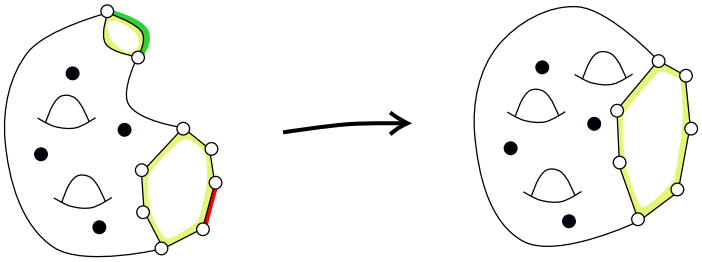


Figure 1.5: If we identify two edges of different holes belonging to the same component, then the holes merge, adding one unit to the genus of the map. As in Figure 1.3, the perimeter of the new hole is the sum of the perimeters of the initial holes minus 2.

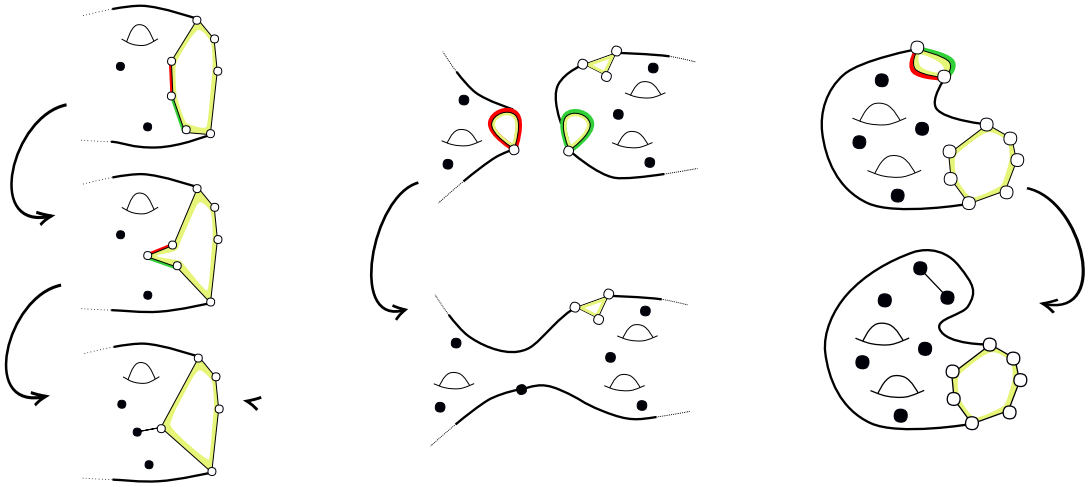


Figure 1.6: Left: As a special case of Figure 1.4, if we identify two neighboring edges on the same hole then one of the two holes created has perimeter 0: we have formed a true vertex. The two other ways to create vertices: If we identify (middle) two loops (holes of perimeter 1) then we create one vertex and if we identify (right) the two edges of a hole of length 2 then we create two vertices.

proved a more general statement than Euler's formula namely the fact that $\chi(\mathbf{M}_{\mathcal{P},\omega}) = \chi(S_n) = \chi(S_0) = 0$: Euler's formula follows in the case when S_n is connected.

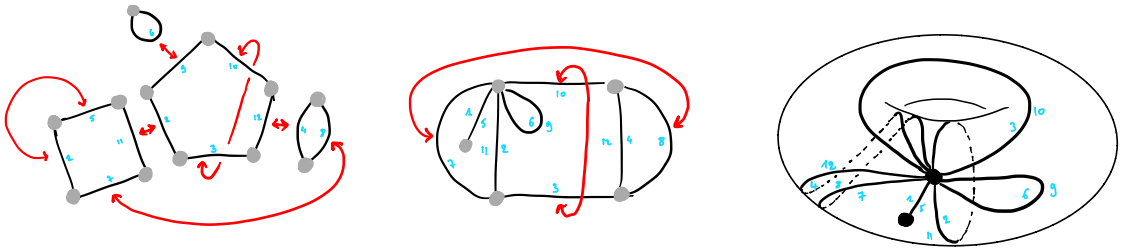
1.1.2 Other definitions of maps

Permutation and duality

Let $\mathfrak{m}$ be a labeled map with n edges. One can associate with $\mathfrak{m}$ two permutations of $\mathfrak{S}_{2n}$ over $\{1, 2, \ldots, 2n\}$: one **involution** α **without fixed point** encoding the pairing of the half-edges

labeled by $\{1, 2, \ldots, 2n\}$ and one permutation ϕ whose cycles are the indices of the half-edges we encounter in clockwise order around the polygons (the faces of the map). For example the permutations α and ϕ associated with the labeled map of Figure 1.1 are

$$\alpha = (1,5)(7,8)(2,11)(6,9)(3,10)(4,12) \quad \text{and} \quad \phi = (1,5,11,7)(6)(2,9,10,12,3)(4,8).$$



The connectedness of the map is then equivalent to the fact that the subgroup of $\mathfrak{S}_{2n}$ generated by α and ϕ acts transitively over $\{1, 2, \ldots, 2n\}$. We leave the reader check that $\mathfrak{m} \to (\alpha, \phi)$ is an encoding of the map :

Definition 1.3 (Maps as permutations)**.** *A labeled map with n edges can equivalently be seen as a pair of permutations $(\alpha, \phi) \in \mathfrak{S}_{2n} \times \mathfrak{S}_{2n}$ such that α is an involution without fixed points and $\langle \alpha, \phi \rangle$ acts transitively over $\{1, 2, \ldots, 2n\}$.*

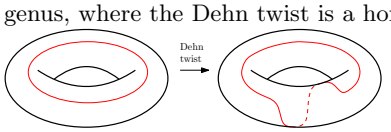
Embedded graphs

Let us give a more "topological" definition of maps which is perhaps the most natural but also the least easy to manipulate. Consider again the topological gluing of polygons encoded by a map $\mathfrak{m}$, then the trace of the edges on the resulting surface is an embedding of the graph $\mathrm{Graph}(\mathfrak{m})$. It is furthermore a **proper embedding** in the sense that the edges do not cross (except at their endpoints) and that the faces (the connected components of the complement of the edges) are all homeomorphic to disks.

Definition 1.4 (Maps as embedded graphs)**.** *A rooted map of genus $g \geq 0$ is an equivalence class of proper embeddings of graphs in torus of genus g given with a distinguished oriented edge (the root edge), and where two embeddings are identified if they differ by a homeomorphism of the surface preserving the orientation and the root edge.*

In the case of genus 0 we speak of **planar maps**, and the previous definition is the one that is usually presented first[3]. The important combinatorial property of embedded graphs is

[3]In the case of the sphere $g = 0$, the homeomorphisms preserving the orientation are all homotopic to the identity and we usually say "... identified if they differ by a continuous deformation of the surface" rather than by a homeomorphism of the surface preserving the orientation and the root edge. This is not true anymore in higher genus, where the Dehn twist is a homeomorphism not homotopic to the identity:



that on top of their graph structure they inherit a cyclic orientation of each half-edges around each vertex and this structure is preserved by homeomorphisms that preserve the orientation. Another way to express this is that a rooted map is a connected graph together with a system of cyclic orientations of the edges around each vertex, and with a distinguished oriented edge.

1.1.3 Duality

The **dual** of a (rooted or labeled) map $\mathfrak{m}$ is the map $\mathfrak{m}^\dagger$ obtained by exchanging the roles of faces and vertices but keeping the same incidence relations. Graphically, $\mathfrak{m}^\dagger$ can be obtained by placing a vertex in each face of $\mathfrak{m}$ and linking two adjacent faces by an edge.

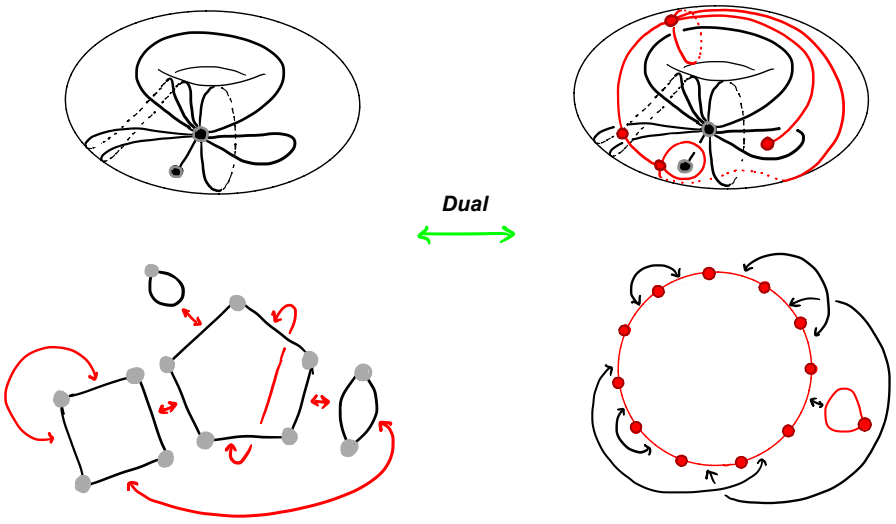


Figure 1.7: The dual of the map presented in Figure 1.1. On the right, the convention for transferring the root edge from the primal to the dual map.

If the map is (labeled or) rooted, the root edge (or the labelings of the half-edges) is transferred canonically to the dual map using the following rule:

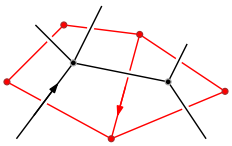


Figure 1.8: The convention for transferring the root edge from the primal to the dual map. Starting from the root edge in the primal map (in black), go around one step around the face adjacent on the right and then take the dual edge (in red) oriented towards the center of that face.

In the interpretation of a map as a pair of permutations (α, ϕ) taking the dual is an easy operation: consider $\sigma = \alpha\phi$ whose cycles represent the half-edges ending to each vertex in

counterclockwise order, then the map associated to (α, σ) is $\mathfrak{m}^\dagger$. In particular since $\alpha^2 = Id$, we also have $\phi = \alpha\sigma$ and the roles of ϕ and σ are indeed symmetric in the definition of the map. It is easy to see topologically (or from Euler's formula) that the dual map $\mathfrak{m}^\dagger$ has the same genus as $\mathfrak{m}$.

1.2 Geometry and topology of uniform maps

In this section we enumerate maps with n edges via two different techniques: an analytic approach using generating functions and a more probabilistic approach which enables us to understand very well the geometry of $\mathbb{M}_n$ a uniform random rooted map with n edges as $n \to \infty$.

1.2.1 Enumeration "à la Tutte"

Let m_n be the number of rooted maps with n edges and $\mathbf{M}(z) = \sum_{n\geq 0} m_n z^n$ for its (formal) generating function. It will be convenient in what follows to consider that there is a unique "**vertex map**" with a single vertex, hence $m_0 = 1$ (with no edge and 1 face).

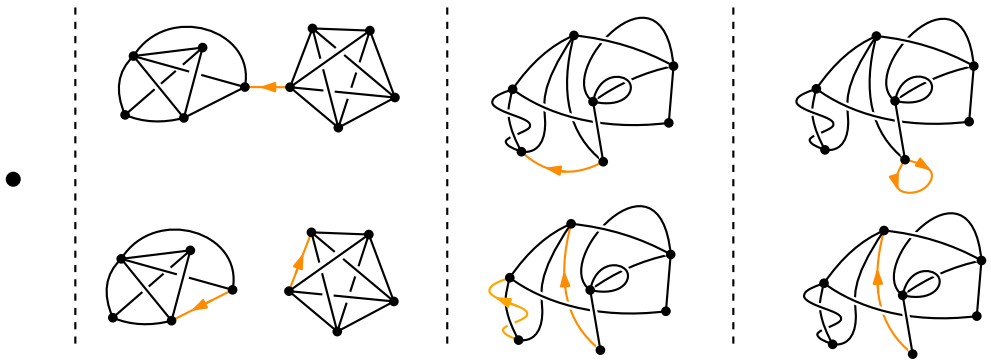


Figure 1.9: Decomposition of rooted maps according to the removal of the root edge. From left to right: the case where the map is the vertex map and the case where the removal of the root edge disconnects the map. When the removal of the root edge does not disconnect the map, the root edge can be recover by its two corners, unless those tow corners are the same in which case there are two possible orientations.

One natural idea is to see them as (connected) graphs with cyclic orientation and to look for a recursion, and the most natural thing to do is to **remove the root edge**. See Figure 1.9. If $n \geq 1$, then the removal of the root edge can yield two different situations: either it disconnects the maps into two parts with n_1 and n_2 edges with $n_1 + n_2 = n - 1$ or it stays connected. On the event where the removal of the root edge disconnects the map, the two remaining maps can be canonically rooted respectively at the first edge outgoing of the target of the root edge in cyclic order and the first edge outgoing of the origin of the root edge in cyclic order. If it does not disconnect the map then the resulting map with $n-1$ edges is canonically endowed with *two* distinguished oriented edges $\vec{e}_1$ and $\vec{e}_2$. In the particular case where the root edge is a loop

whose extremities lie in the same corner of the map minus the loop, then $\vec{e}_1 = \vec{e}_2$ but there are two possible orientations for the root loop. Since any rooted map with $n-1$ edges has exactly $2(n-1)$ oriented edges we get from the above bijective description that for $n \geq 1$

$$m_n = \sum_{n_1+n_2=n-1} m_{n_1} m_{n_2} + (2n-1) m_{n-1}.$$

Equivalently, $m_0 = 1$ and the above display is equivalent to the following equation for the formal power series $\mathbf{M}(z)$:

$$\mathbf{M}(z) = 1 + z\big(\mathbf{M}(z)\big)^2 + 2z^2 \frac{\partial}{\partial z}\mathbf{M}(z) + z\mathbf{M}(z).$$

Actually, the series $\mathbf{M}(z)$ has a radius of convergence equal to zero and so the previous line should be seen as an equation for formal power series. It is an exercise to see from these relations that $m_n \sim 2n(2n-1)!! = 2n \times (2n-1) \times (2n-3) \times \cdots \times 3 \times 1$ and we can also compute the first few coefficients $\mathbf{M}(z) = 1 + 2z + 10z^2 + 74z^3 + 706z^4 + \ldots$ which follow A000698 in Sloane online encyclopedia for integer sequences.

1.2.2 Uniform maps are almost uniform permutations

Let us now present another approach to the (asymptotic) enumeration of maps based on the permutation representation. We denote by $\mathcal{I}_{2n} \subset \mathfrak{S}_{2n}$ the subset of involutions without fixed points (i.e. product of n non overlapping transpositions). Clearly, we have $\#\mathcal{I}_{2n} = (2n-1)!!$. As remarked above, a pair $(\alpha, \phi) \in \mathcal{I}_{2n} \times \mathfrak{S}_{2n}$ does not necessarily encode a labeled map: for $n = 2$, the "map" associated with the permutation $\alpha = (12)(34)$ and $\phi = (1)(2)(3)(4)$ consists of two disjoint loops, but we will see that this situation is marginal.

Theorem 1.5 (*Uniform maps are almost uniform permutations*)

*Let $\mathcal{C}_{2n} = \{(\alpha, \phi) \in \mathcal{I}_{2n} \times \mathfrak{S}_{2n} : (\alpha, \phi)$ encodes a **connected** labeled map$\}$, so that $\mathcal{C}_{2n}$ is in bijection with labeled maps with n edges and $\#\{$rooted maps with n edges$\} = \frac{1}{(2n-1)!}\#\mathcal{C}_{2n}$. Then we have the asymptotic expansion*

$$\frac{\#\mathcal{C}_{2n}}{(2n)!(2n-1)!!} = 1 - \frac{1}{2n} + O(1/n^2).$$

In particular if $(A_n, F_n) \in \mathcal{I}_{2n} \times \mathfrak{S}_{2n}$ is the pair of permutations associated with a uniform labeled map with n edges and if (α_n, ϕ_n) is uniformly distributed over $\mathcal{I}_{2n} \times \mathfrak{S}_{2n}$ then we have

$$\mathrm{d_{TV}}\big((A_n, S_n); (\alpha_n, \phi_n)\big) \sim \frac{1}{2n} \text{ as } n \to \infty,$$

where $\mathrm{d_{TV}}$ is the total variation distance.

Proof. Let $(\alpha_n, \phi_n) \in \mathcal{I}_{2n} \times \mathfrak{S}_{2n}$ be uniformly distributed. If (α_n, ϕ_n) does not yield a connected map, this means that the subgroup generated by α_n and ϕ_n does not act transitively on $\{1, 2, \ldots, 2n\}$ or equivalently that $\{1, 2, \ldots, 2n\}$ can be partitioned into two non-empty subsets I and J, such that both I and J are stable by α_n and ϕ_n. By partitioning according to the smallest

stable subset containing 1 we obtain the following recurrence relation

$$\#\mathcal{C}_{2n} = (2n-1)!!(2n)! - \sum_{\ell=0}^{n-2} \binom{2n-1}{2\ell+1} \#\mathcal{C}_{2(\ell+1)}\,(2n-2\ell-3)!!(2n-2\ell-2)!.$$

Writing $c_n = \#\mathcal{C}_{2n}/(2n-1)!$ for the number of rooted maps with n edges and $\xi_n = (2n-1)!!$ the above recurrence is equivalent to $c_n = 2n \cdot \xi_n - \sum_{\ell=1}^{n-1} c_\ell \xi_{n-\ell}$ which yields, after iterating, the closed formula for c_n:

$$c_n = \sum_{i \geq 1} (-1)^{i+1} \sum_{\substack{k_1+k_2+\cdots+k_i=n \\ k_i \geq 1}} 2k_1 \cdot \xi_{k_1} \xi_{k_2} \ldots \xi_{k_i}.$$

The terms $2n\xi_n$ and $-2(n-1)\xi_{n-1}\xi_1$ give $(2n-1)!!(2n-1+O(1/n))$ while the total sum of the absolute values of the other terms is easily seen to be of order $O(1/n) \times (2n-1)!!$. This proves the first claim of the theorem. The second one is a trivial consequence since (A_n, S_n) is uniformly distributed over $\mathcal{C}_{2n}$. □

Geometric and topological properties of a uniform map. Let $\mathbb{M}_n$ be a random labeled map uniformly distributed over all labeled maps with n edges. Using the above theorem we deduce that the statistical properties of the faces (number, degrees, etc) of $\mathbb{M}_n$ is approximately the same as the statistical properties of the cycles of a random uniform permutation $\phi_n \in \mathfrak{S}_{2n}$. The latter is very well-know and here are a few direct consequences (see the following exercise):

- the number of faces of $\mathbb{M}_n$ is approximately $\log n$ and obeys a central limit theorem,
- the number of faces of degree $1, 2, 3, \ldots$ in $\mathbb{M}_n$ converge jointly towards independent Poisson random variables of mean $1, 1/2, 1/3, \ldots$,
- the large face degrees, once rescaled by $1/(2n)$, converge towards the standard Poisson–Dirichlet partition,
- the degree of the root face (i.e. of a size-biased pick) converges, once rescaled by $1/(2n)$, towards a uniform variable on $[0, 1]$.

Since $\mathbb{M}_n$ is self-dual, the same results hold for $\mathbb{M}_n^\dagger$ the dual map of $\mathbb{M}_n$. Actually we even have

$$\left(\frac{\#\mathcal{V}(\mathbb{M}_n) - \log n}{\sqrt{\log n}}, \frac{\#\mathcal{F}(\mathbb{M}_n) - \log n}{\sqrt{\log n}}, \frac{\mathcal{G}\backslash(\mathbb{M}_n) - \frac{n}{2} + \log n}{\sqrt{\log n}} \right) \xrightarrow[n\to\infty]{(d)} \left(\mathcal{N}_1, \mathcal{N}_2, -\frac{\mathcal{N}_1 + \mathcal{N}_2}{2} \right), \tag{1.2}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent standard Gaussian random variables. The convergences of the first and second components alone follow from the discussion above. The perhaps surprising phenomenon is that the number of vertices and faces of $\mathbb{M}_n$ are asymptotically independent: we shall not prove it here and refer to [61] for details. The previous convergence is automatic given the first two and Euler's formula (1.2).

Exercise 1.6. Let ϕ_n be a uniform permutation of $\mathfrak{S}_n$. Let $X_i \in \{0,1\}$ be independent Bernoulli random variables such that $\mathbb{P}(X_i = 1) = \frac{1}{n-i+1}$. We write $0 = \theta_0^{(n)} \leq \theta_1^{(n)} \leq \theta_2^{(n)}, \dots$ the indices such that $X_i = 1$ (in particular $X_n = 1$).

1. By considering the length of the cycle containing 1 in ϕ_n show that
$$(\theta_i^{(n)} - \theta_{i-1}^{(n)})_{i\geq 0}$$
has the same distribution as the cycle lengths of ϕ_n.

2. Deduce that $\#\mathrm{Cycles}(\phi_n) = \sum_{i=1}^n X_i$ and that the latter is close to $\mathrm{Poisson}(\log n)$ in total variation distance. In particular prove the central limit theorem:
$$\frac{\#\mathrm{Cycles}(\phi_n) - \log n}{\sqrt{\log n}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1).$$

3. If $C_1^{(n)}, C_2^{(n)}, \dots$ are the decreasing lengths of the cycles of ϕ_n prove that
$$\left(\frac{C_i(n)}{n}\right)_{i\geq 1} \xrightarrow[n\to\infty]{(d)} \mathbf{PD}(1),$$
where the (standard) **Poisson–Dirichlet distribution** $\mathrm{PD}(1)$ is the probability measure on partitions of 1, i.e. on sequences $x_1 > x_2 > x_3 > \cdots$ such that $\sum x_i = 1$, which is obtained by reordering in decreasing order the lengths $U_1, U_2(1-U_1), U_3(1-U_1)(1-U_2), \dots$ where $(U_i : i \geq 1)$ is a sequence of i.i.d. uniform variables on $[0,1]$.

The above description of the topology of a uniform random map over n edges may convince the reader that the geometry of the later is not so rich: the graph of $\mathbb{M}_n$ is collapsed on $\approx \log n$ vertices and $\mathbb{M}_n$ almost has the maximal possible genus. Actually we even have:

Theorem 1.7 (*The diameter of a random map is* 2 *or* 3)

There exists a constant $\xi \in (0,1)$ such that
$$\lim_{n\to\infty} \mathbb{P}(\mathrm{Diameter}(\mathbb{M}_n) = 3) = 1 - \lim_{n\to\infty} \mathbb{P}(\mathrm{Diameter}(\mathbb{M}_n) = 2) = \xi.$$

We will not give the proof of this result here but only indicate that the diameter of $\mathbb{M}_n$ is ruled by *two* phenomena: the large degree vertices *and* the repartition of the small degree vertices. Anyhow, such a theorem dampens hopes of having non trivial scaling limit of the graph structure of $\mathbb{M}_n$.

1.3 Exploring random maps with prescribed faces and a conjecture

In view of the previous section, where the geometry of uniform random maps has been fairly well understood, one can ask whether there are some models of random maps (with genus unfixed) whose geometric behavior is very different. We will see that this is not the case for a large class of maps.

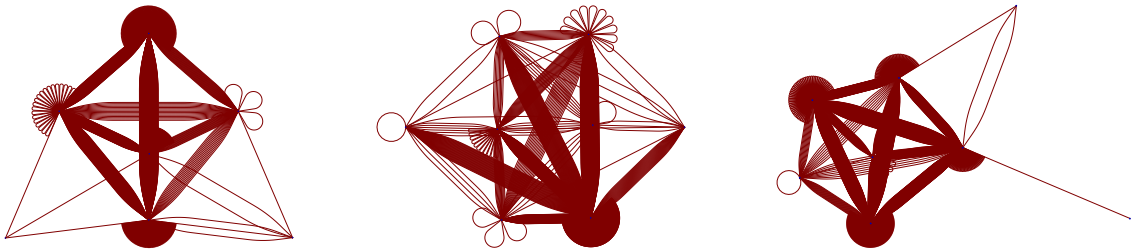


Figure 1.10: Three samples of the graph structure of a uniform random map (genus unfixed) with 2000 edges. We see that the graph is highly connected with few vertices carrying many multiple edges and loops.

1.3.1 Random gluing of prescribed polygons

Instead of choosing a map uniformly at random among all (labeled or rooted) maps with n edges one may want to prescribe the sequence $\mathcal{P} = \{p_1, p_2, \dots, p_k\}$ of perimeters of its faces and consider the random labeled map $\mathbf{M}_{\mathcal{P}}$ obtained by uniformly labeling polygons whose perimeters are given by $\mathcal{P}$ and gluing their sides pairwise in a uniform manner, see Figure 1.1. More precisely this means that the pairing ω is chosen uniformly at random among all involutions of $\{1, 2, \dots, 2n\}$ without fixed points.

Remark 1.8. By Theorem 1.5 it is easy to see that up to an error in total variation distance of $O(1/n)$, a uniform labeled map $\mathbb{M}_n$ can be seen as a uniform gluing of polygons, where the perimeters of the polygons are themselves random (independent of the gluing) and are distributed according to the length of cycles in a uniform permutation of $\mathfrak{S}_{2n}$.

Obviously, the gluing $\mathbf{M}_{\mathcal{P}}$ may not be connected, but if $\mathcal{P}$ does not have too many 1-gons and 2-gons then it is: A sequence $(\mathcal{P}_n)_{n\geq 1}$ of configurations of polygons with total perimeter $2n$ is said to be *good* if

$$\frac{\#1\text{-gons}(\mathcal{P}_n)}{\sqrt{n}} \to 0, \quad \text{and} \quad \frac{\#2\text{-gons}(\mathcal{P}_n)}{n} \to 0, \quad \text{as } n \to \infty. \tag{1.3}$$

We will see later that for good configurations $\mathbf{M}_{\mathcal{P}_n}$ is connected with high probability as $n \to \infty$. One example is obtained by gluing $2n/3$ (provided n is divisible by 3) triangles in a uniform fashion to obtain a surface with n edges in total. In this case, the random map $\mathbf{M}_{\mathcal{P}_n}$ is very different from a uniform map $\mathbb{M}_n$ simply because $\mathbf{M}_{\mathcal{P}_n}$ has $2n/3$ triangular faces whereas $\mathbb{M}_n$ has roughly $\log n$ faces and only a few of them are triangles. However, it turns out that the *graphs* associated with $\mathbf{M}_{\mathcal{P}_n}$ and to $\mathbb{M}_n$ share universal properties:

Theorem 1.9 (*Poisson–Dirichlet universality*)

If $(\mathcal{P}_n)_{n\geq 1}$ is a good sequence of configurations then $\mathbf{M}_{\mathcal{P}_n}$ is connected with high probability as $n \to \infty$ and furthermore if $D_1, \dots, D_{V_n}$ denotes the degrees of the vertices of $\mathsf{Graph}(\mathbf{M}_{\mathcal{P}_n})$ ranked in decreasing order then we have

$$\text{(number of vertices)} \qquad \frac{V_n - \log n}{\sqrt{\log n}} \xrightarrow[n\to\infty]{} \mathcal{N}(0,1),$$

$$\text{(Poisson–Dirichlet paradigm)} \qquad \left(\frac{D_i}{2n}\right)_{i\geq 0} \xrightarrow[n\to\infty]{} \text{Poisson} - \text{Dirichlet partition}.$$

The above theorem can be seen as a strong universality property for the graphs associated to $\mathbf{M}_{\mathcal{P}}$ regardless of the prescribed degrees of the faces as long as they do not contain too many loops or bigons. We even conjecture that the universality goes much further: *the law* of the random graph $\mathbf{G}_{\mathcal{P}}$ associated to $\mathbf{M}_{\mathcal{P}}$ is always the same (in a strong sense) regardless of the configuration of polygons we started with and is close to the law of the random graph obtained from a *uniform random map* with $|\mathcal{P}|$ edges. However, it is easy to see from Euler's formula that the number of vertices of $\mathbf{M}_{\mathcal{P}}$ has the same parity as $|\mathcal{P}| + \#\mathcal{P}$ and so the proper conjecture needs to deal with this parity constraint: Let $\mathbb{G}_n = \mathsf{Graph}(\mathbb{M}_n)$ be the random graph structure of a uniform random labeled map on n edges, and denote by $\mathbb{G}_n^{\text{odd}}$ and $\mathbb{G}_n^{\text{even}}$ the random graph $\mathbb{G}_n$ conditioned respectively on having an odd or even number of vertices.

Open Question 1.10 (Universality for $\mathbf{G}_{\mathcal{P}}$ from [61])**.** *Let $(\mathcal{P}_n)_{n\geq 1}$ be a good sequence of configurations and write $\epsilon_n \in \{\text{even}, \text{odd}\}$ be the parity of $n + \#\mathcal{P}_n$. Then we have*

$$\mathrm{d_{TV}}\left(\mathbf{G}_{\mathcal{P}_n}, \mathbb{G}_n^{\epsilon_n}\right) \to 0, \qquad \text{as } n \to \infty.$$

Appart from Theorem 1.9, a compelling evidence for the above question is the result of Chmutov & Pittel [79], which asserts that when all the polygon's perimeters are larger than 3 then up to an error of $O(1/n)$ in total variation distance, the degree distribution of $\mathbf{G}_{\mathcal{P}_n}$ is the same as that of $\mathbb{G}_n^{\epsilon_n}$. The goal of the rest of this section is to explain the idea of the proof of Theorem 1.9 without going into details. It is based on an exploration procedure of $\mathbf{M}_{\mathcal{P}}$ which we give the flavor of what we will be doing in the rest of these lecture notes for random planar maps.

1.3.2 Peeling explorations of $\mathbf{M}_{\mathcal{P}}$

In this section we explain how to explore in a "dynamical" fashion the random discrete surface $\mathbf{M}_{\mathcal{P}}$. The idea is to use the construction of $\mathbf{M}_{\mathcal{P}}$ step-by-step as in the proof of Theorem 1.2.

Let $\mathcal{P}$ be a configuration of labeled polygons with total perimeter $2n$ and ω be a pairing of $\{1, 2, ..., 2n\}$. Recall from the proof of Theorem 1.2 the construction of $S_0 \to S_1 \to \cdots \to S_n = \mathbf{M}_{\mathcal{P},\omega}$ where we glued edges pairwise to construct iteratively the surface $\mathbf{M}_{\mathcal{P},\omega}$. On top of $\mathcal{P}$ and the gluing ω, the sequence $S_0 \to S_1 \to \cdots \to S_n$ depends on an **algorithm** called the peeling algorithm

which is simply a way to decide in which order we perform the identifications of the half-edges. More precisely, assume that we have a function $\mathcal{A}$ which associate with each discrete surface S_i and edge $\mathcal{A}(S_i) \in \partial^* S_i$ on its active boundary.

Definition 1.11 (Peeling exploration). *The peeling exploration of $\mathbf{M}_{\mathcal{P},\omega}$ with algorithm $\mathcal{A}$ is the sequence*

$$S_0^{\mathcal{A}} \to S_1^{\mathcal{A}} \to \cdots \to S_n^{\mathcal{A}} = \mathbf{M}_{\mathcal{P},\omega}$$

obtained by starting with $S_0^{\mathcal{A}}$, the initial configuration made of the labeled polygons whose perimeters are prescribed by $\mathcal{P}$, and to go from $S_i^{\mathcal{A}}$ to $S_{i+1}^{\mathcal{A}}$ we perform the identification of the edge $\mathcal{A}(S_i)$ together with its partner in the paring ω.

When ω is uniform the sequence $(S_i^{\mathcal{A}} : 0 \le i \le n)$ is a (inhomogeneous) Markov chain:

Proposition 1.12. *If the gluing ω is uniformly distributed over all pairings of $\{1, 2, ..., 2n\}$ (and independent of the labeled polygons $\mathcal{P}$) then the exploration $(S_i^{\mathcal{A}})_{0 \le i \le n}$ is an inhomogeneous Markov chain whose probability transitions are described as follows: Conditionally on $S_i^{\mathcal{A}}$ and on $\mathcal{A}(S_i^{\mathcal{A}})$ we pick E_i uniformly at random among the $2(n-i)-1$ edges of $\partial^* S_i^{\mathcal{A}} \backslash \{\mathcal{A}(S_i^{\mathcal{A}})\}$ and identify $\mathcal{A}(S_i^{\mathcal{A}})$ with E_i.*

Proof. It suffices to notice by induction that at each step $i \ge 0$ of the exploration, conditionally on S_i, the pairing $\tilde{\omega}$ of the (unexplored) edges of $\partial^* S_i^{\mathcal{A}}$ is uniform. Hence, if an edge $\mathcal{A}(S_i^{\mathcal{A}})$ is picked independently of $\tilde{\omega}$ then its partner is a uniform edge on $\partial^* S_i^{\mathcal{A}} \backslash \{\mathcal{A}(S_i^{\mathcal{A}})\}$. □

The strength of the above proposition is that we can use *different* algorithms $\mathcal{A}$ to explore the *same* random surface $\mathbf{M}_{\mathcal{P}}$ and get *different* type of information.

1.3.3 Examples of peeling explorations

Let us give a few examples of peeling explorations and a few results that were proven using them without going into details. Probably the most obvious peeling algorithm is the following one:

Uniform peeling

For $0 \le i \le n$, given the discrete surface S_i, we pick the next edge to peel uniformly at random on $\partial^* S_i$ (independently of the past operations and of the gluing of the edges).

The cautious reader may have noticed that the above algorithm $\mathcal{A}$ is not deterministic, but as long as the randomness involved is independent of the gluing the conclusion of Proposition 1.12 holds true. Although very natural, this algorithm has not been studied in depth in the literature. Notice e.g. that the Markov chain this exploration induces on the set of perimeters of the holes of S_i is very appealing: If $\{p_1, \dots, p_k\}$ is the configuration of the perimeters at time i, the next state is obtained by first sampling independently two indices $I, J \in \{1, 2, \dots, k\}$ proportionally to $p_1, \dots, p_k$. If $I \ne J$ then we replace the two numbers p_I and p_J with the single one $p_I + p_J - 2$. If $I = J$ we replace p_I with a uniform splitting of $p_I - 2$ into $\{\{0, p_I - 2\}, \{1, p_I - 2\}, \dots, \{p_I - 2, 0\}\}$.

This is a discrete version of the split-merge dynamic which preserves the Poisson–Dirichlet law considered in [102, 179] except that we have a deterministic "erosion" of -2 at each step in the above dynamic. We are thus led to the following open question:

Open Question 1.13. *Assume $(\mathcal{P}_n)_{n\geq 1}$ is a good sequence of configurations and perform the exploration of $\mathbf{M}_{\mathcal{P}_n}$ using the uniform peeling. If $\mathrm{P}_1^{(n)}, \dots, \mathrm{P}_n^{(n)}$ is the sequence of the perimeters of the exploration $S_0 \to S_1 \to \cdots \to S_n = \mathbf{M}_{\mathcal{P}_n}$, can we prove that $\frac{1}{n}\mathrm{P}_i^{(n)}$ converges in law towards the standard Poisson–Dirichlet partition for large n and i?*

Here is another fairly natural peeling algorithm:

PEELING THE MINIMAL HOLE

For $0 \leq i \leq n$, given S_i, the next edge to peel is the edge which belongs to a *hole of minimal perimeter* and if there are multiple choices we pick the edge having the minimal label.

This algorithm is the key to prove the first two points of Theorem 1.9 (connectedness of $\mathbf{M}_{\mathcal{P}_n}$ with high probability and the central limit theorem for the number of vertices). The main idea being that using this algorithm we quickly arrive at a situation where S_i is made of a single hole and that very few vertices have been created so far. At that time, the rest of the exploration is just a folding of a *unique* face on itself known as *unicellular maps*.

Here is another peeling algorithm which explores the 1-neighborhood of a given vertex. Once all the edges adjacent to this vertex have been discovered, we choose a new vertex on the hole of the current surface and iterate. This algorithm is used to prove the last point of Theorem 1.9.

PEELING VERTICES

Given the initial configuration of labeled polygons $S_0 \equiv \mathcal{P}$ we pick a "red" vertex $\mathfrak{R}_0 \in \partial^* S_0$ uniformly at random. Inductively, given the discrete surface S_i with a distinguished "red" point $\mathfrak{R}_i \in \partial^* S_i$, we peel the edge lying immediately on the left of $\mathfrak{R}_i$ to get S_{i+1}. If during the peeling step, the red vertex has been swallowed by the process then we resample $\mathfrak{R}_{i+1} \in \partial^* S_{i+1}$ uniformly at random (independently of the past and of the gluing). Otherwise $\mathfrak{R}_{i+1}$ canonically results of $\mathfrak{R}_i$.

We let the reader imagine other type of explorations and to deduce interesting geometric consequences they imply on $\mathbf{M}_{\mathcal{P}}$!

Conclusion: impose topological constraints! In this chapter, we have seen that several models of random maps with n edges obey the "Poisson–Dirichlet paradigm": they form random graphs with roughly $\log n$ vertices whose rescaled degrees follow the Poisson–Dirichlet partition. If we even believe in Conjecture 1.10, then according to Theorem 1.7 the diameter of those random graphs is asymptotically 2 or 3. The reader interested in geometry might be disappointed by such a behavior... The above phenomenon appears because the mean degree of the graph blows up. Indeed, when we do not impose a restriction on the genus of the random maps we consider,

then they naturally tend to have almost the maximal genus and very few vertices concentrating all the edges.

To prevent the above phenomenon, we will henceforth impose topological constraint on our maps by restricting to PLANAR maps, i.e. maps whose genus is 0 which exhibit a much richer geometry

Bibliographical notes. We refer the reader to standard textbooks [110, 137, 165] for more on the definitions of maps, see in particular [165] for a careful treatment of the "obvious" topological lemmas. We borrow also a lot from the nice exposition [72]. The reader eager to learn more on the cycle structure of uniform permutation and other logarithmic structure is referred to [21]. Building random surface by gluing randomly polygons has been considered by many authors including Brooks & Makover [50] in the case of the gluing of triangles ($p_i = 3$) and later studied by Pippenberg & Schleich [169] and Chmutov & Pittel [79] when $p_i \geq 3$, see also [70]. Of course, the model is closely related to the *configuration model* although the latter is focused on the graph structure and does not take into account the more rigid combinatorics of maps. The rest of this chapter is largely adapted from [61]. We offer a drink or a box of chocolate to the reader who will prove (or disprove!) Conjecture 1.10. A possible line of research is to extend the result of this chapter to the non-orientable case and by allowing the presence of boundaries as in [109].

Open Question 1.14. *Consider the a uniform gluing of $2N$ triangles (or more generally of a sequence of polygons) by pairing a subset of their sides (i.e. by choosing a uniform involution, which may have fixed points) and possibly allowing non-orientable gluings. What can be said about the non-orientable map we obtain? Describe the geometry of the non-paired edges.*

Chapter II : Why are planar maps exceptional?

In this chapter, we review a few of the exceptional properties of planar maps inspired by the beautiful post on Mathoverflow "Why are planar graphs exceptional"[1]. In particular, we gather a few applications of Euler's formula (Platonic solids, Fáry theorem, the five and six colors theorems). We also present briefly the theory of circle packings which is a means to represent a (simple) planar map faithfully in the plane.

2.1 Finite and infinite planar maps

2.1.1 Finite planar maps

Definition 2.1. *A map* $\mathfrak{m}$ *is planar if its genus is* 0.

From the previous chapter, such a map can equivalently be seen as:

- a finite connected (multi-)graph properly embedded in the plane (or in the sphere) viewed up to homeomorphisms that preserve the orientation,
- a gluing of finitely many polygons (the faces of the map) along their edges so that the manifold produced in this way is a topological sphere,
- a finite connected (multi-)graph with a system of orientations giving rise to a planar orientation.

The above equivalent definitions are summarized in Figure 2.1

Recall the notation $\mathsf{Edges}(\mathfrak{m}), \mathsf{Vertices}(\mathfrak{m})$ and $\mathsf{Faces}(\mathfrak{m})$ for the set of edges, vertices and faces of $\mathfrak{m}$. Recall also that the degree $\deg(f)$ of a face f (we sometimes also say the perimeter of f) is the number of edges incident to this face with the convention that when an edge is lying completely inside a face it is counted twice in the degree. Similarly, the degree $\deg(x)$ of a vertex x is the number of edges adjacent to x, where loops attached to x are counted twice. We remind the reader that the graph structure of $\mathfrak{m}$ contains strictly less information than the map itself, see Figure 2.2.

Planar maps are more rigid than planar graphs since they are given with an embedding (equivalently a planar orientation) whereas planar graphs only *possess* such an embedding. This

[1] https://mathoverflow.net/questions/7114/why-are-planar-graphs-so-exceptional

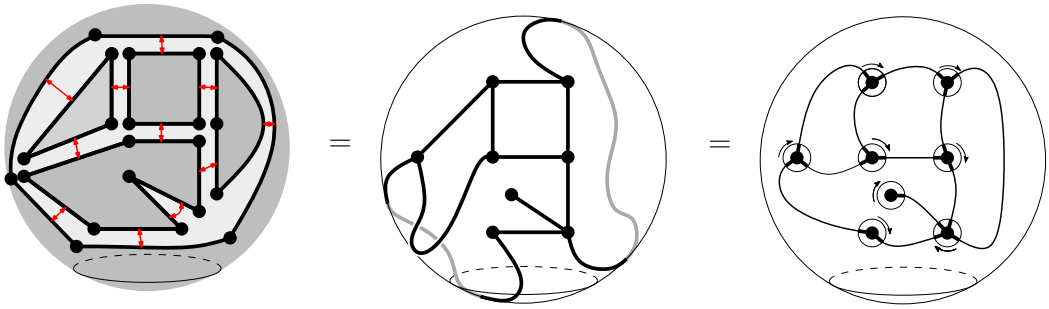


Figure 2.1: The same planar map seen: [left] as a gluing of polygons, [center] as an equivalence class of embeddings of a finite planar graph in the sphere, [right] as a graph with cyclic orientations around vertices.

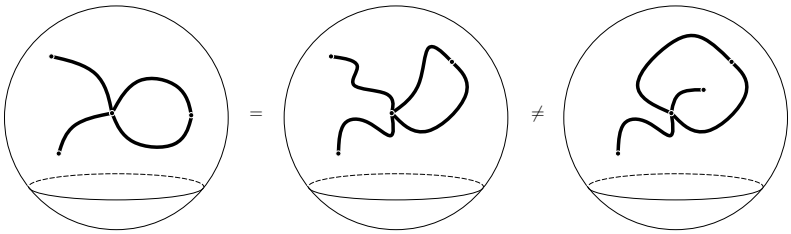


Figure 2.2: The same underlying planar graph can yield different planar maps.

rigidity enables us to enumerate planar maps more easily than planar graphs and this is mainly why we will consider maps instead of graphs. For a complete rigidity, we will only consider *rooted* maps, that are maps given with one distinguished oriented edge called the *root edge* which we denote by $\vec{e}$. The **origin vertex** ρ of $\vec{e}$ is the root vertex (also called the origin) and the face incident on the right of $\vec{e}$ is the root face f_r of the map. Equivalently, the rooting of a planar map can be obtained by distinguishing a **corner** of the map, i.e. an angular sector between two consecutive edges around a vertex. Once rooted, maps have no non-trivial symmetry. From now on, all the maps considered are planar and rooted. We denote by

$$\mathcal{M} := \text{ the set of all (rooted) finite planar maps.}$$

In the following a generic planar map will be denoted by $\mathfrak{m} \in \mathcal{M}$. Also recall that there exists a unique "vertex map", denoted by †, which is made of a unique vertex, no edge, one face, and is thus planar. A **simple** map is a map in which multiple edges or loops are forbidden.

2.1.2 Local topology and infinite maps

In these lecture notes, we will also deal with **infinite maps**. The proper definition of infinite map comes hand in hand with the concept of local topology which we first present.

If $\mathfrak{m}$ is a map and $r \in \{0, 1, 2, 3, \ldots\}$, we denote by $[\mathfrak{m}]_r$ and call it the ball of radius r in

$\mathfrak{m}$, the map spanned by all the edges of $\mathfrak{m}$ which have an endpoint at graph distance distance less than or equal to $r-1$ of the origin ρ of $\mathfrak{m}$. The map $[\mathfrak{m}]_r$ inherits the planar orientation from $\mathfrak{m}$ and is indeed a planar map rooted at the root edge of $\mathfrak{m}$ as soon as $r \geq 1$. When $r = 0$ we put $[\mathfrak{m}]_0 = \dagger$ the "vertex-map". Notice the compatibility relation for the restriction operator $[\mathfrak{m}]_r = [[\mathfrak{m}]_R]_r$ for any $r \leq R$.

Definition 2.2 (Local distance). *The local distance on the set $\mathcal{M}$ of all finite planar maps is defined for $\mathfrak{m}, \mathfrak{m}' \in \mathcal{M}$ by*

$$\mathrm{d}_{\mathrm{loc}}(\mathfrak{m}, \mathfrak{m}') \quad = \quad \Big(1 + \sup\{r \geq 0 : [\mathfrak{m}]_r = [\mathfrak{m}']_r\}\Big)^{-1}. \tag{2.1}$$

We leave the reader check that this is indeed a distance. Note that the space $(\mathcal{M}, \mathrm{d}_{\mathrm{loc}})$ is not complete. One way to complete it is to introduce $\overline{\mathcal{M}}$, the space of all coherent sequences of maps

$$\overline{\mathcal{M}} = \Big\{(m_0, m_1, m_2, \dots) \quad \text{with } m_i \in \mathcal{M} \quad \text{and} \quad [m_i]_r = m_r, \quad \forall i \geq r\Big\}.$$

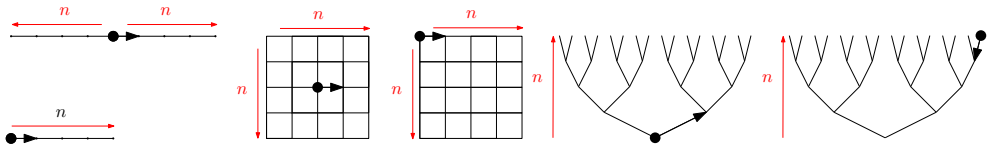
If we extend $\mathrm{d}_{\mathrm{loc}}$ to $\overline{\mathcal{M}}$ by setting $[(m_i)_{i\geq 0}]_r = m_r$ then the injection $\mathcal{M} \to \overline{\mathcal{M}}$ sending $\mathfrak{m}$ to $([\mathfrak{m}]_r)_{r\geq 0}$ is an isometry, and $(\overline{\mathcal{M}}, \mathrm{d}_{\mathrm{loc}})$ is now complete for this distance (see below). We shall abuse notation and still write $\mathfrak{m}$ for the elements of $\overline{\mathcal{M}}$ and speak of planar maps. An element $\mathfrak{m} \in \overline{\mathcal{M}}$ for which $[\mathfrak{m}]_r$ is not eventually stationary is an *infinite planar map*.

Proposition 2.3. *The space $(\overline{\mathcal{M}}, \mathrm{d}_{\mathrm{loc}})$ is Polish (metric, separable and complete). Furthermore, a subset $A \subset \overline{\mathcal{M}}$ is relatively compact (its closure is compact) if and only if for every $r \geq 0$*

$$\#\{[\mathfrak{m}]_r : \mathfrak{m} \in A\} < \infty \qquad \textit{or equivalently} \qquad \sup\{\deg(x) : x \in \mathsf{Vertices}([\mathfrak{m}]_r), \mathfrak{m} \in A\} < \infty.$$

Proof. It is easy to see that $\mathrm{d}_{\mathrm{loc}}$ is a distance. The separation is granted since $\mathcal{M}$ is dense in $\overline{\mathcal{M}}$ and countable. Completeness. If $(\mathfrak{m}_n)$ is a Cauchy sequence for $\mathrm{d}_{\mathrm{loc}}$ then for every r, the ball $[\mathfrak{m}_n]_r$ stabilizes to a certain map $\mathfrak{m}_r^*$. By coherence we have $[\mathfrak{m}_{r'}^*]_r = \mathfrak{m}_r^*$ for any $r' \geq r$ and so we can define a unique possibly infinite map $\mathfrak{m}_\infty \in \mathcal{M}$ by $[\mathfrak{m}_\infty]_r = \mathfrak{m}_r^*$. It is then clear that $\mathfrak{m}_n \to \mathfrak{m}_\infty$ for $\mathrm{d}_{\mathrm{loc}}$. Characterization of the compacts. The condition in the theorem is clearly necessary for A to be relatively compact for otherwise there exist $r_0 \geq 0$ and a sequence $(\mathfrak{m}_n)$ in A that are all at distance $\varepsilon = \frac{1}{1+r_0}$ from each other. Such a sequence cannot admit a convergent subsequence. Conversely, a subset A satisfying the condition of the theorem is easily seen to be pre-compact (hence relatively compact by completeness) for $\mathrm{d}_{\mathrm{loc}}$: just cover it with balls of radius $1/r$ centered on each element of $\{[\mathfrak{m}]_r : \mathfrak{m} \in A\}$. We leave the equivalent condition to the reader. □

Exercise 2.4. Compute the $\mathrm{d}_{\mathrm{loc}}$ limit of the following 6 sequences of planar maps.



It is easy to check that if $(\mathbb{P}_n)_{n\geq 0}$ and $\mathbb{P}_\infty$ are probability measures on $\overline{\mathcal{M}}$ then $\mathbb{P}_n \to \mathbb{P}_\infty$ in distribution for the local distance if and only if for any $r \geq 0$ and any fixed planar map b_0 we have

$$\mathbb{P}_n([\mathfrak{m}]_r = b_0) \xrightarrow[n\to\infty]{} \mathbb{P}_\infty([\mathfrak{m}]_r = b_0).$$

In other words, convergence in distribution for the local distance is equivalent to convergence in distribution of the ball of radius r, for any $r \geq 0$. Beware though, the convergence of the probabilities $\mathbb{P}_n([\mathfrak{m}]_r = b_0)$ as $n \to \infty$ is not sufficient to imply convergence in distribution because tightness is missing (these probabilities could all converge to 0 for example). The following remark, which is based on Fatou's lemma, will be useful:

Remark 2.5. Suppose that for some laws $\mathbb{P}_n, \mathbb{P}_\infty$ on $\overline{\mathcal{M}}$ we have

$$\liminf_{n\to\infty} \mathbb{P}_n([\mathfrak{m}]_r = b_0) \geq \mathbb{P}_\infty([\mathfrak{m}]_r = b_0),$$

for any $r \geq 0$ and any $b_0 \in \mathcal{M}$, then $\mathbb{P}_n \to \mathbb{P}_\infty$ in distribution for the local topology. Indeed, since we are working with probability measures we have $\mathbb{P}_n([\mathfrak{m}]_r = b_0) = 1 - \sum_{b\neq b_0} \mathbb{P}_n([\mathfrak{m}]_r = b)$ and so by Fatou's lemma for non-negative series we have

$$\limsup_{n\to\infty} \mathbb{P}_n([\mathfrak{m}]_r = b_0) \leq 1 - \sum_{b\neq b_0} \liminf_{n\to\infty} \mathbb{P}_n([\mathfrak{m}]_r = b) \leq 1 - \sum_{b\neq b_0} \mathbb{P}_\infty([\mathfrak{m}]_r = b) = \mathbb{P}_\infty([\mathfrak{m}]_r = b_0),$$

which combined with the lower bound gives the desired convergence.

The concept of vertices and edges is easily extended to infinite maps. With a little care, this can also be done for faces, but an infinite map $\mathfrak{m}$ may have faces of infinite degrees. For example, the line graph of length $2n$ rooted in the middle converges locally towards the map made of an infinite line separating two faces of infinite degrees. We will not give a precise definition and rather use the reader's intuition for these concepts which should be clear when dealing with infinite maps of the plane or the half-plane which we define below.

Exercise 2.6. Suppose that we are given a metric space $(\mathcal{E}, \delta)$ such that for any $x \in \mathcal{E}$ and for any $r \in \{0, 1, 2, 3, \dots\}$, there is a notion of *restriction of radius* r in x that we denote by

$$[x]_r \in \mathcal{E}$$

satisfying the following properties:

- (compatibility) for any $x \in \mathcal{E}$ and any $r' \geq r$ we have $[[x]_{r'}]_r = [x]_r$,
- (continuity) for any $r \geq 0$ the map $x \mapsto [x]_r$ is continuous for δ,
- (injectivity) for any $r \geq 0$, the map $x \in \mathcal{E} \mapsto ([x]_r)_{r\geq 0}$ is injective,
- (separation) for any $r \geq 0$, the set $\{[x]_r : x \in \mathcal{E}\}$ is separable and complete for δ.

The set of coherent restrictions $\overline{\mathcal{E}} = \{(x_0, x_1, ...) \in \mathcal{E}^{\mathbb{N}} : [x_{r'}]_r = x_r, \forall r' \geq r\}$ endowed with the local distance $\mathrm{d}((x_i), (y_i)) = (1 + \sup\{r : x_r = y_r\})^{-1}$ is Polish.

2.1.3 Infinite maps of the plane and the half-plane

The above definition of infinite planar maps is equivalent to an (orientation preserving) gluing of possibly infinitely many polygons so that the resulting surface is planar[2]. Beware, infinite planar maps cannot be seen as proper embeddings of infinite graphs in the plane/sphere seen up to continuous deformations, see Figure 2.3 below.

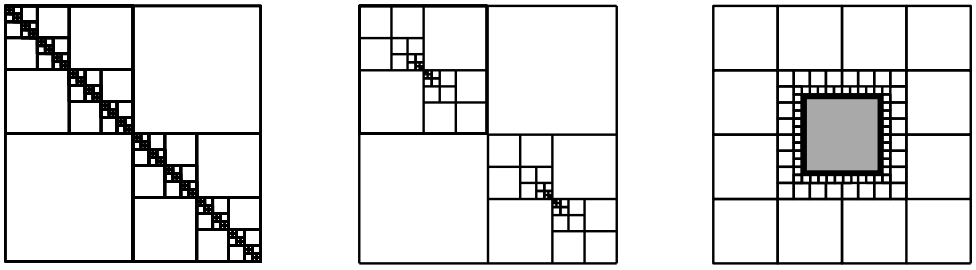


Figure 2.3: Three examples of infinite maps, the left-most one has infinitely many ends, the center one has two ends whereas the right-most one has only one end (the centered region in gray is not a face). In particular, the right-most map can be drawn on the sphere after shrinking the gray region (and the latter can not be obtained by applying a homeomorphism of the right picture).

For one-ended maps however, things are easier. Recall that an infinite locally finite multigraph $\mathfrak{g}$ has **one end**, if for any finite subgraph $\mathfrak{h} \subset \mathfrak{g}$ there is exactly one infinite component in $\mathfrak{g}\backslash\mathfrak{h}$. Coming back to maps (the number of ends of a map is the number of ends of its graph) we have:

Proposition 2.7. *Infinite planar maps with one end can be seen as equivalence classes (for orientation preserving homeomorphisms of the plane) of proper embeddings of infinite one-ended planar graphs on the plane $\mathbb{R}^2$ such that every compact of $\mathbb{R}^2$ intersects only finitely many edges of the embedding.*

Proof. It should be clear that an (equivalence class of) embedding such as the one above defines an infinite planar map with only one end since any finite set in the map is contained in a compact set of the embedding and its complement contains at most one unbounded region. Reciprocally, if $\mathfrak{m}$ is an infinite planar map with one end, then for every r recall that $[\mathfrak{m}]_r$ is the ball of radius r and write $\overline{[\mathfrak{m}]}_r$ for the hull of the ball of radius r obtained by filling-in all the finite components of $\mathfrak{m}\backslash[\mathfrak{m}]_r$. Denote the vertices which are at distance r from the origin of $\mathfrak{m}$ and located on the boundary of $\overline{[\mathfrak{m}]}_r$ by $\partial\overline{[\mathfrak{m}]}_r$. We then claim that it is possible to draw $\mathfrak{m}$ on $\mathbb{R}^2$ in such a way that the vertices of $\partial\overline{[\mathfrak{m}]}_r$ are drawn on the circle of center $(0,0)$ and radius r, and that all the

[2]See [171] for the classification of two-dimensional non-compact surfaces. In our case, the polygons may also have an infinite perimeter. In particular, a surface coming from the gluing of infinitely many polygons is planar, if any subsurface obtained by keeping finitely many polygons has no genus.

edges and vertices of $\overline{[\mathfrak{m}]_r}$ are inside that circle. Such an embedding is indeed of the required form. □

Using the previous proposition, it is legitimate to call infinite planar maps with one end *infinite maps of the plane* and we almost do so after splitting this group into two further subclasses: An infinite map with one end can have 0 or 1 face of infinite degree. If it has one such face it can be drawn on $\mathbb{R}\times\mathbb{R}_+$ by saying the face of infinite degree contains the half-plane $\mathbb{R}\times\mathbb{R}_-$.

Definition 2.8 (Maps of the plane and half-plane)**.** A map of the plane *is a rooted infinite planar map with only one end such that all faces are of finite degree.* A map of the half-plane *is a rooted infinite planar map with one end such that the root face is of infinite degree.*

2.2 Euler's formula and applications

Let us come back to finite planar maps and derive a few consequences of Euler's formula. In the planar case, Theorem (1.2) reads

$$\#\mathsf{Vertices}(\mathfrak{m}) + \#\mathsf{Faces}(\mathfrak{m}) - \#\mathsf{Edges}(\mathfrak{m}) \quad = \quad 2, \tag{2.2}$$

for any finite planar map $\mathfrak{m}$. The notion of face is not well-defined for planar graphs as it may depend on its planar embedding. However, we see from Euler's formula that the number of faces does not depend on the embedding but only on the underlying graph structure.

Exercise 2.9. Using Euler's formula show that the complete graph K_5 on 5 vertices (with an edge between any pair of vertices) and the graph $K_{3,3}$ made of 3 black vertices and 3 white vertices such that there is an edge between any pair of black and white vertices are not planar graphs.

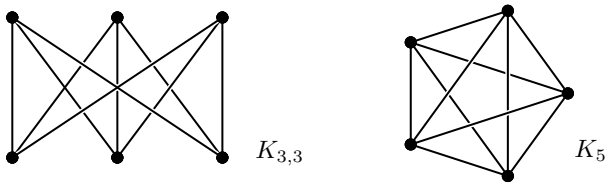


Figure 2.4: The graphs $K_{3,3}$ (on the left) and K_5 (on the right).

The converse of the above exercise is also true: By the famous Kuratowski theorem (1930), a graph is planar if it does not contain K_5 or $K_{3,3}$ as a minor. However, in this course, the planar maps come with their embeddings, and so planarity testing is never an issue.

2.2.1 k-angulations and bipartite maps

Euler's formula is particularly useful when we deal with special classes of planar maps: A **k-angulation** (for $k \geq 3$) is a planar map all the faces of which have degree k. In particular, we speak of **triangulations** in the case $k = 3$ and of **quadrangulations** in the case $k = 4$. Beware,

since we allow multiple edges and loops, a triangle (or a quadrangle) can be folded on itself and look weird at first glance, see Figure 2.5.

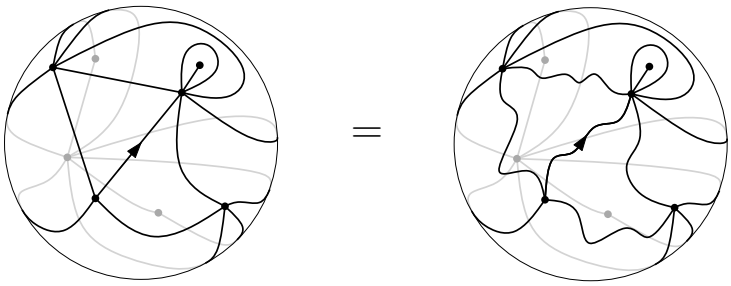


Figure 2.5: A finite triangulation of the sphere. Notice the loop with an inner edge: it is indeed a triangle!

In a finite k-angulation $\mathfrak{m}$, we have $k \cdot \#\mathsf{Faces}(\mathfrak{m}) = 2 \cdot \#\mathsf{Edges}(\mathfrak{m})$, because each edges is incident to two faces (or twice to the same face). This, combined with Euler's formula, gives an affine relation between the number of vertices, edges and faces of $\mathfrak{m}$ (depending on k, there are congruence constraints).

Definition 2.10 (Bipartite planar maps)**.** *A planar map is bipartite if one can color its vertices in two colors (black and white say) so that two neighboring vertices do not share the same color. Equivalently, a planar map is bipartite if and only if all its faces have even degree.*

The equivalent between the two points of the definition is left as an exercise for the reader. As we will see later on, bipartite planar maps are in a sense more regular than general maps and their enumeration formulas are nicer. The special case of quadrangulations deserves a particular attention because of the so-called "Tutte's bijection":

Recall the construction of the dual map $\mathfrak{m}^\dagger$ from Section 1.1.3. This construction can be modified to yield a bijection between, on the one hand, the set of all quadrangulations with n faces, and on the other hand, the set of all planar maps with n edges. The one-to-one correspondence is given as follows: If $\mathfrak{m}$ is a planar map with n edges, then in each face of $\mathfrak{m}$ we put an extra point that we link to all (corners of) the vertices adjacent to this face, see Figure 2.6. We then erase all the edges of $\mathfrak{m}$ and we are left with a quadrangulation $\mathfrak{q}$ with n faces (which is clearly bipartite!). The root edge is transferred from $\mathfrak{m}$ to $\mathfrak{q}$ as depicted on Figure 2.6. As a consequence of this bijection, the number of planar maps with n edges is the same as the number of quadrangulations with n faces, which will turn out to be relatively simple to compute. See Corollary 3.3.

2.2.2 Platonic solids

A well-known application of Euler's formula is the classification of all regular polyhedra, or Platonic solids. Indeed, a regular polyhedron can be seen as a finite map such that the degrees

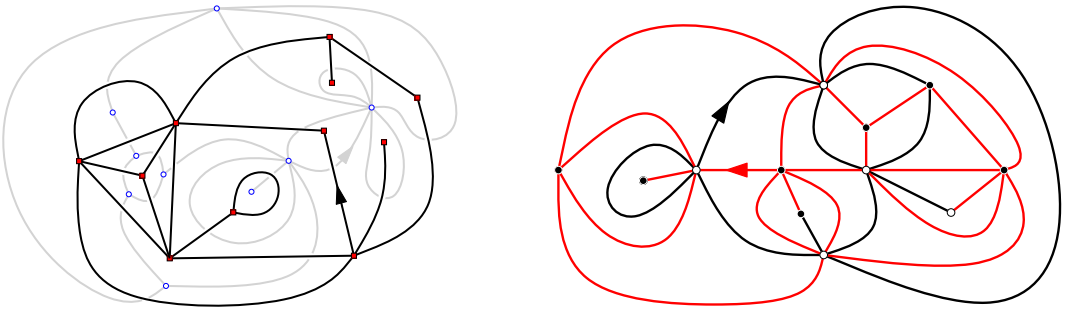


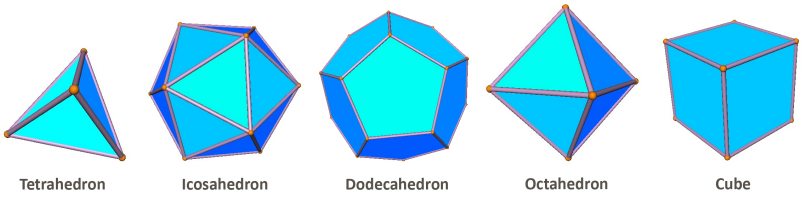
Figure 2.6: Duality between planar maps (left) and between quadrangulations and planar maps (right).

of the vertices and faces are constant. If $\alpha \geq 3$ and $\gamma \geq 3$ denote respectively the common degree of the vertices and faces of the map $\mathfrak{m}$ with v vertices, f faces and e edges then we have

$$\alpha v = 2e, \qquad \gamma f = 2e, \qquad \text{and} \qquad v + f - e = 2.$$

It is easy to see that there are only 5 solutions to these equations giving rise to 5 regular polyhedra described below. Notice the symmetry between the number of vertices and faces which play the same role once exchanged. This is of course a manifestation of duality.

Name	α	γ	e
Tetrahedron	3	3	6
Cube	3	4	12
Octahedron	4	3	12
Dodecahedron	3	5	30
Icosahedron	5	3	30



It is a classical exercise to show that the above Platonic solids do exist that is, can be constructed in three dimensions by gluing identical flat regular polygons together.

2.2.3 Fàry theorem

By definition, planar maps can always be drawn on the plane in a proper way. One can wonder whether it is possible to do this with straight lines. Obvious obstructions are multiple edges or loops, but once these have been forbidden the answer is yes!

Theorem 2.11 (*Fàry 1948*)

Any simple (i.e. without multiple edges nor loops) finite planar map can be properly drawn in the plane with straight edges.

Proof. We can suppose, without loss of generality, that $\mathfrak{m}$ is a (simple) triangulation since adding edges makes the drawing with straight lines even more difficult. The proof is then done by induction on the number of vertices of the triangulation $\mathfrak{m}$. If $\mathfrak{m}$ has only one vertex then

(since loops are forbidden) it corresponds to the vertex map which can be drawn on the plane with no line (in particular straight). If $\mathfrak{m}$ has more than 2 vertices and is simple then we use the following lemma:

Lemma 2.12. *Any finite simple planar map $\mathfrak{m}$ has a vertex of degree less than or equal to 5.*

Proof of the lemma. By Euler's formula we have $v + f - e = 2$ with obvious notations, whereas the edge count gives

$$2e = f_1 + 2f_2 + 3f_3 + \cdots,$$

where f_i is the number of faces of degree i in the map $\mathfrak{m}$. Since $\mathfrak{m}$ is simple, there are no faces of degree 1 or 2 and it follows that $2e \geq 3f$. Combining with Euler's formula we get that $3v - e \geq 6$ or equivalently $\frac{2e}{v} \leq 6 - \frac{12}{v}$. Since $\frac{2e}{v}$ represents the mean degree of a vertex in the map $\mathfrak{m}$, the previous inequality implies the existence of a vertex of degree less than or equal to 5. □

Coming back to the proof of the theorem, we take a vertex v of degree less than 5 in the triangulation $\mathfrak{m}$. Now erase v from the triangulation (as well as its incident edges). By the induction hypothesis, the rest of the map can be drawn on the plane with straight lines. Consider now the vertices to which the vertices v should have been linked. They form a polygonal face of degree less than 5 (we use the fact that $\mathfrak{m}$ is a triangulation here). By the art gallery theorem (see wikipedia) we can place back v inside this face in such a way that it can be linked by straight lines to all its neighboring vertices. □

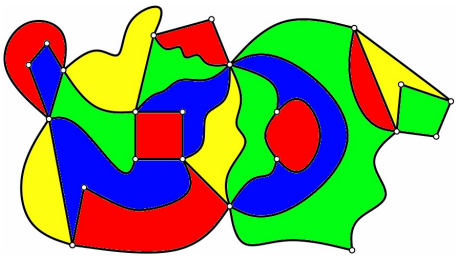
Exercise 2.13. Show that there is no infinite triangulation of the plane whose vertex degrees are bounded by 5. Is there an infinite triangulation of the plane with degrees only in $\{5, 6\}$ apart from the six-regular triangulation? How many such points of degree 5 can there be?

2.2.4 6 − 5 − 4 color theorem

A famous theorem about planar maps is the following:

Theorem 2.14 (*Four colors theorem (Appel & Haken 1976)*)

The faces of any finite planar map can be colored in four colors so that no two adjacent faces share the same color.



Although it looks childish, the proof of the above theorem is extremely difficult and requires the help of a computer to check numerous cases, but a version with 6 or even 5 colors is much easier to get (see Exercise 2.15).

Exercise 2.15. Deduce from Lemma 2.12 that any planar map can be properly colored with 6 colors. Harder: Prove that 5 colors actually suffice.

2.3 Faithful representations of planar maps

A priori, a planar map does not have any canonical representation in the plane (or the sphere) since even in the finite case, it is given as an equivalence class of embeddings. Still one can ask if we can make sense of a "faithful" representation of a map.

2.3.1 Tutte's barycentric embedding

One very natural idea in order to draw a map in the plane, is to use a physical system of springs. More precisely, imagine you are given a simple face of the map, and that we fix the location of the vertices of this face as a convex polygon. Then if we imagine that the edges of the map are made of ideal springs, the system should find an equilibrium point and all the other vertices will be located inside the convex polygon, see Figure 2.7 below.

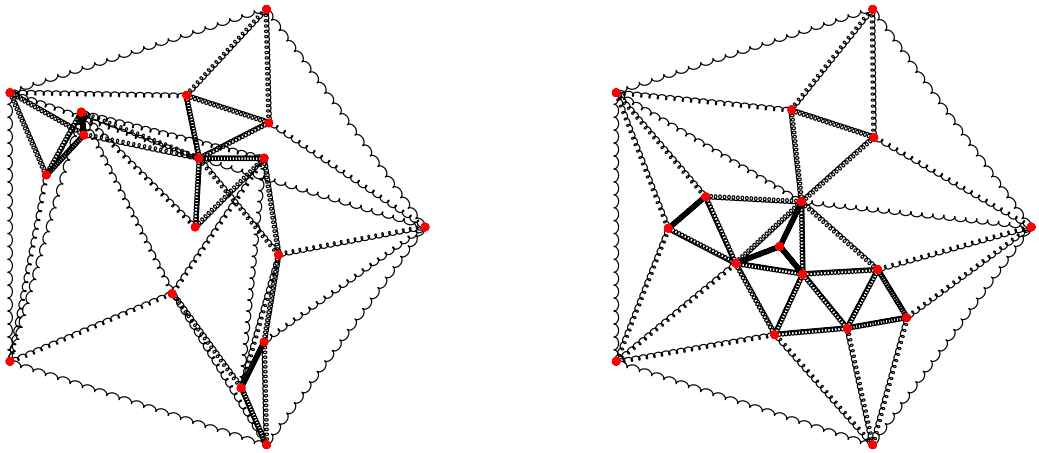


Figure 2.7: [Left] We fix the 5 vertices of a simple face of the map (in blue) in a convex position, and we let the other vertices find their locations [Right] by minimizing the total potential energy of the springs.

Such a representation of the graph is called a **spring embedding** or a **barycentric embedding** since the equilibrium condition of the springs means that the location in $\mathbb{R}^2$ of each vertex (except those fixed as the convex polygon) is the isobarycenter of (the locations of) its neighbors. Of course, such a representation cannot be a proper embedding if there are loops in the graph, and more generally if the removal of one or two vertices may disconnect the graph.

Definition 2.16 (3-connected graphs)**.** *A graph is* 3*-connected if the removal of at most* 2 *vertices cannot disconnect the graph.*

The class of 3-connected planar graphs is interesting from many points of view because they coincide precisely with the 1-skeletons of convex polyhedra in $\mathbb{R}^3$ (Steinitz 1916) and for them there is (almost) no distinction between planar maps and planar graphs: Whitney (1933) proved that every 3-connected planar graph admits a unique planar embedding up to homeomorphism and inversion of the sphere. In the case of 3-connected planar maps, the barycentric or spring embedding is indeed a proper embedding as proved by Tutte:

Theorem 2.17 (*Tutte (1963)*)

If $\mathfrak{m}$ is a 3-connected planar map and $v_1, v_2, v_3, \dots, v_k$ are $k \geq 3$ vertices around a (necessarily simple) face of $\mathfrak{m}$ then the barycentric embedding obtained by fixing $v_1, \dots, v_k$ in convex position exists, is unique, and is a proper embedding.

Exercise 2.18. Deduce Fàry's theorem from the previous result.

We refer to [175, Section 12.2] for the proof of this result. Let us nonetheless make a few comments about its proof. Algorithmically, computing the barycentric embedding of a map is an easy task: as remarked above, this means that each vertex (except those fixed) is the barycenter of its neighbors. We thus have to solve a (rather sparse) linear system which has a unique solution by the classical Dirichlet problem: if $f(v)$ is the location in $\mathbb{R}^2$ of the vertex v, the barycentric condition is equivalent to the fact that the process $f(X_n)$ is a martingale in each coordinate where X_n is the random walk on the graph killed when touching $A := \{v_1, v_2, \dots, v_k\}$. The classical representation of harmonic function shows that for any $v \in \mathsf{Vertices}(\mathfrak{m}) \backslash A$

$$f(v) = \mathbb{E}_v[f(X_\tau)], \qquad \text{where} \qquad \tau = \inf\{i \geq 0 : X_i \in A\}.$$

This shows that the locations of the inner points exists and is unique. The complicated part is to show (using the convex condition on the location of the vertices in A) that this actually produces an embedding in the sense that the edges are non-crossing. See [175, Section 12.2].

2.3.2 Circle packing

We now describe another "faithful" representation of planar maps call the "Circle Packing" representation. This theory was popularized by Thurston in 1985 and we shall rely on it in Chapter 15 when studying random walks on infinite random planar maps. The interested reader should consult [181] or [167] for applications of this wonderful theory.

As in Section 2.2.3 we focus on the case of simple maps where multiple edges and loops are forbidden. We say that a simple map $\mathfrak{m}$ is represented by a **circle packing** if there is a collection $(\mathcal{C}_v : v \in \mathsf{Vertices}(\mathfrak{m}))$ of non overlapping disks in the plane $\mathbb{R}^2$ such that $\mathcal{C}_v$ is tangent to $\mathcal{C}_u$ if and only if u and v are neighbors in $\mathfrak{m}$. Recall that the completed plane $\hat{\mathbb{C}} = \mathbb{R}^2 \cup \{\infty\}$ can be identified with the Riemann sphere $\mathbb{S}^2$ by the stereographic projection from the North pole. This projection transforms circles and lines in $\hat{\mathbb{C}}$ into circles on the Riemann sphere. Recall also that the Möbius group

$$\left\{ z \in \hat{\mathbb{C}} \mapsto \frac{az+b}{cz+d} \right\}$$

acts triply transitively on the Riemann sphere (i.e. we can map any triplet of distinct points to any other triplet of distinct points) and preserves circles.

Theorem 2.19 (*Circle packing theorem. Koebe (1936) Andreev–Thurston (1978)*)



Any finite simple map $\mathfrak{m}$ *admits a circle packing representation on the Riemann sphere. Furthermore if* $\mathfrak{m}$ *is a simple triangulation, then the circle packing is unique up to Möbius transformations.*

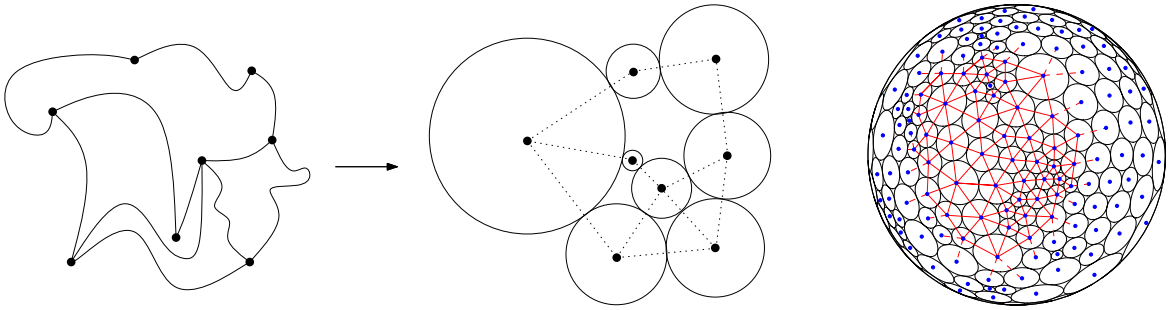


Figure 2.8: On the left, a finite circle packing of a planar map. On the right, a circle packing of a triangulation seen on the sphere $\mathbb{S}_2$.

Sketch of the proof. First, it is easy to see that it suffices to prove the theorem for simple triangulations because we can embed any simple planar map inside a simple triangulation by further triangulating inside each face. Fix a triangulation t and pick a face $f \in \mathsf{Faces}(\mathsf{t})$ that we will see as the exterior face. We will prove that we can construct a circle packing of t such that the three circles corresponding to this outer face are three mutually tangent circles of radius 1, or equivalently that the three vertices of the triangles form an equilateral triangle. The rest of the circles are in-between these three circles. We start with the uniqueness statement.

Uniqueness. Since the Möbius group of the Riemann sphere acts triply transitively we can transform any circle packing into a packing of the above form (with the marked face forming an equilateral triangle and the rest of the vertices inside). Imagine that we are given two packings $\mathcal{P}$ and $\mathcal{P}'$ of the above form, in particular the three exterior circles are of radius 1. We then choose an interior vertex v of the triangulation such that the ratio of the corresponding circles in the packing is maximal i.e.

$$\lambda(v) = \frac{r_{\mathcal{P}}(v)}{r_{\mathcal{P}'}(v)} \quad \text{is maximal.}$$

We then examine the structure of the packing around this circle in $\mathcal{P}$. By dividing all the distances by $\lambda(v)$ we end up with a circle of radius $r_{\mathcal{P}'}(v)$ and such that all the neighboring circles have a radius which is less than the corresponding radius in $\mathcal{P}'$. By an obvious monotonicity property of the angles around a circle we deduce that these new radii must coincide with those in $\mathcal{P}'$ i.e.

$$\lambda(u) = \frac{r_{\mathcal{P}}(u)}{r_{\mathcal{P}'}(u)} = \lambda(v),$$

for all u neighbors of v (for otherwise if the inequality were strict, the neighboring circles would not surround the circle associated with v). Since the graph is connected, we deduce step by step that $\lambda(\cdot)$ is constant and must be equal to 1 by the assumption on the exterior circles. Hence $\mathcal{P} = \mathcal{P}'$.

Existence. We will not prove the existence but just describe an algorithm that can be used (even in practice!) to construct the packing. The idea is to first find all the radii of the circles. Once these radii are found, one can reconstruct the packing step by step by starting from the external face and deploying the circles one by one around the circles already explored (notice that given the radii and the combinatorial layout, we can determine the angles, and for this we crucially use the fact that the underlying map is a triangulation). To find the radii we start with an arbitrary assignment of radii to the vertices of the triangulations except the three vertices of the marked face which have their radii fixed forever to 1. We then examine all the internal vertices in a cyclic order and repeat forever the following adjustment rule: see Figure 2.9.

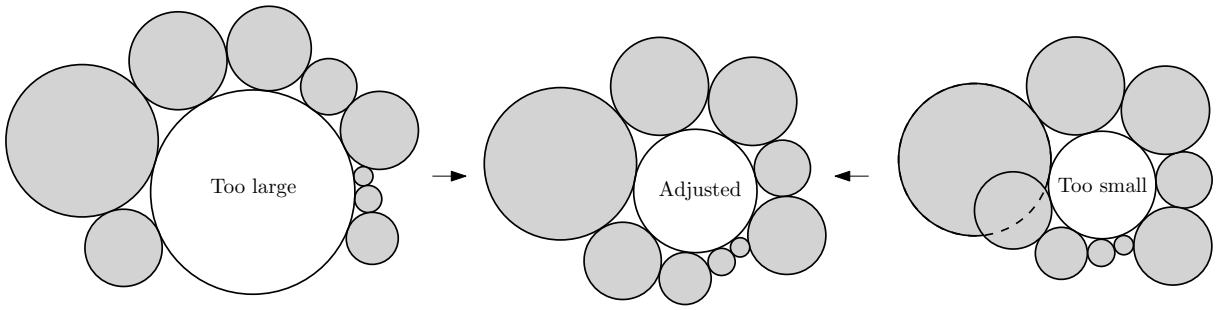


Figure 2.9: Adjustment rule: For an internal vertex v with radius r_v, we examine the radii of the neighbors of v. Using these radii, one can see whether or not placing the circles of the corresponding radii around a circle of radius r_v would close exactly. Most of the time, it will not. But by a monotonicity property, one can always update the radius r_v so that the latter property holds true.

Repeatedly applying this updating rule, it can be proved (but it is not trivial) that this algorithm indeed converges towards the unique fixed point for the right values of the radii for the circles (with the outer three circles normalized) and that these values give rise to a non-degenerate (all the radii are positive) circle packing for the triangulation $\mathfrak{t}$. This can be shown by understanding the monotonicity properties of the "flow of angles" on the graph when looping this rule, see [82]. □

As mentioned above, we will use the theory of circle packings in Chapter 15 when studying random walks on random planar maps. Lipton and Tarjan [146] found a circle-packing-based proof of the separator theorem: If $\mathfrak{m}$ is a planar map with n vertices, then there is a simple closed path made of less than $\sqrt{8n}$ vertices which separates the maps into two components containing less than $2n/3$ vertices each.

There is also a wonderful link between circle packings and the Riemann mapping theorem

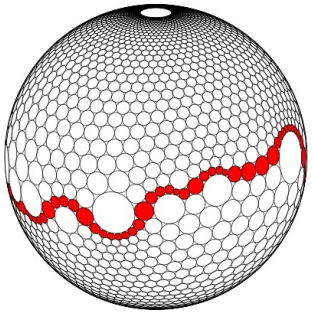


Figure 2.10: A small balanced separator of a planar graph represented by a circle packing on the sphere $\mathbb{S}_2$. Image of Kenneth Stephenson

(initially conjectured by Thurston and proved by Rodin and Sullivan). Imagine that we have a circle packing of a region Ω, with the hexagonal packing say, and that the same combinatorial triangulation structure is circle-packed in the disk (we can always do so by the finite circle packing theorem). We suppose also that the center $z \in \Omega$ is mapped to $0 \in \mathbb{D}$. It turns out that as the maximal radius of the circles goes to 0, the mapping induced by the circle packings approximates a conformal bijection $\Omega \to \mathbb{D}$ such that z is mapped to 0.

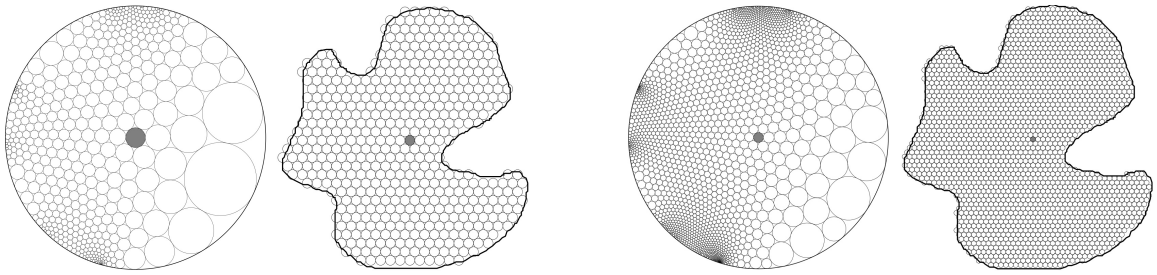


Figure 2.11: Thurston conjecture (Rodin–Sullivan/Schramm theorem): Circle packings can be used to approximate conformal mappings. Images of Kenneth Stephenson.

Bibliographical notes. Most of this chapter can be found in textbooks on planar graphs/maps. The local distance has been introduced in the context of random planar maps by Benjamini & Schramm [29]. Exercise 2.6 is taken from [86]. A solution to Exercise 2.13 can be found in [101, Corollary 1.5]. Circle packing theory was triggered by the influential work of Thurston [184, Chapter 13]. We refer to the book [181] and Saint-Flour notes [167] for many more applications of this wonderful theory, see Chapter XV. The proof of uniqueness in the circle packing theorem via the maximum principle is taken from wikipedia and is due to O. Schramm. Tutte barycentric embedding has been introduced in [186] and we refer to [175, Section 12.2] for more about it.

Chapter III : The miraculous enumeration of bipartite maps

Our goal in this chapter is to enumerate planar maps, quadrangulations say. The basic idea, which goes back to Tutte, is to find a recurrence relation between quadrangulations of different sizes by erasing the root edge to diminish the size of the map as in Section 1.2.1. Unfortunately when doing so, the remaining map is generally not a quadrangulation anymore. The key idea is then to generalize the model, and consider quadrangulations with a boundary, which are now stable under the erasure of the root edge. This method, pioneered by Tutte in the 60's, enabled him to enumerate many classes of planar maps and to find a "miraculous" closed formula for the enumeration of bipartite planar maps: the Slicing Formula (Theorem 3.4). In the rest of these lecture notes we focus on *bipartite* planar maps because their enumeration yields to simpler formula compared to the case of general planar maps. However, most of the material we develop here can be adapted to the general case, and in particular to triangulations to the cost of additional technicalities and heavier notation.

Tutte's equation is very important not only because it leads to exact and asymptotic enumeration of planar maps but also because it is the true spirit of the peeling process. In fact, as we will see later on, the peeling process is *in fine* just a probabilistic way to re-interpret and iterate Tutte's equation.

In the rest of these lecture notes we only deal with bipartite planar maps.

3.1 Maps with a boundary and a target

Let us introduce the class of enhanced maps that we will consider in the rest of this book.

3.1.1 Maps with a boundary.

Recall that if $\mathfrak{m}$ is planar map, the face f_r incident on the right of the root edge is called the root face or sometimes the external face. This enables us to see a planar map as a map with a boundary: the **boundary** $\partial\mathfrak{m}$ is just the contour of the root face, the degree of the root face is in this case also called the **perimeter** of the map and denoted by $2|\partial\mathfrak{m}|$, so that $|\partial\mathfrak{m}|$ is the **half-perimeter** of the map (since we work with bipartite maps, the perimeters are all even numbers and we decided to always divide them by two to avoid parity issues). Notice that the boundary $\partial\mathfrak{m}$ may contain pinch points[1], if it does not, the boundary is said to be **simple** and the map

[1] i.e. a vertex visited more than once during the contour of the face

is ∂-simple (not to confuse with simple maps). For $k \geq 3$, a k-angulation with a boundary is a planar map whose faces are all of degree k except the root face which can be of arbitrary degree (subject to parity constraints). Recall also our convention that $\dagger$ is the only map with a unique vertex and one face, which will be seen as the unique map with a boundary of perimeter 0. The above notions are easily extended to the case of infinite planar maps. For $\ell \geq 0$, we denote by

$$\mathcal{M}^{(\ell)}$$

the set of all finite bipartite planar maps such that the root face has degree 2ℓ (in particular $\mathcal{M}^{(0)} = \{\dagger\}$). We insist on the fact that the root face may not be simple, that is, may contain pinch points. For $n \geq 0$, we also write

$$\mathcal{M}^{(\ell)}[n]$$

for the set of all finite bipartite planar maps having a boundary of perimeter 2ℓ and n edges in total[2].

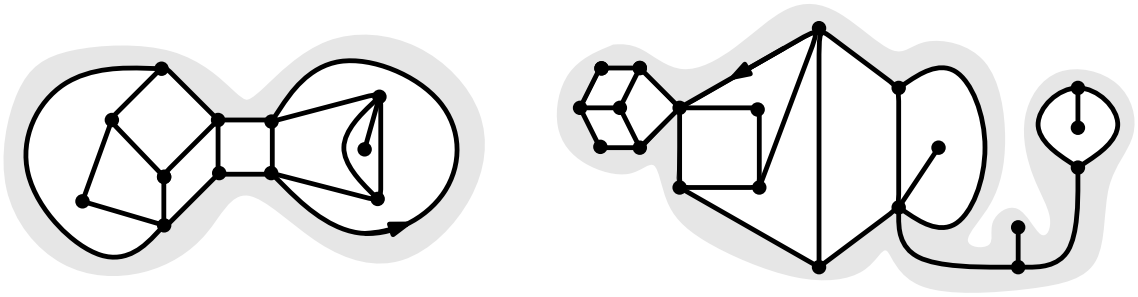


Figure 3.1: A quadrangulation with a simple boundary (on the left) and a quadrangulation with a general boundary (on the right). The root face, or external face, is in light gray.

Remark 3.1 (Maps with an external face of degree 2). There is a unique planar map with a boundary of perimeter 2 (necessarily simple) and no internal face which we shall call the **edge map**: it is made of a single oriented edge. Notice that if $\mathfrak{m}$ is a bipartite map then the root edge cannot be a loop (since the vertices of its endpoints cannot be colored in black and white). If we cut along the root edge and "unzip it", we transform any bipartite map $\mathfrak{m}$ into a planar map with a boundary of perimeter 2 and at least one internal face, see Figure 3.2 below.

By using this transformation we can always suppose that (finite or infinite) planar maps are in fact maps with a boundary of perimeter 2.

3.1.2 Maps with a target

For reasons that will become clearer in Theorem 3.12, we will also need the notion of a planar map with a "target". This target will be either a distinguished face of perimeter $2p$ different

[2]In these lecture notes, the notion of a "size" of a map is its number of edges. But the results can be adapted to the case where we have chosen number of vertices or faces instead.

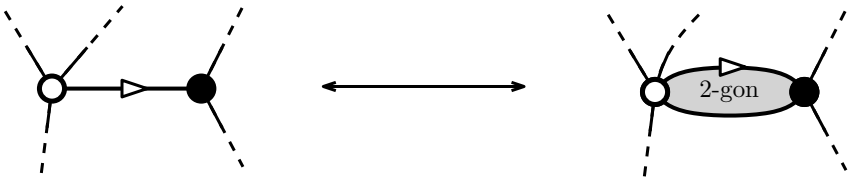


Figure 3.2: Transforming a bipartite planar map (with at least one face) into a bipartite planar map of $\mathcal{M}^{(1)}$ (with at least one inner face).

from the root face, or a distinguished vertex which we will see as a face of perimeter 0. More precisely, a **planar map with a target** $\mathfrak{m}_* = (\mathfrak{m}, *)$ is just a (finite or infinite bipartite) planar map $\mathfrak{m}$ given either with a distinguished vertex $* \in \mathsf{Vertices}(\mathfrak{m})$ (in which case we will sometimes write $* = \bullet$) or a distinguished face $*$ different from the root face (in which case we will write $* = \square$). The most important case is when the target is actually a face of degree 2.

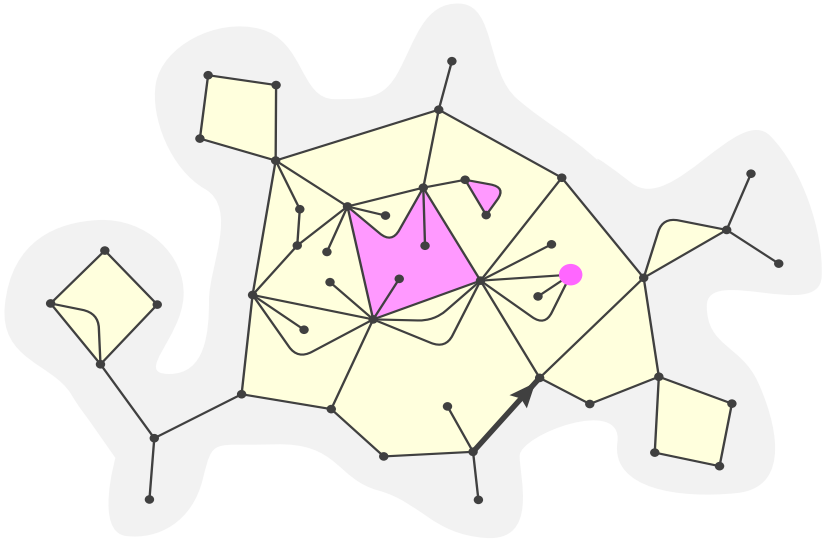


Figure 3.3: A bipartite planar map with a boundary (the root face is in light gray) of perimeter $2\ell = 40$ with three possible different targets (in pink): a face of degree 8, a face of degree 2, and a vertex (seen as a face of degree $p = 0$ in the notation)

We denote accordingly

$$\mathcal{M}_p^{(\ell)} \quad \text{and} \quad \mathcal{M}_p^{(\ell)}[n]$$

the set of all finite (rooted bipartite planar) maps with a boundary of perimeter 2ℓ and a target face of perimeter $2p$ (again when $p = 0$, we agree this is a vertex) and the same set restricted to maps having n edges. Note in particular that the transformation of Remark 3.1 yields a 1-to-n mapping $\mathcal{M}_1^{(1)}[n+1] \to \mathcal{M}^{(1)}[n]$ for all $n \geq 1$, see (3.5). Also, if we distinguish an oriented edge

on the target face (so that the target face is on its right), then we can exchange the roles of the root and target face, see (3.6).

3.2 Counting planar maps & Tutte's equation

We now present the results of Tutte on the enumeration of bipartite planar maps and begin with the simpler case of quadrangulations to help the reader grasp the general idea.

3.2.1 The case of quadrangulations

Let us denote by $\mathcal{Q}^{(\ell)}[n]$ the set of all quadrangulations with n edges and a boundary of perimeter 2ℓ. Recall that when $n = \ell = 0$, the set $\mathcal{Q}^{(0)}[0] = \{\dagger\}$ only contains the vertex map. The idea of Tutte's equation is to write a relation between the $\mathcal{Q}^{(\ell)}[n]$'s by erasing the root edge. Let us describe this decomposition via a figure:

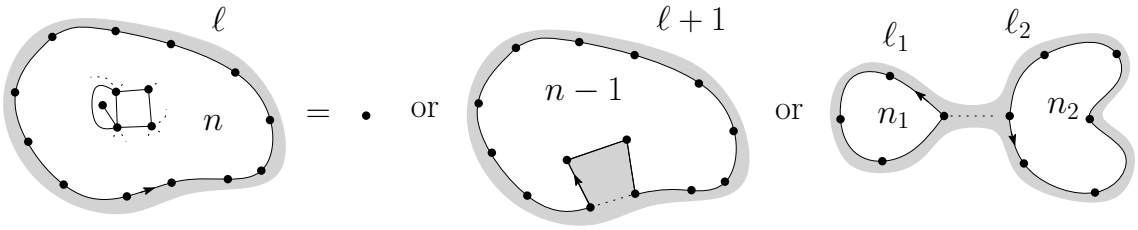


Figure 3.4: Tutte's decomposition.

In words, this decomposition says that a map of $\mathcal{Q}^{(\ell)}[n]$ is either the vertex map (if both $n = \ell = 0$) or we have the following alternative:

- if after erasing the root edge, the map stays connected, then after a proper rooting, we can associate with it an element of $\mathcal{Q}^{(\ell+1)}[n-1]$, where necessarily $\ell + 1 \geq 2$,
- if it does not, then erasing the root edge splits the map into two elements of $\mathcal{Q}^{(\ell_1)}[n_1]$ and $\mathcal{Q}^{(\ell_2)}[n_2]$ respectively (we use a trivial convention for rooting those maps), with $n_1+n_2 = n-1$ and $\ell_1 + \ell_2 = \ell - 1$.

After introducing the generating function (which can be seen at first as a formal power series)

$$Q(g,z) = \sum_{n\geq 0,\ell\geq 0} g^n z^\ell \#\mathcal{Q}^{(\ell)}[n],$$

the former equation becomes Tutte's equation

$$Q(g,z) = 1 + \frac{g}{z}\big(Q(g,z) - [z^0]Q(g,z) - z[z^1]Q(g,z)\big) + gz\big(Q(g,z)\big)^2. \tag{3.1}$$

Notice the terms $1 = [z^0]Q(g,z)$ and $[z^1]Q(g,z)$, which represent the terms in z^0 and z^1 in Q, must be subtracted since the quadrangulations for which the erasure of the root edge leaves a

connected part must be of half-perimeter at least 2 (without these terms, the equation would be very easy to solve!). At first glance, this equation may seem impossible to solve, because we have one equation at our disposal but two unknowns Q and $[z^1]Q(g,z)$. But this is not the case:

Exercise 3.2. Prove that the above equation characterizes Q as a formal power series in g and z: In other words, for any $n, \ell \geq 0$ fixed, Equation (3.1) enables us to compute recursively the number of quadrangulations with a boundary of perimeter 2ℓ and n edges.

Tutte and Brown have developed the so-called quadratic method to solve "explicitly" such equations. This method was later generalized by Bousquet-Mélou & Jehanne. We will not enter the details of this technique which could be the subject of entire books. In the case of quadrangulations, the generating function Q is explicit and is amenable to exact coefficient extractions: The number of quadrangulations with n inner faces and perimeter 2ℓ is

$$\#\mathcal{Q}^{(\ell)}[2n+\ell] = \frac{(2\ell)!}{\ell!(\ell-1)!}3^n\frac{(2n+\ell-1)!}{(n+\ell+1)!n!} \quad \text{and} \quad \#\mathcal{Q}^{(0)}[0] = 1. \tag{3.2}$$

By specifying the previous display with $\ell = 1$, using Remark 3.1 and the bijection presented in Section 2.2.1 we deduce (see the entry A000168 in Sloane's on-line encyclopedia of integer sequences):

Corollary 3.3. *The number of rooted planar maps with n edges is equal to $3^n \cdot \frac{2\cdot(2n)!}{(n+2)!n!}$ for $n \geq 0$.*

3.2.2 Boltzmann maps and Tutte slicing formula

The reader may be amazed by the exact closed and simple formulae one obtained for quadrangulations and general maps in the previous section. Actually, the miracle is deeper and happens when enumerating any kind of *bipartite* maps. Here is the result:

Theorem 3.4 (*Tutte's slicing formula*)

The number $S(\ell_1, \ell_2, \dots, \ell_k)$ of bipartite unrooted planar maps with k distinguishable faces numbered $1, 2, \dots, k$ of degrees $2\ell_1, 2\ell_2, \dots, 2\ell_k$ each carrying a distinguished corner is equal to

$$S(\ell_1, \ell_2, \dots, \ell_k) = \frac{(e-1)!}{v!}\prod_{i=1}^{k}\frac{(2\ell_i)!}{\ell_i!(\ell_i-1)!},$$

where $e = \sum \ell_i$ and $v = 2 + e - k$ respectively are the number of edges and vertices of such a map.



Remark 3.5. Notice the striking similarities with the enumeration of plane trees (maps with one face) with prescribed vertex degrees (Theorem A.9), which oddly, was proved a bit later by Tutte together with Harary and Prins.

Remark 3.6. Taking $\ell_1 = \ell, \ell_2 = 2, \ell_3 = 2, \dots, \ell_{n+1} = 2$ in the above formula, after dividing by $(n!)4^n$ to kill the symmetry between the quadrangles and remove the distinguished corners (except the distinguished one on the face of degree 2ℓ), we recover the formula (3.2).

Tutte's slicing formula can be taken as an enumerative "black-box" in these lecture notes. There are nowadays several routes to prove this wonderful result (see Appendix B) but Tutte's achievement was a tour de force at the time: roughly speaking Tutte "guessed" the formula in the theorem and proved it by induction using the equations coming from erasing the root edge (try it yourself to grasp the difficulty...).

3.3 Formulas for disk partition functions

In the rest of this section we will re-interpret Theorem 3.4 using multivariate generating functions for bipartite maps with parameters counting the faces of different degrees. In probabilisitic terms, we will speak of **Boltzmann measure** on bipartite maps. The enumerative formulas will be explicit in terms of a random walk with i.i.d. increments of law μ supported by $\{-1, 0, 1, 2, ...\}$ defined from the weight sequence $\mathbf{q}$.

Boltzmann measure. Let $\mathbf{q} = (q_k)_{k\geq 1}$ be a non-zero sequence of non-negative real numbers which will be called the **weight sequence** in these pages. We use this sequence to define a σ-finite measure on the set of all finite planar maps by the formula

$$w_{\mathbf{q}}(\mathfrak{m}) \quad = \prod_{f\in \mathsf{Faces}(\mathfrak{m})\setminus\{f_r\}} q_{\deg(f)/2}. \tag{3.3}$$

Notice that *the root face does not receive any weight in the above product.* In particular, if we take the weight sequence $\mathbf{q} = (\mathbf{1}_{k=k_0})_{k\geq 1}$ then $w_{\mathbf{q}}(\mathfrak{m})$ is simply 1 if all the faces apart from the root face are of degree $2k_0$, so the measure is concentrated on $2k_0$-angulations. In the case of maps with a target, if $* = \bullet$, the weight $w_{\mathbf{q}}(\mathfrak{m}_*)$ is simply the weight of $\mathfrak{m}$, and if $* = \square$ it is the weight of $\mathfrak{m}$ where *we remove the factor corresponding to the target face.*

Definition 3.7 (Disk partition functions). *For $\ell, p, n \geq 0$, we introduce the "disk partition functions"*

$$W^{(\ell)}, \quad W^{(\ell)}[n], \quad W_p^{(\ell)}, \quad \text{and} \quad W_p^{(\ell)}[n],$$

the $w_{\mathbf{q}}$-weight of the corresponding sets $\mathcal{M}^{(\ell)}, \mathcal{M}^{(\ell)}[n], \mathcal{M}_p^{(\ell)}, \mathcal{M}_p^{(\ell)}[n]$ of finite planar maps. When we need to make the dependence in $\mathbf{q}$ explicit, we add $(\mathbf{q})$ to those notations. In particular, $W^{(0)}(\mathbf{q}) = W_0^{(0)}(\mathbf{q}) = 1$ but $W_p^{(0)}(\mathbf{q}) = 0$ for $p \geq 1$. The partitions functions $W^{(\ell)}, \dots$ might yield infinite numbers, but they always make sense as formal power series in $q_1, q_2, \dots$.

Exercise 3.8. If $\mathbf{q}$ has at least one $q_i > 0$ for $i \geq 2$ then $W^{(\ell)} > 0$ and $W_p^{(\ell)} > 0$ for all $\ell \geq 1, p \geq 0$.

In the rest of this chapter we will gather enumerative properties on the partition function W that we will use in the rest of the notes. The most important is Theorem 3.12 below which gives a universal formula for $W_p^{(\ell)}$ which builds upon Theorem 3.4. Exactly as in the previous section, those formal series obey some recursive equation, called **Tutte's equation**, obtained by

deleting the root edge:

$$W^{(\ell)} = \sum_{k=1}^{\infty} q_k W^{(\ell+k-1)} + \sum_{\ell_1+\ell_2=\ell-1} W^{(\ell_1)} W^{(\ell_2)}, \quad \ell \geq 1, \tag{3.4}$$

and $W^{(0)} = 1$. As in Exercise 3.2, we let the reader check that the above identities completely characterize the formal power series $W^{(\ell)}(\mathbf{q})$.

3.3.1 Admissibility

Coming back to Boltzmann measures, we will say that the weight sequence $\mathbf{q}$ is admissible if the partition functions W make sense as finite real numbers:

Definition 3.9 (Admissibility). *The weight sequence* $\mathbf{q} = (q_k : k \geq 1)$ *is called* admissible *if the disk partition functions* $W_p^{(\ell)}(\mathbf{q})$ *and* $W^{(\ell)}(\mathbf{q})$ *are finite for all* $\ell, p \geq 0$.

Exercise 3.10. Show that $\mathbf{q}$ is admissible if and only if $W_p^{(\ell)}$ is finite *for one pair* $\ell \geq 1, p \geq 0$. The reader may wonder at this point whether the weaker assumption of finiteness of $W^{(\ell)}$ or just of $W^{(1)}$ is equivalent to admissibility. It is indeed the case (see Corollary 3.15) but we do not know any "elementary" proof of this fact.

Recall from Remark 3.1 the simple but important interpretation of the partition function $W_1^{(\ell)}[n+1]$: a map with $n+1$ edges and with a target face of perimeter 2, can after contraction of the target be seen as a map with a distinguished edge among e edges in total and so for all $\ell \geq 1, e \geq 1$ we have

$$W_1^{(\ell)}[n+1] \quad = \quad n \cdot W^{(\ell)}[n]. \tag{3.5}$$

Also recall that if we distinguish an oriented edge on the target face then we can exchange the roles of the root and target face. This shows that

$$p \cdot W_p^{(\ell)}[n] = \ell \cdot W_\ell^{(p)}[n]. \tag{3.6}$$

The next result, based on Tutte's slicing formula, is the enumerative cornerstone of these lecture notes. It gives a practical criterion for deciding admissibility of a weight sequence and furnishes a formula for computing disk partitions functions. To present it, let us first introduce the function $f_{\mathbf{q}}$:

$$f_{\mathbf{q}}(x) = 1 + \sum_{k=1}^{\infty} q_k \binom{2k-1}{k} x^k.$$

If the equation $f_{\mathbf{q}}(x) = x$ has a positive solution, we shall always denote by $Z_{\mathbf{q}}$ the smallest one and set $c_{\mathbf{q}} = 4Z_{\mathbf{q}}$ (notice $Z_{\mathbf{q}} > 1$ and so $c_{\mathbf{q}} > 4$). In the above case, the function $g_{\mathbf{q}}(z) = \frac{f_{\mathbf{q}}(z \cdot Z_{\mathbf{q}})}{f_{\mathbf{q}}(Z_{\mathbf{q}})}$ is the generating function of a probability measure on $\{0, 1, 2, ...\}$ (of mean less than or equal to 1) which we denote by $(\mu(k-1) : k \geq 0)$ so that μ is a probability measure on $\{-1, 0, 1, ...\}$

of non-positive mean. We shall always denote $(Y_n)_{n\geq 0}$ a μ-random walk[3] started from 0 : it is a skip-free descending random walk which does not drift towards $+\infty$ (see Section A.2.2 for background on fluctuations theory for one-dimensional random walks).

Remark 3.11. Anticipating a few definitions, the probability distribution $(\mu(k-1) : k \geq 0)$ associated to $g_{\mathbf{q}}$ can be interpreted as the offspring distribution of the tree obtained from the Janson–Stefanson coding of the Bouttier–Di Francesco–Guitter mobile associated to the **q**-Boltzmann map with a target vertex. See Section B.3.2.

Theorem 3.12 (***Admissibility criterion and universal form for disk partition functions***)

The sequence $\mathbf{q}$ *is admissible if and only if the equation* $f_{\mathbf{q}}(x) = x$ *has a positive solution. In this case, with the above notation we have for every* $\ell, p, n \geq 1$

$$\begin{aligned} W_p^{(\ell)}[n] &= \frac{\ell}{2}\binom{2\ell}{\ell}\binom{2p}{p}(Z_{\mathbf{q}})^{\ell+p}\frac{1}{n}\mathbb{P}(Y_n = -\ell - p) \\ &= \frac{\ell}{2(\ell+p)}\binom{2\ell}{\ell}\binom{2p}{p}(Z_{\mathbf{q}})^{\ell+p}\,\mathbb{P}(\tau_{-\ell-p} = n), \end{aligned}$$

where τ_{-x} *is the hitting time of* $-x$ *by the* μ*-random walk* (Y) *with i.i.d. increments of law* μ*. If* $p = 0$*, the factor* $1/2$ *in the above formula disappears and we have for every* $\ell, n \geq 1$

$$W_0^{(\ell)}[n] = \frac{\ell}{n}\binom{2\ell}{\ell}(Z_{\mathbf{q}})^{\ell}\,\mathbb{P}(Y_n = -\ell) = \binom{2\ell}{\ell}(Z_{\mathbf{q}})^{\ell}\,\mathbb{P}(\tau_{-\ell} = n).$$

In the rest of these lecture notes we will use the following functions to simplify our notation. For $\ell \geq 0$, define

$$h^{\downarrow}(\ell) = 2^{-2\ell}\binom{2\ell}{\ell}, \tag{3.7}$$

the exponentially-corrected central binomial coefficient. We put $h^{\downarrow}(\ell) = 0$ for $\ell \leq -1$. In particular, $h^{\downarrow}$ vanishes on $\mathbb{Z}_{<0}$ and $h^{\downarrow}(\ell) \sim 1/\sqrt{\pi\ell}$ for large ℓ. Also for for $p \geq 0$ we put for $\ell \neq -p$

$$h_p^{\downarrow}(\ell) = h^{\downarrow}(\ell)h^{\downarrow}(p)\frac{\ell}{\ell+p}, \tag{3.8}$$

with the convention that $h_p^{\downarrow}(-p) = 1$. In particular, when $\mathbf{q}$ is admissible, since $\sum k\mu_k \leq 0$ the stopping times $\tau_{-\ell-p}$ are almost surely finite so that after summing over $n \geq 1$ we deduce the universal formula

$$W_p^{(\ell)} = \frac{1}{2}h_p^{\downarrow}(\ell)c_{\mathbf{q}}^{\ell+p} \quad \text{and} \quad W_0^{(\ell)} = h_0^{\downarrow}(\ell)c_{\mathbf{q}}^{\ell} \tag{3.9}$$

which only depends on $\mathbf{q}$ through $c_{\mathbf{q}}$.

[3]Here and later, if ξ is probability measure on $\mathbb{Z}$, we consider $X_1, X_2, \ldots$ i.i.d. random variables of law ξ and form the process $(x_0 + X_1 + \ldots + X_n)_{n\geq 0}$ which we call a ξ-random walk started from x_0, see Chapter 15.

Proof. Let us start with the case $\ell, p \geq 1$. Recall that $W_p^{(\ell)}[n]$ is the $\mathbf{q}$-weight of planar maps with a boundary of perimeter 2ℓ, with a target face of perimeter $2p$ and n edges in total. Starting with the definition of $S(\ell, p, \ell_1, \dots, \ell_k)$ we obtain that

$$W_p^{(\ell)}[n] = \sum_{\ell_1+\cdots+\ell_k=n-\ell-p} S(\ell, p, \ell_1, \dots, \ell_k) \frac{1}{2p} \frac{1}{2\ell_1 2\ell_2 \dots 2\ell_k} \frac{1}{\mathrm{Sym}(\ell_1, \ell_2, \dots, \ell_k)} q_{\ell_1} \cdots q_{\ell_k},$$

we divide out by $2p2\ell_1 \dots 2\ell_k$ in order to remove the distinguished corner in each face (except the root face) and where the symmetry factor is $\mathrm{Sym}(\ell_1, \ell_2, \dots, \ell_k) = \prod_{i\geq 1} k_i!$ with k_i for the number of faces of degree $2i$ (except the root face and the distinguished face). Hence, thanks to Tutte's slicing formula (Theorem 3.4) we have

$$W_p^{(\ell)}[n] = \frac{(2\ell)!}{\ell!(\ell-1)!} \frac{(2p)!}{p!(p-1)!} \frac{1}{2p} \sum_{k_1,k_2,\dots} \frac{(n-1)!}{v!} \prod_{i\geq 1} \left(\binom{2i-1}{i} q_i \right)^{k_i} \frac{1}{k_i!},$$

where the sum runs over all k_i's so that $n = \sum ik_i + \ell + p$ is the number of edges, and where $v = n - \sum k_i$ is the number of vertices. At this point, we remark that the sum in the previous display can be written as

$$\begin{aligned}
\sum_{k_1,k_2,\dots} \frac{(n-1)!}{v!} \prod_{i\geq 1} \left(\binom{2i-1}{i} q_i \right)^{k_i} \frac{1}{k_i!} &= \sum_{k_1,k_2,\dots} \frac{1}{n} \frac{n!}{v! \prod k_i!} \prod_{i\geq 1} \left(\binom{2i-1}{i} q_i \right)^{k_i} \\
&= \frac{1}{n} \sum_{\substack{k_1,k_2,\dots \\ \sum ik_i = n-p-\ell}} \frac{n!}{v! \prod k_i!} \prod_{i\geq 1} \left(\binom{2i-1}{i} q_i \right)^{k_i} \\
&= \frac{1}{n} [z^{n-p-\ell}] \left(1 + \sum_{i\geq 1} \binom{2i-1}{i} q_i z^i \right)^n,
\end{aligned}$$

where we recall that $[z^n]g(z)$ is the coefficient in front of z^n in the power series $g(z)$. Notice then that for every $\alpha, \beta > 0$ we can write:

$$\begin{aligned}
W_p^{(\ell)}[n] &= \frac{\ell}{2} \binom{2\ell}{\ell} \binom{2p}{p} \frac{1}{n} [z^{n-\ell-p}] \big(f_{\mathbf{q}}(z)\big)^n \\
&\underset{\forall \beta > 0}{=} \frac{\ell}{2} \binom{2\ell}{\ell} \binom{2p}{p} \frac{1}{n} \beta^n [z^{n-\ell-p}] \left(\frac{f_{\mathbf{q}}(z)}{\beta} \right)^n \\
&\underset{\forall \alpha > 0}{=} \frac{\ell}{2} \binom{2\ell}{\ell} \binom{2p}{p} \frac{1}{n} \beta^n \alpha^{\ell+p-n} [z^{n-\ell-p}] \left(\frac{f_{\mathbf{q}}(\alpha \cdot z)}{\beta} \right)^n. \qquad (3.10)
\end{aligned}$$

If $f_{\mathbf{q}}(z) = z$ admits a positive solution, we let $Z_{\mathbf{q}}$ be the smallest one and put $\alpha = \beta = Z_{\mathbf{q}}$ in the last display so that $\frac{f_{\mathbf{q}}(Z_{\mathbf{q}} z)}{Z_{\mathbf{q}}}$ is the generating function of a probability distribution $(\mu(k-1) : k \geq 0)$. If $J_1, J_2, \dots$ are iid random variables on $\{-1, 0, 1, 2, \dots\}$ of law $(\mu(k) : k \geq -1)$ then the last display can be written as

$$W_p^{(\ell)}[n] = \frac{\ell}{2} \binom{2\ell}{\ell} \binom{2p}{p} (Z_{\mathbf{q}})^{\ell+p} \frac{1}{n} \mathbb{P}(J_1 + \cdots + J_n = -\ell - p).$$

Since the random walk $(J_1 + \cdots + J_n)_{n\geq 0}$ is skip-free descending, by Kemperman formula (Proposition A.7) the last display can be reduced to

$$W_p^{(\ell)}[n] \quad = \quad \frac{\ell}{2(\ell+p)}\binom{2\ell}{\ell}\binom{2p}{p}\,(Z_{\mathbf{q}})^{\ell+p}\,\mathbb{P}(\tau_{-\ell-p}=n),$$

where τ_{-x} is the first hitting time of $-x$ by the random walk with increments J_i. This proves the enumerative formula when $f_{\mathbf{q}}$ has a positive fixed point (by summing over n, this implies that $\mathbf{q}$ is admissible). Reciprocally, suppose that $\mathbf{q}$ is admissible. Take any $\alpha > 0$ so that $f_{\mathbf{q}}(\alpha) < \infty$ and set $\beta = f_{\mathbf{q}}(\alpha)$. We deduce from (3.10), Kemperman's formula and after summing over n that

$$W_p^{(\ell)} = \frac{\ell}{2}\binom{2\ell}{\ell}\binom{2p}{p}\alpha^{\ell+p}\mathbb{E}[(\beta/\alpha)^{\tilde{\tau}_{-\ell-p}}] < \infty.$$

As above, $\tilde{\tau}_x$ is the first hitting time of $-x$ by the random walk with independent increments of law $(\tilde{\mu}(k+1) : k \geq -1)$ with $(\tilde{\mu}(k) : k \geq 0)$ has generating function $\tilde{g}_{\mathbf{q}}(z) = f_{\mathbf{q}}(\alpha z)/f_{\mathbf{q}}(\alpha)$. In particular $\mathbb{E}[(\beta/\alpha)^{\tilde{\tau}_{-1}}] < \infty$. On the other hand, applying the Markov property at the first step of the walk, we deduce that $y = \mathbb{E}[(\beta/\alpha)^{\tilde{\tau}_{-1}}]$ satisfies the following equation

$$y = \frac{\beta}{\alpha}\tilde{g}_{\mathbf{q}}(y) = \frac{\beta}{\alpha}f_{\mathbf{q}}(\alpha y)/f_{\mathbf{q}}(\alpha) = \frac{f_{\mathbf{q}}(\alpha y)}{\alpha}.$$

This implies that αy is a fixed point of $f_{\mathbf{q}}$ as desired. The case $p = 0$ is similar (the pointing transforms the factor $v!$ into $(v-1)!$) and is left to the reader. □

3.4 Getting our hands on $W^{(\ell)}$

The above Theorem 3.12 gives a universal formula for $W_p^{(\ell)}$ depending on a single parameter $c_{\mathbf{q}}$. However, we will soon need an access to the asymptotic behavior of $W^{(\ell)}$ which is a bit more complicated, and less universal than its analog with a target (see Chapter 5). This technical section can be skipped at first reading.

3.4.1 Towards an expression for $W^{(\ell)}$

With the notation of Theorem 3.12 we have

$$W^{(\ell)} \underset{(3.5)}{=} \sum_{n\geq 1}\frac{1}{n}W_1^{(\ell)}[n+1] \underset{\text{Thm.3.12}}{=} \frac{1}{2}h_1^{\downarrow}(\ell)c_{\mathbf{q}}^{\ell+1}\cdot\mathbb{E}\left[\frac{1}{\tau_{-\ell-1}-1}\right]. \tag{3.11}$$

Although this expression depends on a rather involved manner on $\mathbf{q}$ via $\tau_{-\ell-1}$, we can first deduce a useful strong ratio limit theorem without any pain:

Lemma 3.13 (Strong ratio limit for $W^{(\ell)}$). *If $\mathbf{q}$ is an admissible weight sequence, then we have*

$$\lim_{\ell\to\infty}\frac{W^{(\ell+1)}}{W^{(\ell)}} = c_{\mathbf{q}}.$$

Proof. By the display before the lemma, it suffices to prove that we have

$$\frac{\mathbb{E}\left[\frac{1}{\tau_{-x}-1}\right]}{\mathbb{E}\left[\frac{1}{\tau_{-x-1}-1}\right]} \underset{x\to\infty}{\sim} \frac{\mathbb{E}\left[\frac{1}{\tau_{-x}}\right]}{\mathbb{E}\left[\frac{1}{\tau_{-x-1}}\right]} \xrightarrow[x\to\infty]{} 1,$$

where we recall that $\tau_{-x} \geq x$ is the (almost surely finite) hitting time of $-x \in \{..., -3, -2, -1\}$ by a random walk with i.i.d. increments of law μ. By applying the Markov property at time τ_{-x} we can write $\tau_{-x-1} = \tau_{-x} + \tilde{\tau}_{-1}$ where $\tilde{\tau}_{-1}$ is independent of τ_{-x} and has the same distribution as τ_{-1}. Since $\tau_1 < \infty$ almost surely, for any $\varepsilon > 0$, we can find $A > 0$ so that $\mathbb{P}(\tilde{\tau}_{-1} \leq A) \geq 1 - \varepsilon$ and since $\tau_{-x} \geq x$ we can choose x large enough so that $\frac{1}{y+A} \geq \frac{1-\varepsilon}{y}$ for any $y \geq x$. Using this we have

$$(1-\varepsilon)^2\mathbb{E}\left[\frac{1}{\tau_{-x}}\right] \leq (1-\varepsilon)\mathbb{E}\left[\frac{1}{\tau_{-x}+A}\right] \leq \mathbb{E}\left[\frac{1}{\tau_{-x}+\tilde{\tau}_{-1}}\mathbf{1}_{\tilde{\tau}_{-1}\leq A}\right] \leq \mathbb{E}\left[\frac{1}{\tau_{-x-1}}\right] \leq \mathbb{E}\left[\frac{1}{\tau_{-x}}\right],$$

and this entails our goal. □

To get an explicit expression of $W^{(\ell)}$ we shall use the following lemma:

Lemma 3.14. *Let $(Y_n)_{n\geq 0}$ be a skip-free descending random walk which does not drift towards $+\infty$ with i.i.d. increments J_i. Then for $x \geq 2$ with the above notation we have*

$$\mathbb{E}\left[\frac{1}{\tau_{-x}-1}\right] = -\mathbb{E}\left[\frac{J}{x+J}\right].$$

Proof. We write

$$\begin{aligned}
\mathbb{E}\left[\frac{\tau_{-x}}{\tau_{-x}-1}\mathbf{1}_{\tau_{-x}=n}\right] &= \frac{n}{n-1}\frac{x}{n}\mathbb{P}(Y_n = -x) && \text{by Kemperman's formula}\\
&= \frac{x}{n-1}\mathbb{E}[\mathbb{P}(Y_{n-1} = -x - J_1 \mid J_1)] && \text{by Markov property at time 1}\\
&= \mathbb{E}\left[\frac{x}{x+J_1}\mathbb{P}(\tilde{\tau}_{-x-J_1} = n-1 \mid J_1)\right] && \text{by Kemperman's formula (again).}
\end{aligned}$$

Since $x \geq 2$, we have $\tau_{-x} \geq 2$ and $\tau_{-x-J_1} \geq 1$ almost surely. Hence, by summing over $n \geq 2$ we deduce that

$$\mathbb{E}\left[\frac{\tau_{-x}}{\tau_{-x}-1}\right] = \mathbb{E}\left[\frac{x}{x+J}\right],$$

and the desired formula follows after subtracting 1 on both sides. □

Combining the above lemma with (3.11) it follows, after a few easy manipulations using generating functions, that for any $\ell \geq 1$ we have

$$W^{(\ell)} = \frac{1}{2}h_1^{\downarrow}(\ell)c_{\mathfrak{q}}^{\ell+1} \cdot \mathbb{E}\left[\frac{-J}{J+\ell+1}\right] = \frac{\ell}{\ell+1}\binom{2\ell}{\ell}\int_0^{Z_{\mathfrak{q}}} \mathrm{d}v \left(f_{\mathfrak{q}}(v) - v f_{\mathfrak{q}}'(v)\right) v^{\ell-1}, \tag{3.12}$$

and this will be used in the next section to compute exactly $W^{(\ell)}$ when $f_{\mathfrak{q}}$ has a nice form.

3.4.2 Back to the admissibility criterion

Another (useful) way to arrive at (3.12) is to make the weight sequence vary. Indeed, if $\mathbf{q}$ is an admissible weight sequence, for $u \in (0,1)$ we introduce the weight sequence $\mathbf{q}_u$ defined by $(q_u)_k = q_k u^k$. It is again admissible, since we decrease the weights. If $\mathfrak{m}_* \in \mathcal{M}_1^{(\ell)}$, it is easy to see (recalling that we do not put any weight on the root face nor on the target face) that $w_{\mathbf{q}_u}(\mathfrak{m}_*) = w_{\mathbf{q}}(\mathfrak{m}_*)u^{|\mathfrak{m}|-\ell-1}$ where $|\mathfrak{m}|$ is the number of edges of the map $\mathfrak{m}$. Hence we have

$$\begin{aligned} W_1^{(\ell)}[n+1](\mathbf{q}_u) &= \sum_{\mathfrak{m}^* \in \mathcal{M}_1^{(\ell)}[n+1]} w_{\mathbf{q}}(\mathfrak{m}^*) u^{n-\ell} \\ &\underset{(3.5)}{=} nu^{n-\ell} \sum_{\mathfrak{m} \in \mathcal{M}^{(\ell)}[n]} w_{\mathbf{q}}(\mathfrak{m}) = nu^{n-\ell} \cdot W^{(\ell)}[n](\mathbf{q}). \end{aligned} \tag{3.13}$$

Noticing that $f_{\mathbf{q}_u}(z) = f_{\mathbf{q}}(uz)$, one can use Theorem 3.12 and the exact expressions of $h_1^{\downarrow}(\ell)$ to recover (3.12) by integrating over $u \in (0,1)$. Let us deduce a corollary of the previous displays, which says that the admissibility definition could have equivalently been based on finiteness of $W^{(\ell)}$ rather than on the finiteness of $W_p^{(\ell)}$.

Corollary 3.15. *Let $\mathbf{q}$ be a weight sequence such that $W^{(\ell)}$ is finite for some $\ell \geq 1$ (as opposed to the stronger condition $W_p^{(\ell)} < \infty$). Then $\mathbf{q}$ is admissible in the sense of Definition 3.9.*

Proof. Let $\mathbf{q}$ be as in the statement and $u \in (1/2, 1)$. Recall (3.13) and notice that $nu^{n-\ell}$ is bounded (independently of n) so that by summing over n we deduce that if $W^{(\ell)}(\mathbf{q}) < \infty$ then $W_1^{(\ell)}(\mathbf{q}_u) < \infty$. By Exercise 3.10, this shows that $\mathbf{q}_u$ is indeed admissible in the sense of Definition 3.9. Hence, by Theorem 3.12, for every $u \in (1/2, 1)$ there is a solution to $f_{\mathbf{q}_u}(z) = f_{\mathbf{q}}(uz) = z$. Taking limit as $u \to 1$ it is easy to see that $f_{\mathbf{q}}(z) = z$ also admits a solution, and $\mathbf{q}$ is admissible thanks to Theorem 3.12. □

3.5 Examples

In this section, we give a few examples of weight sequences which we will use throughout these lecture notes. We also include the case of triangulations, which although not bipartite, is relatively simple and enables the reader to extend our future results to the case of random triangulations. Some of the weight sequences below are *critical*: the reader is referred to the following Chapter 7 for definition. The section is not meant to be entirely read at first sight.

3.5.1 $2p$-angulations

The most obvious weight sequence is

$$\mathbf{q} = (\xi \cdot \mathbf{1}_{k=p})_{k \geq 0}, \quad \text{for } \xi > 0,$$

which gives a weight ξ^n to any $2p$-angulation with n inner faces. In this case, the function $f_{\mathbf{q}}(z)$ is thus equal to $1 + \binom{2p-1}{p} \xi \cdot z^p$. It is easy to see that $f_{\mathbf{q}}(z) = z$ has a solution (and so $\mathbf{q}$ is admissible)

as long as $\xi \le \xi_{\mathrm{crit},2p}$ where

$$\xi_{\mathrm{crit},2p} = \left(\frac{p}{p-1}\right)^{-p} \frac{p!(p-1)!}{(p-1)(2p-1)!}.$$

In particular $\xi_{\mathrm{crit},4} = \frac{1}{12}, \xi_{\mathrm{crit},6} = \frac{2}{135}, \xi_{\mathrm{crit},8} = \frac{27}{8960}, \dots$. When $\xi = \xi_{\mathrm{crit},2p}$ the weight sequence is still admissible and becomes critical. We have then

$$Z_{\mathbf{q}} = \frac{p}{p-1}, \quad c_{\mathbf{q}} = \frac{4p}{p-1}.$$

Moreover, a specialization of Tutte's slicing formula easily gives the number of $2p$-angulations with a boundary of perimeter 2ℓ with n faces. We can also compute exactly the partition functions $W^{(\ell)}$ when $\xi = \xi_{\mathrm{crit}}$ using (3.11) and (3.12) since in this case $f_{\mathbf{q}}(v) = 1 + (p-1)^{p-1}p^{-p}v^k$, so the integrals are easily evaluated. This yields the remarkable formula

$$W^{(\ell)} = \underbrace{\left(\frac{p}{p-1}\right)^{\ell}\binom{2\ell}{\ell}}_{c_{\mathbf{q}}^{\ell}h^{\downarrow}(\ell)} \frac{p}{(\ell+1)(\ell+p)}.$$

3.5.2 Uniform bipartite maps

Another natural weight sequence is $\mathbf{q} = (\alpha \cdot \beta^k)_{k\ge 1}$ which gives weight $\alpha^f \beta^{2n}$ to any bipartite map with $2n$ edges and f inner faces. In this case, we have $f_{\mathbf{q}}(z) = 1 + \frac{\alpha}{2}(\frac{1}{\sqrt{1-4\beta z}} - 1)$, and $\mathbf{q}$ is admissible if and only if (α, β) is below the (lowest) line defined by

$$1 + 6(-2+\alpha)\beta + 8(-2+\alpha)^3\beta^3 - 3(4+\alpha)(-4+5\alpha)\beta^2 = 0.$$

In the critical case when (α, β) lies on this line we have $1 + \beta + \beta Z_{\mathbf{q}}(-6 + Z_{\mathbf{q}}(-3 + 16\beta Z_{\mathbf{q}})) = 0$. In particular, if $\alpha = 1$, we must have $\beta \le 1/8$, and if $\beta = 1/8$ then we have $Z_{\mathbf{q}} = \frac{3}{2}$. This case corresponds to critical uniform bipartite maps for which $W^{(\ell)}$ is again explicit in terms of special functions (we refrain from giving its expression).

3.5.3 Triangulations

Although not bipartite, triangulations form a very natural class of maps. This is the nicest class of maps after the bipartite case since they also possess explicit enumerative formulas. We shall gather a few enumerative results in this section for the reader eager to extend the results of these lecture notes to random triangulations (in particular to the famous Uniform Infinite Planar Triangulation). We denote here $\mathcal{T}^{(\ell)}$ the set of all (finite rooted planar) triangulations with a boundary of perimeter $\ell \ge 0$.

In the context of triangulations, the root-transform of Remark 3.1 is changed into another, slightly more complicated surgical operation transforming any triangulation of the sphere into a triangulation with a boundary of perimeter 1: Let $\mathfrak{t}$ be a triangulation, if we split the root edge

of a $\mathfrak{t}$ into a double edge, then add a loop inside the region bounded by this double edge, and root the resulting triangulation on this new loop in clockwise direction, we obtain a triangulation with a (necessarily simple) boundary of perimeter 1. This yields a bijection between triangulations of the 1-gon with n vertices and triangulations of the sphere with n vertices. The number of

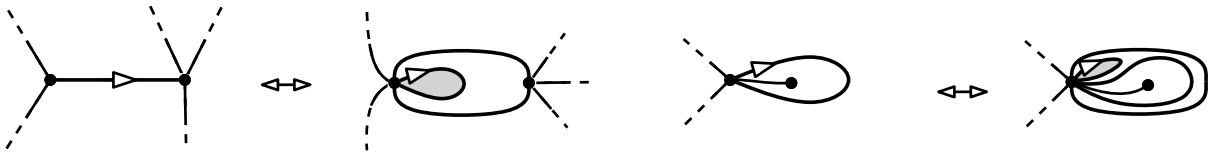


Figure 3.5: Transforming the root edge of a triangulation into a loop (in the right part, the case where the initial root edge is a loop) provides a bijection between triangulations of the sphere and triangulations of the 1-gon.

triangulations of the sphere with $2n$ faces is explicit and equals (see entry A002005 in Sloane's on-line encyclopedia of integer sequences):

$$\frac{2^{2n+1}(3n)!!}{(n+1)!(n+2)!!}, \quad n \geq 1.$$

In particular the Boltzmann measure putting weight ξ on each *triangle* is admissible iff

$$\xi \leq \xi_{\mathrm{trig}} = \frac{1}{\sqrt[4]{432}}.$$

In the critical case when $\xi = \xi_{\mathrm{trig}}$, the generating function $\mathrm{T}(y) = \sum_{\ell \geq 1} \#\mathcal{T}^{(\ell)} y^\ell$ can be explicitly computed from Tutte's equation $\mathrm{T} = 1 + (y\mathrm{T})^2 + \xi_{\mathrm{trig}} \frac{\mathrm{T} - 1 - y[y^1]\mathrm{T}}{y}$. The calculation of $[y^1]\mathrm{T} = 3^{3/4} - 3^{5/4}/2$ can be performed using the explicit formula above. We find that T has a radius of convergence equal to $1/c_{\mathbf{q}}$ where

$$c_{\mathbf{q}} = \sqrt{6 + 4\sqrt{3}},$$

and we find the exact generating function

$$\mathrm{T}\left(\frac{y}{c_{\mathbf{q}}}\right) = \sum_{k \geq 1} \mathcal{T}^{(\ell)} y^\ell = \frac{2}{\left(y(\sqrt{3}-1)\right)^3} \left((3-\sqrt{3})y - 1 + (1-y)\sqrt{(1-y)(1+(2\sqrt{3}-3)y)} \right).$$

3.5.4 Canonical stable maps

We end this section by giving an explicit weight sequence $\mathbf{q}$ with an asymptotic behaviour $q_k \sim c\kappa^{k-1}k^{-a}$ and which is furthermore critical (see the forthcoming Definition/Theorem 5.4). We will call the associated random maps the "canonical maps of type $a \in (3/2; 5/2]$". For $a \in (3/2; 5/2)$ we consider the weight sequence given by

$$q_k = c\kappa^{k-1} \frac{\Gamma(\frac{1}{2} - a + k)}{\Gamma(\frac{1}{2} + k)} \mathbf{1}_{k \geq 2}, \qquad \kappa = \frac{1}{4a-2}, \qquad c = \frac{-\sqrt{\pi}}{2\,\Gamma(3/2-a)}. \tag{3.14}$$

Notice that this weight sequence is term-wise continuous as $a \to 5/2$ taking the value $q_k = \frac{1}{12}\mathbf{1}_{k=2}$, which corresponds exactly to critical quadrangulations.

Lemma 3.16. *For $a \in (3/2; 5/2]$ the weight sequence (3.14) is admissible (and critical). Furthermore we have $c_{\mathbf{q}} = 4a - 2$ and $W^{(\ell)}$ is given by*

$$W^{(\ell)} = \frac{1}{2}(4a-2)^{\ell+1} c \frac{\Gamma(1/2 - a - \ell)}{\Gamma(1/2 - \ell)}, \quad \ell \geq 0.$$

Proof. It is elementary (but tedious) to check that the function $f_{\mathbf{q}}(z)$ is given by

$$\frac{\gamma z - 2z^2 + \gamma\left(1 - \frac{2z}{\gamma}\right)^{a+\frac{1}{2}}}{\gamma - 2z},$$

where $\gamma = 2a - 1$. From there, one checks that $f_{\mathbf{q}}(z) = z$ indeed has a solution for $z = a - \frac{1}{2}$ and furthermore $z \mapsto f_{\mathbf{q}}(z)$ and $z \mapsto z$ are tangent at that point (this will later show that the weight sequence is critical). The computation of $W^{(\ell)}$ is performed using (3.11) and (3.12) as before. □

We gather here a few computations of scaling constants in the case of critical $2p$-angulations, critical triangulations, critical uniform bipartite maps and (critical) canonical stable maps of parameter $a \in (3/2; 5/2]$. Of course, the meaning of (some of) these constants is unclear at this stage but we will see that they enable us to compute various quantities related to volume growth or percolations thresholds.

	$2p$-angulations	Unif. Bipartite	Triang.	(3.14), $a \in (2; 5/2]$
$c_{\mathbf{q}}$	$\frac{4p}{p-1}$	6	$\sqrt{6+4\sqrt{3}}$	$4a-2$
$\mathsf{e}_{\mathbf{q}}$	$4^{p-1}\left(\binom{2p-2}{p-1}\right)^{-1}$	$\frac{7}{2}$	$1+\frac{1}{\sqrt{3}}$	$1+\frac{1}{2a-4}$
$\mathsf{p}_{\mathbf{q}}$	$\frac{p-1}{2\sqrt{\pi}}$	$\frac{9}{4\sqrt{\pi}}$	$\left(\frac{3}{8\pi(\sqrt{3}-1)}\right)^{1/2}$ (*)	$\frac{\Gamma(a-1/2)}{2\sqrt{\pi}}$
$\mathbb{P}(\hat{\mathcal{G}}_r > 0)$	$\frac{p}{4(p-1)}\left(2^{2p-1}\frac{(p-1)!(p-1)!}{(2p-1)!} - 2\right)^2$	$\frac{3}{8}$	$\frac{1}{4\sqrt{3}}$	$\frac{2a-1}{8(a-1)^2}$

(*) in the case of triangulations, the constant $\mathsf{p}_{\mathbf{q}}$ is computed for the perimeter and not the half-perimeter. We also recall the relation with other constants used in the notes which hold for the above examples:

$$Z_{\mathbf{q}} = \frac{c_{\mathbf{q}}}{4}, \quad \hat{\mathsf{e}}_{\mathbf{q}} = \mathsf{e}_{\mathbf{q}} = 1 + 2\mathsf{g}_{\mathbf{q}} = 1 + 2\hat{\mathsf{g}}_{\mathbf{q}}, \quad \mathsf{b}_{\mathbf{q}} \equiv \mathsf{b}_{\mathbf{q}}^{\text{edges}} = \frac{2}{c_{\mathbf{q}}\,\mathsf{p}_{\mathbf{q}}\sqrt{\pi}}, \quad \mathbb{P}(\hat{\mathcal{G}}_r > 0) = \hat{c}_{\mathbf{q}}^{-1}.$$

Bibliographical notes. The enumeration of planar maps is still a very active field of research which started with the pioneer works of Tutte (mostly based on Tutte's equation) in his series of papers "A census of ...". In particular Tutte slicing formula is taken from [185], see also [39, 81, 83, 177] for alternative proofs. We advise the reading of [47] for a gentle introduction to

manipulation of generating functions and enumeration of maps. Although theoretically accessible using Tutte's equation, the enumeration results presented in this section are more efficiently proved (and understood) using bijections between maps and labeled trees (Schaeffer's bijection [178] in the case of quadrangulations and its extension by Bouttier–Di Francesco–Guitter [48] in the case of bipartite maps), see Appendix B. In particular, the measure μ appearing in Theorem 3.12 is connected to the underneath tree structure of those maps (see Remark 3.11). The formalism of this chapter is largely adapted from Marckert & Miermont [150] and later revisited by Budd [51, 53, 55]. Our point of view is nonetheless slightly more probabilist than those references. The universal forms in Theorem 3.12 were observed in [53] in the pointed case, or in [51] and [81], see also [110]. Corollary 3.15 has been observed by L. Chen and L. Richier. Lemma 3.14 is taken from [77, Theorem 2]. Uniform bipartite maps have been studied in [1] and the enumeration of triangulations is taken e.g. from Krikun [135]. The "canonical discrete stable" weight sequences of Section 3.5.4 have been introduced in [9].

Part II:
Peeling explorations

In this part, we introduce the Markovian exploration of random Boltzmann maps called the peeling process. After defining it for finite random maps, we use it to construct infinite random maps of the half-plane and of the plane as local limits of finite random planar maps. The study of the asymptotic properties of the peeling process induces a classification on the weight sequences.

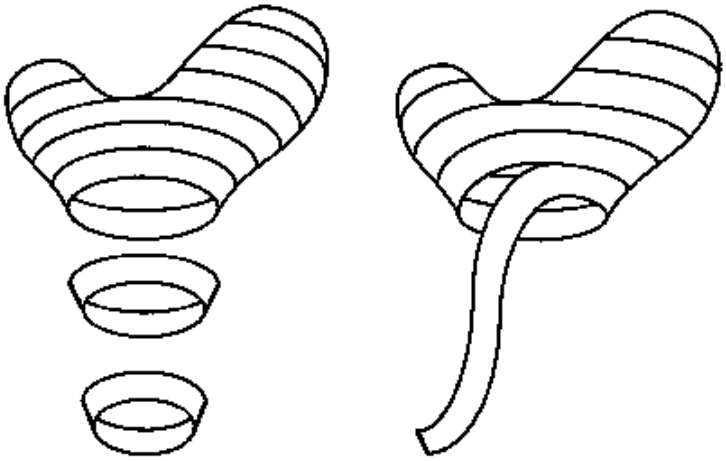


Figure 3.6: Original figures from the paper of Watabiki [190].

Chapter IV : Peeling of finite Boltzmann maps

In this chapter, we define what we mean by peeling a deterministic (bipartite) planar map. We then compute the law of the Markov chains given by peeling explorations of finite $\mathbf{q}$-Boltzmann planar maps and show in the next chapter that a natural random walk hides behind such explorations. We also develop another notion of exploration called *simple peeling* process based on maps with simple boundaries. Although more complicated than the peeling process, the simple peeling process will be useful later on when studying site percolation on random maps.

> The reader eager to manipulate planar maps in order to get a better understanding of the notions developed in this chapter is warmly encouraged to play with the wonderful (and free!) software developed by Timothy Budd available here:
>
> *http://hef.ru.nl/~tbudd/software/*

4.1 Peeling processes

4.1.1 Gluing maps with a boundary

Let $\mathfrak{m}$ be a (rooted bipartite) planar map and recall that $\mathfrak{m}^\dagger$ stands for its dual map whose vertices are the faces of $\mathfrak{m}$ and whose edges are dual to those of $\mathfrak{m}$. The origin of $\mathfrak{m}^\dagger$ is the root face f_r incident on the right of the root edge of $\mathfrak{m}$. Let $\mathfrak{e}^\circ$ be a finite connected subset of edges of $\mathfrak{m}^\dagger$ such that the origin of $\mathfrak{m}^\dagger$ is incident to $\mathfrak{e}^\circ$ (the letter "$\mathfrak{e}$" stands for **explored**). We associate with $\mathfrak{e}^\circ$ a planar map $\mathfrak{e}$ which, roughly speaking, is obtained by gluing the faces of $\mathfrak{m}$ corresponding to the vertices adjacent to $\mathfrak{e}^\circ$ along the (dual) edges of $\mathfrak{e}^\circ$, see Figure 4.1. The resulting map, rooted at the root edge of $\mathfrak{m}$, is a finite (rooted bipartite) planar map with several distinguished faces $h_1, \dots, h_k \in \mathsf{Faces}(\mathfrak{e})$ that correspond to the connected components of $\mathfrak{m}^\dagger \backslash \mathfrak{e}^\circ$. These distinguished faces are called the **holes** of $\mathfrak{e}$. Notice that the holes are simple, meaning that there is no pinch-point on their boundaries, and these boundaries do not have vertices in common (but a hole could share part of its boundary with the root face). Such an object will be called a planar map with holes. See Figure 4.1 below.

We say that $\mathfrak{e}$ is a **submap** of $\mathfrak{m}$ and write

$$\mathfrak{e} \subset \mathfrak{m}$$

since $\mathfrak{m}$ can be obtained back from $\mathfrak{e}$ by gluing inside each hole h_i of $\mathfrak{e}$ a (uniquely defined) bipartite planar map $\mathfrak{u}_i$ of perimeter $\deg(h_i)$ –the letter $\mathfrak{u}$ stands for *unexplored*. To perform this

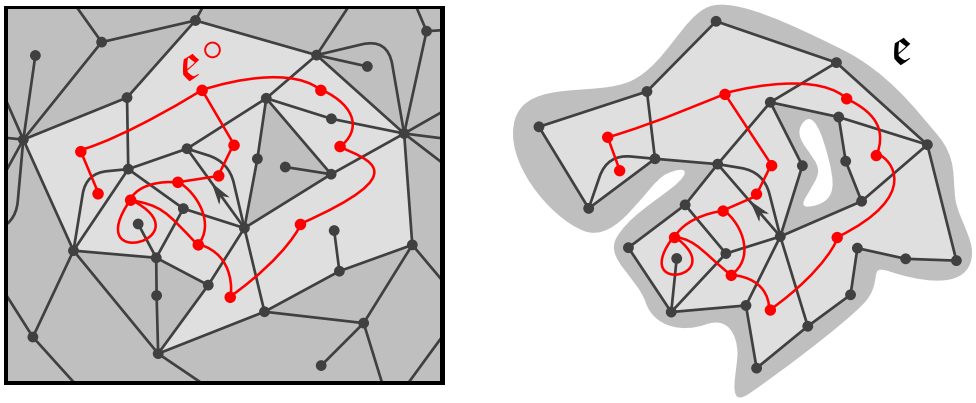


Figure 4.1: Illustration of the duality between connected subsets of edges on the dual map and their associated submaps on the primal lattice.

gluing operation, we implicitly assume that an edge has been distinguished on the boundary of each hole h_i of $\mathfrak{e}$, on which we glue the root edge of $\mathfrak{u}_i$. We will not mention this further, since these edges can be arbitrarily chosen using a deterministic procedure given $\mathfrak{e}$. Notice that after this gluing operation, it might happen that a few pairs of edges on the boundary of a given hole of $\mathfrak{e}$ get identified because the boundary of $\mathfrak{u}_i$ may not be simple, see Figure 4.2 below.

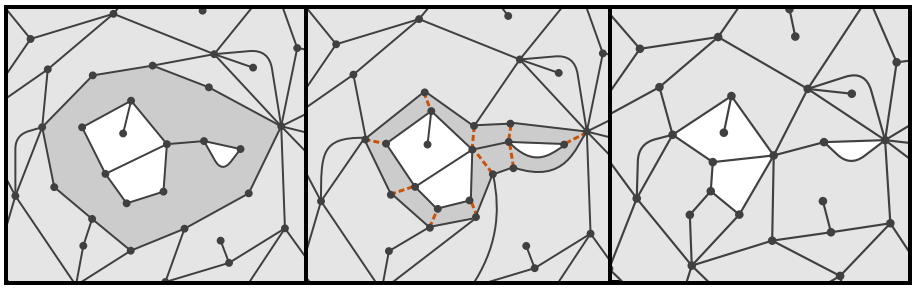


Figure 4.2: Illustration of the gluing operation. Notice that two edges on the boundary of the hole get identified during the gluing.

It is easy to see that this operation is rigid in the sense that if $\mathfrak{e} \subset \mathfrak{m}$, then the maps $(\mathfrak{u}_i)_{1\leq i\leq k}$ are uniquely defined (in other words, if one glues different maps inside a given planar map with holes, one gets different maps after the gluing procedure). This definition even makes sense when $\mathfrak{e}$ is a finite map and $\mathfrak{m}$ is an infinite map. Conversely, if $\mathfrak{e} \subset \mathfrak{m}$, one can recover $\mathfrak{e}^\circ$ in a unique way as the set of all the dual edges between faces of $\mathfrak{e}$ which are not holes.

This discussion shows that there are two points of view on submaps of $\mathfrak{m}$ which are equivalent: either submaps can be seen as objets of the type of $\mathfrak{e}^\circ$ (which are connected components of edges containing the origin in $\mathfrak{m}^\dagger$), or as planar maps $\mathfrak{e} \subset \mathfrak{m}$ with holes (possibly none) which may be filled-in to obtain $\mathfrak{m}$. In these lecture notes, we will mostly work with the second point of view.

In the case when $\mathfrak{e}^\circ$ is empty, by convention we shall agree it contains one point (the root

face) and no dual edges; then we abuse notation and say that $\mathfrak{e}$ is equal to the root face f_r. In fact, when doing so, we see the root face of $\mathfrak{m}$ as a map with one hole made of one simple face of degree $\deg(f_r)$ and the corresponding hole of the same perimeter.

4.1.2 Peeling process

A peeling exploration is a means to explore a planar map $\mathfrak{m}$ edge after edge. If $\mathfrak{e} \subset \mathfrak{m}$ is a planar map with holes, the **active boundary** of $\mathfrak{e}$ is the union of all the edges adjacent to the holes of $\mathfrak{e}$. We denote it by

$$\partial^* \mathfrak{e}$$

(not to be confused with the boundary $\partial\mathfrak{e}$ of the map, i.e. the contour of the root face) and its *half-perimeter* by $|\partial^*\mathfrak{e}|$. Formally, a peeling exploration depends on a function $\mathcal{A}$, called the **peeling algorithm**, which associates with any planar map with holes $\mathfrak{e}$ an edge of $\partial^*\mathfrak{e} \cup \{\dagger\}$, where $\dagger$ is a cemetery point which we interpret as the will to end the exploration. In particular, if $\mathfrak{e}$ has no hole, we must have $\mathcal{A}(\mathfrak{e}) = \dagger$.

Intuitively speaking, given the peeling algorithm $\mathcal{A}$, the peeling process of a (bipartite) planar map $\mathfrak{m}$ is a way to iteratively explore $\mathfrak{m}$ by starting from its boundary and by discovering a piece of $\mathfrak{m}$ by **peeling an edge** determined by the algorithm $\mathcal{A}$. If $\mathfrak{e} \subset \mathfrak{m}$ is a planar map with holes and a is an edge of $\partial^*\mathfrak{e}$ or $a = \dagger$, the planar map with holes $\mathsf{Peel}(\mathfrak{e}, a, \mathfrak{m})$ obtained by peeling a is defined as follows. Let F_a be the face of $\mathfrak{m}$ adjacent to a (provided that $a \neq \dagger$) on the other side of a with respect to $\mathfrak{e}$. Then there are three possibilities, see Figure 4.3:

- Either $a = \dagger$ and $\mathsf{Peel}(\mathfrak{e}, \dagger, \mathfrak{m}) = \mathfrak{e}$.
- *Event* C_k*:* the face F_a is not a face of $\mathfrak{e}$ (it is a face inside the hole to which a belongs) and has degree $2k$. Then $\mathsf{Peel}(\mathfrak{e}, a, \mathfrak{m})$ is obtained by gluing F_a on a without performing the possible identifications of its other edges inside $\mathfrak{m}$.
- *Event* G_{k_1,k_2}*:* the face F_a is actually a face of $\mathfrak{e}$. In this case, the edge a is identified in $\mathfrak{m}$ with another edge a' on the boundary of the same hole where k_1 (resp. k_2) is half of the number of edges on the boundary of the hole strictly between a and a' when turning in clockwise order around the hole (resp. k_2 is the half-perimeter of the other hole), and $\mathsf{Peel}(\mathfrak{e}, a, \mathfrak{m})$ is the map after this identification in $\mathfrak{e}$.

When $k_1 > 0$ and $k_2 > 0$, note that the event G_{k_1,k_2} results in the splitting of a hole into two holes. If $k_1 = 0$ or $k_2 = 0$ the corresponding hole of perimeter 0 is actually a vertex of the map since by our convention the vertex map is the only one map of perimeter 0. In particular, the event $\mathsf{G}_{0,0}$ results in the disappearance of a hole.

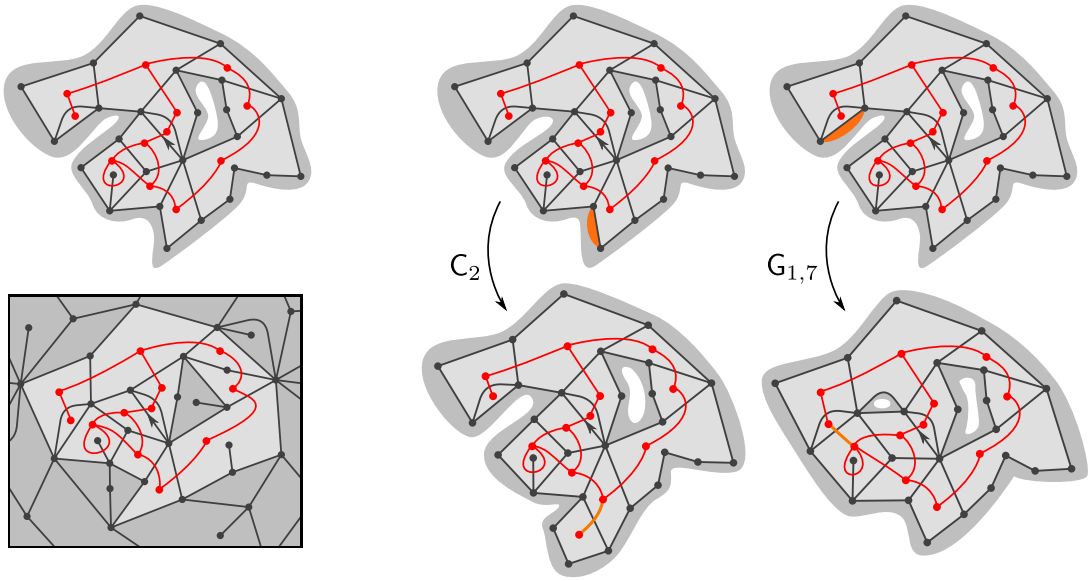


Figure 4.3: Illustration of the different peeling events. On the left column is represented the submap $\mathfrak{e} \subset \mathfrak{m}$ as well as its associated dual version $\mathfrak{e}^\circ$ in red. The center and right columns represent two different peeling events, the edge to be peeled is in thick orange. The event C_2 occurs in the central column, whereas the event $G_{1,7}$ occurs in the right column.

Definition 4.1 (Peeling exploration)**.** *If $\mathfrak{m}$ is a (finite or infinite) planar map, the peeling exploration of $\mathfrak{m}$ with algorithm $\mathcal{A}$ is the sequence of planar maps with holes*

$$\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m},$$

such that the map $\mathfrak{e}_0$ is the root face f_r seen as a map with hole and for every $i \geq 0$

$$\mathfrak{e}_{i+1} = \mathsf{Peel}(\mathfrak{e}_i, \mathcal{A}(\mathfrak{e}_i), \mathfrak{m}).$$

Observe that if $\mathfrak{e}_i \neq \mathfrak{e}_{i-1}$ with $i \geq 1$, then $\mathfrak{e}_i$ has exactly i internal edges. The maps $\mathfrak{e}_i$ are clearly deterministic functions (depending on i and $\mathcal{A}$) of $\mathfrak{m}$. Notice also that although not visible in our notation, the sequence of explored maps $(\mathfrak{e}_i)$ depends obviously on the underlying map, but also on the peeling algorithm $\mathcal{A}$. It should be clear in the following which are the statements valid for all peeling explorations and those for specific ones.

Remark 4.2. One can alternatively represent a peeling exploration $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m}$ as the associated sequence of growing connected subset of edges $(\mathfrak{e}_i^\circ)_{i\geq 0}$ of the dual map $\mathfrak{m}^\dagger$, such that $\mathfrak{e}_{i+1}^\circ$ is obtained from $\mathfrak{e}_i^\circ$ by adding one edge of $\mathfrak{m}^\dagger$ (unless the exploration has stopped) specified by the algorithm, see Figure 4.3. We will however mostly use the first point of view.

Exercise 4.3. Work out the peeling exploration $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_{14} = \mathfrak{m}$ (and the associated

events C_k, G_{k_1,k_2}) when the associated subsets of edges $\mathfrak{e}_i^\circ$ are those containing the dual edges whose index is less than or equal to i in Figure 4.4

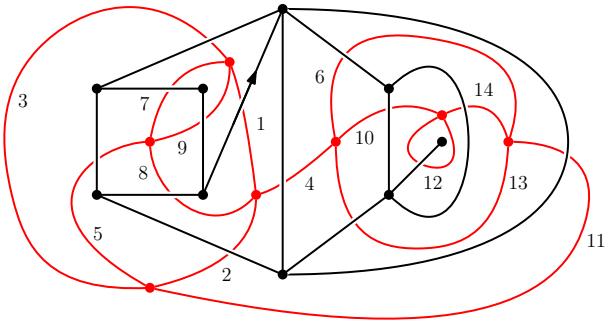


Figure 4.4: Figure for Exercise 4.3 .



Remark 4.4 (A castle interpretation of the peeling, by Jean-François Le Gall). Consider each face of a planar map as a room in a castle having only one floor. The edges incident to a face are the walls bounding this room, and in each such wall there is a door leading to another room (face), which may be the same if both sides of the edge are incident to the same face.

Suppose then that an explorer discovers the castle by opening successively the different doors. Initially the explorer sits in the root face, and only sees this room, and thus the degree of the face corresponding the number of walls/doors he sees (it is important to realize that if both sides of a wall belong to the same room, the explorer sees them as different walls until the moment he opens a door in this wall). At the first step, the explorer chooses one of the doors he sees, and opens it, so that he discovers a new room in which he again sees a certain number of walls and closed doors (as mentioned earlier, there is also the possibility that the door that is opened leads to the same room the explorer started from). After k steps, the explorer has discovered certain rooms and opened certain doors in these rooms. At the next step, among the closed doors that the explorer can still see in the rooms he has discovered, he chooses one of them (according to the peeling algorithm) and opens it, leading either to a new room (not yet explored) or to a room that he had already visited. One can view the information recorded by the explorer after k steps as a planar map with faces corresponding to the rooms already visited and "holes" with simple boundaries corresponding to unexplored regions: in that planar map, edges whose both sides are incident to a white face correspond to opened doors, and the other edges, which have only one side incident to a white face, correspond to closed doors. This is the state $\mathfrak{e}_k$ of the peeling process at step k.

4.1.3 Peeling process with a target and filled-in explorations

In what follows, we will most of the time explore maps with a distinguished target and concentrate our attention the evolution of the hole containing the target by considering *filled-in explorations*.

Let us fix a peeling algorithm $\mathcal{A}$. If $\mathfrak{m}_* = (\mathfrak{m}, *)$ is a map where $* \in \{\bullet, \square, \infty\}$ is either a distinguished vertex, or a distinguished face different from the root face, or the only end of the map $\mathfrak{m}$ if it is infinite and one-ended, then for any submap $\mathfrak{e} \subset \mathfrak{m}$, the pointing of $\mathfrak{m}$ enables to distinguish a unique hole of $\mathfrak{e}$, namely the one containing $*$, unless $* \in \{\bullet, \square\}$ and the vertex/face $*$ is an inner vertex/face of $\mathfrak{e}$ (an inner vertex of a map with holes is a vertex that is not incident to the active boundary). When exploring a map $\mathfrak{m}_*$ the "point" $*$ will be seen as the *target* of the peeling exploration and we define the **filled-in peeling exploration** of $\mathfrak{m}_*$ with algorithm $\mathcal{A}$ as the sequence

$$\overline{\mathfrak{e}}_0 \subset \overline{\mathfrak{e}}_1 \subset \cdots \subset \overline{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{m},$$

of submaps with *at most one hole* where $\overline{\mathfrak{e}}_0$ is the root face of $\mathfrak{m}$ and for each $i \geq 0$, the map $\overline{\mathfrak{e}}_{i+1}$ is obtained by peeling the edge $\mathcal{A}(\overline{\mathfrak{e}}_i)$ and by immediately filling-in the holes that do not contain $*$ that the peeling step may have created. In the case $* \in \{\bullet, \square\}$, if at the i-th peeling step we discover the target $*$, then $\overline{\mathfrak{e}}_{i+1} = \mathfrak{m}$ and the exploration stops. If $* = \infty$, the exploration may go on forever.

4.2 Law of the peeling under the Boltzmann measures

When applied to a random map whose underlying distribution is of product form (the so-called Boltzmann distribution, see below), the peeling exploration $(\mathfrak{e}_i : i \geq 0)$ turns out to be a Markov chain whose probability transitions are explicit in terms of the disk partitions functions $W^{(\ell)}$ and $W_p^{(\ell)}$.

4.2.1 q-Boltzmann maps

As we shall need to work with many different laws on (rooted bipartite planar, possibly infinite) maps (with holes, possibly with target and possibly given with a distinguished hole), it will be convenient to adopt canonical notation: The space of maps is always equipped with the Borel σ-field for the (extended) local topology. We shall denote a generic map by $\mathfrak{m}$ or by $\mathfrak{m}_* = (\mathfrak{m}, *)$ when it has a target. When a peeling algorithm $\mathcal{A}$ is fixed we denote by $(\mathfrak{e}_i)_{i\geq 0}$ and $(\overline{\mathfrak{e}}_i)_{i\geq 0}$ respectively for the peeling and filled-in peeling explorations of the underlying planar map $\mathfrak{m}$ with algorithm $\mathcal{A}$. The canonical filtration generated by the exploration will be denoted by $(\mathcal{F}_n : n \geq 0)$. Let us now present our basic probability measures on finite maps which are the **q-Boltzmann distributions** with or without target.

Definition 4.5 (**q**-Boltzmann maps)**.** *If* $\mathbf{q}$ *is an admissible weight sequence, for any* $\ell \geq 1$ *we define* $\mathbb{P}^{(\ell)}$*, the* $\mathbf{q}$-Boltzmann distribution *on* $\mathcal{M}^{(\ell)}$ *giving a weight*

$$\mathbb{P}^{(\ell)}(\mathfrak{m}) = \frac{w_{\mathbf{q}}(\mathfrak{m})}{W^{(\ell)}}$$

to any planar map with a boundary of perimeter 2ℓ*, and similarly for* $p \geq 0$*, the* $\mathbf{q}$*-Boltzmann distribution with target on* $\mathcal{M}_p^{(\ell)}$ *giving weight*

$$\mathbb{P}_p^{(\ell)}(\mathfrak{m}_*) = \frac{w_{\mathbf{q}}(\mathfrak{m}_*)}{W_p^{(\ell)}}$$

to any planar map with a boundary of perimeter 2ℓ *and a target of perimeter* $2p$*. As usual, we shall denote by* $\mathbb{E}^{(\ell)}$ *and* $\mathbb{E}_p^{(\ell)}$ *the associated expectations and we notice that the dependence in* $\mathbf{q}$ *is implicit.*

For probabilistic convenience, we also write $\mathfrak{M}_p^{(\ell)}$ or $\mathfrak{M}^{(\ell)}$ for a random variable which under the generic probability measure $\mathbb{P}$ has law $\mathbb{P}_p^{(\ell)}$ resp. $\mathbb{P}^{(\ell)}$. We call them **random Boltzmann maps with or without target**. In other words, for any positive measurable function f, we have the following two ways of writing the same expectation:

$$\mathbb{E}_p^{(\ell)}[f(\mathfrak{m}_*)] = \mathbb{E}[f(\mathfrak{M}_p^{(\ell)})] \quad \text{and } \mathbb{E}^{(\ell)}[f(\mathfrak{m})] = \mathbb{E}[f(\mathfrak{M}^{(\ell)})].$$

4.2.2 **q**-Boltzmann maps without target

In this section, we compute the law of the peeling exploration under $\mathbb{P}^{(\ell)}$. To do this, we define the following **probability transitions in the Boltzmann case**: For any $\ell \geq 1, k \geq 1$ and $k_1, k_2 \geq 0$ such that $k_1 + k_2 + 1 = \ell$, we define

$$b^{(\ell)}(k) = q_k \frac{W^{(\ell+k-1)}}{W^{(\ell)}}, \qquad b^{(\ell)}(k_1, k_2) = \frac{W^{(k_1)} W^{(k_2)}}{W^{(\ell)}}.$$

The fact that $b^{(\ell)}$ defines probability transitions, that is for all $\ell \geq 1$

$$1 = \sum_{k \geq 1} b^{(\ell)}(k) + \sum_{k_1 + k_2 + 1 = \ell} b^{(\ell)}(k_1, k_2)$$

is equivalent to Tutte's equation (3.4).

Proposition 4.6 (Law of the peeling process under the Boltzmann distribution)**.** *Fix* $\ell \geq 1$*, a peeling algorithm* $\mathcal{A}$*, and let* $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m}$ *be the peeling exploration of* $\mathfrak{m}$ *with algorithm* $\mathcal{A}$ *under* $\mathbb{P}^{(\ell)}$*.*

- (Spatial Markov Property) *For any* $n \geq 0$*, conditionally on* $\mathfrak{e}_n$*, the maps filling-in the holes of* $\mathfrak{e}_n$ *inside* $\mathfrak{m}$ *are independent* $\mathbf{q}$*-Boltzmann maps with the proper perimeters.*

- *The transition probabilities of the Markov chain* $(\mathfrak{e}_n)_{n\geq 0}$ *are as follows: Conditionally on* $\mathfrak{e}_n$ *and provided that* $\mathcal{A}(\mathfrak{e}_n) \neq \dagger$*, if we denote by* $\mathcal{L}_n$ *the half-perimeter of the hole on which* $\mathcal{A}(\mathfrak{e}_n)$ *is selected, then the events* C_k *and* G_{k_1,k_2} *(where* $k \geq 1$ *and* $k_1 + k_2 + 1 = \mathcal{L}_n$ *with* $k_1, k_2 \geq 0$*) occur respectively with probabilities*

$$b^{(\mathcal{L}_n)}(k) \quad \text{and} \quad b^{(\mathcal{L}_n)}(k_1, k_2).$$

Proof. Let us consider the first step of the peeling process assuming that $\mathcal{A}(\mathfrak{e}_0) \neq \dagger$. By the rigidity of the gluing operation, the event C_k happens if and only if the map $\mathfrak{m}$ is obtained from the gluing of a map of perimeter $2\ell + 2k - 2$ onto the map $\mathfrak{e}_0$ to which we glued a face of degree $2k$ on $\mathcal{A}(\mathfrak{e}_0)$, hence we have

$$\mathbb{P}^{(\ell)}(\mathsf{C}_k \mid \mathcal{F}_0, \mathcal{A}(\mathfrak{e}_0)) = \frac{1}{W^{(\ell)}} \cdot q_k \sum_{\mathfrak{m}_1 \in \mathcal{M}^{(\ell+k-1)}} w_{\mathbf{q}}(\mathfrak{m}_1) = q_k \frac{W^{(\ell+k-1)}}{W^{(\ell)}} = b^{(\ell)}(k).$$

Furthermore, the above calculation shows that conditionally on the above event, the map $\mathfrak{m}_1$ filling-in the hole of $\mathfrak{e}_1$ is distributed according to $\mathbb{P}^{(\ell+k-1)}$. Similarly and again by rigidity, the event G_{k_1,k_2} where $k_1 + k_2 + 1 = \ell$ happens if and only if the map $\mathfrak{m}$ is obtained by first identifying the edge $\mathcal{A}(\mathfrak{e}_0)$ with the edge of the same hole located $2k_1$ steps on its left and then gluing two maps of respective perimeters $2k_1$ and $2k_2$ into the two holes created (recall that when $k_i = 0$ then we just glue a vertex in). The same calculation shows that

$$\mathbb{P}^{(\ell)}(\mathsf{G}_{k_1,k_2} \mid \mathcal{F}_0, \mathcal{A}(\mathfrak{e}_0)) = \frac{1}{W^{(\ell)}} \sum_{\mathfrak{m}_1 \in \mathcal{M}^{(k_1)}} w_{\mathbf{q}}(\mathfrak{m}_1) \sum_{\mathfrak{m}_2 \in \mathcal{M}^{(k_2)}} w_{\mathbf{q}}(\mathfrak{m}_2) = b^{(\ell)}(k_1, k_2).$$

Again, conditionally on the above event, an easy extension of the previous calculation shows that the maps $\mathfrak{m}_1$ and $\mathfrak{m}_2$ filling-in the two holes of $\mathfrak{e}_1$ are independent and respectively distributed according to $\mathbb{P}^{(k_1)}$ and $\mathbb{P}^{(k_2)}$. This proves both points of the proposition for $n = 0, 1$ and $n = 0$ respectively. Using the spatial Markov property, we can then propagate the result in each hole of $\mathfrak{e}_1$ to get the proposition. □

4.2.3 $\mathbf{q}$-Boltzmann maps with target

We now proceed to similar calculations in the case of Boltzmann planar maps with target where we use a filled-in peeling exploration with target $(\overline{\mathfrak{e}}_i : i \geq 0)$. In particular $|\partial^* \overline{\mathfrak{e}}_n|$ is the half-perimeter of the only hole of $\overline{\mathfrak{e}}_n$ (if there is no hole we put it to 0). Recall the definition of the events C_k and G_{k_1,k_2}. In the case of the peeling of the unique hole of $\overline{\mathfrak{e}}_n$, we shall write G_{*,k_2} for the event G_{k_1,k_2} where the hole on the left of the peeled edge becomes the next hole of $\overline{\mathfrak{e}}_{n+1}$ and similarly for $\mathsf{G}_{k_1,*}$.

If $* = \bullet$ and if the perimeter of the hole of $\overline{\mathfrak{e}}_n$ is 2ℓ then the peeling stops on the events $\mathsf{G}_{*,\ell-1}$ and $\mathsf{G}_{\ell-1,*}$. If $* = \square$ and the target face has perimeter $2p$ the events $\mathsf{G}_{*,\ell-1}$ and $\mathsf{G}_{\ell-1,*}$ are impossible but the event C_p may end the peeling exploration if the new discovered face is the

target face. We shall denote this event $\mathsf{C}_p^{\text{stop}}$. We now introduce the **transitions probabilities for the distinguished hole in the case of Boltzmann with target distribution**: For any $\ell \geq 1, p \geq 0, k \geq 1$ and $k_1, k_2 \geq 0$ such that $k_1 + k_2 + 1 = \ell$ we put

$$b_p^{(\ell)}(k) = q_k \frac{W_p^{(\ell+k-1)}}{W_p^{(\ell)}}, \qquad b_p^{(\ell)}(*, k_1) = b_p^{(\ell)}(k_1, *) = \frac{W^{(k_1)} W_p^{(k_2)}}{W_p^{(\ell)}}, \qquad \text{and } b_p^{(\ell)}(\square) = \frac{W^{(\ell+p-1)}}{W_p^{(\ell)}}.$$

Again the fact that $b_p^{(\ell)}$ defines probability transitions for all $\ell, p \geq 0$ is equivalent to Tutte's equation with a target (recall that the target face does not contribute to the weight of the map).

Proposition 4.7 (Law of the peeling process with target under the Boltzmann distribution)**.** *Fix $\ell \geq 1, p \geq 0$, a peeling algorithm $\mathcal{A}$, and let $\overline{\mathfrak{e}}_0 \subset \overline{\mathfrak{e}}_1 \subset \cdots \subset \overline{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{m}$ be the filled-in peeling exploration of $\mathfrak{m}_*$ with algorithm $\mathcal{A}$ under $\mathbb{P}_p^{(\ell)}$.*

- (Spatial Markov Property) *For any $n \geq 0$, conditionally on $\overline{\mathfrak{e}}_n$ the map filling-in the hole of $\overline{\mathfrak{e}}_n$ inside $\mathfrak{m}$ has law $\mathbb{P}_p^{(|\partial^* \overline{\mathfrak{e}}_n|)}$.*
- *The transition probabilities of the Markov chain $(\overline{\mathfrak{e}}_n)_{n \geq 0}$ are as follows: Conditionally on $\overline{\mathfrak{e}}_n$ and provided that $\mathcal{A}(\overline{\mathfrak{e}}_n) \neq \dagger$, the events $\mathsf{C}_k, \mathsf{C}_p^{\text{stop}}$, $\mathsf{G}_{k,*}$ and $\mathsf{G}_{*,k}$ (where $0 \leq k \leq |\partial^* \overline{\mathfrak{e}}_n| - 1$) occur respectively with probabilities*

$$b_p^{(|\partial^* \overline{\mathfrak{e}}_n|)}(k), \quad b_p^{(|\partial^* \overline{\mathfrak{e}}_n|)}(\square), \qquad b_p^{(|\partial^* \overline{\mathfrak{e}}_n|)}(k, *) \quad \text{and} \quad b_p^{(|\partial^* \overline{\mathfrak{e}}_n|)}(*, k),$$

and conditionally on each event, the holes filled-in during the exploration are independent $\mathbf{q}$-Boltzmann maps (without target) of the proper perimeters.

Proof. This is mutatis mutandis the same proof as for Proposition 4.6. □

4.3 Simple submaps and simple peeling explorations

This section presents another notion of peeling process based on maps with simple boundaries (as opposed to maps with a general boundary in the previous section) which will be seldom used in these lecture notes. This process has a more complicated combinatorial description and is (as of today) not as well understood as the preceding peeling process. The reader may skip this section at first reading.

To create a new notion of peeling exploration, we first need a new notion of submap.

4.3.1 Maps with simple boundary

Recall that a face f of a (bipartite rooted planar) map $\mathfrak{m}$ is **simple** if it contains no pinch points, i.e. if we do not visit a given vertex more than once during the contour of the face f in clockwise order, see Figure 3.1. *We will add a hat "$\hat{}$" to our usual notation when dealing with simple boundary analogs.* E.g. we denote by $\hat{\mathcal{M}}^{(\ell)}$ the set of all finite (bipartite rooted planar maps)

∂-simple maps of perimeter 2ℓ and $\hat{W}^{(\ell)}$ for the total $w_{\mathbf{q}}$-weight of this set. We write $\hat{\mathbb{P}}^{(\ell)}$ for the simple $\mathbf{q}$-Boltzmann measure on $\hat{\mathcal{M}}^{(\ell)}$ obtained by normalizing the $w_{\mathbf{q}}$-weights and denote by $\hat{\mathfrak{M}}^{(\ell)}$ a random planar map of law $\hat{\mathbb{P}}^{(\ell)}$.

Notice that in the case $\ell = 1$ we have $\mathcal{M}^{(1)} = \hat{\mathcal{M}}^{(1)}$ since in the bipartite case, a face of degree 2 is necessarily simple, so that $W^{(1)} = \hat{W}^{(1)}$. We refer to the forthcoming Chapter 9 for more on the enumeration of Boltzmann maps with a simple boundary and their local limits.

4.3.2 Simple submaps

Let $\hat{\mathfrak{m}}$ be a map with a simple boundary. Recalling the notions introduced in Section 4.1, if $\mathfrak{e}^\circ$ is a connected subset of the dual map containing f_{r}, we associated with it a submap $\mathfrak{e} \subset \hat{\mathfrak{m}}$. We will now associate with it a different notion of submap which we call a **simple submap**. More precisely, we shall cut along all the edges of $\hat{\mathfrak{m}}$ whose dual edges are not in $\mathfrak{e}^\circ$ but we will keep the identifications of vertices in $\hat{\mathfrak{m}}$ contrary to what we were doing to get $\mathfrak{e}$. See Figure 4.5 below.

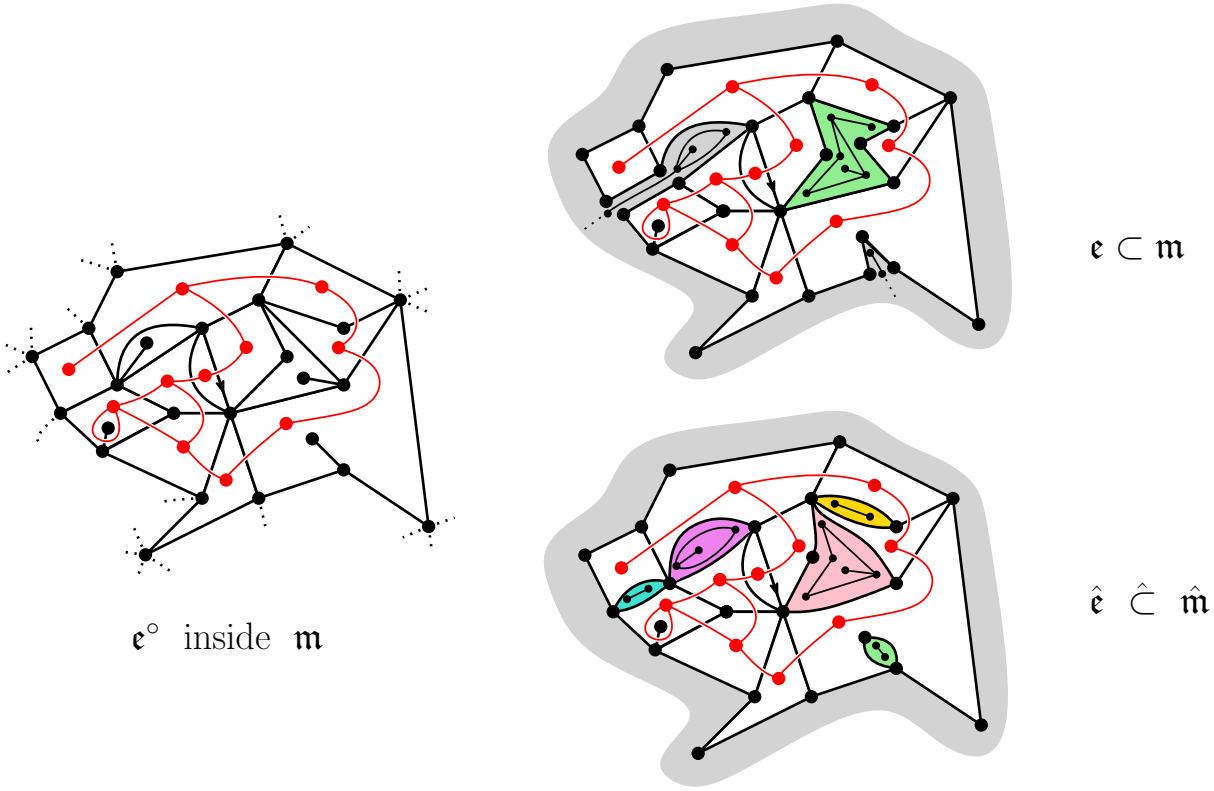


Figure 4.5: Illustration of the two notions of submaps. On the left, part of a map $\hat{\mathfrak{m}}$ with a simple boundary of perimeter 2 together with a connected subset $\mathfrak{e}^\circ$ of its dual. On the right column, the submap and simple submap $\mathfrak{e}$ and $\hat{\mathfrak{e}}$ obtained by cutting along all edges not in $\mathfrak{e}^\circ$ and keeping or not the identification of vertices inside $\hat{\mathfrak{m}}$.

We will say that $\hat{\mathfrak{e}}$ is a *simple submap* of $\hat{\mathfrak{m}}$ in the sense that $\hat{\mathfrak{m}}$ can be obtained by filling-in the holes of $\hat{\mathfrak{e}}$ with maps with a *simple boundary* of matching perimeters. We then write

$$\hat{\mathfrak{e}} \quad \hat{\subset} \quad \hat{\mathfrak{m}}.$$

Exactly as in Section 4.1, it is easy to see that this operation is rigid (in the sense that the maps filling-in the holes of $\hat{\mathfrak{e}}$ are uniquely defined) and that this definition makes sense when $\hat{\mathfrak{m}}$ is an infinite map. At this point, it is important to stress the differences between this notion of submap and the one encountered in Section 4.1: In Section 4.1, the submaps $\mathfrak{e}$ had a general boundary, the holes were simple and they did not share any vertices. They were filled-in with maps with general boundaries. In the current notion, the submaps $\hat{\mathfrak{e}}$ have a simple boundary and the holes are again simple. They can share vertices (but the associated set $\mathfrak{e}^\circ$ should be connected). They are filled-in with maps with a simple boundary.

4.3.3 Simple peeling exploration

With the above notions at hand, recall from Section 4.1 and Remark 4.2, that given an algorithm $\mathcal{A}$, an exploration of a map $\mathfrak{m}$ is a sequence of submaps $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{m}$ associated with the growing dual connected subsets $(\mathfrak{e}_i^\circ)_{i\geq 0}$ in which $\mathfrak{e}_{n+1}^\circ$ is obtained by adding (the dual edge) of $\mathcal{A}(\mathfrak{e}_n)$ inside $\mathfrak{m}$ (provided that $\mathcal{A}(\mathfrak{e}_n) \neq \dagger$). Taking the associated simple submaps instead of the submaps, we can similarly define simple peeling explorations:

Definition 4.8 (Simple peeling explorations). *If $\hat{\mathfrak{m}}$ is a (finite or infinite) planar map with a simple root face, the simple peeling exploration of $\hat{\mathfrak{m}}$ with algorithm $\mathcal{A}$ is the sequence of simple submaps*

$$\hat{\mathfrak{e}}_0 \quad \hat{\subset} \quad \hat{\mathfrak{e}}_1 \quad \hat{\subset} \quad \cdots \quad \hat{\subset} \quad \hat{\mathfrak{e}}_n \quad \hat{\subset} \quad \cdots \quad \hat{\subset} \quad \mathfrak{m},$$

associated to the growing sequences $(\mathfrak{e}_i^\circ)_{i\geq 0}$ where $\mathfrak{e}_0^\circ$ is the root face and $\mathfrak{e}_{n+1}^\circ$ is obtained by adding the edge dual to $\mathcal{A}(\hat{\mathfrak{e}}_n)$ inside $\mathfrak{m}$.

Contrary to the peeling exploration of Section 4.1 the moves $\hat{\mathfrak{e}}_n \to \hat{\mathfrak{e}}_{n+1}$ cannot be classified into two types of moves (events C_k and G_{k_1,k_2}) because many different topological situations can occur, see Figure 4.6. This is the main reason why the simple peeling process is more complicated to study compared to the peeling process of Section 4.1.

4.3.4 Law of the simple peeling under the Boltzmann measure

Similarly to Proposition 4.6 one can compute the law of simple peeling exploration $(\hat{\mathfrak{e}}_n : n \geq 0)$ under $\hat{\mathbb{P}}^{(\ell)}$ with algorithm $\mathcal{A}$. We again get a Markov chain, whose probability transitions are as follows: Conditionally on the past exploration and on the edge to peel, if the hole to which $\mathcal{A}(\mathfrak{e}_n)$ belongs has perimeter 2ℓ, then any transition of the form of Figure 4.6 happens with probability

$$\frac{1}{\hat{W}^{(\ell)}} q_k \prod_i \hat{W}^{(\ell_i)},$$

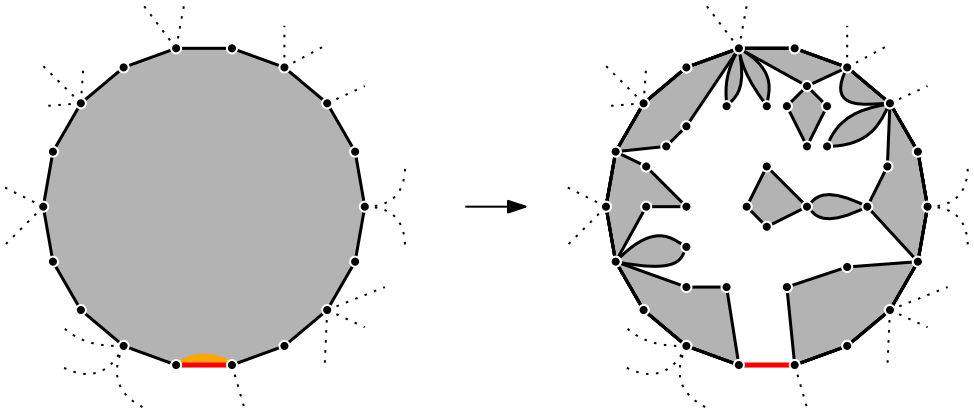


Figure 4.6: An example of move from $\hat{\mathfrak{e}}_n$ to $\hat{\mathfrak{e}}_{n+1}$, where a hole with perimeter 18 is split into 14 new holes of perimeter $6, 2, 6, 6, 2, 2, 4, 4, 2, 2, 6, 2, 4, 6$ after discovering a new face of perimeter 38.

where k is the half-perimeter of the face we reveal and ℓ_i are the half perimeters of the holes created. If $\ell = 1$, we might not reveal any face and glue the two sides of the 2-gon together and closing it. This happens with probability $1/\hat{W}^{(1)}$. Furthermore, conditionally on the past exploration all the holes are filled-in independently with ∂-simple Boltzmann maps with the correct perimeter.

Case of triangulations and quadrangulations. As of today, the simple peeling process has mainly been studied in the case of triangulations or quadrangulations where there are just a few topologically different moves and where the simple disk partitions functions $\hat{W}^{(\ell)}$ are known explicitly, see [91, Section 6]. See Chapter 9 for more details about the elusive connections between the peeling and the simple peeling process in general.

Bibliographical notes. The peeling process was first used in the physics literature by Watabiki [190] without a precise justification. Some form of it can even be observed in applied computer science [176]. A rigorous version of the peeling process and its Markovian properties was given by Angel [13] in the case of the Uniform Infinite Planar Triangulation (UIPT). In the terminology of these lecture notes, the peeling process used by Angel [13] is the simple peeling exploration on triangulations. It is well suited to studying planar maps with low face degrees (such as triangulations or quadrangulations) but much harder to study for general Boltzmann maps, see Chapter 9. The (general) peeling process we considered in the first part of this chapter and which we mainly use in the rest of the notes was recently introduced in [53]: based on maps with general boundaries whose enumeration is more universal, it enables us to make a unified treatment of all Boltzmann measures contrary to the simple peeling process. The presentation of the beginning of the section is adapted from [34]. The setup for the simple peeling process

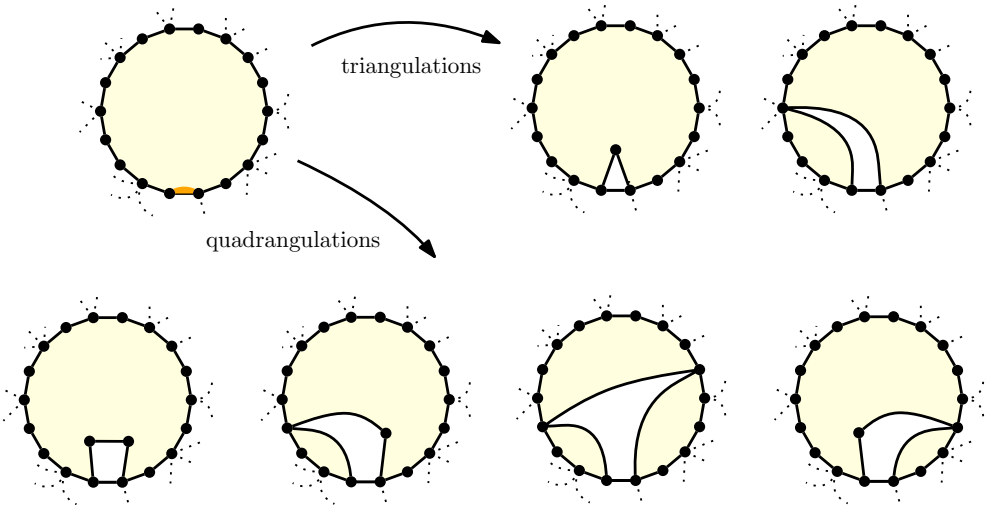


Figure 4.7: In the case of triangulations (alas non bipartite), the simple peeling steps can be classified in two shapes. In the case of quadrangulations, we need four different shapes. In the case of k-angulations, classifying the "topologically" different shapes of a simple peeling step necessitates 2^{k-2} different shapes (as the number of ways to choose which among the available $k-2$ vertices will be sticked on the boundary of the hole).

is copied from [56]. They are many more "exploration processes" that can be used to encode planar maps and most of them rely on introducing some sort of "spanning tree" in the map, see e.g. [166] and [180]. Apart from the Bouttier–Di Francesco–Guitter construction (Appendix B) we shall not describe those constructions in these notes.

Chapter V : Classification of weight sequences

In this chapter, we uncover the role of an underlying random walk in the peeling process of random Boltzmann maps. We will study the possible probabilistic behaviors of this random walk thus introducing a classification of the weight sequences $\mathbf{q}$ echoing the enumeration results of Chapter 3. We will see in the next part that those different behaviors are attached to different types of infinite planar maps we can define from the Boltzmann measure. Some background on one-dimensional random walks is recalled in Appendix A. The reader eager to jump to later chapters just needs to read the definition of ν and the statement of Theorem 5.4.

In this chapter, $\mathbf{q}$ is an admissible weight sequence.

5.1 The ν-random walk

In this section we introduce the step distribution ν on $\mathbb{Z}$ which is obtained as the limit distribution of the increment of the half-perimeter of a very large hole during a peeling step.

5.1.1 The step distribution ν

Imagine that we fix a peeling algorithm $\mathcal{A}$ and make it run on a Boltzmann planar map with a very large boundary. Using the explicit form of the probability transitions $b_p^{(\ell)}$ (in the target case) or $b^{(\ell)}$ (in the non-target case) it follows from Theorem 3.12 (in the target case) and Lemma 3.13 (in the non-target case) that these transitions admit a limit:

$$\text{for } k \geq 1, \quad \lim_{\ell\to\infty} b^{(\ell)}(k) = \lim_{\ell\to\infty} b_p^{(\ell)}(k) \quad = \quad q_k\, c_{\mathbf{q}}^{k-1} \tag{5.1}$$

and

$$\begin{aligned} \text{for } k \geq 0, \quad \lim_{\ell\to\infty} b^{(\ell)}(k, \ell-k-1) = \lim_{\ell\to\infty} b_p^{(\ell)}(k,*) \quad &= \quad W^{(k)}\, c_{\mathbf{q}}^{-k-1}, \\ \lim_{\ell\to\infty} b^{(\ell)}(\ell-k-1, k) = \lim_{\ell\to\infty} b_p^{(\ell)}(*,k) \quad &= \quad W^{(k)}\, c_{\mathbf{q}}^{-k-1}. \end{aligned} \tag{5.2}$$

This leads us to introduce the following step distribution which, roughly speaking, encodes the limiting increment of the half-perimeter of a large hole after a peeling step under $\mathbb{P}^{(\ell)}$ as $\ell \to \infty$:

Definition 5.1 (The step distribution ν). *Let ν be the measure on $\mathbb{Z}$ defined by*

$$\nu(k) = \begin{cases} q_{k+1} c_{\mathbf{q}}^k & \text{for } k \geq 0 \\ 2W^{(-1-k)} c_{\mathbf{q}}^k & \text{for } k \leq -1, \end{cases}$$

A priori, this measure is a sub-probability measure because it is obtained as a limit of probability measures (by Fatou's lemma). However, using Proposition 4.7 and Theorem 3.12 it is easy to see that the half-perimeter of the unique hole in a filled-in exploration under $\mathbb{P}_p^{(\ell)}$ evolves as a Markov chain whose probability transitions can be rewritten as

$$\mathbb{P}_p^{(\ell)}\left(|\partial^*\bar{\mathfrak{e}}_{n+1}| = (m+k)\,\middle|\,|\partial^*\bar{\mathfrak{e}}_n| = m\right) = \nu(k)\frac{h_p^{\downarrow}(m+k)}{h_p^{\downarrow}(m)}, \quad k \in \mathbb{Z} \tag{5.3}$$

where we recall that $h_p^{\downarrow}$ was introduced in (3.8). For this statement to hold true, in the case $* = \square$ and the filled-in peeling discovered the target face of perimeter $2p$, *it is necessary to agree that the half-perimeter process drops to* $-p$. The form of these probability transitions is a particular case of a ***h*-transform** of the ν-random walk killed upon touching $\mathbb{Z}_{\leq 0}$ in the sense of Doob, see Section A.3.1. We will use this to prove that actually ν has mass one:

Lemma 5.2. *The measure ν is an aperiodic probability measure on $\mathbb{Z}$.*

Proof. Recall by Fatou's lemma that ν is of mass less than or equal to 1 since it is a limit of transition probabilities. With our definition of $h_p^{\downarrow}(\ell)$, using the fact that $\sum_{p\geq 0} 2^{-2p}\binom{2p}{p}x^p = \frac{1}{\sqrt{1-x}}$ for any $\ell \in \{1,2,3,\ldots\}$ we have

$$\sum_{p\geq 0} h_p^{\downarrow}(\ell) = \ell\, h^{\downarrow}(\ell) \sum_{p\geq 0} 2^{-2p}\binom{2p}{p}\frac{1}{\ell+p} = \ell\, h^{\downarrow}(\ell)\int_0^1 \mathrm{d}x \frac{x^{\ell-1}}{\sqrt{1-x}} = 2\ell\, h^{\downarrow}(\ell)\int_0^{\pi/2} \cos^{2\ell-1}(\theta)\mathrm{d}\theta = 1$$

using the change of variable $x = \cos^2(\theta)$ and the classic computation of Wallis' integrals. We also have $\sum_{p\geq 0} h_p^{\downarrow}(\ell) = 1$ when $\ell \leq 0$ given our convention of $h_p^{\downarrow}(-p) = 1$. Equation (5.3) for $p = 0, 1, 2, 3\ldots$ tells us that $h_p^{\downarrow}(\ell) = \sum_{k\in\mathbb{Z}} \nu(k) h_p^{\downarrow}(\ell+k)$ for any $\ell \geq 1$ and any $p \geq 0$. Summing those equations for $p \geq 0$ we deduce using the previous computation that

$$\underbrace{\sum_{p\geq 0} h_p^{\downarrow}(\ell)}_{=1} = \sum_{k\geq 0} \nu(k) \underbrace{\sum_{p\geq 0} h_p^{\downarrow}(\ell+k)}_{=1},$$

and ν is indeed of mass 1. □

In the rest of these lecture notes we shall denote by $(S_n : n \geq 0)$ a random walk with independent increments distributed according to ν. We assume that under the probability $\mathbf{P}^{(\ell)}$ this walk starts from $\ell \geq 0$.

5.1.2 Probabilistic interpretation of the $h_p^{\downarrow}$-transformation

As seen in (5.3), the half-perimeter process of a filled-in exploration with target under the **q**-Boltzmann measure with target can be seen as a $h_p^{\downarrow}$-transform of the ν-walk (S) killed upon touching $\mathbb{Z}_{\leq 0}$. This can be recast as follows:

Proposition 5.3. *For any peeling algorithm, under $\mathbb{P}_p^{(\ell)}$ with $\ell \geq 1$ and $p \geq 0$, the Markov chain $(|\partial^* \bar{\mathfrak{e}}_n| : n \geq 0)$ has the law of the random walk (S) started from ℓ and conditioned to hit $\mathbb{Z}_{\leq 0}$ for the first time at $-p$ in a finite time and killed afterwards. In particular,*

$$h_p^{\downarrow}(\ell) = \mathbf{P}^{(\ell)}\left(\tau_{-p} = \tau_{\mathbb{Z}_{\leq 0}} < \infty\right),$$

where $\tau_A = \inf\{i \geq 0 : S_i \in A\}$ is the first entrance time of the random walk (S) in $A \subset \mathbb{Z}$ (and where we wrote $-p$ instead of $\{-p\}$ by abuse of notation).

Proof. Let us compute first $\mathbf{P}^{(\ell)}(\tau_{-p} = \tau_{\mathbb{Z}_{\leq 0}} < \infty)$ where $\mathbb{Z}_{\leq 0} = \{\dots, -2, -1, 0\}$. Since $h_p^{\downarrow}(S_{n\wedge\tau_{\mathbb{Z}_{\leq 0}}})$ is a bounded martingale (recall that $h_p^{\downarrow}(k)$ decreases like $1/\sqrt{k}$ when $k \to \infty$) we have by the optional sampling theorem:

$$h_p^{\downarrow}(\ell) = \mathbf{E}^{(\ell)}\left[h_p^{\downarrow}(S_{n\wedge\tau_{\mathbb{Z}_{\leq 0}}})\right] \quad \xrightarrow[n\to\infty]{} \quad \mathbf{E}^{(\ell)}\left[h_p^{\downarrow}(S_{\tau_{\mathbb{Z}_{\leq 0}}})\mathbf{1}_{\tau_{\mathbb{Z}_{\leq 0}}<\infty}\right] = \underbrace{h_p^{\downarrow}(-p)}_{=1} \cdot \mathbf{P}^{(\ell)}(\tau_{-p} = \tau_{\mathbb{Z}_{\leq 0}} < \infty).$$

Now Equation (A.2) shows that for any possible particular trajectory Traj starting from ℓ and ending at $-p$ without touching $\mathbb{Z}_{\leq 0}\backslash\{-p\}$, the probability to observe such a trajectory for $(|\partial^* \bar{\mathfrak{e}}_n|)_{n\geq 0}$ under $\mathbb{P}_p^{(\ell)}$ is equal to

$$\frac{h_p^{\downarrow}(-p)}{h_p^{\downarrow}(\ell)} \cdot \mathbf{P}^{(\ell)}\left((S_n)_{0\leq n\leq \tau_{-p}} = \mathrm{Traj}\right),$$

and the conclusion follows from the two displays above. □

In particular it follows from the previous proposition and Section A.3.2 that $h_0^{\downarrow} = h^{\downarrow}$ is the pre-renewal function of the walk (S) and is in particular harmonic on $\mathbb{Z}_{>0}$ as we saw above. On the way we deduce that the ν-random walk (S) cannot drift towards $+\infty$ since we have

$$\mathbf{P}^{(\ell)}(\tau_{\mathbb{Z}_{\leq 0}} < \infty) = \sum_{p\geq 0} \mathbf{P}^{(\ell)}(\tau_{-p} = \tau_{\mathbb{Z}_{\leq 0}} < \infty) = \sum_{p\geq 0} h_p^{\downarrow}(\ell) = 1,$$

by the computation used in Lemma 5.2 (alternatively, we can use Section A.3.2 and the fact that the renewal function of the walk is unbounded). Following Section A.3.2, it is natural to ask whether the renewal function (the discrete primitive of $h^{\downarrow}$) is harmonic too, or equivalently whether or not (S) drifts towards $-\infty$. This will induce an important dichotomy on weight sequences.

5.2 Critical weight sequences

In this section we introduce the important notion of critical weight sequence. We will see in the next theorem that this notion manifests itself in different contexts: behavior of the μ-random walk, behavior of ν-random walk, volume of Boltzmann maps, behavior of the peeling process... and we will see later that it enables us to define several models of infinite planar maps.

5.2.1 Equivalent definitions of criticality

For $\ell \geq 1$ we introduce

$$h^{\uparrow}(\ell) = h^{\downarrow}(\ell-1) + h^{\downarrow}(\ell-2) + \cdots + h^{\downarrow}(0) = 2\ell h^{\downarrow}(\ell) = 2\ell 2^{-2\ell}\binom{2\ell}{\ell}, \tag{5.4}$$

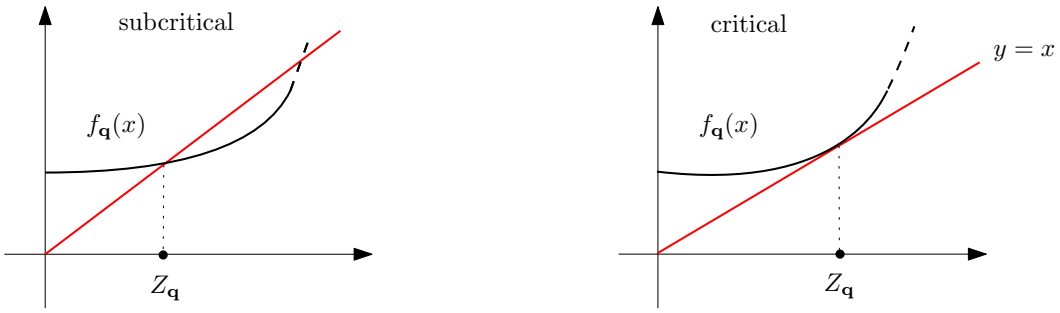
and $h^{\uparrow}(\ell) = 0$ otherwise. The shift of -1 in the definition is done on purpose to stick to the notation in Appendix A. This is the *renewal function* for the ν-random walk, see Appendix A.

Theorem 5.4 (*Definition(s) of criticality*)

Let $\mathbf{q}$ *be an admissible weight sequence, then the following propositions are equivalent and if they hold the weight sequence* $\mathbf{q}$ *is* ***critical****, otherwise the weight sequence is* ***subcritical****:*

(i) With the notation of Theorem 3.12, the measure μ *is centered or equivalently the graphs of the function* $x \mapsto f_{\mathbf{q}}(x)$ *and* $x \mapsto x$ *are tangent at* $Z_{\mathbf{q}}$*, i.e.*

$$f'_{\mathbf{q}}(Z_{\mathbf{q}}) = 1.$$



(ii) The function $h^{\uparrow}$ *is harmonic for the* ν*-random walk* (S) *killed on* $\mathbb{Z}_{\leq 0}$*,*

(iii) The ν*-random walk* (S) *oscillates,*

(iv) The expected volume (number of edges) of a $\mathbf{q}$*-Boltzmann map with target is infinite, i.e.* $\int \mathrm{d}\mathbb{P}_p^{(\ell)}(\mathfrak{m})|\mathfrak{m}| = \infty$*, or equivalently the variance of the volume of a* $\mathbf{q}$*-Boltzmann map without target is infinite.*

Remark 5.5. It follows from Proposition A.10 that when $\mathbf{q}$ is subcritical then $h^{\uparrow}$ is super-harmonic for the walk (S) killed on $\mathbb{Z}_{\leq 0}$, and the walk (S) drifts towards $-\infty$ whereas μ has a strictly negative mean.

Example 5.6. From the calculation performed in Section 3.5 we see that $2p$-angulations (as well as triangulations) with the critical weight, or critical uniform bipartite maps or the "canonical" maps of type $a \in (3/2; 5/2]$ are actually critical.

Proof. The equivalence between *(ii)* and *(iii)* is given by Proposition A.10 (and the remark just before it since $h^{\uparrow}$ is unbounded) since we know that $h^{\uparrow}$ is the renewal function of (S). To see

$(i) \iff (ii)$ let us write explicitly the two equations defining $Z_{\mathbf{q}}$ and the tangency at $Z_{\mathbf{q}}$:

$$1 + \sum_{k\geq 1} q_k \binom{2k-1}{k} Z_{\mathbf{q}}^k = Z_{\mathbf{q}}, \qquad \text{(admissibility)} \tag{5.5}$$

$$\sum_{k\geq 1} q_k \binom{2k-1}{k} Z_{\mathbf{q}}^{k-1} k = 1, \qquad \text{(criticality)}. \tag{5.6}$$

Easy calculations using $c_{\mathbf{q}} = 4Z_{\mathbf{q}}$ and the definitions of $h^\uparrow, h^\downarrow$ and of ν show that (5.5) and (5.6) are respectively equivalent the harmonicity of $h^\downarrow$ and $h^\uparrow$ at the point 1 i.e. to

$$\sum_{k\geq -1} \nu(k) h^\downarrow(k+1) = h^\downarrow(1) \quad \text{and} \quad \sum_{k\geq 0} \nu(k) h^\uparrow(k+1) = h^\uparrow(1).$$

Hence $(ii) \Rightarrow (i)$. Actually, as soon as $h^\uparrow$ is harmonic at 1, using the harmonicity of $h^\downarrow$, we deduce that $h^\uparrow$ is also harmonic on $\mathbb{Z}_{>0}$, see the proof of Proposition A.10. Hence $(i) \Rightarrow (ii)$. Finally let us see why $(i) \iff (iv)$. By Theorem 3.12, we see that the volume of a $\mathbf{q}$-Boltzmann map with target is infinite if and only if $\mathbb{E}[\tau_{-\ell-p}] = \infty$ where $\tau_{-\ell-p}$ is the hitting time of $-\ell - p$ by the skip-free μ-walk (Y) started from 0. For such walks, it is classical (e.g. by Wald' identity) that the descending ladder times have infinite expectation if and only if the walk is centered. See (A.1). The claim on $\mathbf{q}$-Boltzmann map without target follows from (3.5). □

We will see later in this chapter that the asymptotics for $W^{(\ell)}$ are very different in the subcritical and in the critical cases. This, in turns, changes dramatically the geometry of the underlying Boltzmann random maps.

5.2.2 $h^\uparrow$-transform

Notice a slight but important difference between $h^\downarrow = h_0^\downarrow$ and $h^\uparrow$: although both harmonic for the ν-walk on $\{1, 2, 3, \ldots\}$ we have

$$h^\uparrow(0) = 0 \quad \text{whereas} \quad h^\downarrow(0) = 1.$$

This changes the behavior of the h-transforms: with obvious notations, on the one hand $S^\downarrow$ will eventually hit 0, whereas on the other hand $S^\uparrow$ will always stay positive and even drift to infinity (Proposition A.11). Indeed, if $\mathbf{q}$ is a critical weight sequence, by the above theorem and Section A.3.2, the $h^\uparrow$-transform $(S^\uparrow)$ of the walk (S) started from $\ell \geq 1$ can be interpreted as the walk (S) conditioned to stay positive forever (in Section A.3 we conditioned to walk to stay non-negative, notice the shift of 1 here).

5.3 Discrete stable weight sequences

In this section we derive precise asymptotics for the disk partition functions in the case of subcritical weight sequences or in the case of critical weight sequences with well-behaved weights $\mathbf{q}$. The different power-tail behaviors of $\mathbf{q}$ will eventually lead to different "universality classes"

for the geometry of random maps or for the behavior of percolation or random walk on them. There are classified by an exponent $a \in [3/2; 5/2]$ and correspond to the classic Brownian/stable paradigms in probability theory. These refinements can be skipped at first reading.

In the following, we will always suppose that $\mathbf{q}$ is admissible and we recall the notation $c_{\mathbf{q}}/4 = Z_{\mathbf{q}}$ for the smallest solution to $f_{\mathbf{q}}(x) = x$. We have seen in Theorem 3.12 that the behavior of the partition function $W_p^{(\ell)}$ is universal and only depends on $c_{\mathbf{q}}$. In Section 3.4.1 we got an exact expression for $W^{(\ell)}$ whose asymptotics was dictated by the behavior of $f_{\mathbf{q}}$ near $Z_{\mathbf{q}}$. In particular, starting from (3.12) and performing the change of variable $v = \phi(u)$ where ϕ the inverse function of $z \mapsto z/f_{\mathbf{q}}(z) : [0, Z_{\mathbf{q}}] \to [0, 1]$, we arrive after a few manipulations at

$$W^{(\ell)} = \frac{\ell}{\ell+1}\binom{2\ell}{\ell}\int_0^1 \mathrm{d}u \frac{(\phi(u))^{\ell+1}}{u^2} \underset{\ell\to\infty}{\sim} \frac{4^\ell}{\sqrt{\pi\ell}}\int_0^1 \mathrm{d}u\ (\phi(u))^{\ell+1}, \quad \ell \geq 1. \tag{5.7}$$

This form will be useful in what follows when we apply classical Tauberian theorems and the Laplace method. Recall also the definition of the measure μ just before Theorem 3.12. We already made a dichotomy in Theorem 5.4 between the critical case where the function $f_{\mathbf{q}}(z)$ and z are tangent at $Z_{\mathbf{q}}$ and the subcritical case where their derivatives differ. More precisely when $s \uparrow Z_{\mathbf{q}}$ we have

$$f_{\mathbf{q}}(s) = Z_{\mathbf{q}} - (1-\kappa)(Z_{\mathbf{q}} - s) + o(Z_{\mathbf{q}} - s) \quad \text{with } 0 < \kappa < 1 \qquad \text{(subcritical case)} \tag{5.8}$$

$$f_{\mathbf{q}}(s) = Z_{\mathbf{q}} - (Z_{\mathbf{q}} - s) + o(Z_{\mathbf{q}} - s) \qquad \text{(critical case).} \tag{5.9}$$

In this section we will focus on the critical case and will furthermore investigate according to the following behaviors:

$$f_{\mathbf{q}}(s) = Z_{\mathbf{q}} - (Z_{\mathbf{q}} - s) + \kappa(Z_{\mathbf{q}} - s)^{a-1/2} + o(Z_{\mathbf{q}} - s)^{a-1/2}, \quad \text{as } s \uparrow Z_{\mathbf{q}} \text{ for some } \kappa > 0 \tag{5.10}$$

Notice that the range of a is constrained: since $f'_{\mathbf{q}}(Z_{\mathbf{q}}) = 1 < \infty$ we must have $a > 3/2$ and because $f''_{\mathbf{q}}(Z_{\mathbf{q}}) \neq 0$ we must have[1] $a \leq 5/2$. In all these cases we introduce the constant

$$\mathrm{p}_{\mathbf{q}} := \frac{\kappa Z_{\mathbf{q}}^{a-3/2}}{2\sqrt{\pi}}\Gamma\left(a + \frac{1}{2}\right) > 0.. \tag{5.11}$$

5.3.1 Subcritical case: $a = \frac{3}{2}$

We first come back to the case when $\mathbf{q}$ is a subcritical weight sequence. We recall a few of the characterizations given by Theorem 5.4 and complete them with tail asymptotics for ν:

Proposition 5.7 (Weight sequence of type $a = \frac{3}{2}$)**.** *The following propositions are equivalent:*

(i) The weight sequence $\mathbf{q}$ *is a subcritical weight sequence, also called sequence of type* $a = \frac{3}{2}$.

[1] In general a series expansion (5.10) with $a > 5/2$ is not necessarily in contradiction with $f''_{\mathbf{q}}(Z_{\mathbf{q}}) < \infty$ but it is an easy exercise to see that since $f_{\mathbf{q}}$ is a power series with positive coefficients we must indeed have $a \leq 5/2$.

(ii) *We have* (5.8),

(iii) *The measure* μ *has strictly negative mean* $-\kappa \in (-1, 0)$,

(iv) $W^{(\ell)} \sim \frac{p_{\mathbf{q}}}{2} \cdot c_{\mathbf{q}}^{\ell+1}\, \ell^{-3/2}$ *as* $\ell \to \infty$ *and in particular* $\nu(-k) \sim p_{\mathbf{q}} \cdot k^{-3/2}$ *as* $k \to \infty$,

(v) *The expected variance of the volume of a* $\mathbf{q}$*-Boltzmann map is finite,*

where the constants κ *and* $p_{\mathbf{q}}$ *are related by* (5.11). *In this case, we have also* $\nu([k, \infty)) = o(k^{1-3/2})$.

Remark 5.8. Since in the subcritical case we have $\nu(-k) \sim p_{\mathbf{q}} \cdot k^{-3/2}$, in particular, if the measure ν admits a first moment, then necessarily $\mathbf{q}$ is critical. In this case, since the walk (S) must oscillate by Theorem 5.4, we must have $\sum_{k\in\mathbb{Z}} k\,\nu(k) = 0$. In other words, if ν has a first moment, then it is centered!

Proof. The equivalence between $(i), (ii), (iii)$ and (v) is done in Theorem 5.4. Given (5.7), the definition of $p_{\mathbf{q}}$ and $h^{\downarrow}(\ell) \sim 1/\sqrt{\pi\ell}$, the equivalence between (ii) and (iv) follows from an application of Karamata Tauberian theorem, see [42, Theorem 1.7.1']. To prove that $\nu([k, \infty)) = o(k^{-1/2})$ notice that since $f'_{\mathbf{q}}(Z_{\mathbf{q}})$ is finite we have

$$\sum_{k\geq 1} k \binom{2k-1}{k} q_k Z_{\mathbf{q}}^k < \infty \Rightarrow \sum_{k\geq 1} \sqrt{k} q_k c_{\mathbf{q}}^k < \infty \Rightarrow \sum_{k\geq 1} \sqrt{k}\nu(k) < \infty,$$

and the claim follows. □

Due to the appearance of the exponent $a = \frac{3}{2}$ in the above statements and to unify later results, we decide to rename subcritical weight sequences as **weight sequences of type** $a = \frac{3}{2}$. Notice that if $\mathbf{q}$ is critical then Laplace method using (5.7) and (5.9) implies that

$$\nu(-k) = o(k^{-3/2}). \tag{5.12}$$

In other words, the heaviest polynomial decay for the negative tail of ν happens when $\mathbf{q}$ is subcritical.

5.3.2 Critical generic case: $a = \frac{5}{2}$

The next generic situation is that of **weight sequences of type** $5/2$:

Proposition 5.9 (Type $a = \frac{5}{2}$). *The following propositions are equivalent:*

(i) *The weight sequence* $\mathbf{q}$ *is a critical weight sequence with* $f''_{\mathbf{q}}(Z_{\mathbf{q}}) < \infty$,

(ii) *We have* (5.10) *with* $a = \frac{5}{2}$,

(iii) *The measure* μ *is centered and has finite variance* $\sigma^2 = 2Z_{\mathbf{q}}\kappa$,

(iv) $W^{(\ell)} \sim \frac{p_{\mathbf{q}}}{2} \cdot c_{\mathbf{q}}^{\ell+1}\, \ell^{-5/2}$ *as* $\ell \to \infty$ *and in particular* $\nu(-k) \sim p_{\mathbf{q}} \cdot k^{-5/2}$ *as* $k \to \infty$,

where the constants κ *and* $\mathsf{p}_{\mathbf{q}}$ *are related by* (5.11). *In this case, we have also* $\nu([k,\infty)) = o(k^{1-5/2})$. *In this case, such a weight sequence* $\mathbf{q}$ *is called a weight sequence of type* $5/2$.

Proof. As above the equivalence between (ii) and (iv) follows from (5.7) and Tauberian theorems. The equivalence between (i), (ii) and (iii) is an easy exercise for inverse of generating series with positive coefficients. The fact that $\nu([k,\infty)) = o(k^{-3/2})$ follows from $f''_{\mathbf{q}}(Z_{\mathbf{q}}) < \infty$ as in the previous proof. □

Notice that as soon as $\mathbf{q}$ is admissible we always have $f_{\mathbf{q}}(s) - Z_{\mathbf{q}} \geq \mathsf{c} \cdot (Z_{\mathbf{q}} - s)^2$ as $s \uparrow Z_{\mathbf{q}}$ for some $\mathsf{c} > 0$. It follows from (5.7) and Laplace's method that

$$W^{(\ell)} \geq \mathsf{c}' \cdot c_{\mathbf{q}}^{\ell} \cdot \ell^{-5/2} \tag{5.13}$$

for some constant c' for ℓ large enough. In other words, the lightest polynomial decay for the negative tail of ν happens when $\mathbf{q}$ is critical of type $a = 5/2$.

5.3.3 Critical non-generic: $a \in (3/2; 5/2)$

We now come to more exotic behaviors for the weight sequence which we name **non-generic:**

Proposition 5.10 (Type $a \in (3/2;5/2)$). *The following propositions are equivalent:*

(i) We have (5.10) *with* $a \in (3/2;5/2)$,

(ii) The measure μ *is centered and* $\mu([k,\infty)) \sim \frac{\kappa \cdot Z_{\mathbf{q}}^{a-3/2}}{|\Gamma(3/2-a)|} k^{-a+1/2}$ *as* $k \to \infty$,

(iii) $W^{(\ell)} \sim \frac{\mathsf{p}_{\mathbf{q}}}{2} \cdot c_{\mathbf{q}}^{\ell+1}\ \ell^{-a}$ *as* $\ell \to \infty$ *and in particular* $\nu(-k) \sim \mathsf{p}_{\mathbf{q}} \cdot k^{-a}$ *as* $k \to \infty$,

(iv) $\nu([k,\infty)) \sim \mathsf{p}_{\mathbf{q}} \cdot \cos(a\,\pi) \cdot \frac{1}{a-1} k^{1-a}$,

where the constants κ *and* $\mathsf{p}_{\mathbf{q}}$ *are related by* (5.11). *In this case, such a weight sequence* $\mathbf{q}$ *is called a weight sequence of type* $a \in (3/2;5/2)$.

Proof. Writing

$$f_{\mathbf{q}}(z) = 1 + \sum_{k\geq 1} \binom{2k-1}{k} q_k z^k = 1 + c_{\mathbf{q}} \sum_{k\geq 0} \nu(k) 4^{-(k+1)} \binom{2k+1}{k+1} \left(\frac{z}{Z_{\mathbf{q}}}\right)^{k+1},$$

the above equivalences all follow from (3.11) and Tauberian theorems for Laplace integrals and generating series, see [42, Theorem 1.7.1'] and [42, Theorem 8.1.6]. □

Remark 5.11. By the above asymptotics, when $\mathbf{q}$ is of type $a \in [3/2;2]$ the probability measure ν does not admit a first moment. However when $a \in (2;5/2]$ it does, and so by Remark 5.8 it is centered.

Remark 5.12 (Probabilistic interpretation in terms of stable laws.). Gathering-up the last propositions and using the definition of domain of attraction of stable laws (see Section A.4.2) we deduce that when $\mathbf{q}$ is of type $a \in (3/2;5/2]$ then

- The measure μ is in the strict domain of attraction of the $a-\frac{1}{2}$ spectrally positive stable law (has finite variance if $a=5/2$),
- The measure ν is in the strict domain of attraction of the $(a-1)$-stable law with positivity index ϱ given by
$$(a-1)(1-\varrho)=\frac{1}{2}.$$

5.3.4 Examples

If $\mathbf{q}$ is finitely supported then it can either be subcritical or critical with $a=\frac{5}{2}$, moreover $\mathbf{q}$ is critical when we cannot anymore increase any q_k while keeping the weights admissible. To escape those generic behaviors, one has to consider fine-tuned weight sequences with polynomial tails. Let us present different ways to get weight sequence of type $a\in(3/2;5/2)$ apart from the explicit example (3.14).

Fine tuning. A way to construct a non-generic critical weight sequence $\mathbf{q}$ of type $a\in(3/2;5/2)$ is to start from a weight sequence $\hat{q}_k$ asymptotic to k^{-a} for $a\in(3/2;5/2)$ when $k\to\infty$ and then modifying it to achieve the desired criticality:

Exercise 5.13. Prove that we can find $c>0$ and $\beta>0$ so that the weight sequence $\mathbf{q}$ defined by $q_k=c\cdot\beta^{k-1}\cdot\hat{q}_k$ is admissible, critical and of type $a\in(3/2;5/2)$.

Link with the $O(\mathfrak{n})$ model. Let us now show that non-generic critical weight sequences of type $a\in(3/2;5/2)$ naturally appear when considering statistical mechanics models, namely the loop $O(\mathfrak{n})$ model on random maps.

A loop-decorated quadrangulation $(\mathfrak{q},\mathbf{l})$ is a planar map whose faces are all quadrangles on which non-crossing loops $\mathbf{l}=(l_i)_{i\geq 1}$ are drawn (see Figure 2 in [45]). For simplicity we consider the so-called rigid model when loops can only cross quadrangles through opposite sides. We define a measure on such configurations by putting

$$W_{h,g,\mathfrak{n}}((\mathfrak{q},\mathbf{l}))=g^{|\mathfrak{q}|}h^{|\mathbf{l}|}\mathfrak{n}^{\#\mathbf{l}},$$

for $g,h>0$ and $n\in(0,2)$ where $|\mathfrak{q}|$ is the number of edges of the quadrangulation, $|\mathbf{l}|$ is the total length of the loops and $\#\mathbf{l}$ is the number of loops. Provided that the measure $W_{h,g,\mathfrak{n}}$ has finite total mass one can use it to define $\mathfrak{M}_{h,g,\mathfrak{n}}$ a random loop-decorated quadrangulations (with a boundary if we wish so). We then consider the **gasket** $\mathfrak{G}_{h,g,\mathfrak{n}}$ of $\mathfrak{M}_{h,g,\mathfrak{n}}$ obtained by pruning off the interiors of the outer-most loops, see Figure 5.1. It is easy to see that $\mathfrak{G}_{h,g,\mathfrak{n}}$ is actually a Boltzmann map for some (complicated) weight sequence depending on $h,g,\mathfrak{n}$.

Fix $\mathfrak{n}\in(0,2)$. For most of the parameters (g,h) the random map $\mathfrak{G}_{h,g,\mathfrak{n}}$ is subcritical (with weight sequence of type $a=3/2$) or "generic critical" (with weight sequence of type $a=5/2$). However, there is a fine tuning of g and h (actually a critical line) for which they are non-generic

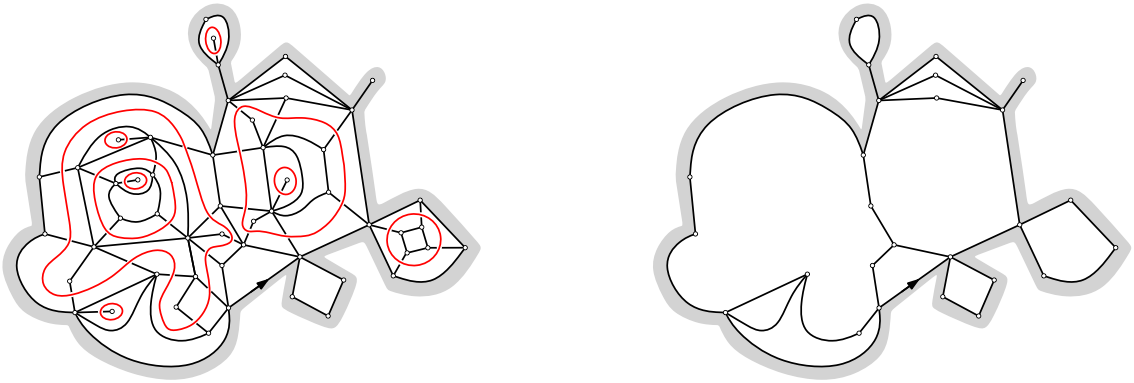


Figure 5.1: Taking the gasket of a loop-decorated quadrangulation.

critical Boltzmann planar maps with weight sequence of type a where

$$a = 2 \pm \frac{1}{\pi} \arccos(\mathfrak{n}/2),$$

see [45, 54]. The case $a = 2 - \frac{1}{\pi}\arccos(\mathfrak{n}/2) \in (\frac{3}{2}; 2)$ is called the **dense phase** because the loops on the gasket are believed in the scaling limit to touch themselves and each other. The case $a = 2 + \frac{1}{\pi}\arccos(\mathfrak{n}/2) \in (2; \frac{5}{2})$ (which occurs at the extremity of the critical line) is called the **dilute phase** because the loops on the gasket are believed to be simple and should avoid each other in the scaling limit.

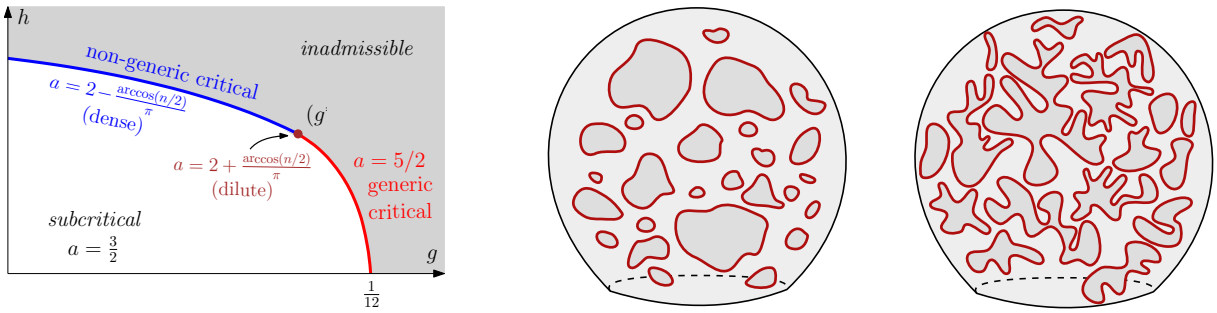


Figure 5.2: The phase diagram in (g, h) for a fixed $\mathfrak{n}$ and a schematic illustration of $\mathfrak{q}$-Boltzmann random planar maps of type $a \in (3/2; 5/2]$ in the dilute (left) and dense phase (right).

Bibliographical comments. The role of an underlying random walk in the peeling process is already present in the works of Angel [13]. The connection with h-transforms was made explicit in [91]. The results of Section 5.1 with the above peeling process and in this generality are almost all due to Budd [53]. Budd also points to an explicit connection between Tutte's equation and Wiener-Hopf factorization of the walk (S) in [51]. The classification of critical weight sequences in this form has first been introduced in [150]. Non-generic critical weight sequences were explicitly

considered in [141] where the link with $O(\mathfrak{n})$ model was proposed based on results in physics' literature. They were later rigorously studied in [45] and [54]. See also [77]. Exercise 5.13 is taken from [141, Proposition 2]. Notice that a more general definition of critical non-generic weight sequences is used in [174, 99] allowing the presence of slowly varying functions. We decided to stick to the case of the strict domain of attraction to spare the reader any avoidable pain.

Part III:
Infinite Boltzmann maps

In this part, we use the peeling process and random walk theory (in particular the conditioning of random walk to stay non-negative) to construct several random infinite Boltzmann maps. Some have the "topology" of the half-plane, some have the "topology" of the plane, with various boundary constraint (simple, general etc). We shall see later on that those infinite planar maps have very different properties at large scale (either tree-like, critical, or hyperbolic). In these notes, when constructing a law $\mathbb{P}$ on infinite maps of the plane or half-plane we shall always follow the same recipe:

1. (**Uniqueness**) Find a spatial Markov property that characterizes the law $\mathbb{P}$.
2. (**Peeling**) Check that the spatial Markov property satisfies "Tutte's equation" and derive the law of the peeling process under $\mathbb{P}$.
3. (**Existence**) Finally construct the law $\mathbb{P}$ using a well-chosen peeling algorithm and check that the spatial Markov property 1) is indeed satisfied.
4. (Optional: **Local limit**) Show that it is the local limit of some random finite maps.

Remark 5.14 (Why studying infinite maps?). When studying properties of large random planar maps, it is convenient to possess a local infinite limit on which asymptotic results are easier to state and sometimes to prove (e.g. using exact independence or simplified calculations). This is the idea of the *objective method* devised by Aldous & Steele [7]. For example, the recurrence/transience dichotomy for simple random walk or the existence of a percolation threshold are examples of properties of infinite maps which can be seen as limit of large scale properties of finite random planar maps.

Chapter VI : Infinite Boltzmann maps of the half-plane

In this chapter, we suppose that $\mathbf{q}$ is an admissible weight sequence. We study random maps sampled according to $\mathbb{P}^{(\ell)}$ and prove that they converge as $\ell \to \infty$ towards a limiting infinite random map of the half-plane (i.e. with one end and with an infinite boundary) which is called the **q-Boltzmann map of the half-plane**. Here is the main result of this chapter:

Theorem 6.1 (*Boltzmann map of the half-plane*)

Let $\mathbf{q}$ *be an admissible weight sequence. For any* $p \geq 0$*, we have the following convergence in distribution for the local topology*

$$\mathbb{P}^{(\ell)} \xrightarrow[\ell\to\infty]{(d)} \mathbb{P}^{(\infty)} \quad \text{and} \quad \mathbb{P}_p^{(\ell)} \xrightarrow[\ell\to\infty]{(d)} \mathbb{P}^{(\infty)}$$

where $\mathbb{P}^{(\infty)}$ *is a distribution supported by infinite rooted bipartite planar maps of the half-plane (see Definition 2.8). A random map* $\mathfrak{M}^{(\infty)}$ *of law* $\mathbb{P}^{(\infty)}$ *is called a* $\mathbf{q}$*-Boltzmann map of the half-plane (or a half-planar Boltzmann map).*

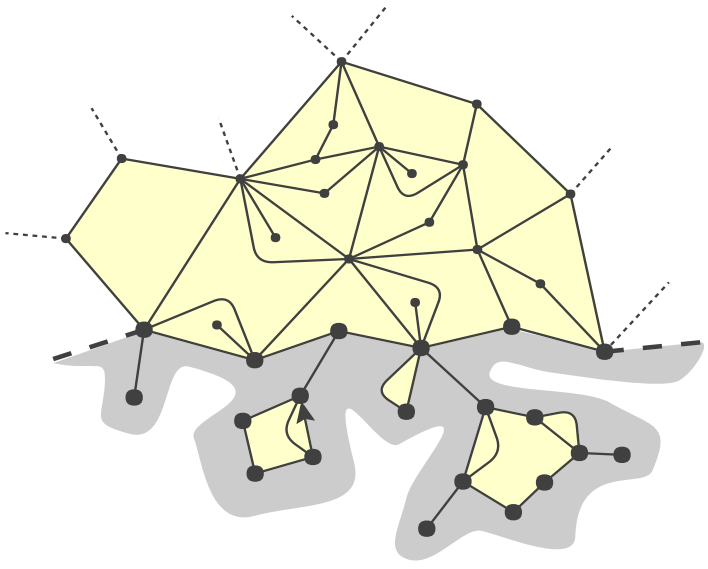


Figure 6.1: A piece of an infinite bipartite map of the half-plane. The infinite face is in light gray.

At first glance, the reader should be quite convinced that the previous theorem follows from the convergence of the probability transitions of the peeling process (5.1) and (5.2), and indeed a proof can be made along these lines. However, according to the general principle of this part, we shall take a slightly different approach and directly construct the law $\mathbb{P}^{(\infty)}$ before proving it is the limit in distribution of $\mathbb{P}^{(\ell)}$ as $\ell \to \infty$.

6.1 The half-planar Boltzmann map

Following the recipe presented at the beginning of this part, we first characterize the law $\mathbb{P}^{(\infty)}$ provided it exists. We then describe the law of (filled-in) peeling explorations under $\mathbb{P}^{(\infty)}$ still conditioned on its existence, and finally construct the law $\mathbb{P}^{(\infty)}$ using a growing Markov chain based on a particular peeling algorithm.

6.1.1 Characterizing $\mathbb{P}^{(\infty)}$

Let us suppose that $\mathbb{P}^{(\infty)}$ is as in Theorem 6.1. Let $\bar{\mathfrak{e}}$ be a map with an infinite boundary, a unique hole of infinite perimeter and otherwise finitely many faces. Although the perimeter of the boundary and of the hole are infinite, since those boundaries coincide except on finitely many edges, we can give a sense to

$$|\partial^* \bar{\mathfrak{e}}| - |\partial \bar{\mathfrak{e}}|, \tag{6.1}$$

the difference between the half-perimeters of the hole and of the boundary face, see Figure 6.2.

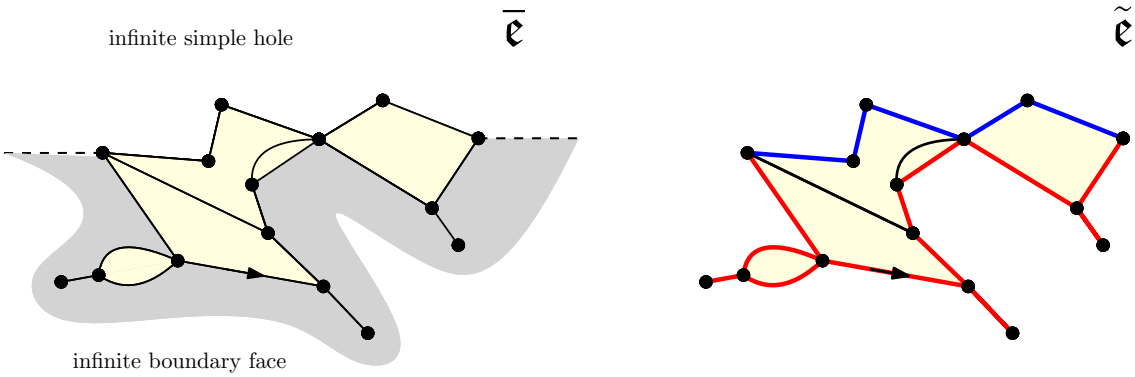


Figure 6.2: On the left, an infinite map $\bar{\mathfrak{e}}$ with an infinite hole. The half-perimeter difference is equal to $|\partial^* \bar{\mathfrak{e}}| - |\partial \bar{\mathfrak{e}}| = (5-15)/2 = -5$. On the right, the associated map $\tilde{\mathfrak{e}}$ with internal (blue) and external (red) boundaries.

If one believes in the convergence $\mathbb{P}^{(\ell)} \to \mathbb{P}^{(\infty)}$ as $\ell \to \infty$, then one can compute $\mathbb{P}^{(\infty)}(\bar{\mathfrak{e}} \subset \mathfrak{m})$. To be precise, let $\tilde{\mathfrak{e}}$ be the finite map obtained from $\bar{\mathfrak{e}}$ by erasing the two infinite connected components made of the edges common to both the boundary and the hole, see Figure 6.2. This map has its boundary split into two connected components: an external boundary to which the

root edge belongs and an internal boundary which must be simple. We write $\tilde{\mathfrak{e}} \Subset \mathfrak{m}$ if the map $\mathfrak{m}$ can be obtained by gluing a map with general boundary on the simple internal boundary of $\tilde{\mathfrak{e}}$. If we denote by p_{int} and p_{ext} respectively the inside and outside half-perimeters of $\tilde{\mathfrak{e}}$, then we have $p_{\mathrm{int}} - p_{\mathrm{ext}} = |\partial^* \overline{\mathfrak{e}}| - |\partial \overline{\mathfrak{e}}|$ and for all ℓ large enough (in particular larger than p_{ext}) we have

$$\begin{aligned}
\mathbb{P}^{(\infty)}(\overline{\mathfrak{e}} \subset \mathfrak{m}) &= \mathbb{P}^{(\infty)}(\tilde{\mathfrak{e}} \Subset \mathfrak{m}) \\
&\underset{\text{hypothesis}}{=} \lim_{\ell\to\infty} \mathbb{P}^{(\ell)}(\tilde{\mathfrak{e}} \Subset \mathfrak{m}) \\
&= w_{\mathfrak{q}}(\overline{\mathfrak{e}}) \frac{W^{\ell-(p_{\mathrm{ext}}-p_{\mathrm{int}})}}{W^{(\ell)}} \\
&\xrightarrow[\ell\to\infty]{\text{Lem. 3.13}} w_{\mathfrak{q}}(\overline{\mathfrak{e}}) \cdot (c_{\mathfrak{q}})^{|\partial^*\overline{\mathfrak{e}}|-|\partial\overline{\mathfrak{e}}|}, \qquad (6.2)
\end{aligned}$$

where we recall that $w_{\mathfrak{q}}(\overline{\mathfrak{e}})$ is the product of $q_{\deg(f)/2}$ over all *inner* faces of $\overline{\mathfrak{e}}$. The above display characterizes the law $\mathbb{P}^{(\infty)}$:

Proposition 6.2. *There is at most one law $\mathbb{P}^{(\infty)}$ supported by infinite maps of the half-plane so that for any finite submap $\overline{\mathfrak{e}}$ of the half-plane with a unique (infinite) hole we have*

$$\mathbb{P}^{(\infty)}(\overline{\mathfrak{e}} \subset \mathfrak{m}) = w_{\mathfrak{q}}(\overline{\mathfrak{e}}) \cdot (c_{\mathfrak{q}})^{|\partial^*\overline{\mathfrak{e}}|-|\partial\overline{\mathfrak{e}}|}. \qquad (6.3)$$

Proof. Assuming existence, let $\mathfrak{M}^{(\infty)}$ be a random map of the half-plane of law $\mathbb{P}^{(\infty)}$. The law of $\mathfrak{M}^{(\infty)}$ is fully characterized by the law of $[\mathfrak{M}^{(\infty)}]_r$ for any $r \geq 0$. But since by hypothesis $\mathfrak{M}^{(\infty)}$ is one-ended almost surely, the previous laws are completely characterized by the probabilities $\mathbb{P}^{(\infty)}(\overline{\mathfrak{e}} \subset \mathfrak{m})$ for finite submaps $\overline{\mathfrak{e}}$ with a unique infinite hole as involved in the proposition. □

6.1.2 Peeling process under $\mathbb{P}^{(\infty)}$

In this section we still *suppose* that the law $\mathbb{P}^{(\infty)}$ exists and obeys Proposition 6.2. We write $\mathfrak{M}^{(\infty)}$ for a random map of law $\mathbb{P}^{(\infty)}$ and derive the law of the filled-in exploration process $\overline{\mathfrak{e}}_0 \subset \overline{\mathfrak{e}}_1 \subset \cdots \subset \overline{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{M}^{(\infty)}$ for a given peeling algorithm $\mathcal{A}$. Notice in particular that the starting configuration $\overline{\mathfrak{e}}_0$ is just the map made of an infinite line separating an infinite hole from the infinite root face (both simple). Fix a submap e, then if $\overline{\mathfrak{e}}_n = e$, the $(n+1)$th filled-in peeling step produces an event C_k iff $e' \subset \mathfrak{m}$ where e' is the map obtained from e by adding a face of degree $2k$ on the edge $\mathcal{A}(e)$. Hence we deduce that conditionally on $\overline{\mathfrak{e}}_n = e$, the probability to see an event C_k during the $(n+1)$th peeling step is equal to

$$\frac{\mathbb{P}^{(\infty)}(e' \subset \mathfrak{m})}{\mathbb{P}^{(\infty)}(e \subset \mathfrak{m})} \underset{\text{Prop.6.2}}{=} \frac{w_{\mathfrak{q}}(e') \cdot (c_{\mathfrak{q}})^{|\partial^* e'|-|\partial e'|}}{w_{\mathfrak{q}}(e) \cdot (c_{\mathfrak{q}})^{|\partial^* e|-|\partial e|}} = q_k c_{\mathfrak{q}}^{k-1}.$$

A similar computation shows that conditionally on $\overline{\mathfrak{e}}_n = e$, the probability of seeing an event of type $\mathsf{G}(\infty, k)$ or $\mathsf{G}(k, \infty)$ during the $(n+1)$th peeling step is equal to

$$W^{(k)} c_{\mathfrak{q}}^{-k-1},$$

and that conditionally on this event, the map filling-in the finite hole (rooted according to a deterministic convention) has law $\mathbb{P}^{(k)}$. These observations lead us to introduce the putative probability transitions: for any $k \geq 0$

$$b^{(\infty)}(k) = q_k c_{\mathbf{q}}^{k-1} \qquad \text{and} \qquad b^{(\infty)}(\infty, k) = b^{(\infty)}(k, \infty) = W^{(k)} c_{\mathbf{q}}^{-k-1}. \tag{6.4}$$

We now "check Tutte's equation", i.e. we verify that the previous definition forms probability transitions in the sense that

$$\sum_{k \geq 0} \left(b^{(\infty)}(k) + b^{(\infty)}(\infty, k) + b^{(\infty)}(k, \infty) \right) = 1.$$

This is indeed a trivial consequence of Lemma 5.2 once we notice that $b^{(\infty)}(k) = \nu(k-1)$ and $b^{(\infty)}(\infty, k) = \frac{1}{2}\nu(-k-1)$ for $k \geq 0$. Summarizing the above discussion yields the following proposition (still provided that the law $\mathbb{P}^{(\infty)}$ satisfying (6.3) does exist):

Proposition 6.3 (Law of the filled-in explorations under $\mathbb{P}^{(\infty)}$)**.** *Suppose $\mathfrak{M}^{(\infty)}$ is a random map of law $\mathbb{P}^{(\infty)}$ which satisfies (6.3). Fix a peeling algorithm $\mathcal{A}$ and let $\bar{\mathfrak{e}}_0 \subset \bar{\mathfrak{e}}_1 \subset \cdots \subset \bar{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{M}^{(\infty)}$ be the filled-in peeling exploration of $\mathfrak{M}^{(\infty)}$ with algorithm $\mathcal{A}$, then:*

- (Spatial Markov Property) *For any $n \geq 0$, conditionally on $\bar{\mathfrak{e}}_n$ the map filling-in the hole of $\bar{\mathfrak{e}}_n$ inside $\mathfrak{M}^{(\infty)}$ has law $\mathbb{P}^{(\infty)}$.*
- *The transition probabilities of the Markov chain $(\bar{\mathfrak{e}}_n)_{n \geq 0}$ are as follows: Conditionally on $\bar{\mathfrak{e}}_n$ and provided that $\mathcal{A}(\bar{\mathfrak{e}}_n) \neq \dagger$, the events C_k, $\mathsf{G}_{k,\infty}$ and $\mathsf{G}_{\infty,k}$ for $k \geq 0$ occur with respective probabilities*

  $$b^{(\infty)}(k), \qquad b^{(\infty)}(k, \infty) \qquad \text{and} \qquad b^{(\infty)}(\infty, k),$$

  and conditionally on these events the possible finite hole created is filled-in with an independent $\mathbf{q}$-Boltzmann map with the correct perimeter.

Proof. The second point follows from the discussion just before Definition 6.4. For the first point, we as usual need a convention, depending on $\bar{\mathfrak{e}}_n$ only, to root the map filling-in its hole. Once this is done the first point follows from the second, or can easily be proved directly from (6.3). We leave the details to the reader. □

Remark 6.4 (The variation of the perimeter is a ν-random walk). Recall from Section 5.1 the definition of the random walk (S) with independent increments distributed according to ν. Then using the above probability transitions, it is plain that

$$(|\partial^* \bar{\mathfrak{e}}_n| - |\partial \bar{\mathfrak{e}}_n|)_{n \geq 0} \text{ under } \mathbb{P}^{(\infty)} \overset{(d)}{=} (S_n)_{n \geq 0} \text{ under } \mathbf{P}^{(0)}. \tag{6.5}$$

6.1.3 Constructing $\mathbb{P}^{(\infty)}$

We will now *construct* the random infinite map of the half-plane $\mathfrak{M}^{(\infty)}$ whose law is $\mathbb{P}^{(\infty)}$. The idea is, of course, to construct $\mathfrak{M}^{(\infty)}$ using a filled-in peeling process whose law has been

determined in the previous proposition. Beware, since our goal is to *construct* the law $\mathbb{P}^{(\infty)}$, one cannot yet speak about filled-in exploration under this law. However, for any algorithm $\mathcal{A}$, the second point of Proposition 6.3 furnishes a Markov chain of growing submaps

$$\overline{\mathbf{E}}_0 \subset \overline{\mathbf{E}}_1 \subset \overline{\mathbf{E}}_2 \subset \cdots,$$

whose probability transitions are ruled by $b^{(\infty)}(\cdot,\cdot)$ and by the algorithm $\mathcal{A}$. An easy computation using the transitions of this Markov chain shows that for any submap $\overline{e}$ of the half-plane with a unique hole we have

$$\mathbb{P}(\overline{\mathbf{E}}_n = \overline{e}) = w_{\mathbf{q}}(\overline{e}) c_{\mathbf{q}}^{|\partial^* \overline{e}| - |\partial \overline{e}|} \mathbf{1}_{\overline{e} \text{ can be obtained after } n \text{ peeling steps using algorithm } \mathcal{A}}. \tag{6.6}$$

Our strategy will be to define $\mathfrak{M}^{(\infty)}$ as the increasing union of the $\overline{\mathbf{E}}_n$'s. However, for certain algorithms $\mathcal{A}$, the increasing union $\cup \overline{\mathbf{E}}_n$ will *not* create a map of the half plane[1] and so we need to choose carefully our peeling algorithm to explore step after step the local structure around the origin in a map. The most obvious of such algorithms is the following:

ALGORITHM $\mathcal{A}_{\text{metric}}$

If $\mathfrak{e} \subset \mathfrak{m}$ we define $\mathcal{A}_{\text{metric}}(\mathfrak{e})$ to be the edge immediately on the left of the vertex x which is the closest inside $\mathfrak{e}$ to the origin of the map among all vertices of $\partial^* \mathfrak{e}$. If there are several choices for x, choose the one we can reach from the origin by a geodesic which turns left as much as possible.

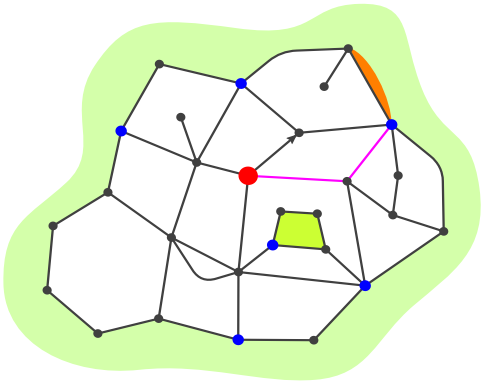


Figure: Illustration of the choice of the edge $\mathcal{A}_{\text{metric}}(\mathfrak{e})$ to peel (in orange). The origin vertex is in red, the two holes of the submap are in green, the vertices on $\partial^* \mathfrak{e}$ minimizing the distance to the origin are in blue. The left-most geodesic towards one of those vertices is in pink (it leaves the root edge on its left).

An important dubious point in the definition of the algorithm $\mathcal{A}_{\text{metric}}$ is that we have chosen x closest to ρ for the *graph distance in* $\mathfrak{e}$ (so that $\mathcal{A}_{\text{metric}}(\mathfrak{e})$ is indeed a function of $\mathfrak{e}$ and not of the underlying explored map). But it is easy to check that if $(\mathfrak{e}_n : n \geq 0)$ is the peeling exploration

[1] If for example we peel at points whose distance from the origin along the boundary increases very fast.

of the map $\mathfrak{m}$ using algorithm $\mathcal{A}_{\text{metric}}$, and if σ_n is the minimal graph distance of vertices of $\partial^* \mathfrak{e}_n$ to ρ, then we have (recall the notation from Section 2.1.2)

$$[\mathfrak{e}_n]_{\sigma_n} = [\mathfrak{m}]_{\sigma_n}.$$

In words, the peeling with algorithm $\mathcal{A}_{\text{metric}}$ roughly discovers metric balls of increasing radii.

Proposition 6.5. *Let $(\overline{\mathbf{E}}_n : n \geq 0)$ be the Markov chain of growing (sub)maps of the half-plane with a unique hole (necessarily of infinite perimeter) whose probability transitions are given by the second point of Proposition 6.3 with the peeling algorithm $\mathcal{A}_{\text{metric}}$. Then*

$$\mathfrak{M}^{(\infty)} := \bigcup_{n \geq 0} \overline{\mathbf{E}}_n,$$

is a random infinite map of the half-plane which satisfies (6.3).

Proof. There are two non-trivial points in the proposition. First, one needs to prove that $\mathfrak{M}^{(\infty)}$ as defined above is indeed a map of the half-plane and second that it satisfies (6.3).

For the first point, the problem that could appear is that some vertex x remains exposed on $\partial^* \overline{E}_n$ forever (i.e. is never swallowed by the process). If that happens, then σ_n the minimal graph distance of vertices of $\partial^* \mathfrak{e}_n$ does not grow to infinity (it is eventually constant) and by definition of our algorithm $\mathcal{A}_{\text{metric}}$, we eventually keep on peeling on the left of some vertex $x' \in \partial^* \overline{\mathfrak{e}}$. This cannot happen a.s., since at each step of the Markov chain, there is a probability at least $b^{(\infty)}(*, 0)$ that during the next step, the Markov chain swallows the point on right of the peeled edge (i.e. this vertex becomes an internal vertex of $\overline{\mathbf{E}}_n$). Hence $\sigma_n \to \infty$ a.s. and so $\mathfrak{M}^{(\infty)}$ is indeed a map of the half-plane almost surely.

We then need to check that $\mathfrak{M}^{(\infty)}$ satisfies (6.3). Fix a submap $\overline{e}_{n_0}$ of the half-plane with a unique hole of infinite perimeter which can be obtained as the result of n_0 steps of a filled-in exploration $\overline{e}_0 \subset \cdots \subset \overline{e}_{n_0}$. Our goal is then to prove that

$$\mathbb{P}(\overline{e}_{n_0} \subset \mathfrak{M}^{(\infty)}) = w_{\mathbf{q}}(\overline{e}_{n_0}) c_{\mathbf{q}}^{|\partial^* \overline{e}_{n_0}| - |\partial \overline{e}_{n_0}|}. \tag{6.7}$$

To do this, we first design another filled-in peeling algorithm $\mathcal{A}'_{\text{metric}}$ so that if $\overline{\mathfrak{e}} = \overline{e}_i$ for some $i < n_0$ then $\mathcal{A}'_{\text{metric}}(\overline{\mathfrak{e}})$ is the edge peeled when passing from $\overline{e}_i$ to $\overline{e}_{i+1}$, otherwise we put $\mathcal{A}'_{\text{metric}}(\overline{\mathfrak{e}}) = \mathcal{A}_{\text{metric}}(\overline{\mathfrak{e}})$. Roughly speaking, the filled-in exploration using algorithm $\mathcal{A}'_{\text{metric}}$ first decides whether or not we have $\overline{e}_{n_0} \subset \mathfrak{m}$ and then performs the metric exploration with $\mathcal{A}_{\text{metric}}$. We denote by $(\overline{E}_n : n \geq 0)$ and $(\overline{E}'_n : n \geq 0)$ the Markov chains of growing maps ruled by the transition $b^{(\infty)}$ of Proposition 6.3 and peeling algorithms $\mathcal{A}_{\text{metric}}$ and $\mathcal{A}'_{\text{metric}}$ respectively. Adapting the above argument we get that

$$\mathfrak{M}^{(\infty),'} = \bigcup_{n \geq 0} \overline{\mathbf{E}}'_n,$$

is almost surely a map of the half-plane. Using (6.6), we have

$$\mathbb{P}(\overline{e}_{n_0} \subset \mathfrak{M}^{(\infty),'}) = \mathbb{P}(\overline{\mathbf{E}}'_{n_0} = \overline{e}_{n_0}) = w_{\mathbf{q}}(\overline{e}_{n_0}) c_{\mathbf{q}}^{|\partial^* \overline{e}_{n_0}| - |\partial \overline{e}_{n_0}|}.$$

Hence, it suffices to prove that $\mathfrak{M}^{(\infty)} = \mathfrak{M}^{(\infty),\prime}$ in law to deduce our goal (6.7). To see this, let us introduce the stopping time τ (respect. τ'), the first time at which the minimal graph distance of vertices of $\partial^*\overline{\mathbf{E}}_n$ (resp. $\partial^*\overline{\mathbf{E}}'_n$) to ρ is larger than $d \geq 0$, where d is chosen large enough so that the diameter of $\overline{e}_{n_0}$ is strictly smaller than d. By the above consideration, we have $\tau < \infty$ and $\tau' < \infty$ and an easy extension of (6.6) to almost sure finite stopping times shows that for any $\overline{e}$, we have

$$\mathbb{P}(\overline{\mathbf{E}}_\tau = \overline{e}) = w_{\mathbf{q}}(\overline{e}) c_{\mathbf{q}}^{|\partial^*\overline{e}|-|\partial\overline{e}|} \mathbf{1}_{\overline{e} \text{ can be obtained as } \overline{\mathbf{E}}_\tau},$$

$$\mathbb{P}(\overline{\mathbf{E}}'_{\tau'} = \overline{e}) = w_{\mathbf{q}}(\overline{e}) c_{\mathbf{q}}^{|\partial^*\overline{e}|-|\partial\overline{e}|} \mathbf{1}_{\overline{e} \text{ can be obtained as } \overline{\mathbf{E}}'_{\tau'}}.$$

But by our choice of d and the definitions of $\tau, \tau', \mathcal{A}_{\text{metric}}$ and $\mathcal{A}'_{\text{metric}}$, a moment's thought shows that the possible outcomes of $\overline{\mathbf{E}}_\tau$ or $\overline{\mathbf{E}}'_{\tau'}$ *are the same*. The last display then shows that $\overline{\mathbf{E}}_\tau = \overline{\mathbf{E}}'_{\tau'}$ in distribution and by letting $d \to \infty$ we deduce that $\bigcup_{n\geq 0} \overline{\mathbf{E}}'_n = \mathfrak{M}^{(\infty),\prime} = \mathfrak{M}^{(\infty)} = \bigcup_{n\geq 0} \overline{\mathbf{E}}_n$ in law as desired. □

6.1.4 $\mathbb{P}^{(\infty)}$ as the weak limit of $\mathbb{P}^{(\ell)}$ as $\ell \to \infty$

Now that $\mathbb{P}^{(\infty)}$ is constructed, proving Theorem 6.1 is a piece of cake:

Proof of Theorem 6.1. With the notation of (6.2), we have thanks to Lemma 3.13 and Proposition 6.2

$$\mathbb{P}^{(\ell)}(\tilde{\mathfrak{e}} \Subset \mathfrak{m}) \xrightarrow[\ell\to\infty]{\text{Lem. 3.13}} \mathbb{P}^{(\infty)}(\tilde{\mathfrak{e}} \Subset \mathfrak{m}) = \mathbb{P}^{(\infty)}(\overline{\mathfrak{e}} \subset \mathfrak{m}).$$

Since by Proposition 6.2 the right-hand side characterizes the law of $\mathbb{P}^{(\infty)}$, this is sufficient to imply the weak convergence for the local topology. In the case of $\mathbb{P}_p^{(\ell)}$, by restricting to the event where the target face does not belong to $\tilde{\mathfrak{e}}$ we can write

$$\mathbb{P}_p^{(\ell)}(\tilde{\mathfrak{e}} \Subset \mathfrak{m}) \geq w_{\mathbf{q}}(\overline{\mathfrak{e}}) \frac{W_p^{\ell-(p_{\text{ext}}-p_{\text{int}})}}{W_p^{(\ell)}} \xrightarrow[\ell\to\infty]{(3.9)} \mathbb{P}^{(\infty)}(\tilde{\mathfrak{e}} \Subset \mathfrak{m}),$$

and this is sufficient to imply the desired weak convergence by (a variation on) Remark 2.5. □

6.2 Basic properties

In this section, we prove basic properties of $\mathfrak{M}^{(\infty)}$, the $\mathbf{q}$-Boltzmann map of the half-plane of law $\mathbb{P}^{(\infty)}$: stationary and ergodicity with respect to re-rooting along the boundary, and the study of cut-points along the boundary.

6.2.1 Translation invariance and ergodicity

Let $\theta\mathfrak{M}^{(\infty)}$ be the map obtained by re-rooting $\mathfrak{M}^{(\infty)}$ on the edge immediately on the right of the root edge on the root face.

Proposition 6.6. *The shift operator θ preserves the law $\mathbb{P}^{(\infty)}$ and is ergodic.*

Proof. In the finite setting (with obvious notation), it is obvious that $\theta\mathbb{P}^{(\ell)} = \mathbb{P}^{(\ell)}$ simply because the re-rooting operation is a bijection on the set of all finite bipartite planar maps with a boundary of perimeter 2ℓ. Taking the limit as $\ell \to \infty$ for the local topology (Theorem 6.1) gives the first point of the proposition. For the second point, we will prove the stronger statement that the shift θ is mixing. Let $\bar{\mathfrak{e}}$ be a submap of the half-plane with a unique hole. From Proposition 6.2, if k is larger than twice the total perimeter of the pruned map $\tilde{\mathfrak{e}}$ (see Figure 6.2) then we have by (6.3)

$$\mathbb{P}\left(\bar{\mathfrak{e}} \subset \mathfrak{M}^{(\infty)} \text{ and } \bar{\mathfrak{e}} \subset \theta^k \mathfrak{M}^{(\infty)}\right) = \left(w_{\mathbf{q}}(\bar{\mathfrak{e}}) \cdot c_{\mathbf{q}}{}^{|\partial^* \bar{\mathfrak{e}}| - |\partial \bar{\mathfrak{e}}|}\right)^2 = \mathbb{P}\left(\bar{\mathfrak{e}} \subset \mathfrak{M}^{(\infty)}\right) \mathbb{P}\left(\bar{\mathfrak{e}} \subset \theta^k\, \mathfrak{M}^{(\infty)}\right),$$

and this is sufficient to imply ergodicity of the shift operator for the local topology. □

As a corollary of the previous proposition, any function of $\mathfrak{M}^{(\infty)}$ that is invariant under translation of the root edge is almost surely constant. Examples of such properties are: recurrence of the underlying graph, value of Bernoulli percolation thresholds... or as we will see below, existence of infinitely many cut-points separating the origin from ∞.

6.2.2 Cut-edges and cut-points

Recall that the boundary of $\mathfrak{M}^{(\infty)}$ is not assumed to be simple. We call **cut-edges** (resp. **cut-points**) the edges (resp. vertices) on the boundary of $\mathfrak{M}^{(\infty)}$ which separate the origin ρ from ∞ (recall that $\mathfrak{M}^{(\infty)}$ is one-ended). A cut-corner is an angular sector of the external face associated with a cut-point. See Figure 6.3. We denote by $\mathcal{C}_{\text{edge}}, \mathcal{C}_{\text{point}}$ and $\mathcal{C}_{\text{corner}}$ the number of cut-edges, cut-points and cut-corners in the map $\mathfrak{M}^{(\infty)}$.

Proposition 6.7 (Criterion for infinitely many cut-points)**.** *We have*

$$(\mathcal{C}_{\text{edge}} = \infty) \iff (\mathcal{C}_{\text{point}} = \infty) \iff (\mathcal{C}_{\text{corner}} = \infty) \iff \sum_{k \geq 0} k\nu(-k) = \infty.$$

*When the above conditions are satisfied we will say that we are in the **dense phase** (and in the **dilute phase** otherwise), see Section 5.3.4 for explanation of the terminology.*

Proof. The proof is based on the following peeling algorithm which discovers the cut-corners one after the other:

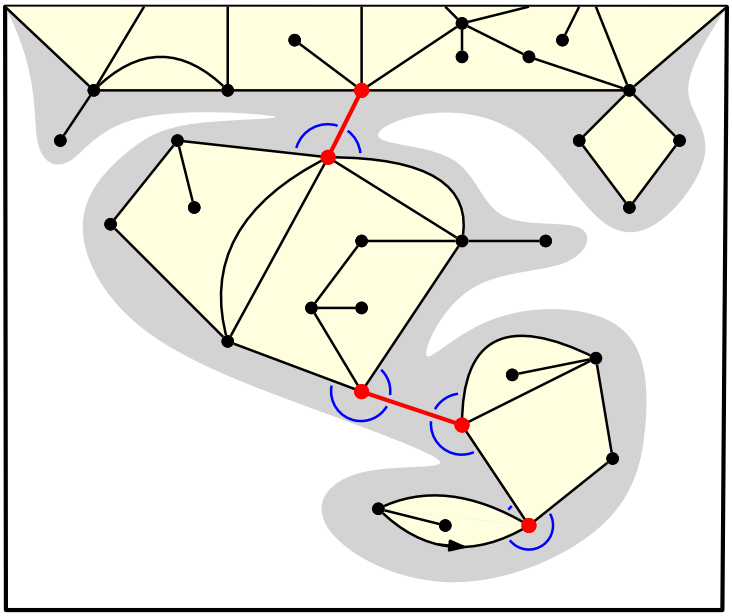


Figure 6.3: The first few cut-points, cut-edges (in red) and cut corners (in blue) of $\mathfrak{M}^{(\infty)}$.

ALGORITHM $\mathcal{C}_{\text{corners}}$

If $\mathfrak{e} \subset \mathfrak{m}$, we define $\mathcal{A}_{\text{corners}}(\mathfrak{e})$ to be the edge immediately on the right of the first edge on the left of the root edge that is still part of the initial boundary of $\mathfrak{m}$. See figure below.

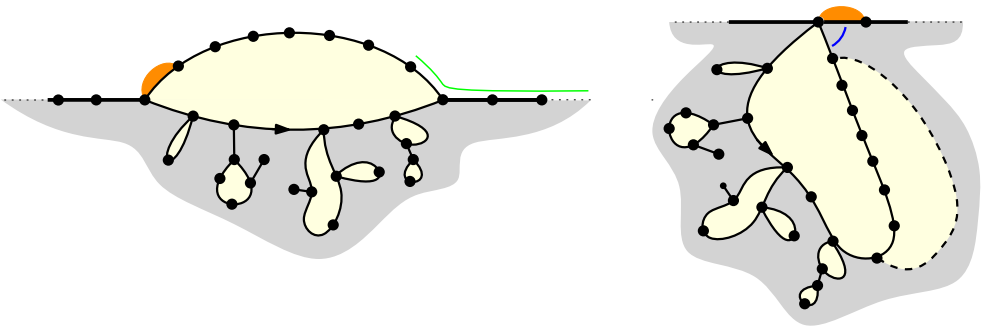


Figure: Illustration of algorithm $\mathcal{A}_{\text{corners}}$. The edge to peel is in orange. If it is identified with an edge belonging to the green part, then this creates a cut-corner in the map, as depicted in the right picture.

If the above algorithm $\mathcal{A}_{\text{corners}}$ is used to explore the map (in its filled-in version), then right cut-corners will be found sequentially when we identify the peeling edge with another edge and that the entire explored region is disconnected from the hole by a single vertex, see the above figure. We claim that those times in the exploration process correspond to the discovering of a cut-corner on the right boundary of $\mathfrak{M}^{(\infty)}$, see Figure 6.4.

By the spatial Markov property (Proposition 6.3 Item 1 extended to finite stopping times),

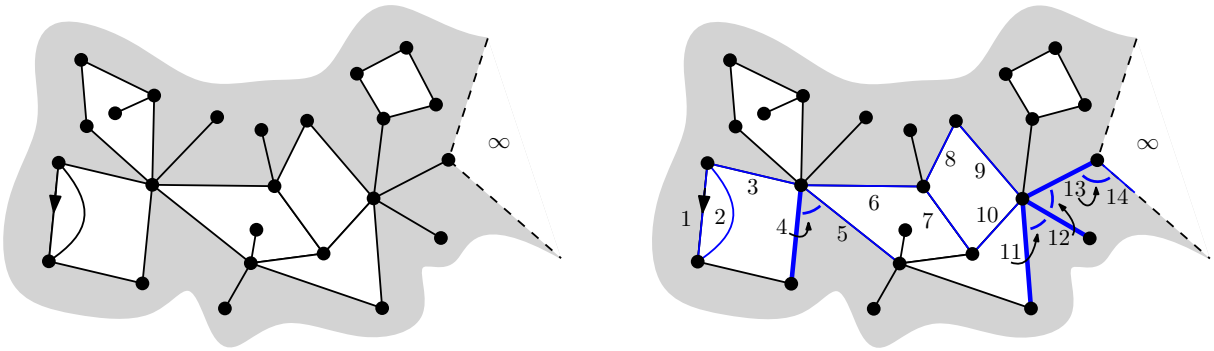


Figure 6.4: Illustration of the exploration using algorithm $\mathcal{A}_{\text{corners}}$. On the left picture we see a piece of a half-planar map with cut-points. In the right picture, we highlighted in blue the edges peeled by the algorithm chronologically. The heavier blue lines yield the discovery of a cut-corner which are materialized by crosses.

once we found a new right cut-corner, then the remaining number of right cut-corners is distributed as in a fresh copy of $\mathfrak{M}^{(\infty)}$. In other words, the number of right cut-corners has a geometric distribution. Notice also, that once a right cut-corner is found, if the next peeling event identifies the two edges surrounding the cut-point just discovered then we create a cut-edge in $\mathfrak{M}^{(\infty)}$. By these remarks, we have $\{\mathcal{C}_{\text{edge}} = \infty\} = \{\mathcal{C}_{\text{point}} = \infty\} = \{\mathcal{C}_{\text{corner}} = \infty\}$ up to null events and this occurs with probability one if and only if $\mathbb{E}[\mathcal{C}_{\text{edge}}] = \infty$. To compute the latter, we consider the boundary of $\mathfrak{M}^{(\infty)}$ seen as unexplored, which we identify with $\mathbb{Z}$. For any $i \geq 0$ and $j \geq 0$ such that $i + j$ is even, the probability that the edge $\{(-i-1) \to -i\}$ is identified in $\mathfrak{M}^{(\infty)}$ with the edge $\{j \to (j+1)\}$ is equal to

$$\frac{1}{2}\nu\left(-\frac{i+j}{2} - 1\right).$$

Hence we have

$$\mathbb{E}[\mathcal{C}_{\text{edge}}] = \sum_{\substack{i\geq 0, j\geq 0 \\ i+j \text{ even}}} \frac{1}{2}\nu\left(-\frac{i+j}{2} - 1\right),$$

and this series has the same behavior as $\sum_{k\geq 0} k\,\nu(-k)$ as desired. □

Bibliographical notes. The infinite $\mathbf{q}$-Boltzmann map of the half-plane is probably the random infinite map having the nicest spatial Markov property (reminiscent of the Markov property of Schramm-Loewner evolution in the half-plane) and is a enjoyable probability playground. Such type of random lattices were first introduced by Angel [12] in the case of triangulations with simple boundary to study percolation, we speak of the **U**niform **I**nfinite **H**alf-**P**lanar **T**riangulation or UIHPT or UIHPQ in the quadrangular case, see Chapter 9. The first construction of infinite half-planar maps having a general boundary can be found in [97] in the quadrangulation case. The more general results of this section are adapted from [158]. When $\mathbf{q}$ is subcritical,

the geometry of $\mathfrak{M}^{(\infty)}$ is tree-like and related to the infinite Brownian CRT (although we do not know of any proper statement of this fact in the literature), see [170]. However, when $\mathbf{q}$ is critical, the geometry is much more complex and is still the subject of intense studies, see e.g. [23, 67, 68, 120, 170]. Invariance and ergodicity of the translation operator along the boundary is adapted from [18] and the dichotomy for presence of infinitely many cut-points has already been noticed by Richier [174] in the case of regular varying weight sequences. The proof of Proposition 6.7 is adapted from [57, Theorem 4]. A version of random half-planar triangulation coupled with (critical) Ising model is studied in [78].

Chapter VII : Infinite Boltzmann maps of the plane

In this chapter, we introduce the infinite Boltzmann maps of the plane of which the famous **U**niform **I**nfinite **P**lanar **Q**uadrangulation (UIPQ for short) is an example. Contrary to the half-planar case, we will need to deal with *critical* weight sequences (see Definition-Theorem 5.4).

In the rest of this chapter we shall assume that $\mathbf{q}$ *is critical.*

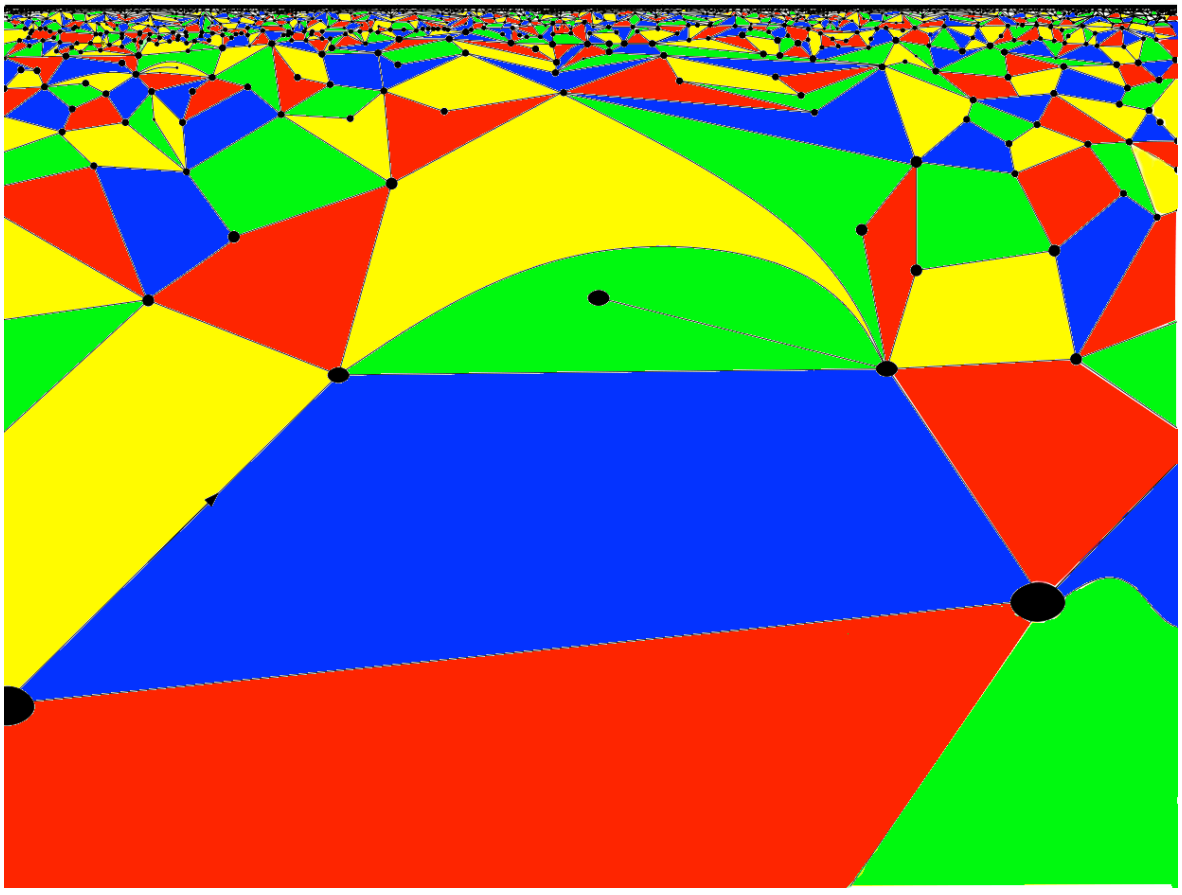


Figure 7.1: An artistic representation of the UIPQ, which is a random map distributed according to the measure $\mathbb{P}_\infty$ when $\mathbf{q} = (12^{-1}\mathbf{1}_{k=2})$ corresponding to the model of critical random quadrangulations.

Here is the main result of this chapter:

Theorem 7.1 (*Boltzmann map of the plane*)

Let $\mathbf{q}$ *be an admissible and critical weight sequence. For any* $\ell \geq 1$*, we have the following convergence in distribution for the local topology*

$$\mathbb{P}_p^{(\ell)} \xrightarrow[p\to\infty]{(d)} \mathbb{P}_\infty^{(\ell)} \quad \text{and} \quad \mathbb{P}^{(\ell)}(\cdot \mid |\mathfrak{m}| = n) \xrightarrow[n\to\infty]{(d)} \mathbb{P}_\infty^{(\ell)}$$

where the last convergence holds along the values of n *for which* $\mathbb{P}^{(\ell)}(|\mathfrak{m}| = n) > 0$*. The distribution* $\mathbb{P}_\infty^{(\ell)}$ *is supported by infinite bipartite maps of the plane such that the external face is of degree* 2ℓ*. This is the so-called infinite* $\mathbf{q}$*-Boltzmann distribution of the plane with a boundary of perimeter* 2ℓ*.*

As expected, we shall write $\mathfrak{M}_\infty^{(\ell)}$ for a random map of law $\mathbb{P}_\infty^{(\ell)}$. When $\ell = 1$, we recall from Remark 3.1 that a (rooted bipartite finite or infinite) map with a root face of degree 2 can be seen after contraction of that face just as a rooted map. Hence the measure $\mathbb{P}_\infty^{(1)}$ can also be seen as a probability distribution over the set of rooted infinite bipartite planar map of the plane. We shall call this random lattice the **q-Boltzmann map of the plane** and drop the exponent (1) and write $\mathfrak{M}_\infty$ respectively $\mathbb{P}_\infty$ instead of $\mathfrak{M}_\infty^{(1)}$ and $\mathbb{P}_\infty^{(1)}$.

Remark 7.2 (UIPQ). In the case $\mathbf{q} = (12^{-1}\mathbf{1}_{k=2})_{k\geq 1}$, corresponding to critical Boltzmann quadrangulations (see Section 3.5), the measure $\mathbb{P}_\infty$ on quadrangulations of the plane is the law of the Uniform Infinite Planar Quadrangulation defined in [74, 133, 154]. In the triangular case (although not bipartite, our techniques goes through) we speak of the UIPT for Uniform Infinite Planar Triangulation defined by Angel & Schramm [20].

7.1 Infinite Boltzmann maps of the plane

Fix $\ell \geq 1$. In this section, we construct the law $\mathbb{P}_\infty^{(\ell)}$ and then prove that it appears as weak limits for the local distance of various finite random Boltzmann maps. To build those laws, we follow the general recipe sketched at the beginning of this part and already applied in the previous chapter.

7.1.1 Characterizing $\mathbb{P}_\infty^{(\ell)}$

Let us suppose first that $\mathbb{P}_p^{(\ell)} \to \mathbb{P}_\infty^{(\ell)}$ as $p \to \infty$ and deduce a characterization of this law. If $\bar{\mathfrak{e}}$ is a submap with boundary of half-perimeter $|\partial\bar{\mathfrak{e}}| = \ell$ and with a unique hole of half-perimeter $|\partial^*\bar{\mathfrak{e}}|$ then we have

$$\begin{aligned}\mathbb{P}_\infty^{(\ell)}(\bar{\mathfrak{e}} \subset \mathfrak{m}) &\underset{\text{hypothesis}}{=} \lim_{p\to\infty} \mathbb{P}_p^{(\ell)}(\bar{\mathfrak{e}} \subset \mathfrak{m}) \\ &= \lim_{p\to\infty} \left(\mathbb{P}_p^{(\ell)}(\bar{\mathfrak{e}} \subset \mathfrak{m} \text{ and the target face is in } \bar{\mathfrak{e}}) + w_{\mathbf{q}}(\bar{\mathfrak{e}}) \cdot \frac{W_p^{(|\partial^*\bar{\mathfrak{e}}|)}}{W_p^{(\ell)}} \right).\end{aligned} \tag{7.1}$$

The first term $\mathbb{P}_p^{(\ell)}(\bar{\mathfrak{e}} \subset \mathfrak{m}$ and the target face is in $\bar{\mathfrak{e}})$ is eventually null for large p (larger than the maximal degree of a face of $\bar{\mathfrak{e}}$). So by Theorem 3.12 and the explicit form of the functions $h_p^{\downarrow}$, we thus guess that

$$\mathbb{P}_\infty^{(\ell)}(\bar{\mathfrak{e}} \subset \mathfrak{m}) \stackrel{?}{=} w_{\mathbf{q}}(\bar{\mathfrak{e}}) \cdot \frac{c_{\mathbf{q}}^{|\partial^*\bar{\mathfrak{e}}|} h^\uparrow(|\partial^*\bar{\mathfrak{e}}|)}{c_{\mathbf{q}}^{\ell} h^\uparrow(\ell)},$$

where the function $h^\uparrow$ is the renewal function of the ν-walk, see (5.4). A straightforward adaptation of Proposition 6.2 shows that this is indeed sufficient to characterize the law $\mathbb{P}_\infty^{(\ell)}$:

Proposition 7.3. *For any $\ell \geq 1$, there is at most one law $\mathbb{P}_\infty^{(\ell)}$ on infinite maps of the plane with a boundary of perimeter 2ℓ, so that for any finite submap $\bar{\mathfrak{e}}$ with boundary of perimeter 2ℓ and a unique hole we have*

$$\mathbb{P}_\infty^{(\ell)}(\bar{\mathfrak{e}} \subset \mathfrak{m}) \quad = \quad w_{\mathbf{q}}(\bar{\mathfrak{e}}) \cdot c_{\mathbf{q}}^{|\partial^*\bar{\mathfrak{e}}|-|\partial\bar{\mathfrak{e}}|} \cdot \frac{h^\uparrow(|\partial^*\bar{\mathfrak{e}}|)}{h^\uparrow(|\partial\bar{\mathfrak{e}}|)}. \tag{7.2}$$

7.1.2 Peeling process under $\mathbb{P}_\infty^{(\ell)}$

Following our general strategy, we suppose that the law $\mathbb{P}_\infty^{(\ell)}$ exists and obeys (7.2). We then derive the law of its filled-in peeling process $(\bar{\mathfrak{e}}_n : n \geq 0)$ for a given algorithm $\mathcal{A}$. Indeed, following the same lines as in the previous chapter, we can use (7.2) to see that conditionally on $\bar{\mathfrak{e}}_n$ then the next peeling step is an event C_k or $\mathsf{G}_{(*,k)}$ or $\mathsf{G}_{(k,*)}$ according to the following probabilities:

$$b_\infty^{(\ell)}(k) = c_{\mathbf{q}}^{k-1} q_k \frac{h^\uparrow(\ell+k-1)}{h^\uparrow(\ell)}, \quad \text{and} \quad b_\infty^{(\ell)}(*,k) = b_\infty^{(\ell)}(k,*) = c_{\mathbf{q}}^{-k-1} W^{(k)} \frac{h^\uparrow(\ell-k-1)}{h^\uparrow(\ell)}, \tag{7.3}$$

where $\ell = |\partial^*\bar{\mathfrak{e}}_n| \geq 1$ and $k \geq 0$. We now "check Tutte's equation", i.e. we verify that those probability transitions indeed sum-up to 1. By definition of ν, this is equivalent to the fact that the function $h^\uparrow$ is harmonic for the ν-random walk on $\{1, 2, ...\}$ which is the case by Theorem 5.4 item (ii) since $\mathbf{q}$ is supposed to be critical. In terms of the peeling process we deduce:

Proposition 7.4 (Law of the filled-in explorations under $\mathbb{P}_\infty^{(\ell)}$)**.** *Fix $\ell \geq 1$. Suppose $\mathfrak{M}_\infty^{(\ell)}$ is a random map of law $\mathbb{P}_\infty^{(\ell)}$ which satisfies (7.2). Fix a peeling algorithm $\mathcal{A}$ and let $\bar{\mathfrak{e}}_0 \subset \bar{\mathfrak{e}}_1 \subset \cdots \subset \bar{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{M}_\infty^{(\ell)}$ be the filled-in peeling exploration of $\mathfrak{M}_\infty^{(\ell)}$ with algorithm $\mathcal{A}$, then:*

- (Spatial Markov Property) *For any $n \geq 0$, conditionally on $\bar{\mathfrak{e}}_n$, the map filling-in the hole of $\bar{\mathfrak{e}}_n$ inside $\mathfrak{M}_\infty^{(\ell)}$ has law $\mathbb{P}_\infty^{(|\partial^*\bar{\mathfrak{e}}_n|)}$.*
- *The transition probabilities of the Markov chain $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ are as follows: Conditionally on $\bar{\mathfrak{e}}_n$ and provided that $\mathcal{A}(\bar{\mathfrak{e}}_n) \neq \dagger$, the events C_k, $\mathsf{G}_{k,*}$ and $\mathsf{G}_{*,k}$ for $k \geq 0$ occur with respective probabilities*

  $$b_\infty^{(|\partial^*\bar{\mathfrak{e}}_n|)}(k), \qquad b_\infty^{(|\partial^*\bar{\mathfrak{e}}_n|)}(k,*) \qquad \text{and} \qquad b_\infty^{(|\partial^*\bar{\mathfrak{e}}_n|)}(*,k),$$

  and conditionally on these events, the possible finite hole created is filled-in with an independent $\mathbf{q}$-Boltzmann map of the correct perimeter.

Remark 7.5. Notice that the process of the half-perimeter of the hole $(|\partial^*\overline{\mathfrak{e}}_n| : n \geq 0)$ during such an exploration has the law of the $h^{\uparrow}$-transform of the ν-random walk S started from ℓ, in other words $(|\partial^*\overline{\mathfrak{e}}_n| : n \geq 0)$ under $\mathbb{P}_{\infty}^{(\ell)}$ has the law of $S^{\uparrow}$ under $\mathbf{P}^{(\ell)}$.

7.1.3 Constructing $\mathbb{P}_{\infty}^{(\ell)}$

Exactly as in the previous chapter, one can now *construct* the law $\mathbb{P}_{\infty}^{(\ell)}$ using a Markov chain of growing submaps based on the transition probabilities identified in the previous proposition applied with the algorithm $\mathcal{A}_{\text{metric}}$ of Section 6.1.3:

Proposition 7.6. *Let $(\overline{\mathbf{E}}_n : n \geq 0)$ be the Markov chain whose probability transitions are given by Proposition 7.4 for the peeling algorithm $\mathcal{A}_{\text{metric}}$ started with a boundary of perimeter 2ℓ. Then*

$$\mathfrak{M}_{\infty}^{(\ell)} := \bigcup_{n\geq 0} \overline{\mathbf{E}}_n,$$

is a random infinite map of the plane which satisfies (7.2).

Proof. The proof is mutatis mutandis that of Proposition 6.5. To see that the union of the $\overline{\mathbf{E}}_n$'s indeed forms a map of the plane, one needs to show that every vertex on the boundary is eventually swallowed. This follows from the fact that at each step, if $|\partial^*\overline{\mathfrak{e}}_n| \geq 2$, an event of type $\mathrm{G}_{(*,0)}$ can swallow the point on the right of peeled edge with probability at least

$$b_{\infty}^{(|\partial^*\overline{\mathfrak{e}}_n|)}(*,0) = c_{\mathfrak{q}}^{-1}\frac{h^{\uparrow}(|\partial^*\overline{\mathfrak{e}}_n|-1)}{h^{\uparrow}(|\partial^*\overline{\mathfrak{e}}_n|)} \geq c_{\mathfrak{q}}^{-1}\inf_{\ell>1}\frac{h^{\uparrow}(\ell-1)}{h^{\uparrow}(\ell)} > 0.$$

If $|\partial^*\overline{\mathfrak{e}}_n| = 1$ this vertex can be swallowed in two steps. We conclude as in Proposition 6.5 that $\mathfrak{M}_{\infty}^{(\ell)}$ is almost surely a map of the plane. To see that it satisfies (7.2), we proceed as in Proposition 6.5 and show that $\bigcup_{n\geq 0}\overline{\mathbf{E}}_n$ has the same law as $\bigcup_{n\geq 0}\overline{\mathbf{E}}'_n$ where the Markov chain $\overline{\mathbf{E}}'$ is now driven by a variation of the algorithm $\mathcal{A}_{\text{metric}}$ designed to first check whether $\overline{e}_{n_0} \subset \mathfrak{m}$ for some fixed submap $\overline{e}_{n_0}$. The adaptation is left to the reader. □

7.1.4 $\mathbb{P}_{\infty}^{(\ell)}$ as the limit of maps conditioned to be large

As in the half-plane case, once the law of the infinite Boltzmann map of the plane is constructed, showing that it is a local limit in distribution of finite Boltzmann maps is rather easy:

Proof of Theorem 7.1. Let us treat first the case of $\mathbb{P}_p^{(\ell)}$ when $p \to \infty$. Recalling (7.1) we have

$$\mathbb{P}_p^{(\ell)}(\overline{\mathfrak{e}} \subset \mathfrak{m}) \geq w_{\mathfrak{q}}(\overline{\mathfrak{e}}) \cdot \frac{W_p^{(|\partial^*\overline{\mathfrak{e}}|)}}{W_p^{(|\partial\overline{\mathfrak{e}}|)}} \xrightarrow[p\to\infty]{\text{Thm. 3.12 \& (5.4)}} w_{\mathfrak{q}}(\overline{\mathfrak{e}}) \cdot c_{\mathfrak{q}}^{|\partial^*\overline{\mathfrak{e}}|-|\partial\overline{\mathfrak{e}}|} \cdot \frac{h^{\uparrow}(|\partial^*\overline{\mathfrak{e}}|)}{h^{\uparrow}(|\partial\overline{\mathfrak{e}}|)} \underset{(7.2)}{=} \mathbb{P}_{\infty}^{(\ell)}(\overline{\mathfrak{e}} \subset \mathfrak{m}).$$

By Remark 2.5 this is sufficient to imply the weak convergence $\mathbb{P}_p^{(\ell)} \to \mathbb{P}_{\infty}^{(\ell)}$ for the local topology as $p \to \infty$. To treat the case of $\mathbb{P}^{(\ell)}(\cdot \mid |\mathfrak{m}| = n)$ we use the same calculation together with the following enumeration result. Recall that $W^{(\ell)}[n]$ is the $w_{\mathfrak{q}}$-weight of all planar maps with external face of degree 2ℓ and n edges.

Lemma 7.7. *Let* $\mathbf{q}$ *be an admissible and critical weight sequence. For any* $\ell, \ell' \geq 1$ *and for any* $n_0 \geq 0$ *we have*

$$\frac{W^{(\ell)}[n]}{W^{(\ell')}[n-n_0]} \xrightarrow[n\to\infty]{} \frac{h^{\uparrow}(\ell)\cdot c_{\mathbf{q}}^{\ell}}{h^{\uparrow}(\ell')\cdot c_{\mathbf{q}}^{\ell'}},$$

where the limit is taken along the integers n *for which both* $W^{(\ell)}[n]$ *and* $W^{(\ell')}[n-n_0]$ *are non zero.*

Proof. Recall from Theorem 3.12 that

$$W_1^{(\ell)}[n] = \frac{\ell}{n}\binom{2\ell}{\ell} Z_{\mathbf{q}}^{\ell+1}\cdot \mathbb{P}(Y_n = -\ell - 1),$$

where $(Y_n : n \geq 0)$ is a skip-free descending random walk with i.i.d. increments of law μ. Since $\mathbf{q}$ is supposed to be critical, the increment distribution μ is centered and in particular (Y) is recurrent. We can apply the strong ratio limit theorem (Theorem A.19 and the remark following it) and get that for any $k_0, a, b \in \mathbb{Z}$ we have

$$\lim_{n\to\infty} \frac{\mathbb{P}(Y_{n-k_0} = a)}{\mathbb{P}(Y_n = b)} = 1,$$

where the convergence holds along the values of n for which the above probabilities are non-zero. Coming back to our setup, thanks to the above displays we deduce that

$$\begin{aligned}\frac{W^{(\ell)}[n]}{W^{(\ell')}[n-n_0]} &\overset{(3.5)}{=} \frac{n-n_0}{n}\frac{W_1^{(\ell)}[n+1]}{W_1^{(\ell')}[n-n_0+1]} \\ &= \underbrace{\frac{\mathbb{P}(Y_{n+1} = -\ell-1)}{\mathbb{P}(Y_{n-n_0+1} = -\ell'-1)}\frac{(n-n_0)(n-n_0+1)}{n(n+1)}}_{\xrightarrow[n\to\infty]{} 1}\cdot \underbrace{\frac{\ell\binom{2\ell}{\ell}Z_{\mathbf{q}}^{\ell}}{\ell'\binom{2\ell'}{\ell'}Z_{\mathbf{q}}^{\ell'}}}_{\frac{h^{\uparrow}(\ell)c_{\mathbf{q}}^{\ell}}{h^{\uparrow}(\ell')c_{\mathbf{q}}^{\ell'}}}.\end{aligned}$$

□

7.2 Basic properties

We now study basic properties of the random lattice $\mathfrak{M}_\infty$ sampled according to $\mathbb{P}_\infty$ for a fixed admissible and critical weight sequence. Contrary to the half-planar case where the translation of the root edge along the boundary was an obvious invariant shift operation, here the translation of the root edge must occur *along a simple random walk* to preserve the distribution.

7.2.1 Stationarity and reversibility

For our purposes, it is more convenient to see the **simple random walk** on a planar map as a process taking values in the set of oriented edges rather than in the set of vertices:

Definition 7.8. *If $\mathfrak{m}$ is a finite or infinite rooted planar map, a simple random walk on $\mathfrak{m}$ is a Markov process taking values in the space of oriented edges of $\mathfrak{m}$, starting with the root edge, and such that if $\vec{e}$ is the current state, then the next oriented edge is chosen uniformly among all the oriented edges outgoing from the target vertex of $\vec{e}$ independently of the past path. We denote by $\mathrm{SRW}_{\mathfrak{m}}$ its law.*

This definition is equivalent to a walk on the vertices which jumps iteratively to a uniform neighbor in the case of simple maps, but in general, the presence of loops and of multiple edges make the transitions between vertices a little bit more complicated. In the case of a random rooted map $\mathfrak{M}$ with underlying law $\mathbb{P}$, we will speak of the random walk $(\vec{E}_k : k \geq 0)$ on $\mathfrak{M}$ under the law

$$\int \mathbb{P}(\mathrm{d}\mathfrak{m}) \int \mathrm{SRW}_{\mathfrak{m}}(\mathrm{d}(\vec{e}_k) : k \geq 0).$$

In other words, we first sample the map $\mathfrak{M}$ and then perform the random walk $(\vec{E}_k : k \geq 0)$ starting from its root edge.

Proposition 7.9 (Stationarity and reversibility)**.** *If $(\vec{E}_k : k \geq 0)$ is the random walk on $\mathfrak{M}_\infty$ then for any $k \geq 0$ we have*

$$\mathrm{Reroot}\left(\mathfrak{M}_\infty; \vec{E}_k\right) \overset{(d)}{=} \mathfrak{M}_\infty,$$

and moreover the map $\mathrm{Reroot}\left(\mathfrak{M}_\infty; \overleftarrow{E}_0\right)$ in which we reverse the orientation of the root edge also has the same law as $\mathfrak{M}_\infty$.

This result gives a sense to "invariance under re-rooting" in $\mathfrak{M}_\infty$. More precisely, we can define a shift operation θ that acts on the space of (non rooted) planar maps given with an infinite oriented path by erasing the first step of the path:

$$\theta(\mathfrak{m}, (\vec{e}_n)_{k\geq 0}) = (\mathfrak{m}, (\vec{e}_k)_{k\geq 1}),$$

and the above result shows that θ leaves $\int \mathbb{P}_\infty(\mathrm{d}\mathfrak{m}) \int \mathrm{SRW}_{\mathfrak{m}}$ invariant. Notice that this re-rooting property cannot hold under $\mathbb{P}_\infty^{(\ell)}$ with $\ell \geq 1$ since the root face has a prescribed degree.

Proof. The proposition is easily obtained by passing to the limit in the finite setting. Indeed, if $\mathfrak{M}[n]$ of law $\mathbb{P}^{(1)}(\cdot \mid |\mathfrak{m}| = n)$ is a $\mathbf{q}$-Boltzmann map conditioned to have n edges (provided that n is chosen so that this has a sense), then it is plain that if conditionally on $\mathfrak{M}[n]$, we resample an oriented edge $\vec{E}$ of $\mathfrak{M}[n]$ uniformly at random, then $\mathrm{Reroot}(\mathfrak{M}[n], \vec{E})$ has the same law as $\mathfrak{M}[n]$. Since the uniform distribution on the set of oriented edges of $\mathfrak{M}[n]$ is the stationary distribution for $\mathrm{SRW}_{\mathfrak{M}[n]}$, it follows that for every $k \geq 0$, if $\vec{E}_k$ is the kth edge encountered by a simple random walk on $\mathfrak{M}[n]$ then we have

$$\mathrm{Reroot}(\mathfrak{M}[n], \vec{E}_k) \overset{(d)}{=} \mathfrak{M}[n].$$

The display of Proposition 7.9 follows by taking local limits as $n \to \infty$ using Theorem 7.1. The reversibility is obtained by a similar argument. □

Remark 7.10 (Unimodularity and the mass-transport principle). In general, random rooted graphs that satisfy the properties in Proposition 7.9 are called **stationary and reversible random graphs**. The notion of stationary and reversible random graph is closely related to the notion of **unimodular random graph** and the related **mass-transport principle**: if $(G, \vec{e})$ is a random rooted graph which is stationary and reversible then we have

$$\mathbb{E}\left[\sum_{\vec{a}\in\vec{\mathrm{E}}(G)} f(G,\vec{e},\vec{a})\right] = \mathbb{E}\left[\sum_{\vec{a}\in\vec{\mathrm{E}}(G)} f(G,\vec{a},\vec{e})\right],$$

where $f(G,\vec{e},\vec{a})$ should be interpreted as a flux from $\vec{e}$ to $\vec{a}$ in the graph G so that the mean flux outgoing from the root edge is equal to the mean flux ingoing into $\vec{e}$, see [6, 24, 86].

7.2.2 Ergodicity

Proposition 7.11 (Ergodicity). *The random map $\mathfrak{M}_\infty$ is ergodic in the two equivalent senses:*

- *the shift θ rerooting along the simple random walk is ergodic,*
- *any event invariant (up to sets of probability zero) with respect to local modifications around the root edge has probability* 0 *or* 1*.*

Proof. We will not prove the equivalence of the two statements above and refer to [89, Proposition 10] and [149, Theorem 5.1] for details (to be accurate [89] deals with bi-infinite paths which are constructed using reversibility and the previous proposition). We will directly prove the second item in the case of $\mathfrak{M}_\infty$. Let A be an event invariant with respect to local modifications. If $(\bar{\mathfrak{e}}_n : n \geq 0)$ is a filled-in exploration of $\mathfrak{M}_\infty$ algorithm $\mathcal{A}_{\text{metric}}$ generating the filtration $(\mathcal{F}_n)$ then by the spatial Markov property we have

$$\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_n] = \phi(|\partial^*\bar{\mathfrak{e}}_n|),$$

for some function ϕ because A does not depend on the explored part since we can modify it without affecting the realization of A. In other words, the function ϕ is a bounded harmonic function for the process $(|\partial^*\bar{\mathfrak{e}}_n| : n \geq 0)$ which we recall has the same law as $(S_n^{\uparrow} : n \geq 0)$, the ν-random walk conditioned to stay positive. In particular, both $h^{\uparrow}$ and $\phi \cdot h^{\uparrow}$ are harmonic functions for the (oscillating) ν-random walk on $\{1, 2, 3, ...\}$, and by [104, Theorem 1] we deduce that ϕ is constant. Since $\cup\bar{\mathfrak{e}}_n = \mathfrak{M}_\infty$ by Proposition 7.6, the martingale convergence theorem implies $\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_n] \to \mathbf{1}_A$ almost surely as $n \to \infty$, the event A must be deterministic. □

In fact, one can prove that if Z and Z' are two $h^{\uparrow}$-transforms of the ν-random walk S started respectively from $1 \leq \ell < \ell'$, then we can couple Z and Z' on the same probability space so that $Z_i = Z'_i$ for all i large enough. This coupling implies a stronger form of ergodicity than the one mentioned above.

Bibliographical notes. The $\mathbf{q}$-Boltzmann maps of the plane can be somehow seen as the analog of the infinite Kesten tree [130] in the theory of random trees. The convergence towards infinite maps of the plane has first been proved in the case of triangulations in the pioneer paper of Angel & Schramm [20] followed by Krikun [134] in the case of quadrangulations. Later Schaeffer-type constructions of the Uniform Infinite Planar Quadrangulation (whose law is that of $\mathbb{P}_\infty$ when $\mathbf{q} = (12^{-1}\mathbf{1}_{k=2})$) were given [74, 154, 96] and this opened the door to the more general convergence of Theorem 7.1 which was proved in this generality in [43] using the Bouttier–Di Francesco–Guitter bijection, see also [182] for the non-bipartite case. The proof presented here (only based on peeling) is new. The convergence $\mathbb{P}_p^{(\ell)} \to \mathbb{P}_\infty^{(\ell)}$ is due to Timothy Budd [51]. The use of strong ratio limit theorem to local convergence is inspired by Abraham & Delmas [2] working in the case of random trees. Stationarity of infinite planar maps along the random walk (Proposition 7.9) is instrumental in many works and was already used in the case of the UIPT by Angel & Schramm. Ergodicity of those lattices in the present generality seems to be new. See [6, 24] for more about links between unimodular random graphs and stationary/reversible random graphs.

Chapter VIII : Hyperbolic random maps

In this chapter, we introduce "hyperbolic" random planar maps which are infinite random maps of the plane or of the half-plane. Contrary to the previous cases, it is still open in general to show that those infinite lattices are local limit of finite random maps (in high genus), but we can nevertheless introduce and study those infinite random graphs for themselves.

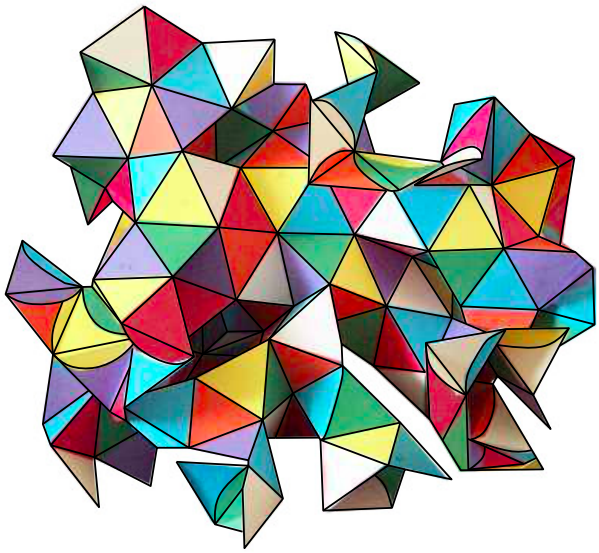


Figure 8.1: A random triangulation in high genus whose local limits are the hyperbolic random triangulations by the result of Budzinski and Louf [64].

In the rest of this chapter we assume that $\mathbf{q}$ *is subcritical and satisfies* ***Cramér condition*** *(see Definition A.17) i.e.*

$$\exists \omega > 1, \quad \sum_{k\in\mathbb{Z}} \omega^k \nu(k) = 1, \qquad \text{(Cramer)}.$$

Under the above hypotheses, by Remark 5.5, the ν-random walk S drifts towards $-\infty$. Recall from Section A.3.5 that the ω in Cramér's hypothesis is unique provided it exists, and that $\tilde{\nu}(k) = \nu(k)\omega^k$ is the associated step distribution. The $\tilde{\nu}$-walk $\tilde{S}$ then drifts towards $+\infty$ and has

a bounded renewal function $\tilde{h}^{\uparrow}$ equal to

$$\tilde{h}^{\uparrow}(p) = \sum_{i=0}^{p-1} \omega^{-i} h^{\downarrow}(i) = \sum_{i=0}^{p-1} (4\omega)^{-i} \binom{2i}{i} \xrightarrow[p\to\infty]{} \sqrt{\frac{\omega}{\omega-1}}.$$

Theorem 8.1 (*Hyperbolic maps of the half-plane and plane*)

If $\mathbf{q}$ *is subcritical and satisfies Cramér's condition, then one can construct laws* $\mathbb{H}^{(\infty)}$ *and* $\mathbb{H}_{\infty}^{(\ell)}$ *for* $\ell \geq 1$ *supported respectively by infinite maps of the half-plane and of the plane (with boundary of perimeter* 2ℓ*) characterized by*

$$\mathbb{H}^{(\infty)}(\bar{\mathfrak{e}} \subset \mathfrak{m}) = w_{\mathbf{q}}(\bar{\mathfrak{e}}) \cdot (c_{\mathbf{q}}\omega)^{|\partial^*\bar{\mathfrak{e}}|-|\partial\bar{\mathfrak{e}}|}, \tag{8.1}$$

$$\mathbb{H}_{\infty}^{(\ell)}(\bar{\mathfrak{e}} \subset \mathfrak{m}) = w_{\mathbf{q}}(\bar{\mathfrak{e}}) \cdot (c_{\mathbf{q}}\omega)^{|\partial^*\bar{\mathfrak{e}}|-|\partial\bar{\mathfrak{e}}|} \frac{\tilde{h}^{\uparrow}(|\partial^*\bar{\mathfrak{e}}|)}{\tilde{h}^{\uparrow}(|\partial\bar{\mathfrak{e}}|)}, \tag{8.2}$$

for any submap $\bar{\mathfrak{e}}$ *with one hole and the proper perimeter.*

We shall write $\mathfrak{H}_{\infty}^{(\ell)}$ and $\mathfrak{H}^{(\infty)}$ for random maps of law $\mathbb{H}_{\infty}^{(\ell)}$ respectively $\mathbb{H}^{(\infty)}$, and call them "hyperbolic $\mathbf{q}$-Boltzmann maps of the plane or of the half-plane". As usual when $\ell = 1$ we drop the notation exponent (1) and write directly $\mathfrak{H}_{\infty}$.

Open Question 8.2. *Are the above random maps weak limits of finite random graphs? More precisely* $\mathbb{H}_{\infty}$ *should be the local limit in distribution of random Boltzmann maps with* n *edges drawn* on the torus of genus $[\alpha n]$ *for an appropriate value of* $\alpha > 0$*. This is proved in the (type I) triangulation case by Budzinski and Louf [64] and for bipartite maps with a tail condition on the weight sequence in [63]. The half-plane analog* $\mathbb{H}^{(\infty)}$ *should be the limit of finite maps in high genus with a boundary, in a suitable regime where the size of the map, the genus of the map and the perimeter of its boundary tend to infinity.*

8.1 Constructions

In this section we prove Theorem 8.1 and build the laws $\mathbb{H}^{(\infty)}$ and $\mathbb{H}_{\infty}^{(\ell)}$. The strategy is again the one used in Chapter 6 or Chapter 7 and is based on : 1) characterizing of the law by a Spatial Markov Property, 2) deriving the law of the filled-in peeling process and 3) constructing the laws via an appropriate growing Markov chain.

Uniqueness by Spatial Markov Property. Exactly as in Propositions 6.2 or 7.3, it is easy to check that (8.1) and (8.2) characterize the laws $\mathbb{H}^{(\infty)}$ and $\mathbb{H}_{\infty}^{(\ell)}$ provided they exist.

Law of the peeling process. We now *suppose* $\mathbb{H}^{(\infty)}$ and $\mathbb{H}_{\infty}^{(\ell)}$ exist and obey (8.1) and (8.2) respectively. Using similar computations as in Chapter 6 we deduce that the filled-in peeling process under $\mathbb{H}^{(\infty)}$ must obey the following transition probabilities respectively for events of

type $\mathrm{C}_k, \mathrm{G}_{\infty,k}$ and $\mathrm{G}_{k,\infty}$

$$b_h^{(\infty)}(k) = q_k(c_{\mathbf{q}}\omega)^{k-1}, \qquad b_h^{(\infty)}(\infty,k) = b_h^{(\infty)}(k,\infty) = W^{(k)}(c_{\mathbf{q}}\omega)^{-k-1}, \tag{8.3}$$

for any $\ell \geq 1, k \geq 0$. We then check "Tutte's equation" i.e. that the above definition yields probability transitions: $\sum_{k\geq 0}\left(b_h^{(\infty)}(k) + b_h^{(\infty)}(\infty,k) + b_h^{(\infty)}(k,\infty)\right) = 1$. Given that $b_h^{(\infty)}(k) = \tilde{\nu}(k-1)$ and $b_h^{(\infty)}(k,\infty) = \frac{1}{2}\tilde{\nu}(-k-1)$, this is equivalent to our Cramér condition

$$\sum_{k\in\mathbb{Z}} \tilde{\nu}(k) = \sum_{k\in\mathbb{Z}} \nu(k)\omega^k = 1. \tag{8.4}$$

As usual, this enables us to deduce the following description of filled-in processes under $\mathbb{H}^{(\infty)}$:

Proposition 8.3 (Law of the filled-in explorations under $\mathbb{H}^{(\infty)}$)**.** *Suppose $\mathfrak{H}^{(\infty)}$ is a random map of law $\mathbb{H}^{(\infty)}$ which satisfies (8.1). Fix a peeling algorithm $\mathcal{A}$ and let $\overline{\mathfrak{e}}_0 \subset \overline{\mathfrak{e}}_1 \subset \cdots \subset \overline{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{H}^{(\infty)}$ be the filled-in peeling exploration of $\mathfrak{H}^{(\infty)}$ with algorithm $\mathcal{A}$, then:*

- (Spatial Markov Property) *For any $n \geq 0$, conditionally on $\overline{\mathfrak{e}}_n$ the map filling-in the hole of $\overline{\mathfrak{e}}_n$ inside $\mathfrak{H}^{(\infty)}$ has law $\mathbb{H}^{(\infty)}$.*
- *The transition probabilities of the Markov chain $(\overline{\mathfrak{e}}_n)_{n\geq 0}$ are as follows: Conditionally on $\overline{\mathfrak{e}}_n$ and provided that $\mathcal{A}(\overline{\mathfrak{e}}_n) \neq \dagger$, the events C_k, $\mathrm{G}_{k,\infty}$ and $\mathrm{G}_{\infty,k}$ for $k \geq 0$ occur with respective probabilities*

  $$b_h^{(\infty)}(k), \qquad b_h^{(\infty)}(k,\infty) \quad \text{and} \quad b_h^{(\infty)}(\infty,k),$$

  and conditionally on these events the possible finite hole created is filled-in with an independent $\mathbf{q}$-Boltzmann map with the correct perimeter.

Remark 8.4. Using the above probability transitions we deduce that

$$(|\partial^*\overline{\mathfrak{e}}_n| - |\partial\overline{\mathfrak{e}}_n|)_{n\geq 0} \text{ under } \mathbb{H}^{(\infty)} \overset{(d)}{=} (\tilde{S}_n)_{n\geq 0} \text{ under } \mathbf{P}^{(0)}. \tag{8.5}$$

We can proceed similarly in the plane case and introduce the peeling transitions for the events of type C_k or $\mathrm{G}_{*,k}$ or $\mathrm{G}_{k,*}$ which we deduce from (8.2):

$$b_{h,\infty}^{(\ell)}(k) = (c_{\mathbf{q}}\omega)^{k-1} q_k \frac{h^{\uparrow}(\ell+k-1)}{h^{\uparrow}(\ell)} \quad \text{and} \quad b_{h,\infty}^{(\ell)}(*,k) = b_{h,\infty}^{(\ell)}(k,*) = (c_{\mathbf{q}}\omega)^{-k-1} W^{(k)} \frac{\tilde{h}^{\uparrow}(\ell-k-1)}{\tilde{h}^{\uparrow}(\ell)}, \tag{8.6}$$

for any $\ell \geq 1$ and $k \geq 0$. As before, one can check that those probability transitions indeed sum-up to 1 and this equivalent to the fact that $\tilde{h}^{\uparrow}$ is harmonic at $\{1,2,3,\ldots\}$ for the $\tilde{\nu}$-random walk. In terms of the peeling process we have:

Proposition 8.5 (Law of the filled-in explorations under $\mathbb{H}_\infty^{(\ell)}$)**.** *Fix $\ell \geq 1$. Suppose $\mathfrak{H}_\infty^{(\ell)}$ is a random map of law $\mathbb{H}_\infty^{(\ell)}$ which satisfies (8.2). Fix a peeling algorithm $\mathcal{A}$ and let $\overline{\mathfrak{e}}_0 \subset \overline{\mathfrak{e}}_1 \subset \cdots \subset \overline{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{H}_\infty^{(\ell)}$ be the filled-in peeling exploration of $\mathfrak{H}_\infty^{(\ell)}$ with algorithm $\mathcal{A}$, then:*

- (Spatial Markov Property) *For any $n \geq 0$, conditionally on $\overline{\mathfrak{e}}_n$ the map filling-in the hole of $\overline{\mathfrak{e}}_n$ inside $\mathfrak{H}_\infty^{(\ell)}$ has law $\mathbb{H}_\infty^{(|\partial^*\overline{\mathfrak{e}}_n|)}$.*

- *The transition probabilities of the Markov chain* $(\overline{\mathfrak{e}}_n)_{n\geq 0}$ *are as follows: Conditionally on* $\overline{\mathfrak{e}}_n$ *and provided that* $\mathcal{A}(\overline{\mathfrak{e}}_n) \neq \dagger$ *the events* C_k, $\mathrm{G}_{k,*}$ *and* $\mathrm{G}_{*,k}$ *for* $k \geq 0$ *occur with respective probabilities*

$$b_{h,\infty}^{(|\partial^*\overline{\mathfrak{e}}_n|)}(k), \qquad b_{\infty}^{(|\partial^*\overline{\mathfrak{e}}_n|)}(k,*) \quad \text{and} \quad b_{h,\infty}^{(|\partial^*\overline{\mathfrak{e}}_n|)}(*,k),$$

and conditionally on these events the possible finite hole created is filled-in with an independent $\mathbf{q}$*-Boltzmann map with the correct perimeter.*

Remark 8.6. Notice that during such explorations, the half-perimeter process $(|\partial^*\overline{\mathfrak{e}}_n| : n \geq 0)$ has the law of the $\tilde{h}^{\uparrow}$-transform of the random walk $\tilde{S}$ started from ℓ, i.e. its version conditioned to stay positive.

Constructing the laws. **Proof of Theorem 8.1.** With the previous propositions at hand, it is easy to adapt the arguments of the previous chapters to construct random infinite maps with the desired laws. More precisely, if $(\overline{\mathbf{E}}_n : n \geq 0)$ is the Markov chain of growing submaps starting from the map with a single hole and boundary face of perimeters 2ℓ (resp. $\ell = \infty$) and whose transitions probabilities are those described in Proposition 8.5 (resp. Proposition 8.3) applied with algorithm $\mathcal{A}_{\text{metric}}$, then one easily check that

$$\bigcup_{n\geq 0} \overline{\mathbf{E}}_n$$

is an infinite planar map of the plane (resp. of the half-plane) which satisfies (8.2) (resp. (8.1)). We leave the details to the reader.

We end this section by asking about the existence of "stable hyperbolic random maps". Indeed, as seen above, one can build hyperbolic random infinite maps as soon as $\mathbf{q}$ is subcritical and satisfies Cramér's condition. But although $\sum \tilde{\nu}(k)k$ is positive, it may be infinite, yielding to a perhaps new universality class of random hyperbolic maps:

Open Question 8.7 (Stable hyperbolic random maps?)**.** *Can one find a subcritical weight sequence* $\mathbf{q}$ *so that* ν *satisfies Cramér's condition and such that the associated law* $\tilde{\nu}$ *has a heavy tail on the right, e.g.* $\sum_{k\geq 0} k\tilde{\nu}(k) = \infty$*? If yes, study the properties of those "heavy-tailed hyperbolic random maps".* Added in proof: *[63] proves that those maps exist.*

8.2 Basic properties

We now prove a few basic properties of the hyperbolic random maps we have just constructed. In particular, we prove isoperimetric estimates which legitimate the adjective "hyperbolic" coined above. Recall that $\mathbf{q}$ is subcritical and satisfies Cramér's condition.

8.2.1 Stationarity, reversibility and ergodicity

Proposition 6.6 and Proposition 7.9 hold in the cases of $\mathfrak{H}^{(\infty)}$ and $\mathfrak{H}_\infty$ by considering re-rooting along the boundary or along a simple random walk. The proof Proposition 6.6 can, mutatis mutandis, be copied in the hyperbolic case:

Proposition 8.8. *The shift operation which translates the root edge (one unit to the right) along the boundary preserves the law $\mathbb{H}^{(\infty)}$ and is ergodic.*

Also, since $\sum_{k\geq 0} k\tilde{\nu}(-k) < \infty$, an adaptation of the proof of Proposition 6.7 shows that, almost surely, there are only finitely many cut-points separating the origin from infinity in $\mathfrak{H}^{(\infty)}$.

Proposition 8.9. *The random map of the plane $\mathfrak{H}_\infty$ is stationary and reversible along the simple random walk. Furthermore, re-rooting along the simple random walk is an ergodic shift.*

Proof. One cannot directly mimic the proof of Proposition 7.9 because we do not know (see Open Question 8.2) that $\mathfrak{H}_\infty$ is a local limit of uniformly rooted finite maps. However, we can prove stationarity directly in the limit. The reversibility is easy since changing the orientation of the root edge does not affect the characterization (8.2). For the stationarity, let $i, r > 0$ and let $\bar{\mathfrak{e}}$ be a submap with one hole and a path $w = (\vec{e}_0, \vec{e}_1, ..., \vec{e}_i)$ such that w could be the result of the first i steps of a random walk strictly inside $\bar{\mathfrak{e}}$ (with the convention that $\vec{e}_0$ is the root edge of $\bar{\mathfrak{e}}$). We denote by $x_0, x_1, ..., x_{i+1}$ the vertices visited by the path. Furthermore, we assume that $\bar{\mathfrak{e}}$ is the hull of the ball of radius r around w in the sense that $\bar{\mathfrak{e}}^\circ$ it is made of all the faces containing a vertex at graph distance (inside $\bar{\mathfrak{e}}$) smaller than or equal to $r-1$ from the set $\{x_0, x_1, ..., x_{i+1}\}$ as well as the finite regions they enclosed. We now ask what is the probability that, inside $\mathfrak{H}_\infty$, the first i steps of the walk correspond to w and that the hull of radius r around these is $\bar{\mathfrak{e}}$:

$$\begin{aligned}\mathbb{P}(\overrightarrow{E}_k = \vec{e}_k, \forall k \leq i \text{ and } \bar{\mathfrak{e}} \subset \mathfrak{H}_\infty) &= \int \mathbb{H}_\infty(\mathrm{d}\mathfrak{m}) \mathbf{1}_{\bar{\mathfrak{e}}\subset\mathfrak{H}_\infty} \int \mathrm{SRW}_{\mathfrak{m}}(\mathrm{d}(\vec{\mathfrak{e}}_n) : n \geq 0) \mathbf{1}_{\vec{\mathfrak{e}}_k = \vec{e}_k, \forall k\leq i} \\ &\underset{(6.3)}{=} w_{\mathbf{q}}(\bar{\mathfrak{e}})(c_{\mathbf{q}}\omega)^{|\partial^*\bar{\mathfrak{e}}|-1} \frac{\tilde{h}^{\uparrow}(|\partial^*\bar{\mathfrak{e}}|)}{\tilde{h}^{\uparrow}(1)} \cdot \prod_{k=1}^{i} \deg(x_k)^{-1}.\end{aligned}$$

We now remark that the last probability is the same if we replace $(\bar{\mathfrak{e}}, w)$ with the same submap $\bar{\mathfrak{e}}$ and the reversed path $\overleftarrow{w} = (\overleftarrow{e_i}, ..., \overleftarrow{e_0})$ –this does not affect the hull of the ball of radius r around the path–. Since r is arbitrary this proves that $(\mathfrak{H}_\infty; \overrightarrow{E}_0, ..., \overrightarrow{E}_i)$ and $(\mathfrak{H}_\infty; \overleftarrow{E}_i, ..., \overleftarrow{E}_0)$ indeed have the same law. The ergodicity of the re-rooting along the simple random walk is proved exactly as in Proposition 7.9 and follows from the uniqueness (up to multiplicative constant) of harmonic functions on $\mathbb{Z}_{>0}$ for the $\tilde{\nu}$-random walk. □

8.2.2 Anchored expansion

In this section we shall restrict to subcritical weight sequences $\mathbf{q}$ which are finitely supported.

Under the above assumption, Cramér's condition and the existence of ω is automatic and $m = \sum k\tilde{\nu}(k) \in \mathbb{R}_+^*$. Furthermore $\tilde{\nu}$ has exponential moments. We will establish a first "hyperbolic" property for the random maps $\mathfrak{H}_\infty$ and $\mathfrak{H}^{(\infty)}$, namely that the infimum ratio perimeter/volume over all reasonable subsets is positive, which is the fingerprint of a hyperbolic behavior. Since our graphs are genuinely random the **Cheeger constant** which is the infimum of perimeter/volume over *all* subsets is necessarily zero (every possible pattern appears somewhere in the map). We shall thus work with an anchored version of this notion.

Definition 8.10. *Let $\mathfrak{g}$ be an infinite connected, locally finite, multigraph and $\rho \in V(\mathfrak{g})$. The **anchored (edge) expansion constant** of $\mathfrak{g}$ is*

$$\text{Cheeger}^\bullet(\mathfrak{g}) = \inf\left\{\frac{\|\partial A\|}{\|A\|} : A \subset \mathfrak{g},\ \textit{connected}, \rho \in A\right\},$$

*where $\|\partial A\|$ denotes the number of edges having one extremity in A and one in A^c, and $\|A\|$ is the sum of all the vertex degrees in A. If $\text{Cheeger}^\bullet(\mathfrak{g}) > 0$ we say that $\mathfrak{g}$ is anchored **non-amenable** (otherwise it is anchored amenable).*

Theorem 8.11 (*Anchored non-amenability*)

Suppose $\mathfrak{q}$ has finite support and is subcritical. Then $\mathfrak{H}_\infty$ and $\mathfrak{H}^{(\infty)}$ as well as their duals $\mathfrak{H}_\infty^\dagger$ and $\mathfrak{H}^{(\infty),\dagger}$ (in the last case, we remove the vertex dual to the infinite face) almost surely are anchored non-amenable.

Proof. We first work on the dual map $\mathfrak{H}_\infty^\dagger$. Notice that in the definition of $\text{Cheeger}^\bullet$ we can restrict ourself to connected subsets $\overline{\mathfrak{e}}^\circ \subset \mathfrak{H}_\infty^\dagger$ having only one hole (because filling-in the finite holes decreases the boundary size and increases the volume). If $\overline{\mathfrak{e}} \subset \mathfrak{H}_\infty$ denotes the associated submap, recalling that $|\partial^*\overline{\mathfrak{e}}|$ is the half-perimeter of its hole and $|\overline{\mathfrak{e}}|$ is its number of edges, then with the above notation we have

$$\|\partial\overline{\mathfrak{e}}^\circ\| = 2|\partial^*\overline{\mathfrak{e}}| \quad \text{and } \|\overline{\mathfrak{e}}^\circ\| = 2|\overline{\mathfrak{e}}| - 2|\partial^*\overline{\mathfrak{e}}|,$$

where we recall that $|\overline{\mathfrak{e}}|$ is the number of edges of $\overline{\mathfrak{e}}$. Fix $m \geq 1$ large and let us evaluate, by a first moment method, the probability that there exists $\overline{\mathfrak{e}} \subset \mathfrak{H}_\infty$ such that $|\overline{\mathfrak{e}}| > m \cdot |\partial^*\overline{\mathfrak{e}}|$:

$$\begin{aligned}
\mathbb{P}\Big(\exists \overline{\mathfrak{e}} \subset \mathfrak{H}_\infty : |\overline{\mathfrak{e}}| > m|\partial^*\overline{\mathfrak{e}}|\Big) &\leq \mathbb{E}\Big[\#\left\{\overline{\mathfrak{e}} \subset \mathfrak{H}_\infty : |\overline{\mathfrak{e}}| > m|\partial^*\overline{\mathfrak{e}}|\right\}\Big] \\
&= \sum_{p\geq 1}\sum_{n>mp}\sum_{\overline{\mathfrak{e}}\in\mathcal{M}_p^{(1)}[n]} \mathbb{P}(\overline{\mathfrak{e}} \subset \mathfrak{H}_\infty) \\
&\underset{(8.2)}{\leq} C\sum_{p\geq 1}\sum_{n>mp}\sum_{\overline{\mathfrak{e}}\in\mathcal{M}_p^{(1)}[n]} w_{\mathfrak{q}}(\overline{\mathfrak{e}}) \cdot (c_{\mathfrak{q}}\omega)^p,
\end{aligned}$$

where we just used the fact that the function $\tilde{h}^\uparrow$ is bounded above and below in order to bound above the ratio in (8.2) by a constant $C > 0$. Now notice that $\sum_{\overline{\mathfrak{e}}\in\mathcal{M}_p^{(1)}[n]} w_{\mathfrak{q}}(\overline{\mathfrak{e}}) = W_p^{(1)}[n]$ so that we can use Theorem 3.12 to deduce

$$W_p^{(1)}[n] \leq \text{Cst} \cdot c_{\mathfrak{q}}^p \cdot \mathbb{P}\big(J_1 + \ldots + J_n \geq -n\varepsilon\big),$$

where J_i are i.i.d. random variables with law μ and where we can take $\varepsilon = 2/m$ for n large enough. From our assumptions on $\mathbf{q}$, the random variables J_i have bounded support and *mean strictly negative*. Hence, by a standard large deviation argument, when ε is small enough (i.e. by choosing m large enough) we have $\mathbb{P}\big(J_1 + \cdots + J_n \geq -n\varepsilon\big) \leq \alpha^n$ for some $\alpha \in (0,1)$. Regluing the pieces together, we deduce that

$$\begin{aligned}\mathbb{P}\Big(\exists \overline{\mathfrak{e}} \subset \mathfrak{H}_\infty : |\overline{\mathfrak{e}}| > (m+1)|\partial^* \overline{\mathfrak{e}}|\Big) &\leq C' \sum_{p\geq 1} (c_{\mathbf{q}}^2 \omega)^p \sum_{n > mp} \alpha^n \\ &\leq C'' \sum_{p\geq 1} (c_{\mathbf{q}}^2\omega)^p \alpha^{mp} = C'' \sum_{p\geq 1} (\alpha^m c_{\mathbf{q}}^2 \omega)^p.\end{aligned}$$

Since $\alpha \in (0,1)$, by further increasing m if necessary, we can make the last series as small as we wish. This indeed implies that $\mathbb{P}(\mathsf{Cheeger}^\bullet(\mathfrak{H}_\infty^\dagger) = 0) = 0$. The case of $\mathfrak{H}_\infty^{(\ell),\dagger}$ or $\mathfrak{H}^{(\infty),\dagger}$ is similar and is left as an exercise. To deduce the case of the primal lattices $\mathfrak{H}_\infty$ and $\mathfrak{H}^{(\infty)}$ we rely on the next exercise. □

Exercise 8.12. If $\mathfrak{m}$ is an infinite map of the plane or of the half-plane with only bounded face degrees (except the root face in the half-plane case), then $\mathsf{Cheeger}^\bullet(\mathfrak{m}^\dagger) > 0$ implies $\mathsf{Cheeger}^\bullet(\mathfrak{m}) > 0$ (where we always remove the infinite dual vertex in the dual of a map of the half-plane).

Bibliographical notes. Hyperbolic random maps have first been constructed in the half-planar case (with a simple boundary) by Angel & Ray [18] as a family of random triangulations of the half-plane satisfying a certain Markov property. Their rough properties were studied in [170] and in particular Theorem 8.11 is inspired from it. Those results have been extended to the plane topology in [89] (still in the case of triangulations) and Budzinski [60, 59] performed a detailed analysis of those random triangulations. The laws $\mathbb{H}_\infty^{(\ell)}$ first appeared in [58] in this generality. Proposition 8.9 is adapted from [89, Proposition 9]. The recent breakthrough by Budzinski & Louf [64] shows that, in the triangulation case, those infinite maps are local limit of uniform triangulations in ultra-high genus, see Open Question 8.2 for a more general conjecture. *Added in proof:* Open Questions 8.2 and 8.7 are addressed in [63].

Chapter IX : Simple boundary, yet a bit more complicated

In this chapter, we focus on maps with a simple boundary which we call ∂-simple maps to lighten the prose[1]. We have seen in Section 4.3 that that we can develop an associated notion of simple exploration. Although more complicated combinatorially, those simple explorations will be useful later when studying site percolation on random planar maps (Section 11.4). As in Chapter 6, we will define the random infinite planar ∂-simple maps which are obtained as local limit of finite planar ∂-simple maps. For this, we shall need to restrict to critical weight sequences. Recall that we add a hat " $\widehat{\ }$ " to our usual notation to denote their analogs for ∂-simple maps.

In the rest of this chapter we shall assume that $\mathbf{q}$ *is critical.*

Here is the main result of this chapter:

Theorem 9.1 (*Local limit of ∂-simple maps*)

Let $\mathbf{q}$ *be a* critical *weight sequence. We have the following local convergence:*

$$\hat{\mathbb{P}}^{(\ell)} \xrightarrow[\ell\to\infty]{(d)} \hat{\mathbb{P}}^{(\infty)},$$

where $\hat{\mathbb{P}}^{(\infty)}$ *is supported by maps of the half-plane with a simple boundary.*

The reader will not be surprised to learn that we write $\hat{\mathfrak{M}}^{(\infty)}$ for a random variable of law $\hat{\mathbb{P}}^{(\infty)}$ that we call the $\mathbf{q}$-Boltzmann map of the half-plane with a simple boundary. To prove the above theorem, we shall go through enumeration of ∂-simple maps which is notably more difficult than for general boundaries. We leave the case when $\mathbf{q}$ is subcritical open for further research:

Open Question 9.2. *What happens to the results of this chapter if* $\mathbf{q}$ *is subcritical (one can first suppose that it is finitely supported so that Cramér's condition holds)? Prove an analog of the strong ratio limit for* $\hat{W}^{(\ell)}$ *in this case (Proposition 9.4). What about Theorem 9.1?*

9.1 Enumeration of maps with a simple boundary

In this section, we gather the basics about *∂-simple* maps and in particular prove our key enumeration formula Proposition 9.4 which is the analog of Lemma 3.13 in the general boundary

[1] ...and not to confuse with *simple maps* which are maps where multiple edges or loops are forbidden.

case. We again focus on bipartite maps to keep a certain unity in the lecture notes, but the methodology easily extends to the general case.

9.1.1 The core decomposition

Recall the basic definitions and notation from Section 4.3, in particular that $\hat{W}^{(\ell)}$ is the total $w_{\mathbf{q}}$-weight of all finite ∂-simple maps of perimeter 2ℓ. If $\mathfrak{m}$ is a map with a general boundary, one can consider the ∂-simple map obtained by keeping the simple boundary component carrying the root edge, see Figure 9.1. The ∂-simple component carrying the root edge will be called the *Core* and denoted by $\mathsf{Core}(\mathfrak{m})$. This decomposition yields a way to relate $\hat{W}^{(\ell)}$ to their non-simple analogs:

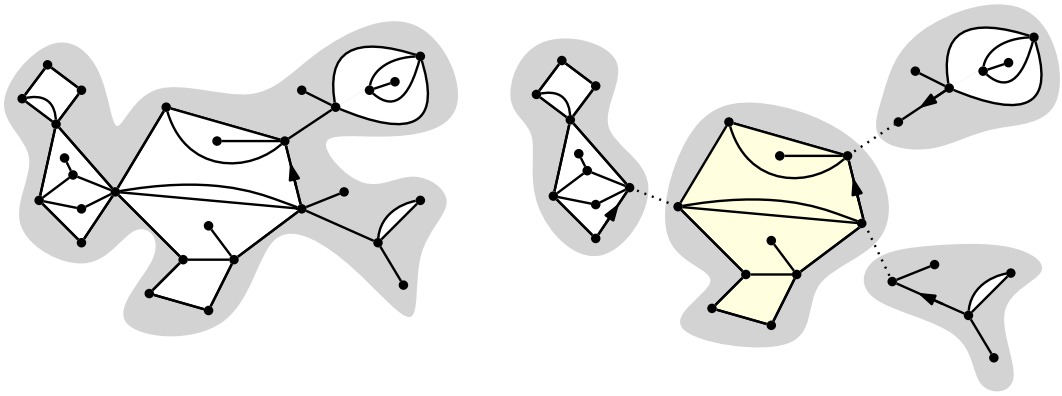


Figure 9.1: The decomposition of a map with a general boundary. On the left, a map $\mathfrak{m}$ with a general boundary. On the right, we see the ∂-simple component of the root edge (in yellow) together with the maps with general boundaries attached to it.

Proposition 9.3. *Let* $\mathbf{q}$ *be a weight sequence. If* $\mathrm{W}(z) = \sum_{\ell\geq 0} W^{(\ell)} z^\ell$ *and* $\hat{\mathrm{W}}(z) = \sum_{\ell\geq 0} \hat{W}^{(\ell)} z^\ell$ *then we have the following equality between (formal) power series:*

$$\hat{\mathrm{W}}\left(z(\mathrm{W}(z))^2\right) = \mathrm{W}(z).$$

Recall from Lemma 3.13 that the radius of convergence of W is $\frac{1}{c_{\mathbf{q}}}$, and $1 < \mathrm{W}(1/c_{\mathbf{q}}) = \frac{c_{\mathbf{q}}}{2}\sum_{k\geq 0} \nu(-k-1) \leq \frac{2}{c_{\mathbf{q}}}$. It follows from the previous proposition that the radius of convergence of $\hat{\mathrm{W}}$ is *at least* $\frac{1}{\hat{c}_{\mathbf{q}}}$ where

$$\hat{c}_{\mathbf{q}} = \frac{c_{\mathbf{q}}}{\mathrm{W}^2(\frac{1}{c_{\mathbf{q}}})}, \tag{9.1}$$

and that $\hat{\mathrm{W}}(1/\hat{c}_{\mathbf{q}}) = \mathrm{W}(1/c_{\mathbf{q}})$. In the following we write $\mathrm{W}_c = \mathrm{W}(1/c_{\mathbf{q}})$ and similarly $\hat{\mathrm{W}}_c = \hat{\mathrm{W}}(1/c_{\mathbf{q}})$ to simplify notation. We shall prove in this section the following key asymptotic enumeration, which will eventually follow from the strong ratio limit theorem (Theorem A.19):

Proposition 9.4 (Strong ratio limit for $\hat{W}^{(\ell)}$). *Let* $\mathbf{q}$ *be an admissible* critical *weight sequence. Then we have*

$$\lim_{\ell\to\infty} \frac{\hat{W}^{(\ell+1)}}{\hat{W}^{(\ell)}} = \hat{c}_{\mathbf{q}}.$$

Remark 9.5. For subcritical weight sequences we may have $\lim_{\ell\to\infty} \frac{\hat{W}^{(\ell+1)}}{\hat{W}^{(\ell)}} > \hat{c}_{\mathbf{q}}$ and this is intimately related to the fact that the perimeter of the core of $\mathfrak{M}^{(\ell)}$ may have an exponential tail as $\ell \to \infty$. This is the reason why we restricted to critical weight sequences in this chapter.

Notice that since $\mathbf{q}$ is critical, it is easy to define the infinite $\mathbf{q}$-Boltzmann map of the plane with simple boundary of perimeter 2ℓ (whose law we denote by $\hat{\mathbb{P}}_\infty^{(\ell)}$) by conditioning $\mathfrak{M}_\infty^{(\ell)}$ to have a simple boundary, which is an event of positive probability. The difference between maps with a simple boundary and maps with general boundary will thus manifest itself more dramatically when dealing with half-planar limits. The rest of this section is devoted to proving Proposition 9.4, and for this we need to develop a peeling algorithm to explore the core of a planar map.

9.1.2 Free Boltzmann map and exploration of the core

In the following, it will be convenient to work with Boltzmann maps with a general boundary whose perimeter is not fixed. We introduce the probability measure

$$\mathbb{P}^{(\mathrm{free})} = \frac{1}{W_c} \sum_{\ell\geq 0} W^{(\ell)} c_{\mathbf{q}}^{-\ell} \cdot \mathbb{P}^{(\ell)}.$$

The random planar map $\mathfrak{M}^{(\mathrm{free})}$ of law $\mathbb{P}^{(\mathrm{free})}$ thus has a random boundary length, but conditionally on this length, is a plain Boltzmann map. By the decomposition of Figure 9.1 we can write

$$\hat{W}^{(\ell)} c_{\mathbf{q}}^{-\ell} (W_c)^{2\ell-1} = \mathbb{P}^{(\mathrm{free})}(|\partial \mathrm{Core}(\mathfrak{m})| = \ell). \tag{9.2}$$

To evaluate the probability in the right-hand side, we design a filled-in peeling algorithm which explores the boundary of the core of a map and computes its perimeter. It is related to the algorithm $\mathcal{A}_{\mathrm{corners}}$ used in the proof of Proposition 6.7 except that we explore a finite map and our target is not at infinity:

ALGORITHM $\mathcal{A}_{\text{core}}$

Start with $\overline{\mathfrak{e}}_0$ to be the root face pointed at $\tilde{\rho}$, the target of the root edge. The first edge to peel is the root edge. Inductively if $\overline{\mathfrak{e}}_n \subset \mathfrak{m}$, the exposed boundary of $\overline{\mathfrak{e}}_n$ is the part of the hole that does not belong to the initial hole of $\overline{\mathfrak{e}}_0$. Supposing that the exposed boundary forms a connected segment, we peel at the left-most edge of it and we fill-in all the holes that do not contain $\tilde{\rho}$. See:

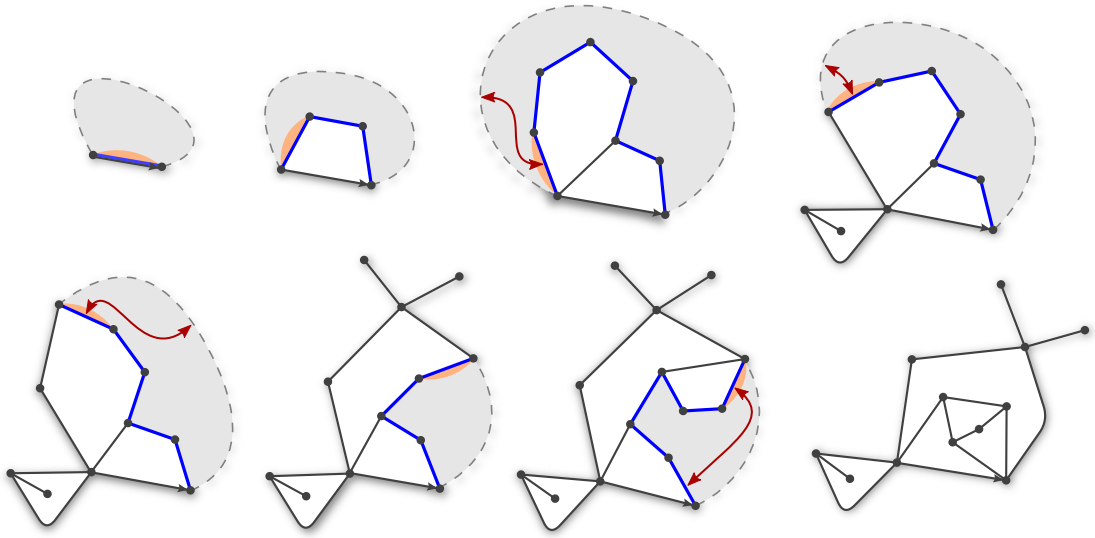


Figure: In the figure, the thick blue line corresponds to the exposed boundary while the dashed line illustrates the initial boundary of $\overline{\mathfrak{e}}_0$. The white region is the explored part. The next edge to peel is indicated in orange. The exploration stops when the length of the exposed boundary drops to 0. This can happen either when the exposed boundary is of length 1 and we identified the peeled edge with another edge of the hole, or if the peeled edge is identified with the right-most edge of the exposed boundary (in blue in the above figure).

Let us clarify a dubious point in the stopping of the exploration: if we identify the peeled edge with the first edge on the right of the exposed boundary, one may think that the exploration is finished since we know the perimeter of core of the map. However, this is because we worked with a map with fixed perimeter: when working with maps with free perimeter on this event "we still do not know what is the perimeter of the core" and so it will be more convient to continue the exploration until the length of the exposed boundary indeed drops to 0. Notice also that if $|\partial\mathfrak{m}| = 0$, i.e. if the underlying map is the vertex map, then there is no exploration process at all.

Let us write $(E_n)_{n\geq 0}$ for the length of the exposed boundary during the exploration with algorithm $\mathcal{A}_{\text{core}}$ so that $E_0 = 1$ and the exploration stops at the first time τ when $E_\tau = 0$. If we denote by D the number of -1 steps that the process (E) performed until time τ, then it is easy to see that

$$\text{the perimeter } 2|\partial\text{Core}(\mathfrak{m})| \text{ is equal to } D+1. \tag{9.3}$$

In the next proposition we will give the law of the process $(E_n : n \geq 0)$ under $\mathbb{P}^{(\text{free})}$. But before this, let us introduce some notation. Recall the measure ν from Definition 5.1 and let us introduce the following measure ζ on $\mathbb{Z}$ which is obtained from ν, roughly speaking, by transforming half of the negative jumps into jumps of -1. More precisely we put

$$\begin{cases} \zeta(2\ell) = \nu(\ell) & \text{for } \ell \geq 0, \\ \zeta(-1) = \frac{1}{2}\nu(\mathbb{Z}_{<0}), & \\ \zeta(-2\ell) = \frac{1}{2}\nu(-\ell) & \text{for } \ell \geq 0. \end{cases}$$

Also, let us introduce the function $H^{\downarrow}$ by setting

$$H^{\downarrow}(\ell) := \mathbb{P}^{(\text{free})}(2|\partial\mathfrak{m}| \geq \ell) = \frac{1}{\mathrm{W}_c} \sum_{2j \geq \ell} W^{(j)} c_{\mathbf{q}}^{-j} = \frac{\nu\left((-\infty, -1 - \ell/2]\right)}{\nu(\mathbb{Z}_{<0})} \tag{9.4}$$

for $\ell \geq 0$ and $H^{\downarrow}(\ell) = 0$ for $\ell < 0$, where we recall that $|\partial\mathfrak{m}|$ is the half-perimeter of the boundary of $\mathfrak{m}$.

Proposition 9.6. *If $(E_n : n \geq 0)$ is the length of the exposed part during the exploration of $\mathfrak{m}$ with algorithm $\mathcal{A}_{\text{core}}$ under $\mathbb{P}^{(\text{free})}$ then $(E_n : n \geq 0)$ is a Markov chain obtained by an h-transform of the random walk with i.i.d. increments of law ζ with respect to the harmonic function $H^{\downarrow}$, started from 1 and killed on $\mathbb{Z}_{\leq 0}$.*

Proof. It is easy to adapt the proofs of Section 4.2.1 to see that conditionally on the past exploration up to time n, if E_n is the length of the exposed boundary then the remaining map has the law of $\mathbb{P}^{(\text{free})}$ conditioned on having perimeter at least E_n. In particular, the conditional probability to perform an event C_k at the next peeling step is equal to

$$q_k c_{\mathbf{q}}^{k-1} \frac{\sum_{2\ell \geq E_n} W^{(\ell+k-1)} c_{\mathbf{q}}^{-\ell-k+1}}{\sum_{2\ell \geq E_n} W^{(\ell)} c_{\mathbf{q}}^{-\ell}} = \zeta(2k-2) \frac{H^{\downarrow}(E_n + 2(k-1))}{H^{\downarrow}(E_n)}.$$

We recognize the form of the $H^{\downarrow}$ transform for the process (E) in this case. Similarly, the conditional probability to perform a peeling step which identifies the peeled edge with another edge not located on the exposed boundary is equal to:

$$c_{\mathbf{q}}^{-1} \frac{\sum_{k \geq 0} W^{(k)} c_{\mathbf{q}}^{-k} \sum_{2j \geq E_n - 1} W^{(j)} c_{\mathbf{q}}^{-j}}{\sum_{2\ell \geq E_n} W^{(\ell)} c_{\mathbf{q}}^{-\ell}} = \sum_{k=0}^{\infty} W^{(k)} c_{\mathbf{q}}^{-k-1} \frac{H^{\downarrow}(E_n - 1)}{H^{\downarrow}(E_n)} = \zeta(-1) \frac{H^{\downarrow}(E_n - 1)}{H^{\downarrow}(E_n)}.$$

Again, we see the $H^{\downarrow}$-transform of the ζ-walk appearing. The last case when the peeled edge is identified with another edge of the exposed boundary is similar and the conditional probability that $E_{n+1} = E_n - 2k$ with $1 \leq k \leq \lfloor E_n/2 \rfloor$ is, as expected, equal to

$$c_{\mathbf{q}}^{-1} \frac{W^{(k-1)} c_{\mathbf{q}}^{-k+1} \sum_{2j \geq E_n - 2k} W^{(j)} c_{\mathbf{q}}^{-j}}{\sum_{2\ell \geq E_n} W^{(\ell)} c_{\mathbf{q}}^{-\ell}} = \zeta(-2k) \frac{H^{\downarrow}(E_n - 2k)}{H^{\downarrow}(E_n)}.$$

□

In particular the function $H^{\downarrow}$ is bounded, equals 1 at 0, vanishes on $\mathbb{Z}_{<0}$ and is harmonic for the ζ-walk on $\mathbb{Z}_{\geq 0}$. By adapting Proposition 5.3, it follows that $H^{\downarrow}$ is the pre-renewal function of the ζ-walk, and we shall write

$$H^{\uparrow}(\ell) = H^{\downarrow}(0) + \cdots + H^{\downarrow}(\ell - 1) \tag{9.5}$$

for the associated renewal function (recall again that we start our walks from 1 instead of 0 as in Chapter A). Recall that S is a random walk started from 1 with i.i.d. increments of law ν, and let us introduce $(R_n : n \geq 0)$ a random walk started from 1 with i.i.d. increments of law ζ. Clearly we have $\nu \leq \zeta$ in terms of stochastic order, and since the random walk S oscillates when $\mathbf{q}$ is critical (Theorem 5.4), it follows that R either oscillates (e.g. if $\mathbf{q}$ is of type $a \in (3/2; 2]$) or drifts towards $+\infty$ (e.g. if $\mathbf{q}$ is of type $a \in (2, 5/2]$ or more generally if $\mathbf{g}_{\mathbf{q}}$ is finite, see Definition 11.1). If $\mathbf{q}$ is subcritical then S drifts towards $-\infty$ and it may (and typically it will) be the case that R also drifts towards $-\infty$ but we excluded this case here.

Proof of Proposition 9.4. As above we write $(E_n : 0 \leq n \leq \tau)$ for the length of the exposed boundary when performing the peeling using algorithm $\mathcal{A}_{\text{core}}$ under the measure $\mathbb{P}^{(\text{free})}$ and D is its number of -1 steps until the first time $\tau \geq 1$ where $E_\tau = 0$. By (9.2) and (9.3), the proposition is proved if we can show that

$$\frac{\mathbb{P}^{(\text{free})}(D = n + 2)}{\mathbb{P}^{(\text{free})}(D = n)} \to 1, \qquad \text{as } n \to \infty \quad \text{(along odd values)}.$$

This will eventually follow from a variation on the strong ratio limit theorem (Theorem A.19) and in particular from the arguments of the elegant proof of Neveu [168]. To connect the above goal to a random walk problem, we use the fact that (E_n) is the $H^{\downarrow}$-transform of the walk (R_n) and we apply Feller's cyclic lemma (Lemma A.4). Using the short hand notation $\mathcal{D}_m = \sum_{i=0}^{m-1} \mathbf{1}_{\Delta R_i = -1}$, we deduce that for $n \geq 1$ and odd we have

$$\begin{aligned}
& \mathbb{P}^{(\text{free})}(D = n, \tau = m) \\
\underset{\text{h-transform}}{=} \; & \mathbb{P}^{(\text{free})}(\mathfrak{m} \neq \dagger) \cdot \frac{H^{\downarrow}(0)}{H^{\downarrow}(1)} \cdot \mathbb{P}(R_m = 0, \mathcal{D}_m = n \;\&\; R_i \geq 1, \forall 0 \leq i \leq m-1) \\
= \; & \mathbb{P}(R_m = 0, \mathcal{D}_m = n \;\&\; R_i \geq 1, \forall 0 \leq i \leq m-1) \\
\underset{\text{Cyclic lemma}}{=} \; & \frac{1}{m}\mathbb{P}\left(R_m = 0, \mathcal{D}_m = n\right).
\end{aligned} \tag{9.6}$$

Recall that $\dagger$ denotes the vertex map. The rough idea is then to argue that $\mathbb{P}^{(\text{free})}(D = n) \approx \mathbb{P}^{(\text{free})}(D = n, \tau = N)$ with $N = [n\zeta(-1)]$ and then use the strong ratio limit theorem on the multidimensional random walk $(R_n, \mathcal{D}_n)$ started at $(1, 0)$ whose i.i.d. increments are $(\Delta R_n, \mathbf{1}_{\Delta R_n = -1})$. More precisely, we will prove in Lemma 9.7 below that for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $|n/m - \zeta(-1)| \leq \delta$ and if $n \geq 1/\delta$ then we have both

$$\mathbb{P}(\mathcal{D}_m = n, R_m = 0) \quad \geq \quad \mathrm{e}^{-\varepsilon n}, \tag{9.7}$$

$$\left|\frac{\mathbb{P}(\mathcal{D}_m = n + 2, R_m = 0)}{\mathbb{P}(\mathcal{D}_m = n, R_m = 0)} - 1\right| \quad \leq \quad \varepsilon, \tag{9.8}$$

where implicitly in the above display and in the rest of the proof, we restrict to values of n, m for which the probabilities considered are positive. Indeed, the random walk $(R, \mathcal{D})$ is not aperiodic since $R_n + \mathcal{D}_n$ is always odd, hence the $n+2$. To come back to $\mathbb{P}(D = n)$, notice that since $\mathcal{D}_m$ is distributed as $\mathsf{Binomial}(m, \zeta(-1))$, easy large deviation estimates show that for the δ fixed above, for all m large enough we have,

$$\mathbb{P}\left(\left|\frac{\mathcal{D}_m}{m} - \zeta(-1)\right| \geq \delta\right) \leq \mathrm{e}^{-cm},$$

for some $c > 0$. Summing over all $m \geq n$ such that $|n/m - \zeta(-1)| \geq \delta$, we deduce using (9.6) that

$$\left|\mathbb{P}^{(\mathrm{free})}(D = n) - \underbrace{\sum_{\substack{m \geq n \\ |\frac{n}{m} - \zeta(-1)| \leq \delta}} \frac{1}{m}\mathbb{P}\left(\mathcal{D}_m = n \text{ and } R_m = 0\right)}_{=:\ \Sigma(n)}\right| \leq \sum_{m \geq n} \mathrm{e}^{-cm} \leq c'\mathrm{e}^{-cn}, \tag{9.9}$$

for some $c' > 0$. In words, the probability $\mathbb{P}^{(\mathrm{free})}(D = n)$ is approximated up to an exponentially small probability by the sum $\Sigma(n)$ in the last display. Thanks to (9.8), $|\Sigma(n+2)/\Sigma(n) - 1| \leq \varepsilon$ since the ratio of each summand is close to 1 (we are neglecting boundary effects here). On the other hand, using (9.6) and (9.7) when $\varepsilon \to 0$, we deduce that $(\mathbb{P}^{(\mathrm{free})}(D = n))^{1/n} \to 1$ as $n \to \infty$. Combining this with (9.9) we deduce that $\mathbb{P}^{(\mathrm{free})}(D = n) \sim \Sigma(n)$ as $n \to \infty$. Consequently, the ratio $\mathbb{P}^{(\mathrm{free})}(D = n+2)/\mathbb{P}^{(\mathrm{free})}(D = n)$ also belongs to $(1 - 2\varepsilon, 1 + 2\varepsilon)$ asymptotically. □

It remains to prove the lemma we used in the course of the proof.

Lemma 9.7. *For every $\varepsilon > 0$, there exists $\delta > 0$, such that if $|n/m - \zeta(-1)| \leq \delta$ and if $n \geq 1/\delta$ then we have both*

$$\mathbb{P}(\mathcal{D}_m = n, R_m = 0) \quad \geq \quad \mathrm{e}^{-\varepsilon n}, \tag{9.10}$$

$$\left|\frac{\mathbb{P}(\mathcal{D}_m = n+2, R_m = 0)}{\mathbb{P}(\mathcal{D}_m = n, R_m = 0)} - 1\right| \quad \leq \quad \varepsilon, \tag{9.11}$$

where we restrict to values of n, m for which the probabilities are positive.

Proof: Fix $\varepsilon > 0$. The first point consists in showing that $\mathbb{P}(\mathcal{D}_m = n, R_m = 0)$ does not decay exponentially fast. For this, it suffices to exhibit a scenario where $\mathcal{D}_m = n$ and $R_m = 0$ which happens with a not too small probability. Notice first that since $\mathcal{D}_m$ has a $\mathsf{Binomial}(m, \zeta(-1))$ distribution, we can find $\delta > 0$ small enough so that if m, n satisfy the requirements of the lemma then

$$\mathbb{P}(\mathcal{D}_{m-1} = n) \geq \mathrm{e}^{-\varepsilon n}.$$

By restricting furthermore to paths having a bound on their R-increments, we can find $A > 0$ large enough so that up to decreasing $\delta > 0$ we have

$$\mathbb{P}(D_{m-1} = n \text{ and } |R_{i+1} - R_i| \leq A, \forall 0 \leq i \leq m-2) \geq \mathrm{e}^{-\varepsilon n},$$

as soon as $|n/m - \zeta(-1)| \le \delta$ and $n \ge 1/\delta$. Now, from [85, Eq (5.13)] the measure ν, hence ζ, always has a polynomial tail on the left, namely $\zeta(-k) \ge ck^{-5/2}$ asymptotically as $k \to \infty$ for some $c > 0$. Hence, in the above scenario, since $R_{m-1} \le Am$, a single large negative jump of R at time m could yield to the value $(n, 0)$ with the additional cost of a polynomially decaying probability. This proves the first point of the lemma.

The second point of the lemma follows from the first point combined with Neveu's proof of the strong ratio limit theorem which we now recall. Fix a possible increment $(a, b) \in \mathbb{Z}^2$ of the walk $(\mathcal{D}, R)$. Following Neveu [168] we write $\mathbb{P}((\Delta\mathcal{D}_0, \Delta R_0) = (a, b) \mid (\mathcal{D}_m, R_m) = (n, 0))$ in two different ways. Introducing N_m the number of increments of $(\mathcal{D}, R)$ equal to (a, b) up to time m and using the permutation symmetry of the increments:

$$\begin{aligned}
& \mathbb{E}\left[\frac{N_m}{m} \,\middle|\, (\mathcal{D}_m, R_m) = (n, 0)\right] && (9.12)\\
\underset{\text{symmetry}}{=} \; & \mathbb{P}((\Delta\mathcal{D}_0, \Delta R_0) = (a, b) \mid (\mathcal{D}_m, R_m) = (n, 0)) && (9.13)\\
\underset{\text{Markov}}{=} \; & \mathbb{P}((\Delta\mathcal{D}_0, \Delta R_0) = (a, b)) && (9.14)\\
& \times \frac{\mathbb{P}((\mathcal{D}_{m-1}, R_{m-1}) = (n - a, -b))}{\mathbb{P}((\mathcal{D}_m, R_m) = (n, 0))}.
\end{aligned}$$

Given the periodicity conditions of the walk $(\mathcal{D}, R)$, to prove the lemma it suffices to show that for any $\eta > 0$ and for any (a, b) in the support of the increment, there exists $\delta > 0$ such that the ratio in the last display lies in $[1 - \eta, 1 + \eta]$ as soon as $|n/m - \zeta(-1)| \le \delta$ and $n \ge 1/\delta$. To see this, we examine the left-hand side of (9.13). Indeed, since N_m has distribution $\mathsf{Binomial}(m, p_{a,b})$ with $p_{a,b} = \mathbb{P}((\Delta\mathcal{D}_0, \Delta R_0) = (a, b))$, an easy large deviation estimate shows that for all m large enough

$$\mathbb{P}\left(\left|\frac{N_m}{p_{a,b} \cdot m} - 1\right| \ge \eta\right) \le \mathrm{e}^{-c_\eta m},$$

for some constant $c_\eta > 0$. Applying this estimate to (9.13) we find that

$$\begin{aligned}
\left|\frac{\mathbb{P}((\mathcal{D}_{m-1}, R_{m-1}) = (n - a, -b))}{\mathbb{P}((\mathcal{D}_m, R_m) = (n, 0))} - 1\right| &= \left|\mathbb{E}\left[\frac{N_m}{p_{a,b} \cdot m} \,\middle|\, (\mathcal{D}_m, R_m) = (n, 0)\right] - 1\right| \\
&\le \eta + \frac{1}{p_{a,b}} \cdot \frac{\mathrm{e}^{-c_\eta m}}{\mathbb{P}((\mathcal{D}_m, R_m) = (n, 0))}.
\end{aligned}$$

Using the first point of the lemma, for fixed $\eta > 0$, we can decrease δ so that $\mathbb{P}((\mathcal{D}_m, R_m) = (n, 0))$ is asymptotically much larger than $\mathrm{e}^{-c_\eta m}$, thus asymptotically bounding the above display by 2η. This finishes the proof of the lemma. □

9.2 Infinite ∂-simple Boltzmann maps of the half-plane

In this section, we introduce the infinite simple Boltzmann maps of the half-plane $\hat{\mathfrak{M}}^{(\infty)}$ which will be the local limit of the Boltzmann maps with a large simple boundary (Theorem 9.1). To do so, we first construct $\widetilde{\mathfrak{M}}^{(\infty)}$, the infinite Boltzmann map of the half-plane (with

a general boundary) conditioned on having an infinite core. Then we simply define $\hat{\mathfrak{M}}^{(\infty)}$ as the infinite core of $\widetilde{\mathfrak{M}}^{(\infty)}$. In the case when $\sum_{k\geq 0} k\nu(-k) < \infty$, we know from Proposition 6.7 that almost surely $\mathfrak{M}^{(\infty)}$ has only finitely many cut-points separating the origin from ∞, and so $\widetilde{\mathfrak{M}}^{(\infty)}$ can alternatively be defined by conditioning $\mathfrak{M}^{(\infty)}$ on the event of positive probability where the root edge is on the infinite core. However, in the case when $\sum_{k\geq 0} k\nu(-k) = \infty$ the last event has probability zero and we shall use a h-transformation. We then describe the law of core decomposition of $\widetilde{\mathfrak{M}}^{(\infty)}$ which is useful to describe the one-step simple peeling exploration under $\hat{\mathfrak{M}}^{(\infty)}$.

9.2.1 Defining $\widetilde{\mathfrak{M}}^{(\infty)}$ and $\hat{\mathfrak{M}}^{(\infty)}$

As usual in this part of the lecture notes, we shall construct a law supported by random maps of the half-plane via three steps: first we characterize the law of the random lattice by a spatial Markov property. We then derive the law of filled-in peeling exploration (after checking "Tutte's equation") and finally *construct* the law by using an appropriate metric exploration. In order to follow this program in our case, we should restrict our attention to so-called *good* (filled-in) explorations that we now define.

Good explorations. Let $\bar{\mathfrak{e}}$ be a submap with an infinite boundary and a unique hole of infinite perimeter (recall that a submap has finitely many faces). We shall say that $\bar{\mathfrak{e}}$ is **good** if the simple connected component carrying the root edge has a part in common with the hole, we call this part the *exposed* boundary, and if the other simple components of $\bar{\mathfrak{e}}$ are attached in the interior of the complement of the exposed boundary on the core. For example, the submaps explored using algorithm $\mathcal{A}_{\text{core}}$ (until its stopping time) are all of this type. More generally, good submaps are those obtained by peeling maps of the half-plane via good algorithms: a peeling algorithm $\mathcal{A}$ is good if the first edge to peel is the root edge and if for each $i \geq 1$, the edge to peel at step i is an edge of the *exposed boundary*, i.e. which *does not* belongs to the initial boundary of $\bar{\mathfrak{e}}_0$. If there are no possible edge to peel, then the algorithm stops.

Proposition 9.8. *There is at most one law $\widetilde{\mathbb{P}}^{(\infty)}$ supported by infinite maps of the half-plane (with general boundary) so that its root core (the simple boundary component containing the root edge) is almost surely infinite and whose law is characterized by*

$$\widetilde{\mathbb{P}}^{(\infty)}(\bar{\mathfrak{e}} \subset \mathfrak{m}) = w_{\mathfrak{q}}(\bar{\mathfrak{e}}) \cdot c_{\mathfrak{q}}{}^{|\partial^*\bar{\mathfrak{e}}|-|\partial\bar{\mathfrak{e}}|} \cdot H^{\uparrow}(p_{\text{exposed}}). \tag{9.15}$$

for any good submap $\bar{\mathfrak{e}}$ of the half-plane with an infinite boundary, and a unique hole and such that the perimeter of the exposed boundary is p_{exposed}. Recall that $|\partial^*\bar{\mathfrak{e}}| - |\partial\bar{\mathfrak{e}}|$ is half *of the perimeter difference.*

Proof. The proof is similar to that of Proposition 6.2 and is left to the reader. □

Following, our general strategy, if $\widetilde{\mathfrak{M}}^{(\infty)}$ is a map of law $\widetilde{\mathbb{P}}^{(\infty)}$ (provided it exists), it is easy to deduce the form of the probability transition for any *good* peeling algorithm. More precisely, if

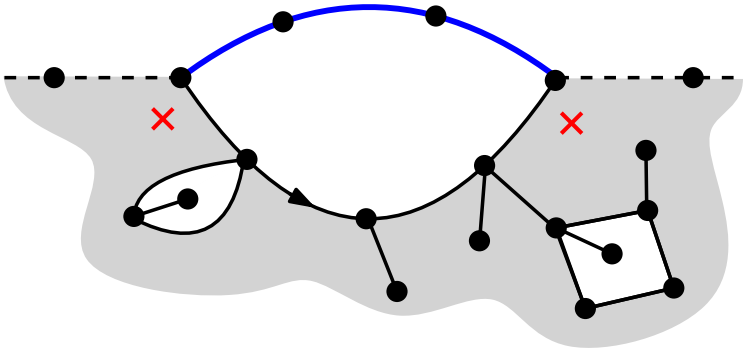


Figure 9.2: A good submap of the half-plane. The exposed boundary is in thick blue line. Notice that no dangling parts are attached to the boundary points of the exposed part.

we peel a good submap of the half-plane with a good algorithm, then the peeling transitions are simply $H^\uparrow$-transforms of the standard peeling transitions in the half-plane, i.e. the probability to witness an event of type $\mathsf{C}_k, \mathsf{G}_{\infty,k}$ or $\mathsf{G}_{k,\infty}$ is given by

$$b^{(\infty)}(k)\frac{H^\uparrow(p_{\text{new}})}{H^\uparrow(p_{\text{old}})}, \quad b^{(\infty)}(\infty,k)\frac{H^\uparrow(p_{\text{new}})}{H^\uparrow(p_{\text{old}})}, \quad \text{and} \quad b^{(\infty)}(k,\infty)\frac{H^\uparrow(p_{\text{new}})}{H^\uparrow(p_{\text{old}})},$$

where p_{old} and p_{new} are respectively the length of the exposed boundary before and after the peeling step. We need now to check Tutte's equation, i.e. to verify that this indeed defines probability transitions:

Lemma 9.9 (Checking Tutte's equation)**.** *The above probability transitions sum-up to* 1 *for any good peeling algorithm and any good submap* $\overline{\mathfrak{e}}$*.*

Proof. Let us consider the situation in which the exposed boundary has length $p \geq 1$ and the peeling algorithm selects the ℓth edge from the left, with $1 \leq \ell \leq p$. Expressing the probabilities above in terms of the law ν we need to check that

$$\sum_{k\geq 0} \nu(k)H^\uparrow(p+2k) + \sum_{k\leq -1} \frac{1}{2}\nu(k)H^\uparrow(p+2k\vee p-\ell) + \sum_{k\leq -1} \frac{1}{2}\nu(k)H^\uparrow(p+2k\vee \ell-1) = H^\uparrow(p). \quad (9.16)$$

We have already seen in Proposition 9.6 that $H^\downarrow$ is the pre-renewal function for the ζ-walk and is thus harmonic. Since the ζ-walk is not drifting towards $-\infty$ it follows from Proposition A.10 that $H^\uparrow$ is also harmonic for the ζ-walk. In our context, this means that the above probability transitions indeed sum-up to 1 *in the particular case of the algorithm* $\mathcal{A}_{\text{core}}$ which peels the left-most edge of the exposed boundary. Therefore (9.16) is satisfied for $\ell = 1$.

Next we compare the left-hand side of (9.16) when peeling the $(\ell+1)$th to that case where

the ℓth edge is peeled. If $1 \le \ell \le p-1$ the difference in the second term is

$$\begin{aligned}
&\sum_{k\le -1} \frac{1}{2}\nu(k)(H^{\uparrow}(p+2k\vee p-\ell-1)-H^{\uparrow}(p+2k\vee p-\ell))\\
&\quad=\sum_{k\le -\ell-1}\frac{1}{2}\nu(k)(H^{\uparrow}(p-\ell-1)-H^{\uparrow}(p-\ell))\\
&\quad\overset{(9.5)}{=}-\sum_{k\le -\ell-1}\frac{1}{2}\nu(k)H^{\downarrow}(p-\ell-1)\overset{(9.4)}{=}-\frac{1}{2}\nu(\mathbb{Z}_{<0})H^{\downarrow}(\ell-1)H^{\downarrow}(p-\ell-1).
\end{aligned}$$

It is canceled by the difference in the third term, which evaluates to $\frac{1}{2}\nu(\mathbb{Z}_{<0})H^{\downarrow}(p-\ell-1)H^{\downarrow}(\ell-1)$. By induction the identity (9.16) follows for all ℓ. □

Given the previous results, one can now *construct* the law $\widetilde{\mathbb{P}}^{(\infty)}$ by a filled-in exploration. For this, we define a *good* variant $\mathcal{A}_{\text{metric,good}}$ of the algorithm $\mathcal{A}_{\text{metric}}$ of Section 6.1.3 by always peeling at edges on the exposed boundary. Then if $(\overline{\mathbf{E}}_n : n \ge 0)$ is the Markov chain of growing good submaps of the half-plane with a unique hole (necessarily of infinite perimeter) whose probability transitions are given by the second point of Proposition 6.3 with the peeling algorithm $\mathcal{A}_{\text{metric,good}}$. Then:

Proposition 9.10. *The map*

$$\widetilde{\mathfrak{M}}^{(\infty)} := \bigcup_{n\ge 0}\overline{\mathbf{E}}_n,$$

is a random infinite map of the half-plane which satisfies (9.15).

Proof. The proof is mutatis mutandis the same as that of Proposition 6.5 and it is easy to see that $\widetilde{\mathfrak{M}}^{(\infty)}$ has a core of infinite perimeter by construction. □

Definition 9.11. *Let* $\mathbf{q}$ *be a critical weight sequence. We define* $\hat{\mathfrak{M}}^{(\infty)}$*, the infinite Boltzmann map of the half-plane with a simple boundary, whose law is denoted by* $\hat{\mathbb{P}}^{(\infty)}$*, as the infinite core of* $\widetilde{\mathfrak{M}}^{(\infty)}$ *i.e.*

$$\hat{\mathfrak{M}}^{(\infty)} := \mathrm{Core}(\widetilde{\mathfrak{M}}^{(\infty)}).$$

Remark 9.12 (Finite gulp case). In the case when $\sum_{k\ge 0} k\nu(-k) < \infty$, by Remark 11.2 the measure ζ has a finite mean and positive drift. It follows that $H^{\uparrow}$ is bounded, converges, and furthermore it is easy to check from the exploration of $\mathfrak{M}^{(\infty)}$ with $\mathcal{A}_{\text{core}}$ that

$$\frac{H^{\uparrow}(1)}{H^{\uparrow}(\infty)} = \mathbb{P}^{(\infty)}(\text{the exploration with } \mathcal{A}_{\text{core}} \text{ does not stop}) = \mathbb{P}(|\partial\mathrm{Core}(\mathfrak{M}^{(\infty)})| = \infty).$$

From this, it is not to hard to deduce that $\widetilde{\mathfrak{M}}^{(\infty)}$ is nothing else but $\mathfrak{M}^{(\infty)}$ conditioned on the event of positive probability $\{|\partial\mathrm{Core}(\mathfrak{M}^{(\infty)})| = \infty\}$. By invariance under translation (Proposition 6.6) one can alternatively define $\widetilde{\mathfrak{M}}^{(\infty)}$ by shifting the root edge of $\mathfrak{M}^{(\infty)}$ on the first edge on the right boundary belonging to an infinite simple component, and biasing by $1/(1+2|\partial C_0|)$ where C_0 is the component dangling on on the left of the root edge in $\widetilde{\mathfrak{M}}^{(\infty)}$. In particular, the probability that the root edge of $\mathfrak{M}^{(\infty)}$ lies on the infinite component with a simple boundary is equal to

$$\mathbb{P}^{(\infty)}(\vec{e}\in\mathrm{Core}(\mathfrak{m})) = \frac{\sum_{\ell\ge 1} W^{(\ell)}c_{\mathbf{q}}^{-\ell}}{\sum_{\ell\ge 1} W^{(\ell)}c_{\mathbf{q}}^{-\ell}(2\ell+1)} = \frac{\mathrm{W}_c}{2c_{\mathbf{q}}^{-1}\mathrm{W}_c' + \mathrm{W}_c}.$$

9.2.2 Simple peeling exploration of $\hat{\mathfrak{M}}^{(\infty)}$

We will now study the filled-in simple peeling process of $\hat{\mathbb{P}}^{(\infty)}$. To develop the theory exactly as in the case of the peeling process (see Section 4.1) all we need is the following simple version of the spatial Markov property.

Proposition 9.13 (Spatial Markov Property under $\hat{\mathbb{P}}^{(\infty)}$). *Recall* (9.1). *Let* $\hat{\mathfrak{e}}$ *be an infinite simple submap with an infinite boundary and a unique infinite hole of infinite perimeter. Denote by* $|\partial^*\hat{\mathfrak{e}}| - |\partial\hat{\mathfrak{e}}|$ *the difference of the hole and boundary half-perimeters of* $\hat{\mathfrak{e}}$. *Then we have*

$$\hat{\mathbb{P}}^{(\infty)}(\hat{\mathfrak{e}} \subset \mathfrak{m}) = \hat{c}_{\mathfrak{q}}^{|\partial^*\hat{\mathfrak{e}}|-|\partial\hat{\mathfrak{e}}|} \cdot w_{\mathfrak{q}}(\hat{\mathfrak{e}}).$$

Furthermore, conditionally on $\hat{\mathfrak{e}} \subset \mathfrak{m}$ *the remaining map with an infinite simple boundary has law* $\hat{\mathbb{P}}^{(\infty)}$.

To prove the above proposition we will rely on (standard) exploration of $\widetilde{\mathfrak{M}}^{(\infty)}$. We first compute the law of the core decomposition under $\widetilde{\mathbb{P}}^{(\infty)}$.

Proposition 9.14 (Core decomposition). *Under* $\widetilde{\mathbb{P}}^{(\infty)}$, *the infinite core and the finite components dangling from it are independent and the later are identically distributed with law* $\mathbb{P}^{(\text{free})}$.

Remark 9.15. In the case $\sum_{k\geq 0} k\nu(-k) < \infty$, combining the previous proposition and Remark 9.12 we deduce that under $\mathbb{P}^{(\infty)}$, the only infinite simple component and the finite parts dangling form it are independent, and the later are i.i.d. of law $\mathbb{P}^{(\text{free})}$ except the component carrying the root edge which has law $\mathbb{P}^{(\text{free})}$ biased by $1 + 2|\partial\mathfrak{m}|$.

Proof. Inside $\widetilde{\mathfrak{M}}^{(\infty)}$, denote by C_i, for $i \in \mathbb{Z}$, the finite components dangling from the infinite core, where C_0 is attached to the origin of the root edge. Fix $j \geq 1$, and let us compute the law of $C_{-j}, C_{-j-1}, \ldots, C_0, C_1, \ldots, C_j$. To do so, we shall explore $\widetilde{\mathfrak{M}}^{(\infty)}$ with a good variant of the metric exploration $\mathcal{A}_{\text{metric}}$ until the minimal distance (inside the explored submap) from the origin to the exposed boundary is larger than $A > j$.

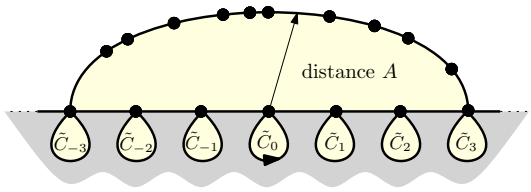


Figure 9.3: If $A > j$ the dangling parts of the core of $\bar{\mathfrak{e}}(A)$ (except possibly the extreme ones) are those dangling from the core of $\widetilde{\mathfrak{M}}^{(\infty)}$.

The algorithm will almost surely stop and output an explored part $\bar{\mathfrak{e}}(A)$. We denote by $(\tilde{C}_i : i \in \mathbb{Z})$ the parts dangling from the core of $\bar{\mathfrak{e}}(A)$, where $\tilde{C}_0$ is attached to the origin of the root edge. It is straightforward using (9.15) to see that $(\tilde{C}_i : -j \leq i \leq j)$ are i.i.d. of law

$\mathbb{P}^{(\text{free})}$. On the other hand, since $\widetilde{\mathfrak{M}}^{(\infty)}$ almost surely has a unique infinite component with simple perimeter (the one carrying the root edge), we have

$$C_i = \tilde{C}_i, \quad \forall -A+1 \le i \le A-1.$$

Since $A > j$ the statement of the proposition follows. □

Proof of Proposition 9.13. It suffices to prove the statement only for the region $\overline{\hat{\mathbf{e}}_1}$ enclosed by the face adjacent on the left of the root edge, and the full statement of the proposition follows by iteration. Fix a map $\mathbf{e}$. We use the definition of $\hat{\mathfrak{M}}^{(\infty)}$ as the core of $\widetilde{\mathfrak{M}}^{(\infty)}$ and scrutinize which events in $\widetilde{\mathfrak{M}}^{(\infty)}$ yield to the event $\{\overline{\hat{\mathbf{e}}_1} = \mathbf{e}\}$ in the proposition. If $2k$ is the degree of the face adjacent to the left of the root edge in $\mathbf{e}$ then clearly the (standard) peeling of the root edge in $\widetilde{\mathfrak{M}}^{(\infty)}$ must discover a face f of degree $2k$. Furthermore, the remaining map $\widetilde{\mathfrak{M}}^{(\infty)}\backslash\mathrm{f}$ must have a core decomposition as in the following Figure 9.4.

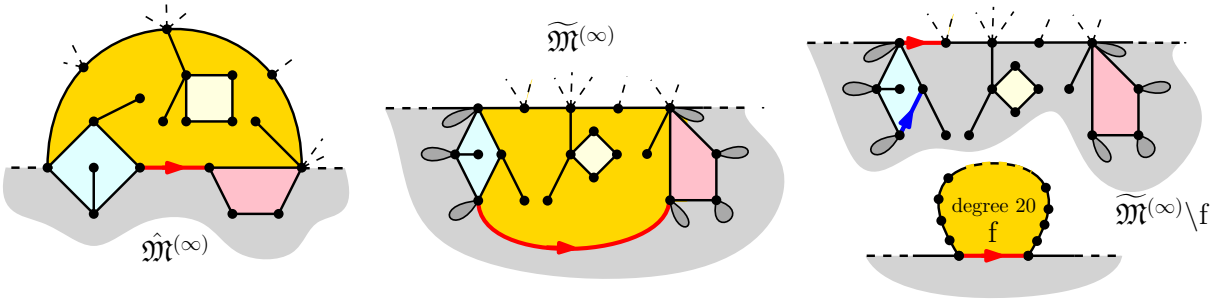


Figure 9.4: On the left, the desired event for $\hat{\mathfrak{M}}^{(\infty)}$. In the center, the corresponding possible events for $\widetilde{\mathfrak{M}}^{(\infty)}$ where the dark gray regions are arbitrary maps with a boundary. On the right, we see the core decomposition of the map $\widetilde{\mathfrak{M}}^{(\infty)}\backslash\mathrm{f}$.

Let us write $\mathfrak{M}[k]$ as a short-hand for the map $\widetilde{\mathfrak{M}}^{(\infty)}\backslash\mathrm{f}$. This map of the half-plane comes with two roots: one on its simple infinite core (in red in Figure 9.4) which is the first one encountered when tracing the contour of the face f in $\widetilde{\mathfrak{M}}^{(\infty)}$, and a second one (in blue in Figure 9.4) to remember where to glue the face f of degree $2k$ to recover $\widetilde{\mathfrak{M}}^{(\infty)}$. These two edges are equal or they have at most $2k-3$ edges in between. The law of $\mathfrak{M}[k]$ is characterized as follows:

Lemma 9.16. *The infinite core of $\mathfrak{M}[k]$ and the parts dangling from it are independent. The latter are identically distributed with law $\mathbb{P}^{(\text{free})}$ except for the component attached to the origin of the (red) root edge. This component C_{root} has law*

$$\mathbb{P}(C_{\text{root}} = \mathfrak{m}) = \frac{1}{\mathrm{W}_c\, H^{\uparrow}(2k-1)} w_{\mathrm{q}}(\mathfrak{m}) \cdot c_{\mathrm{q}}^{-|\partial\mathfrak{m}|} \left((2k-1) \wedge (2|\partial\mathfrak{m}|+1)\right),$$

and conditionally on C_{root} the blue edge of $\mathfrak{M}[k]$ is located uniformly either on the root edge or on one of the $(2k-2 \wedge 2|\partial C_{\text{root}}|)$ edges on the right of the root edge on ∂C_{root}. On the other hand, the core of $\mathfrak{M}[k]$ has law $\tilde{\mathbb{P}}^{(\infty)}$.

Proof of the lemma. We start by establishing a spatial Markov property for $\mathfrak{M}[k]$ with the help of Proposition 9.8. Let $\bar{\mathfrak{e}}$ be a good submap with exposed boundary of length p_{exposed}, equipped with another blue (oriented) edge on the component attached to the origin such that there are at most $2k-3$ between the blue edge and the root edge. Suppose furthermore that the edge e to the right of the blue edge, chosen such that there are precisely $2k-3$ edges in between, is not adjacent to the hole of $\bar{\mathfrak{e}}$. Then $\bar{\mathfrak{e}} \cup \mathfrak{f}$ is the good submap obtained by adding to $\bar{\mathfrak{e}}$ a new edge from the origin of the blue edge to the endpoint of e, and taking this new edge to be the root, see Figure 9.5.

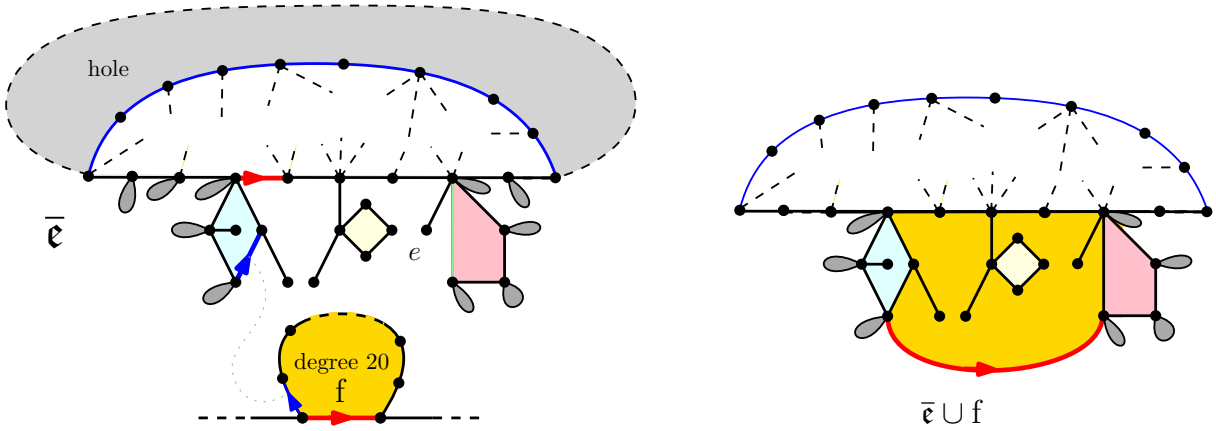


Figure 9.5: Setup of the proof of the lemma. We fix a good submap $\bar{\mathfrak{e}}$ that can appear as a submap of $\mathfrak{M}[k]$, which means that after gluing a face of degree $2k$ on it (with proper starting and ending points) we get a possible submap of $\widetilde{\mathfrak{M}}^{(\infty)}$.

By construction the exposed boundary of $\bar{\mathfrak{e}} \cup \mathfrak{f}$ has length p_{exposed} as well. Then (9.15) implies that

$$\mathbb{P}(\bar{\mathfrak{e}} \subset \mathfrak{M}[k]) = \frac{\widetilde{\mathbb{P}}^{(\infty)}(\bar{\mathfrak{e}} \cup \mathfrak{f} \subset \widetilde{\mathfrak{M}}^{(\infty)})}{\widetilde{\mathbb{P}}^{(\infty)}(\mathfrak{f} \text{ has degree } 2k)} = w_{\mathbf{q}}(\bar{\mathfrak{e}}) \cdot c_{\mathbf{q}}^{|\partial^* \bar{\mathfrak{e}}| - |\partial \bar{\mathfrak{e}}|} \cdot \frac{H^{\uparrow}(p_{\text{exposed}})}{H^{\uparrow}(2k-1)}.$$

The statement of the lemma then follows using the same arguments as in Proposition 9.14. To compute the normalization constant we use that

$$\mathbb{E}^{(\text{free})}[(2|\partial \mathfrak{m}| + 1) \wedge (2k-1)] = \sum_{\ell=0}^{2k-2} \mathbb{P}^{(\text{free})}(2|\partial \mathfrak{m}| \geq \ell) \overset{(9.4)}{=} \sum_{\ell=0}^{2k-2} H^{\downarrow}(\ell) \overset{(9.5)}{=} H^{\uparrow}(2k-1),$$

implying that

$$\sum_{\mathfrak{m}} w_{\mathbf{q}}(\mathfrak{m}) \cdot c_{\mathbf{q}}^{-|\partial \mathfrak{m}|} \left((2k-1) \wedge (2|\partial \mathfrak{m}| + 1)\right) = \mathrm{W}_c \mathbb{E}^{(\text{free})}[(2|\partial \mathfrak{m}| + 1) \wedge (2k-1)] = \mathrm{W}_c H^{\uparrow}(2k-1).$$

□

Coming back to the proof of the proposition, using the above core decomposition of $\mathfrak{M}[k]$ one can explicitly write down the probability inside $\widetilde{\mathfrak{M}}^{(\infty)}$ to produce an event yielding $\{\overline{\hat{\mathfrak{e}}_1} = \mathbf{e}\}$. Suppose the exposed boundary of $\mathbf{e}$ is of length p and the (possibly empty) dangling components

of $\mathbf{e}\setminus \mathfrak{f}$ are $\tilde{C}_0, \tilde{C}_1, \dots, \tilde{C}_p$. Let ℓ be the length of the outer boundary, i.e. the number of edges on the root face that are not shared with the hole, such that $p - \ell = 2(|\partial^*\mathbf{e}| - |\partial\mathbf{e}|)$. Then

$$\begin{aligned}\hat{\mathbb{P}}^{(\infty)}\left(\overline{\hat{\mathbf{e}}_1} = \mathbf{e}\right) &= \mathrm{W}_c^{\ell+1}\,\widetilde{\mathbb{P}}^{(\infty)}(\mathfrak{f} \text{ has degree } 2k)\,\widetilde{\mathbb{P}}^{(\infty)}(C_{\mathrm{root}} = \tilde{C}_0)\prod_{i=1}^{p}\widetilde{\mathbb{P}}^{(\infty)}(C_i = \tilde{C}_i)\\ &= \mathrm{W}_c^{\ell+1}H^{\uparrow}(2k-1)\nu(k-1)\frac{1}{H^{\uparrow}(2k-1)}\prod_{i=0}^{p}\mathbb{P}^{(\mathrm{free})}(\mathfrak{m} = \tilde{C}_i)\\ &= \mathrm{W}_c^{\ell+1}w_{\mathbf{q}}(\mathbf{e})c_{\mathbf{q}}^{k-1}\mathrm{W}_c^{-p-1}\prod_{i=0}^{p}c_{\mathbf{q}}^{-|\partial\tilde{C}_i|} = w_{\mathbf{q}}(\mathbf{e})\left(\frac{c_{\mathbf{q}}}{\mathrm{W}_c^2}\right)^{|\partial^*\mathbf{e}|-|\partial\mathbf{e}|} = w_{\mathbf{q}}(\mathbf{e})\hat{c}_{\mathbf{q}}^{|\partial^*\mathbf{e}|-|\partial\mathbf{e}|},\end{aligned}$$

where the factor of $\mathrm{W}_c^{\ell+1}$ in the first equality takes into account the dangling maps in $\widetilde{\mathfrak{M}}^{(\infty)}$ and in the last equality we used that the length of the outer boundary of $\mathbf{e}\setminus\mathfrak{f}$ is $p+\sum_{i=0}^{p}2|\partial\tilde{C}_i| = \ell+2k-2$. Conditionally on $\overline{\hat{\mathbf{e}}}_1 = \mathbf{e}$ the remaining map is precisely the core of $\mathfrak{M}[k]$, which according to Lemma 9.16 has the law of $\mathfrak{M}[k]$. This proves the proposition for the first step of a simple peeling exploration of $\hat{\mathfrak{M}}^{(\infty)}$. □

9.2.3 $\hat{\mathbb{P}}^{(\infty)}$ as the weak limit of $\hat{\mathbb{P}}^{(\ell)}$

As usual, once the proper framework and properties are in place, deducing Theorem 9.1 is a child's play. It is similar to the proof of Theorem 6.1 using simple exploration in place of standard exploration and Proposition 9.4 in place of Lemma 3.13.

Proof of Theorem 9.1. We follow the same idea as in Section 6.1.1. Let $\mathfrak{s}$ be a finite map with a simple boundary. We suppose that the boundary of $\mathfrak{s}$ is split into two intervals of length p_{ext} and p_{int}, where the part carrying the root edge has length p_{ext}. If $\hat{\mathfrak{m}}$ is a ∂-simple map we write $\mathfrak{s}\hat{\Subset}\hat{\mathfrak{m}}$ if $\hat{\mathfrak{m}}$ is obtained by gluing a ∂-simple map on the part of the boundary of $\mathfrak{s}$ which does not carry the root edge. With this notation we have

$$\hat{\mathbb{P}}^{(\ell)}(\mathfrak{s}\hat{\Subset}\mathfrak{m}) = w_{\mathbf{q}}(\mathfrak{s})\frac{\hat{W}^{(\ell+\frac{1}{2}p_{\mathrm{int}}-\frac{1}{2}p_{\mathrm{ext}})}}{\hat{W}^{(\ell)}} \xrightarrow[\ell\to\infty]{\text{Prop. 9.4}} w_{\mathbf{q}}(\mathfrak{s})\hat{c}_{\mathbf{q}}^{\frac{1}{2}p_{\mathrm{int}}-\frac{1}{2}p_{\mathrm{ext}}} \overset{\text{Prop. 9.13}}{=} \hat{\mathbb{P}}^{(\infty)}(\mathfrak{s}\hat{\Subset}\mathfrak{m}).$$

Since the last probabilities obvious characterize the law of $\hat{\mathbb{P}}^{(\infty)}$, this implies the weak convergence in the local topology. □

9.3 Basic properties

Proposition 6.6, and its proof, are easily adapted to our context and yield:

Proposition 9.17. *The shift operator θ that moves the root edge one unit on the right (or on the left) on the boundary preserves the law $\hat{\mathbb{P}}^{(\infty)}$ and is ergodic.*

Bibliographical notes. As mentioned earlier, random infinite maps of the half-plane with a simple boundary came before (see [12]) their general boundary analogs. The probabilistic aspects of pruning and core decomposition in the case of quadrangulations was developed in [97] relying on the very classical composition scheme recalled in Proposition 9.3 and used heavily in combinatorics, see [49]. Enumeration of Boltzmann maps with simple boundary (in particular in the dense regime) was first attacked by Richier [174] in the case of regular varying weight sequences. The approach based on the simple peeling process and the strong ratio limit theorem is taken from the recent work [56].

Chapter X : Scaling limit for the peeling process

In this chapter, we describe the scaling limit for the evolution of the perimeter and volume processes during filled-in explorations under $\mathbb{P}^{(\ell)}, \mathbb{P}_p^{(\ell)}, \mathbb{P}_\infty^{(\ell)}$ and $\mathbb{P}^{(\infty)}$. To be able to get nice scaling limits for these processes, we shall assume that $\mathbf{q}$ is of type $a \in [3/2; 5/2]$ (see Chapter 5) so that the step distribution ν is in the domain of attraction of an $(a-1)$-stable law.

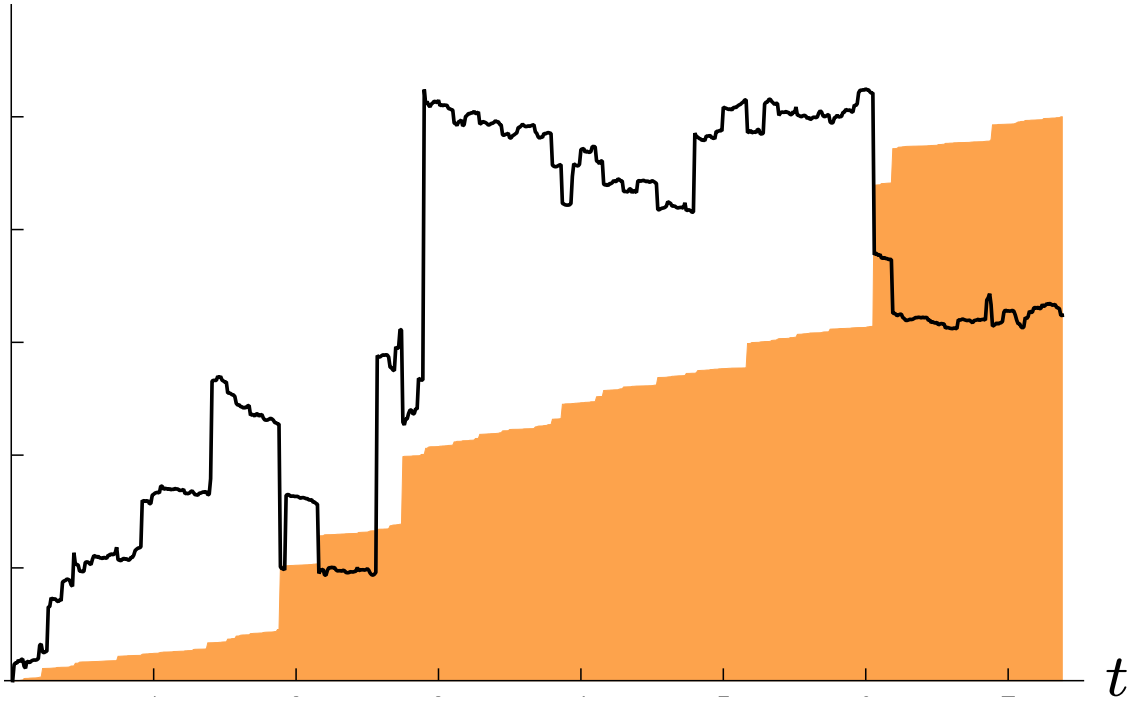


Figure 10.1: Simulation of the processes $\Upsilon_a^{\uparrow}$ (in black) and $\mathcal{V}_a(\Upsilon_a^{\uparrow})$ (in orange) when $a = 2$. These processes describe the scaling limits of the perimeter and volume growth in filled-in explorations under $\mathbb{P}_\infty^{(\ell)}$ when $\mathbf{q}$ is of type $a = 2$. Notice that negative jumps of $\Upsilon_a^{\uparrow}$ correspond to positive jumps of $\mathcal{V}_a(\Upsilon_a^{\uparrow})$.

10.1 Invariance principles for the perimeter process

This section is devoted to the invariance principles for the half-perimeter process of the distinguished hole in a filled-in peeling exploration under $\mathbb{P}^{(\infty)}, \mathbb{P}_p^{(\ell)}$ and $\mathbb{P}_\infty$ when $\mathbf{q}$ is of type

$a \in [3/2; 5/2]$.

10.1.1 The case of the ν-walk S

Invariance principles for random walks with independent increments are standard tools in probability theory, see [126]. To describe them, we start by introducing the limiting processes involved in our case which are stable Lévy processes (we refer to the book of Bertoin [32] for background on Lévy processes).

For $a \in [3/2; 5/2]$, let $(\Upsilon_a(t) : t \geq 0)$ be the $(a-1)$**-stable Lévy process with positivity parameter** $\varrho = \mathbb{P}(\Upsilon_a(t) \geq 0)$ satisfying

$$(a-1)(1-\varrho) = \frac{1}{2}. \tag{10.1}$$

We normalize $(\Upsilon_a(t))_{t\geq 0}$ so that its Lévy measure is equal to

$$\Pi(\mathrm{d}x) = \cos(a\,\pi) \cdot \frac{\mathrm{d}x}{x^a} 1_{x>0} + \frac{\mathrm{d}x}{|x|^a} 1_{x<0}. \tag{10.2}$$

The asymmetry between the negative and positive sides of the Lévy measure is indeed governed by the positive parameter ϱ, see Zolotarev's formula [32, p. 218]. Notice that in the case $a = \frac{5}{2}$ or $a = \frac{3}{2}$, the process Υ_a has no positive jumps. Furthermore, in the case of $a = \frac{3}{2}$, it corresponds to the opposite of the $\frac{1}{2}$-stable subordinator (a pure jump process) and in the case $a = 2$ it is the symmetric Cauchy process without drift. Recall that under $\mathbf{P}^{(0)}$ the process $(S_n)_{n\geq 0}$ is a random walk with independent increments of law ν started from 0; the dependence in $\mathbf{q}$, hence in a, is implicit in what follows.

Proposition 10.1 (Scaling limit for the random walk S)**.** *Let* $\mathbf{q}$ *be of type* $a \in [3/2; 5/2]$ *and recall the definition of* $\mathsf{p}_\mathbf{q}$ *in* (5.11)*. Then under* $\mathbf{P}^{(0)}$ *we have the following convergence in distribution for the Skorokhod topology*

$$\left(\frac{S_{\lfloor nt \rfloor}}{n^{1/(a-1)}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\Upsilon_a(\mathsf{p}_\mathbf{q} \cdot t))_{t\geq 0}.$$

Proof. By classical results (see [126, Chapter VII]) the functional convergence towards the stable Lévy process is granted once we have the convergence of the finite-dimensional marginals or equivalently the convergence in distribution

$$\frac{S_n}{n^{1/(a-1)}} \xrightarrow[n\to\infty]{(d)} \Upsilon_a(\mathsf{p}_\mathbf{q}) \overset{(d)}{=} (\mathsf{p}_\mathbf{q})^{1/(a-1)} \Upsilon_a(1).$$

By the description of the tail asymptotics of ν given in Chapter 5, it follows that when $\mathbf{q}$ is of type a then ν is a probability distribution in the domain of attraction of an $(a-1)$-stable law, see [42, Section 8.3]. This means that $a_n^{-1} S_n - b_n$ converges to an $(a-1)$-stable law for some scaling sequence (a_n) and centering sequence (b_n), see [42, Theorem 8.3.1]. By matching the Lévy measure Π with the tail asymptotics of ν it follows that one can take $a_n = n^{1/(a-1)}$ and the limiting $(a-1)$-stable law as $\Upsilon_a(\mathsf{p}_\mathbf{q})$. It remains to show that the centering sequences (b_n) can

be set to 0. This is always the case when $a \in (3/2; 2)$ since no centering is needed. In the case when $a \in (2; 5/2)$, the fact that the random walk $(S_i)_{i\geq 0}$ has mean 0 (Remark 5.8) shows that the centering sequence can be set to 0 as well and the theorem is proved in those two cases.
In the case of weight sequence of type $a = 2$, the above discussion shows that $n^{-1}S_n - b_n$ converges towards $\Upsilon_2(\mathsf{p_q})$, where by [42, Theorem 8.3.1] we can take

$$b_n = \sum_{k\in\mathbb{Z}} \nu(k)\frac{k}{1+(k/n)^2}. \tag{10.3}$$

To prove that the previous display tends to 0 as $n \to \infty$, we rely on the fact that $h^\uparrow$ is harmonic for S on $\mathbb{Z}_{>0}$. Let us sketch the calculation: for any n (large) we have by harmonicity of $h^\uparrow$

$$h^\uparrow(n) = \sum_{k\geq -n} h^\uparrow(n+k)\nu(k).$$

Recalling that $h^\uparrow(n) \sim \frac{2}{\sqrt{\pi}}\sqrt{n}$ as $n \to \infty$ and from Proposition 5.10 that $\nu([k,\infty)) \sim \nu((-\infty,k]) \sim \mathsf{p_q}/k$ as $k \to \infty$. We deduce that for any $\varepsilon \in (0,1)$ as $n \to \infty$ we have using a Riemann approximatoin

$$\begin{aligned}
1 &= \sum_{k\geq -n} \frac{h^\uparrow(n+k)}{h^\uparrow(n)}\nu(k)\\
\sum_{k<-n} \nu(k) &= \sum_{k\geq -n}\left(\frac{h^\uparrow(n+k)}{h^\uparrow(n)} - 1\right)\nu(k)\\
\frac{\mathsf{p_q}}{n} &\underset{\text{Riemann}}{\sim} \sum_{-\varepsilon n\leq k\leq \varepsilon n}\left(\frac{h^\uparrow(n+k)}{h^\uparrow(n)} - 1\right)\nu(k) + \frac{\mathsf{p_q}}{n}\left(\int_\varepsilon^\infty \frac{\sqrt{1+x}-1}{x^2}\mathrm{d}x + \int_\varepsilon^1 \frac{\sqrt{1-x}-1}{x^2}\mathrm{d}x\right).
\end{aligned}$$

An explicit calculation shows that the sum of the two integrals in the above parenthesis minus 1 can be made smaller than δ by choosing $\varepsilon > 0$ small (although each integral taken separately diverges as $\varepsilon \to 0$). Putting $N = \varepsilon n$ and using $\frac{h^\uparrow(n+k)}{h^\uparrow(n)} - 1 \approx \frac{k}{2n}$ for small ε, we deduce from the previous display that $\limsup_{N\to\infty} \sum_{-N\leq k\leq N} k\,\nu(k) \leq 2\mathsf{p_q}\delta$ and so that

$$\lim_{N\to\infty} \sum_{-N\leq k\leq N} k\,\nu(k) = 0.$$

It is then an exercise to see that this implies that b_n defined in (10.3) tends to 0 as $n \to \infty$ and thus no centering sequence is necessary to get convergence towards our 2-stable law. See [57, Proof of Proposition 2] for an alternative proof in the case $a = 2$. □

10.1.2 The cases of $S^\uparrow$ and $S^\downarrow$

Recall from Section 5.1.2 that $h^\downarrow_p$ (resp. $h^\uparrow$ when $\mathbf{q}$ is critical) is harmonic for the walk S on $\mathbb{Z}_{>0}$ and so we can define the Markov chain $S^\downarrow_p$ (resp. $S^\uparrow$) as the $h^\downarrow_p$-transform (resp. $h^\uparrow$-transform) of S. Those chains start from ℓ under $\mathbf{P}^{(\ell)}$ and they have the same law as the evolution of the half-perimeter of the distinguished hole in filled-in peeling explorations under $\mathbb{P}^{(\ell)}_p$ (resp $\mathbb{P}^{(\ell)}_\infty$). In what follows we shall focus on $p = 0$ and write $S^\downarrow$ for $S^\downarrow_0$. We leave the case when both ℓ and p are large as an open question (or a bibliographical exercise?):

Open Question 10.2 (Peeling of the cylinder). *Prove invariance principles for the process $S_p^{\downarrow}$ under $\mathbb{P}_p^{(\ell)}$ when $p \approx \ell$ both tend to infinity (when $p = \ell$ we are peeling a cylinder). If possible, compute interesting distributional quantities on the limiting process. Some useful information may be obtained with the work [38].*

The limiting processes in the invariance principles for $S^{\uparrow}$ and $S^{\downarrow}$ make appear $\Upsilon_a^{\uparrow}$ and $\Upsilon_a^{\downarrow}$, the versions of the Lévy process Υ_a **conditioned to stay positive** and respectively **to die continuously at** 0 when it enters $\mathbb{R}_-$. These are obtained from continuous h-transforms of Υ_a using the functions $x \mapsto \sqrt{x}$ and $x \mapsto 1/\sqrt{x}$ respectively (see [76, 69] for the definition of these processes). Unless explicitly mentioned, the process $\Upsilon_a^{\uparrow}$ starts from 0 (this indeed has a meaning) and the process $\Upsilon_a^{\downarrow}$ is started from 1. The process $\Upsilon_{3/2}^{\uparrow}$ has no meaning since in the case $a = \frac{3}{2}$ the Lévy process $\Upsilon_{3/2}$ is just the opposite of a subordinator. Combining Proposition 10.1 with the invariance principles proved in [69] shows that:

Proposition 10.3 (Scaling limit for $S^{\uparrow}$ and $S^{\downarrow}$). *If $\mathfrak{q}$ is of type $a \in [3/2; 5/2]$ we have the following convergences in distribution for the Skorokhod topology under $\mathbf{P}^{(\ell)}$ for $\ell \geq 1$, the first one being restricted to $a \neq 3/2$:*

$$\left(\frac{1}{n^{1/(a-1)}} \cdot S^{\uparrow}_{[nt]}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\Upsilon_a^{\uparrow}(\mathfrak{p}_{\mathfrak{q}} \cdot t))_{t\geq 0},$$
$$\left(\frac{1}{\ell} \cdot S^{\downarrow}_{[\ell^{(a-1)}t]}\right)_{t\geq 0} \xrightarrow[\ell\to\infty]{(d)} (\Upsilon_a^{\downarrow}(\mathfrak{p}_{\mathfrak{q}} \cdot t))_{t\geq 0}.$$

10.2 Scaling limit for the volume process

We will now study the volume process in filled-in peeling explorations. Recall that during the filled-in explorations under $\mathbb{P}_p^{(\ell)}, \mathbb{P}^{(\ell)}, \mathbb{P}^{(\infty)}, \mathbb{P}_{\infty}^{(\ell)}, \mathbb{H}_{\infty}$ or $\mathbb{H}^{(\infty)}$ the non-distinguished finite holes created during the peeling are filled-in with independent Boltzmann maps of the proper perimeter. One should first start by studying the renormalized volume of Boltzmann maps with a large perimeter.

10.2.1 Stable limit for the volume of Boltzmann maps

For $a \in [3/2; 5/2]$ we let $\xi_\bullet(a)$ be a positive $1/(a-\frac{1}{2})$-stable random variable with Laplace transform

$$\mathbb{E}[e^{-\lambda \xi_\bullet(a)}] = \exp\left(-(\Gamma(a+1/2)\lambda)^{\frac{1}{a-1/2}}\right). \tag{10.4}$$

We used the above normalization so that $\mathbb{E}[1/\xi_\bullet(a)] = \int_0^\infty dx\, \exp(-x^{1/(a-1/2)})/\Gamma(a+\frac{1}{2}) = 1$, hence we can define a random variable ξ_a after biasing $\xi_\bullet(a)$ by $x \mapsto 1/x$, that is for any $f \geq 0$

$$\mathbb{E}[f(\xi_a)] = \mathbb{E}\left[f(\xi_\bullet(a))\frac{1}{\xi_\bullet(a)}\right].$$

Notice that ξ_a has mean 1. Recall that $|\mathfrak{m}|$ denotes the number of edges of a map $\mathfrak{m}$.

Proposition 10.4. *Suppose that* $\mathbf{q}$ *is of type* $a \in [3/2; 5/2]$. *Then we have*

$$\mathbb{E}^{(\ell)}[|\mathfrak{m}|] \sim \mathrm{b}_{\mathbf{q}} \cdot \ell^{a-1/2} \quad \text{as } \ell \to \infty \qquad \text{where} \qquad \mathrm{b}_{\mathbf{q}} = \frac{1}{2\mathrm{p}_{\mathbf{q}}\sqrt{\pi}}, \tag{10.5}$$

and we have the convergence in distribution under $\mathbb{P}^{(\ell)}$

$$\ell^{-a+\frac{1}{2}}|\mathfrak{m}| \quad \xrightarrow[\ell\to\infty]{(\mathrm{d})} \quad \mathrm{b}_{\mathbf{q}} \cdot \xi_a. \tag{10.6}$$

Remark 10.5. Notice that when $a = 3/2$, the random variable $\xi_{3/2}$ is simply the constant 1, and so the volume (i.e. the number of edges) of $\mathfrak{M}^{(\ell)}$ of law $\mathbb{P}^{(\ell)}$ is roughly proportional to its perimeter. This indicates that the map is folded on its external face and is tree-like.

Remark 10.6. In the above proposition we work with the number of edges to measure the size of a map. However, all the results hold true if one instead measures the volume in terms of the number of faces or vertices of the map provided that we change the constant $\mathrm{b}_{\mathbf{q}} = \mathrm{b}_{\mathbf{q}}^{\text{edges}}$ into

$$\mathrm{b}_{\mathbf{q}}^{\text{vertices}} = \frac{4}{c_{\mathbf{q}}}\mathrm{b}_{\mathbf{q}}^{\text{edges}} \quad \text{and} \quad \mathrm{b}_{\mathbf{q}}^{\text{faces}} = \left(1 - \frac{4}{c_{\mathbf{q}}}\right)\mathrm{b}_{\mathbf{q}}^{\text{edges}}. \tag{10.7}$$

Proof. Using (3.5) and summing over $e \geq 0$ we have

$$\mathbb{E}^{(\ell)}[|\mathfrak{m}|] = \frac{W_1^{(\ell)}}{W^{(\ell)}}. \tag{10.8}$$

Recall the exact expression $W_1^{(\ell)} = \frac{1}{2}c_{\mathbf{q}}^{\ell+1}h_1^{\downarrow}(\ell)$ from (3.9). Combining this with the asymptotics for $W^{(\ell)}$ that we derived in Section 5.3, we get the first statement of the proposition.
To prove the second statement, we first focus on the case when $a \in (3/2; 5/2]$ and suppose for simplicity that we are in the aperiodic case i.e. that $\mathbb{P}^{(\ell)}(|\mathfrak{m}| = n) > 0$ for all n large enough. In this case, we prove the stronger result: for any $x > 0$ we have the local convergence of the "densities"

$$\ell^{-a+1/2}\mathbb{P}^{(\ell)}(|\mathfrak{m}| = [x\ell^{a-1/2}]) \xrightarrow[\ell\to\infty]{} p(x), \tag{10.9}$$

where $p(x)$ is the density of $\mathrm{b}_{\mathbf{q}} \cdot \xi_a$. By Scheffé's lemma this is indeed sufficient to imply the required convergence in distribution. To prove the previous display we start as in the proofs of Lemma 7.7 and we write for $n, \ell \geq 1$

$$\mathbb{P}^{(\ell)}(|\mathfrak{m}| = n) = \frac{W^{(\ell)}[n]}{W^{(\ell)}} \underset{(3.5)}{=} \frac{1}{n}\frac{W_1^{(\ell)}[n+1]}{W^{(\ell)}} \underset{\text{Prop.5.10 and Theo.3.12}}{\sim} \mathrm{Cst} \cdot \frac{\ell^{a+1/2}}{n^2}\mathbb{P}(Y_n = -\ell - 1),$$

where we recall that $(Y_n : n \geq 0)$ is a μ-random walk. Our assumptions on $\mathbf{q}$ imply (by Karamata's Theorem [42, Theorem 8.1.6]) that μ is aperiodic, is centered, and $\mu([n, \infty)) \sim \mathrm{cst} \cdot n^{-a+1/2}$ if $a < 5/2$ and that μ has finite variance if $a = \frac{5}{2}$. In other words, μ is in the strict domain of attraction of

the $a - \frac{1}{2} \in (1, 2]$ stable law and so if we put $n = [x\ell^{a-1/2}]$ we get from Gnedenko's local limit theorem (Theorem A.21) and the previous display that

$$\mathbb{P}^{(\ell)}(|\mathfrak{m}| = [x\ell^{a-1/2}]) \sim \mathrm{Cst} \cdot \ell^{-a+1/2} \frac{\phi_x(-1)}{x^2},$$

where $\phi_x(u)$ is the density of (some) spectrally positive stable Lévy process with index $a - 1/2$ at time $x \geq 0$ and at point $u \in \mathbb{R}$. It is well-known (see [32, Chapter VII]) that the density of the $1/(a - 1/2)$ stable law is obtained up to normalization as $\phi_x(-1)/x$ and so after biasing we have $p(x) \propto \phi_x(-1)/x^2$. Our claim (10.9) follows after a careful check of the normalizing constants using the first point of the proposition. The case when μ is not aperiodic, or the case $a = \frac{3}{2}$ where we in fact have $\ell^{-1} \cdot |\mathfrak{M}^{(\ell)}| \to \mathfrak{b}_{\mathbf{q}}$ in probability are left as exercises for the reader. □

Although not used in this section, the techniques of the previous proof show that the tail of the law of the volume under $\mathbb{P}^{(1)}$ is also characteristic of the type of the weight sequence:

Proposition 10.7 (Tail of the volume)**.** *Let* $\mathbf{q}$ *be a sequence of type* $a \in [3/2; 5/2]$*. The tail of the size (number of edges) of a* $\mathbf{q}$*-Boltzmann map satisfies:*

- *If* $a \in (3/2; 5/2]$ *then we have* $\mathbb{P}^{(1)}(|\mathfrak{m}| = n) \sim c \cdot n^{-2-\frac{1}{a-1/2}}$ *as* $n \to \infty$ *for some* $c > 0$ *where we restrict to values of* n *so that the previous probabilities are non-zero.*
- *If* $a = \frac{3}{2}$ *and if furthermore the degree of the face on the left-hand side of the root edge in* $\mathfrak{M}^{(1)}$ *has an exponential tail, then for all* $n \geq 0$*,*

$$\mathbb{P}^{(1)}(|\mathfrak{m}| = n) \leq \exp(-c \cdot n), \qquad \text{for some } c > 0.$$

Proof. Suppose for simplicity that we are in the case where μ is aperiodic. For the first point, the previous proof shows that

$$\mathbb{P}^{(1)}(|\mathfrak{m}| = n) \sim c \frac{1}{n^2} \mathbb{P}(Y_n = -2),$$

and an application of the local limit theorem proves the desired asymptotic (our stable processes have positive density at 0). In the case $a = 3/2$, the law μ has strictly negative mean, but it could very well have a heavy tail on the right, thus preventing the exponential decay to occur for $\mathbb{P}(Y_n = -2)$. However, if we further assume that the degree of the face on the left-hand side of the root edge in $\mathfrak{M}^{(1)}$ has an exponential tail, we deduce that

$$\mathbb{P}^{(1)}(\text{degree anti} - \text{root face} = 2k) = \frac{q_k W^{(k)}}{W^{(1)}} \leq \exp(-\mathrm{cst} \cdot k), \qquad \text{for cst} > 0.$$

By the asymptotics on $W^{(}k)$ provided by Proposition 5.7, it follows that $q_k c_{\mathbf{q}}^k$ and so $\mu(k)$ decay exponentially fast as $k \to \infty$. The second point then follows from an easy large deviation argument. □

10.2.2 Functional scaling limit for the volume and perimeter processes

We are now able to introduce the joint scaling limit for the perimeter and volume processes during a filled-in peeling exploration. For $a \in [3/2; 5/2]$, if $(\chi_t)_{t\geq}$ is a given càdlàg process we associate with it another process $\mathcal{V}_a(\chi)$ defined as follows. We let $\xi_a^{(1)}, \xi_a^{(2)}, \dots$ be a sequence of independent real random variables distributed as the variable ξ_a of Proposition 10.4. We assume that this sequence is independent of the process χ and for every $t \geq 0$ we set

$$\mathcal{V}_a(\chi)(t) = \sum_{t_i \leq t} \xi_a^{(i)} \cdot |\Delta\chi(t_i)|^{a-\frac{1}{2}} \mathbf{1}_{\Delta\chi(t_i)<0}, \tag{10.10}$$

where $t_1, t_2, \dots$ is a measurable enumeration of the jump times of χ. In general, $\mathcal{V}_a(\chi)$ may be infinite, but since $x \mapsto x^{a-\frac{1}{2}}\mathbf{1}_{x<0}$ integrates the Lévy measure of $(\Upsilon_a(t) : t \geq 0)$ in the neighborhood of 0, it is easy to check that $\mathcal{V}_a(\Upsilon_a)$ is a.s. finite for all $t \geq 0$. It can also be shown using absolute continuity relations with Υ_a that the processes $\mathcal{V}_a(\Upsilon_a^{\uparrow})$ as well as $\mathcal{V}_a(\Upsilon_a^{\downarrow})$ also make sense (in the case $a = \frac{3}{2}$ only $\mathcal{V}_{3/2}(\Upsilon_{3/2}^{\downarrow})$ makes sense since $\Upsilon_{3/2}^{\uparrow}$ is no defined).

Theorem 10.8 (*Scaling limits of the filled-in peeling processes*)

Let $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ be a filled-in exploration process and denote by $|\partial^*\bar{\mathfrak{e}}_n|$ and $|\bar{\mathfrak{e}}_n|$ the half-perimeter of the hole and the number of edges of $\bar{\mathfrak{e}}_n$. Then if $\mathbf{q}$ is of type $a \in [3/2; 5/2]$ we have the following convergences in distribution in the sense of Skorokhod

$$\begin{aligned}
&\text{under } \mathbb{P}^{(\infty)}, && \left(\frac{|\partial^*\bar{\mathfrak{e}}_{[nt]}| - |\partial\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{1}{a-1}}}, \frac{|\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{a-1/2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a, \mathrm{b}_{\mathbf{q}} \cdot \mathcal{V}_a(\Upsilon_a)\right)(\mathrm{p}_{\mathbf{q}} t)\right)_{t\geq 0},\\
&\text{under } \mathbb{P}_{\infty}^{(\ell)}, && \left(\frac{|\partial^*\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{1}{a-1}}}, \frac{|\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{a-1/2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^{\uparrow}, \mathrm{b}_{\mathbf{q}} \cdot \mathcal{V}_a(\Upsilon_a^{\uparrow})\right)(\mathrm{p}_{\mathbf{q}} t)\right)_{t\geq 0},\\
&\text{under } \mathbb{P}_{0}^{(\ell)}, && \left(\frac{|\partial^*\bar{\mathfrak{e}}_{[\ell^{(a-1)}t]}|}{\ell}, \frac{|\bar{\mathfrak{e}}_{[\ell^{(a-1)}t]}|}{\ell^{a-1/2}}\right)_{t\geq 0} \xrightarrow[\ell\to\infty]{(d)} \left(\left(\Upsilon_a^{\downarrow}, \mathrm{b}_{\mathbf{q}} \cdot \mathcal{V}_a(\Upsilon_a^{\downarrow})\right)(\mathrm{p}_{\mathbf{q}} t)\right)_{t\geq 0},
\end{aligned}$$

where the second line is restricted to $a \neq 3/2$ and in the first line $|\partial^*\bar{\mathfrak{e}}| - |\partial\bar{\mathfrak{e}}|$ is defined in (6.1).

Proof. The convergence of the first components in each of the above lines is given by Proposition 10.1 and Proposition 10.3. It thus remains to study the conditional distribution of the second component given the first one. We will do that in the first case since the arguments are the same in the two others cases. To simplify a bit notation we write $P_n = |\partial^*\bar{\mathfrak{e}}_n| - |\partial\bar{\mathfrak{e}}_n|$ and $V_n = |\bar{\mathfrak{e}}_n| - 2P_n$ for the number of *inner* edges of $\bar{\mathfrak{e}}_n$. Recall from the explicit transitions of the filled-in peeling process under $\mathbb{P}^{(\infty)}$ that the number of inner edges in $\bar{\mathfrak{e}}_n$ can be written as

$$V_n = n + \sum_{i=0}^{n-1} \mathcal{X}^{(i)}_{P_i - P_{i+1} - 1},$$

where $\mathcal{X}_j^{(i)}$ for $i \geq 0$ and $j \in \mathbb{Z}$ are independent random variables such that $\mathcal{X}_j^{(i)}$ has the same distribution as the total number of edges inside a $\mathbf{q}$-Boltzmann random map with perimeter $2j$

if $j \geq 0$ and is 0 otherwise. To simplify further we use the notation $\widetilde{\Delta} P_i = P_i - P_{i+1} - 1$ if $\Delta P_i \leq -1$ and 0 otherwise. Fix $\varepsilon > 0$ and set for $k \in \{1, 2, \dots, n\}$

$$V_k^{>\varepsilon} = \sum_{i=0}^{k-1} \mathcal{X}^{(i)}_{\widetilde{\Delta} P_i} \mathbf{1}_{\widetilde{\Delta} P_i > \varepsilon n^{1/(a-1)}}, \quad V_k^{\leq \varepsilon} = \sum_{i=0}^{k-1} \mathcal{X}^{(i)}_{\widetilde{\Delta} P_i} \mathbf{1}_{0 \leq \widetilde{\Delta} P_i \leq \varepsilon n^{1/(a-1)}}. \tag{10.11}$$

We first observe that $n^{-(a-1/2)/(a-1)} \mathbb{E}[V_n^{\leq \varepsilon}]$ is small uniformly in n when ε is small. Indeed, using the bound $\mathbb{E}[\mathcal{X}_\ell] \leq C \ell^{a-1/2}$ provided in Proposition 10.4 we can write

$$\begin{aligned} \mathbb{E}[V_n^{\leq \varepsilon}] \leq C \sum_{i=0}^{n-1} \mathbb{E}\left[|\widetilde{\Delta} P_i|^{a-1/2} \mathbf{1}_{\widetilde{\Delta} P_i \leq \varepsilon n^{1/(a-1)}} \right] &= C n \sum_{k=1}^{\varepsilon n^{1/(a-1)}+1} (k-1)^{a-1/2} \nu(-k) \\ &\leq C' \sqrt{\varepsilon} n^{(a-1/2)/(a-1)}. \end{aligned} \tag{10.12}$$

On the other hand, by Propositions 10.1 and 10.4 together with the fact that $(\Upsilon_a(t))_{t \geq 0}$ does not have jumps of size exactly $-\varepsilon$ almost surely, we deduce that jointly with the convergence of the rescaled perimeter towards Υ_a we have the following convergence in distribution for the Skorokhod topology

$$\left(n^{-\frac{a-1/2}{a-1}} \cdot V^{>\varepsilon}_{[nt]} \right)_{t \geq 0} \xrightarrow[n \to \infty]{(d)} \left(\mathfrak{b}_{\mathfrak{q}} \cdot \mathcal{V}_a^{>\varepsilon}(\Upsilon_a)(\mathfrak{p}_{\mathfrak{q}} t) \right)_{t \geq 0}, \tag{10.13}$$

where the process $\mathcal{V}_a^{>\varepsilon}(\chi)$ is defined as $\mathcal{V}_a(\chi)$ but only keeping the negative jumps of χ of absolute size larger than ε. Then, it is easy to verify that, for every $\delta > 0$ and any $t_0 > 0$ fixed we have

$$\mathbb{P}\Big(\sup_{0 \leq t \leq t_0} |\mathcal{V}_a(\Upsilon_a)(t) - \mathcal{V}_a^{>\varepsilon}(\Upsilon_a)(t)| > \delta \Big) \xrightarrow[\varepsilon \to 0]{} 0.$$

Letting $\varepsilon \to 0$ we can use the previous display, together with (10.13) and (10.12) to deduce the desired convergence in distribution where we replace $|\overline{\mathfrak{e}}_n|$ by its number of *inner* edges, but since the perimeter is of a smaller scale, this does not affect the scaling limit. □

10.2.3 Law of "iterated" logarithm

Theorem 10.8 gives a precise scaling limit in distribution for the perimeter and volume process under filled-in explorations of Boltzmann maps of the plane. We give below a rougher estimate up to logarithmic fluctuations but which holds almost surely over all times.

Lemma 10.9. *Let $\mathbf{q}$ be of type $a \in (3/2; 5/2]$. For any filled-in exploration $(\overline{\mathfrak{e}}_n : n \geq 0)$ under $\mathbb{P}_\infty$, for every $\varepsilon > 0$ almost surely we have eventually*

$$\left(\frac{1}{\log n} \right)^{\frac{1}{a-3/2}+\varepsilon} \leq \frac{|\partial^* \overline{\mathfrak{e}}_n|}{n^{\frac{1}{a-1}}} \leq (\log n)^{\frac{1}{a-3/2}+\varepsilon} \quad \text{and} \quad \left(\frac{1}{\log n} \right)^{\frac{a-1/2}{a-3/2}+\varepsilon} \leq \frac{|\overline{\mathfrak{e}}_n|}{n^{\frac{a-1/2}{a-1}}} \leq (\log n)^{\frac{a-1/2}{a-1}+\varepsilon}.$$

Remark 10.10. A similar result holds in the case of the half-plane topology, but notice that when $a \in (2; 5/2]$, the ν-random walk is recurrent so there is no non trivial deterministic lower bound on $\big| |\partial^* \overline{\mathfrak{e}}_n| - |\partial \overline{\mathfrak{e}}_n| \big|$ valid for all n eventually.

Proof. We first focus on the case when $a \in (3/2; 5/2)$ and start with the half-perimeter process $(|\partial^* \bar{\mathfrak{e}}_n| : n \geq 0)$ which has the law of $(S_n^{\uparrow} : n \geq 0)$ the ν-random walk conditioned to survive. We will rely on Tanaka's construction of the process $S^{\uparrow}$ by concatenating time-space reversal of excursions of S, see Section A.3.4. Recall that under our assumptions, the law ν is in the strict domain of attraction of the $(a-1)$ stable distribution with positivity parameter ρ satisfying (10.1). If $T^{>} = \inf\{k \geq 0 : S_k > 0\}$ is the length of the negative excursion of S then we have

$$\begin{aligned} \mathbb{P}(S_{T^{>}} > x) &\sim c_1 \cdot x^{-(a-3/2)} \\ \mathbb{P}(T^{>} > x) &\sim c_2 \cdot x^{-\frac{a-3/2}{a-1}} \\ \mathbb{P}(\min_{0 \leq i < T^{>}} S_i < -x) &\leq c_3 \cdot x^{-(a-3/2)}, \end{aligned}$$

as $x \to \infty$ for some constants $c_1, c_2, c_3 > 0$. The first two estimates can be found in [69, Remark 1.2, Lemma 2.1] and the last one can be deduced from the second one. Indeed, if the walk S reaches a level below $-x$ before touching $\mathbb{Z}_{>0}$ then using Markov property and Proposition 10.1 we see that with positive probability (independent of x) it stays negative for a time of order x^{a-1}. Hence we have for some $c > 0$

$$\mathbb{P}(T^{>} > x^{a-1}) \geq c \cdot \mathbb{P}(\min_{0 \leq i < T^{>}} S_i < -x),$$

which proves the last estimate, see [55, Proof of Lemma 5.8] for details. We will combine these estimates with the following classical "law of iterated logarithm" for heavy-tailed positive random walk with no mean (see [46, Chapter 3.9]):

Lemma 10.11. *If $Z_n = X_1 + X_2 + \cdots + X_n$ is a random walk whose increments are i.i.d., non-negative, and satisfy $\mathbb{P}(X_i > x) \geq c \cdot x^{-1/\alpha}$, respectively $\mathbb{P}(X_i > x) \leq c \cdot x^{-1/\alpha}$, for $\alpha > 1$ and $c > 0$ then for any $\varepsilon > 0$ we have a.s.*

$$\liminf_{n \to \infty} \frac{Z_n}{n^{\alpha} \log^{-\varepsilon} n} = +\infty \quad \textit{respectively} \quad \limsup_{n \to \infty} \frac{Z_n}{n^{\alpha} \log^{\alpha+\varepsilon} n} = 0.$$

When applied to the above construction, the lemma shows that after concatenating n excursions, then eventually the total length is comprised between $n^{\frac{a-1}{a-3/2}} \log^{-\varepsilon} n$ and $n^{\frac{a-1}{a-3/2}} \log^{\frac{a-1}{a-3/2}+\varepsilon} n$, the current height is in between $n^{\frac{1}{a-3/2}} \log^{-\varepsilon}$ and $n^{\frac{1}{a-3/2}} \log^{\frac{1}{a-3/2}+\varepsilon}$, and the height of the largest excursion is no more than $n^{\frac{1}{a-3/2}} \log^{\frac{1}{a-3/2}+\varepsilon}$. The first result easily follows from these considerations, see Figure 10.2.

Let us now turn our attention to the volume process. It suffices to deal with the number of *inner* edges V_n since the perimeter is of smaller order. Recall that conditionally on the perimeter process, the volume process is obtained by summing the volume of independent Boltzmann maps each time the perimeter produces a negative jump. We first bound the tail of the increments of the number of inner edges V_n in a half-plane exploration. For $x > 0$, we have with an obvious

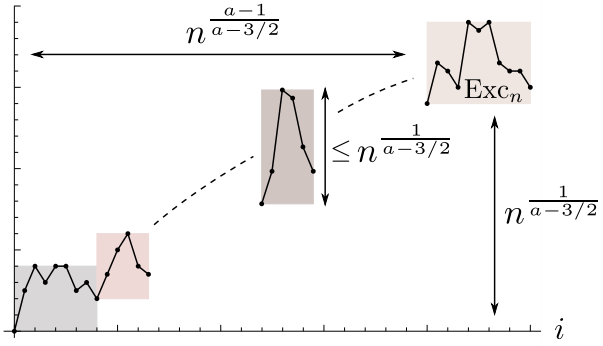


Figure 10.2: Illustration of the proof of Proposition 10.9.

notation

$$\begin{aligned}
\mathbb{P}^{(\infty)}(\Delta V_n > x) &= \sum_{\ell=2}^{\infty} \mathbb{P}^{(\infty)}(\Delta V_n > x \text{ and } \Delta P_n = -\ell) \\
&= \sum_{\ell=2}^{\infty} \mathbb{P}(|\mathfrak{M}^{(\ell-1)}| > x)\, \mathbb{P}^{(\infty)}(\Delta P_n = -\ell) \\
&\underset{\text{Markov}}{\le} \sum_{\ell=2}^{\infty} \left((x^{-1}\mathbb{E}[|\mathfrak{M}^{(\ell-1)}|]) \wedge 1 \right) \nu(-\ell) \\
&\underset{\text{Prop. 5.10 and Prop.10.4}}{\le} c \sum_{\ell=2}^{\infty} \left(\frac{\ell^{a-\frac{1}{2}}}{x} \wedge 1 \right) \ell^{-a} \\
&\le c\, x^{-\frac{a-1}{a-1/2}},
\end{aligned}$$

for some constant $c > 0$ that may vary from line to line. Since ΔV_n under $\mathbb{P}^{(\infty)}$ stochastically bounds from above ΔV_n under $\mathbb{P}_\infty^{(\ell)}$, we can stochastically bound above $(V_n)_{n\ge 0}$ under $\mathbb{P}_\infty$ by a process $(\tilde{V}_n)_{n\ge 0}$ with i.i.d. positive increments with a tail of order $\mathbb{P}(\Delta\tilde{V}_n > x) \sim cx^{-\frac{a-1}{a-1/2}}$. By the above lemma, we deduce that $V_n \le (n \log^{1+\varepsilon} n)^{\frac{a-1/2}{a-1}}$ eventually a.s. For the lower bound we use the fact that V_n dominates its largest jump until time n: On the event when $|\partial^* \overline{\mathfrak{e}}_n| \ge n^{1/(a-1)} \log^{-\frac{1}{a-3/2}-\varepsilon} n$ notice that conditionally on $\mathcal{F}_n$ the probability that

$$\Delta|\partial^* \overline{\mathfrak{e}}_n| \le n^{1/(a-1)} \log^{-\frac{1}{a-3/2}-2\varepsilon} n \text{ and that such a negative jump induces a increase of volume of order } (n^{1/(a-1)} \log^{-\frac{1}{a-3/2}-2\varepsilon} n)^{a-1/2}$$

is bounded from below by $\frac{\log^\delta n}{n}$ for some $\delta > 0$ and all n large enough. Taking $n = 2^k$ and working conditionally on $\{|\partial^* \overline{\mathfrak{e}}_n| \ge n^{1/(a-1)} \log^{-\frac{1}{a-3/2}-\varepsilon} n; \forall n \ge n_0\}$ we deduce by Borel–Cantelli and the first point of the proposition that eventually, the largest jump of (V) before time n is of order at least $(n^{1/(a-1)} \log^{-\frac{1}{a-3/2}-2\varepsilon} n)^{a-1/2}$. This implies our desired lower bound. □

10.3 Scaling limits in the hyperbolic regime

We now study the scaling limit of the perimeter and volume processes during filled-in exploration of the hyperbolic random maps $\mathfrak{H}_\infty$ or $\mathfrak{H}^{(\infty)}$. We shall restrict to the situation when $\tilde{\nu}$ has a first moment so that the law of large number applies. In this case, we have the easy result:

Proposition 10.12 (Scaling limits for the peeling process on hyperbolic random maps)**.** *Suppose that* $\mathfrak{q}$ *is subcritical, satisfies Cramér's condition, and that* $\tilde{\nu}$ *(defined in* (8.4)*) has a finite first moment. If* $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ *is a filled-in peeling exploration under* $\mathbb{H}_\infty$ *or* $\mathbb{H}^{(\infty)}$ *then we have*

$$\left(\frac{|\partial^*\bar{\mathfrak{e}}_{[nt]}| - |\partial\bar{\mathfrak{e}}_{[nt]}|}{n}, \frac{|\bar{\mathfrak{e}}_{[nt]}|}{n}\right)_{t\geq 0} \xrightarrow[n\to\infty]{a.s.} \left(t\cdot\left(\mathrm{p}_{\mathfrak{q}}^{\mathrm{hyp}}, \mathrm{b}_{\mathfrak{q}}^{\mathrm{hyp}}\right)\right)_{t\geq 0},$$

where $\mathrm{p}_{\mathfrak{q}}^{\mathrm{hyp}} := \int \mathrm{d}\tilde{\nu}(x)x$ *and* $\mathrm{b}_{\mathfrak{q}}^{\mathrm{hyp}} := \dfrac{(\sqrt{\omega}-\sqrt{\omega-1})^2}{2\sqrt{(\omega-1)\omega}} + 2\mathrm{p}_{\mathfrak{q}}^{\mathrm{hyp}}$.

Proof. The proposition is very easy to prove in the case of $\mathbb{H}^{(\infty)}$ since both the perimeter and the number of inner edges are (correlated) random walks with i.i.d. increments of law respectively $\tilde{\nu}$ and

$$\sum_{k\geq 1}\tilde{\nu}(-k)\cdot\mathbb{P}^{(k-1)}(|\mathfrak{m}|\in\cdot).$$

Under our hypotheses, those increments have finite mean equal to $\mathrm{p}_{\mathfrak{q}}^{\mathrm{hyp}} := \int \mathrm{d}\tilde{\nu}(x)x$ and

$$\begin{aligned}\sum_{k\geq 1}\tilde{\nu}(-k)\mathbb{E}^{(k-1)}[|\mathfrak{m}|] &\underset{(10.8)}{=} \sum_{k\geq 1}\tilde{\nu}(k)\frac{W_1^{(k-1)}}{W^{(k-1)}}\\ &\underset{\text{def. }\tilde{\nu}\ (8.4)}{=} 2\sum_{k\geq 1}(c_{\mathfrak{q}}\omega)^{-k}W_1^{(k-1)}\\ &\underset{(3.9)}{=} \sum_{k\geq 1}\omega^{-k}h_1^{\downarrow}(k-1) \quad = \quad \frac{(\sqrt{\omega}-\sqrt{\omega-1})^2}{2\sqrt{(\omega-1)\omega}}.\end{aligned}$$

The law of large numbers implies the proposition in the case of filled-in explorations of $\mathfrak{H}^{(\infty)}$ (recall that $|\bar{\mathfrak{e}}_n|$ is equal to the number of inner edges plus twice the half-perimeter). In the case of $\mathfrak{H}_\infty$ the perimeter and the volume processes are not quite random walks with independent increments (because of the conditioning that the perimeter must stay positive). However, since $\tilde{\nu}$ has a positive mean, conditioning $\tilde{S}$ to stay positive is an event of positive probability and thus the previous almost sure convergences also hold in this case. □

10.4 Markovian explorations are always roundish

We finish this chapter by stating a "universal geometric" result about filled-in explorations of $\mathfrak{M}_\infty$ for regular weight sequences: whatever the peeling algorithm chosen to perform a filled-in peeling of $\mathfrak{M}_\infty$, the exploration always reveals about the same portion of the map by time n. This is false for half-plane models since we can peel on far away points on the boundary and it is easy to check that this is also false for hyperbolic maps.

Theorem 10.13 (*Markovian explorations of* $\mathbf{M}_\infty$ *are roundish*)

Fix a critical weight sequence $\mathbf{q}$ *of type* $a \in (\frac{3}{2}, \frac{5}{2}]$. *For any* $\varepsilon > 0$, *there exists an integer* $0 < C_\varepsilon < \infty$ *such that for any peeling algorithms* $\mathcal{A}, \mathcal{B}$, *if* $(\bar{\mathfrak{e}}_n^{\mathcal{A}} : n \geq 0)$ *and* $(\bar{\mathfrak{e}}_n^{\mathcal{B}} : n \geq 0)$ *respectively are the filled-in peeling explorations using algorithms* $\mathcal{A}$ *and* $\mathcal{B}$ *of the same planar map* $\mathfrak{m}$, *then for all* n *large enough*

$$\mathbb{P}_\infty\left(\bar{\mathfrak{e}}_n^{\mathcal{A}} \subset \bar{\mathfrak{e}}_{C_\varepsilon n}^{\mathcal{B}}\right) \geq 1 - \varepsilon.$$

In particular, any filled-in exploration $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ *of* $\mathfrak{M}_\infty$ *will discover the full map i.e*

$$\bigcup_{n\geq 0} \bar{\mathfrak{e}}_n = \mathfrak{M}_\infty, \qquad \textit{almost surely.}$$

We do not give the full proof of this theorem since it relies on a couple of distance/volume estimates which are proved using "Schaeffer-type" construction of random maps (and thus exit the scope of these lecture notes), see [94]. We however prove the easier fact that filled-in explorations discover the whole map, and this already gives a glimpse of the mean idea:

Proof of the second point of the theorem. Fix a peeling algorithm $\mathcal{A}$. One needs to prove that every "vertex" on the boundary of $\bar{\mathfrak{e}}_n$ is eventually swallowed by the peeling process and becomes an interior vertex of $\bar{\mathfrak{e}}_n$. We note that conditionally on the past exploration, if ℓ is the length of the distinguished hole, then for any given vertex v on the boundary of the distinguished hole of $\bar{\mathfrak{e}}_n$ and any edge to peel on that hole there is probability of order

$$\sum_{k=\lceil \ell/2 \rceil}^{\ell-1} b_\infty^{(\ell)}(*,k) \quad \text{or} \quad \sum_{k=\lceil \ell/2 \rceil}^{\ell-1} b_\infty^{(\ell)}(k,*),$$

that the peeling of the edge $\mathcal{A}(\bar{\mathfrak{e}}_n)$ swallows half of the boundary of the distinguished hole, so that the vertex v becomes an inner vertex of $\bar{\mathfrak{e}}_{n+1}$. Using the exact values of $b_\infty^{(\ell)}$ (see Section 7.1.2) as well as the asymptotics for the tail of ν (see Chapter 5) and the expression of $h^\uparrow$ (see (5.4)), it follows that the previous display is of order ℓ^{1-a}. Since the half-perimeter of the only hole of $\bar{\mathfrak{e}}_n$ has the same law as $S^\uparrow$, by a Borel–Cantelli argument the proof of the corollary boils down to proving that almost surely we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{S_n^\uparrow}\right)^{a-1} = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{n^{1/(a-1)}}{S_n^\uparrow}\right)^{a-1} = \infty.$$

We are in the conditions to apply Jeulin's lemma [128, Proposition 4 c]. Since the proof of this lemma is very short and elementary, let us recall it here. We denote by $\mathcal{X}_n = \left(n^{1/(a-1)}/S_n^\uparrow\right)^{a-1}$ and suppose by contradiction that the random series $\sum \frac{1}{n}\mathcal{X}_n$ is bounded by $M \geq 0$ on an event A of probability at least $\varepsilon > 0$. Using the convergence of $\mathcal{X}_n$ towards the strictly positive random variable $1/(\Upsilon_a^\uparrow(\mathfrak{p}_\mathbf{q})^{a-1}$ implied by Proposition 10.3, we deduce that we can find $\delta > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}(\mathcal{X}_n \geq \delta) \geq 1 - \frac{\varepsilon}{2}.$$

Then taking expectation on the event A we deduce that

$$\mathbb{E}\left[\mathbf{1}_A \sum_{n\geq 1} \frac{1}{n}\mathcal{X}_n\right] = \sum_{n\geq 1} \frac{1}{n}\mathbb{E}[\mathbf{1}_A \mathcal{X}_n] \leq M.$$

But using the penultimate display, we have that $\mathbb{E}[\mathbf{1}_A \mathcal{X}_n] \geq \delta \max(0, \varepsilon - \mathbb{P}(\mathcal{X}_n < \delta))$ is asymptotically larger than $\delta\varepsilon/2$. Plugging back into the previous display we find a contradiction because the series $\sum_{n\geq 1} \frac{1}{n}\varepsilon\delta/2$ is obviously not summable! □

Open Question 10.14. *Is the previous theorem true for any critical weight sequence?*

Bibliographical notes. The first estimates for the filled-in simple peeling process were performed by Angel [13] in the case of the UIPT (up to logarithmic factors, which should be compared with Lemma 10.9). They were susequently derived in the case of the UIPQ using a Schaeffer-type construction in [25]. Scaling limits in distribution (Theorem 10.8) were computed in [91] in the case of the filled-in simple peeling process on critical triangulations and quadrangulations. These were later extended by Budd [53] to the present peeling process on $\mathbf{q}$-Boltzmann maps of type $\frac{5}{2}$, and to the case of weight sequence $\mathbf{q}$ of type $[3/2; 2) \cup (2; 5/2]$ in [55]. The case $a = 2$ was subsequently worked out in [57] but the proof presented here is slightly different. Lemma 10.9 is adapted from [55] and Theorem 10.13 is taken from [94]. Scaling limits for the peeling process on free Boltzmann quadrangulation with a boundary are computed in [121]. Proposition 10.12 is adapted from [89, 170] dealing with the case of hyperbolic triangulations.

Part IV:
Percolation(s)

In this part, we study Bernoulli percolations on edges, faces or vertices of our infinite random planar maps and compute in each case the (almost sure) percolation thresholds as well as a few critical exponents. Our main tool will (of course) be a peeling process tailored to the exploration of each type of percolation. We will first study the case of the half-plane $\mathfrak{M}^{(\infty)}$ where the computations are the nicest and then move on to infinite planar, finite or hyperbolic Boltzmann maps. The case of site percolation will require the use of the simple peeling process. Many questions are left open.

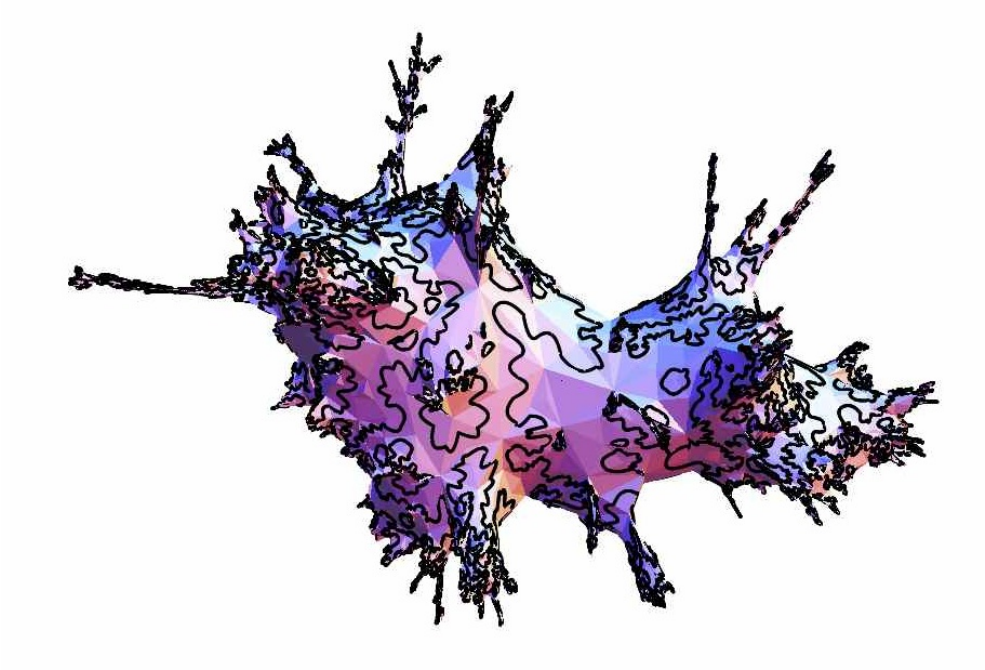


Figure 10.3: A random triangulation (alas not bipartite!) of the sphere together with the interfaces induced by a site percolation on it.

Chapter XI : Percolation thresholds in the half-plane

In this chapter, we deal with random maps $\mathfrak{M}^{(\infty)}$ of the half-plane with law $\mathbb{P}^{(\infty)}$.

In this chapter, we study percolations (face, dual-face, bond and site percolations) on random Boltzmann maps $\mathfrak{M}^{(\infty)}$ of the half-plane of law $\mathbb{P}^{(\infty)}$ for some admissible weight sequence $\mathbf{q}$. Contrary to the cases we considered so far, the peeling process will be driven by the percolation of the map which is itself random (until now, we have considered only deterministic peeling algorithms). We will first see that such explorations are indeed allowed. We then introduce the mean *gulp* and *exposure* which are very important geometric quantities from which the percolation thresholds are computed via universal formulas. The face and bond percolations can be treated using well-designed peeling algorithms whereas the site percolation will require us to use the *simple* peeling exploration.

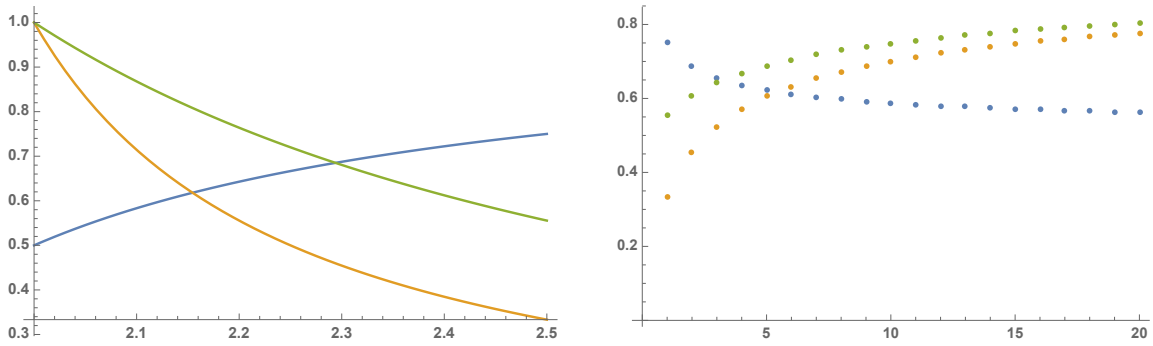


Figure 11.1: Site (green), Bond (orange) and Face (blue) percolation thresholds for the canonical discrete stable maps of parameter $a \in (2, 5/2]$ (on the left) and for critical $2p$-angulations (on the right), see Section 3.5.

Recall from Proposition 6.7 that if $\sum_{k\geq 0} k\,\nu(-k) = \infty$ then $\mathfrak{M}^{(\infty)}$ almost surely contains infinitely many cut points/edges and so the percolation thresholds for site/bond/face percolation are trivially 1. We shall then implicitly exclude this case in this chapter and focus on the dilute phase $\sum_{k\geq 0} k\,\nu(-k) < \infty$. By Proposition 5.7, this in particular implies that $\mathbf{q}$ is critical and if $\mathbf{q}$ is of type a then $a \in (2; 5/2]$.

11.1 Prerequisites

11.1.1 Randomized peeling process

In the previous chapters, we have studied the peeling process for deterministic algorithms $\mathcal{A}$ which picked the next edge to peel as a deterministic function of the explored part (filled-in or not). In the coming chapters, we will need to considered **randomized algorithms**: our maps will be "**decorated**" by a random process (e.g. a percolation that is a coloring of its faces/vertices/edges, a labeling of its edges, etc) and the next edge to peel may depend on the explored map $\mathfrak{e}$ together with its decoration. The crux will be to check that the spatial Markov property (see e.g. Proposition 6.3) is preserved for the "decorated" maps so that in the half-plane case at least the peeling steps are still i.i.d.

11.1.2 Mean gulp and exposure

Let $\mathbf{q}$ be an admissible weight sequence. Recall Definitions 5.1 and 6.4 as well as Remark 6.4 describing the law of the changes of the half-perimeter during a peeling step of $\mathfrak{M}^{(\infty)}$.

Definition 11.1 (Gulps and exposure). *The **gulps** and **exposure** of a peeling step of $\mathfrak{M}^{(\infty)}$ (with weight sequence $\mathbf{q}$) are the random variables $(\mathcal{G}_l, \mathcal{E}, \mathcal{G}_r)$ respectively denoting the number of edges swallowed on the left of the peeled edge, exposed by the possible new face, and swallowed on the right of the peeled edge. I.e. we have $\mathcal{G}_l = (2k+1)$ on events of type $\mathsf{C}_{k,\infty}$ and $\mathcal{E} = (2k-1)$ on events of type C_k and symmetrically for $\mathcal{G}_r$. The **mean gulp and exposure** are*

$$\mathrm{e}_{\mathbf{q}} = \mathbb{E}[\mathcal{E}] = \sum_{k\geq 1} b^{(\infty)}(k)(2k-1) = \sum_{k\geq 0} \nu(k)(2k+1),$$

$$\mathrm{g}_{\mathbf{q}} = \mathbb{E}[\mathcal{G}_r] = \mathbb{E}[\mathcal{G}_l] = \sum_{k\geq 0} b^{(\infty)}(\infty, k)(2k+1) = \frac{1}{2}\sum_{k\geq 1} \nu(-k)(2k-1).$$

Remark 11.2. The mean exposure and gulp may be infinite: when $\mathbf{q}$ is of type a with $a \in (3/2; 2]$, both $\mathrm{e}_{\mathbf{q}}$ and $\mathrm{g}_{\mathbf{q}}$ are infinite. When $\mathbf{q}$ is of type $a = \frac{3}{2}$, the mean gulp is always infinite whereas the mean exposure may be finite. By Theorem 5.4 (and the remark following it), the walk (S) cannot drift towards ∞ hence we cannot have $\mathrm{e}_{\mathbf{q}} = \infty$ and $\mathrm{g}_{\mathbf{q}} < \infty$. Finally, when ν has a first moment it must be centered (Remark 5.8), hence when $\mathrm{e}_{\mathbf{q}}, \mathrm{g}_{\mathbf{q}} < \infty$ we must have:

$$2\mathrm{g}_{\mathbf{q}} + 1 = \mathrm{e}_{\mathbf{q}}. \tag{11.1}$$

11.2 Face percolation

If $\mathfrak{m}$ is a planar map, a **face percolation** of $\mathfrak{m}$ is a coloring of its faces in black and white. This can be modeled by a function $\eta : \mathsf{Faces}(\mathfrak{m}) \to \{0, 1\}$ where 0 represents white faces and 1 represents black faces. We focus here on **Bernoulli percolation** where conditionally on $\mathfrak{m}$, the colors $(\eta(\mathrm{f}) : \mathrm{f} \in \mathsf{Faces}(\mathfrak{m}))$ are i.i.d. black (i.e. 1) with probability p and white (i.e. 0) otherwise.

Two faces are **adjacent in face percolation** if they share an edge in common and if they have the same color, the **clusters** are the connected components of adjacent black faces in the face percolation. As usual in percolation theory we are interested in knowing whether or not there exists an infinite cluster.

In the case when $\mathfrak{m}$ is a map of the half-plane we should change our definition a little bit and introduce a special boundary condition: the external infinite face is colored in white, except for a little region on the right of the root edge, see Figure 11.2. To be rigorous we imagine that we split the initial root edge in order to create a black face of degree 2 on its right (this face thus becomes the root face of the map). Then all the remaining inner faces of the map $\mathfrak{m}$ are



Figure 11.2: The initial boundary condition for face percolation. The unexplored region is the top part of the picture.

independently of each other colored in black with probability $p \in (0,1)$ and in white otherwise. Denote by $\mathrm{P}_{\mathfrak{m}}^{p,\mathrm{face}}$ the probability measure on Bernoulli face percolations of $\mathfrak{m}$ obtained this way.

Theorem 11.3 (*Face percolation thresholds in the half-plane*)

Suppose $\mathbf{q}$ *is an admissible weight sequence and put*

$$p_{c,\mathrm{face}} = \frac{1}{2}\left(1 + \frac{1}{2\mathrm{g}_{\mathbf{q}} + 1}\right)$$

if both $\mathrm{g}_{\mathbf{q}}$ *and* $\mathrm{e}_{\mathbf{q}}$ *are finite, otherwise* $p_{c,\mathrm{face}} = 1$. *Then almost surely for* $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{face}}(\mathrm{d}\eta)$ *there is no infinite black cluster for the* p*-Bernoulli face percolation if* $p \leq p_{c,\mathrm{face}}$ *whereas there is a unique infinite black cluster if* $p > p_{c,\mathrm{face}}$.

To prove the above theorem, we will use a particular (filled-in) peeling exploration that discovers the underlying map along the face percolation interface.

11.2.1 Annealed threshold and exploration of face percolation

Towards Theorem 11.3, we will first compute an "annealed" percolation threshold and then use ergodicity to turn it into a quenched one. More precisely, if we denote by $\mathcal{C}$ the cluster of the little black region on the right of the root edge we prove:

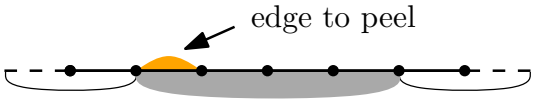
Proposition 11.4 (Annealed threshold). *With the notation of Theorem 11.3, we have for* $p \in (0,1)$

$$\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{face}}(\mathrm{d}\eta)\mathbf{1}_{\{|\mathcal{C}|=\infty\}} = \begin{cases} 0 & \text{if } p \leq p_{c,\mathrm{face}}, \\ > 0 & \text{if } p > p_{c,\mathrm{face}}. \end{cases}$$

Here is the crux, the filled-in peeling algorithm for face percolation on $\mathfrak{M}^{(\infty)}$:

ALGORITHM $\mathcal{A}_{\text{face}}$

Suppose that $\overline{\mathfrak{e}}_n \subset \mathfrak{m}$ is such that along the unique hole of infinite perimeter, the inner faces of $\overline{\mathfrak{e}}_n$ form a connected black component and that all other faces incident to the boundary are white. We speak of white-black-white boundary condition, see figure below. If so, the next edge to peel $\mathcal{A}_{\text{face}}(\overline{\mathfrak{e}}_n)$ is the edge on the boundary which is the left-most on the black component of the boundary. If the black component is empty, the process stops.



submap explored so far

The nice deterministic feature of this algorithm is that after one step it (and after filling-in the possible finite hole created) the explored map $\overline{\mathfrak{e}}_{n+1}$ we obtain is again of the above form and one can thus iterate the exploration, see Figure 11.3 below. Futhermore as long as the black boundary survives, all its incident faces are part of the black cluster of the root face and we have the following deterministic lemma:

Lemma 11.5. *The origin cluster $\mathcal{C}$ is infinite if and only if the exploration of the map with Algorithm $\mathcal{A}_{\text{face}}$ goes on forever.*

Proof. The proof should be clear from Figures 11.3 and 11.4. In words, if the exploration goes on forever then a moment's thought shows that we must discover an infinite number of black faces, and they are all in the cluster of the black root face. Reciprocally, if the process stops, that means that an event of type $\mathsf{G}_{\infty,k}$ has completely swallowed the current "black" boundary thus caging the cluster of the origin into white faces. Since the holes we filled-in during the exploration are all finite, $\mathcal{C}$ must be finite. □

On the probabilistic side, if the underlying Bernoulli face percolated half-plane map $(\mathfrak{m}, \eta)$ is sampled according to $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta)$, then it is a simple matter to check that the decorated version of the spatial Markov property (Proposition 6.3) holds, namely, for any filled-in peeling algorithm $\mathcal{A}$ that may depend on the percolated explored map $(\overline{\mathfrak{e}}, \eta|_{\overline{\mathfrak{e}}})$ then for any $n \geq 0$ the remaining infinite submap after n steps of exploration has law $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta)$.

Proof of Proposition 11.4. With the ingredients at hand, the proof is easy. During the exploration with algorithm $\mathcal{A}_{\text{face}}$, we write B_n for the number of edges of $\partial^* \overline{\mathfrak{e}}_n$ adjacent to a black face. In particular, if $B_n = 0$ then the peeling algorithm stops. By the above lines under $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta)$, the (decorated) peelings steps are i.i.d., and so the process $(B_n)_{n\geq 0}$ is a random walk started from 1, killed and set to 0 when it touches $\{..., -2, -1, 0\}$, with i.i.d. increments of law

$$-1 - \mathcal{G}_r + \epsilon\mathcal{E},$$

where $(\mathcal{E}, \mathcal{G}_r)$ are the exposure and right-gulp of a peeling step (see Definition 11.1) and ϵ is an independent Bernoulli variable so that $\mathbb{P}(\epsilon = 1) = p$. Suppose first that $\mathsf{g}_\mathsf{q}, \mathsf{e}_\mathsf{q} < \infty$, then the

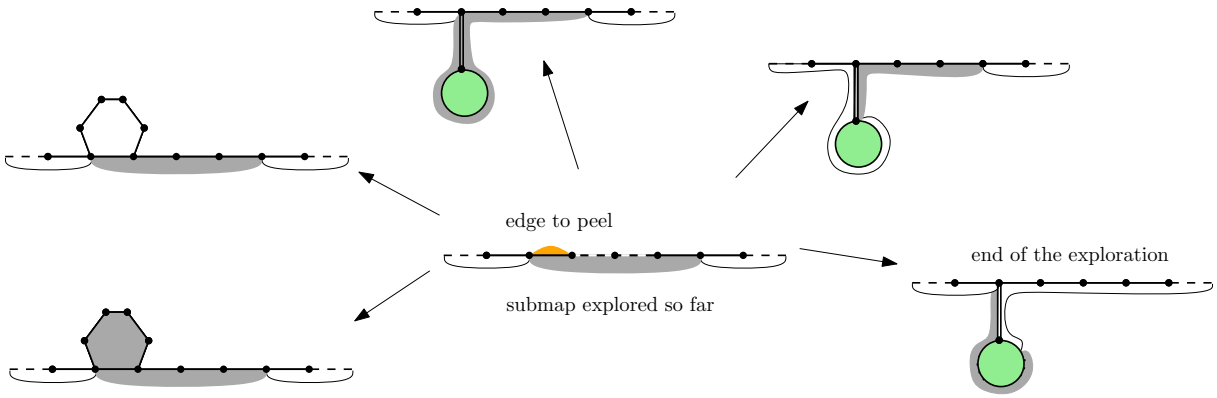
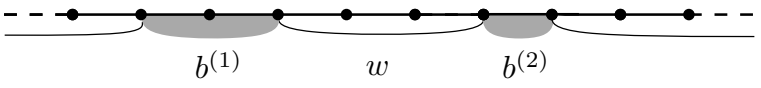


Figure 11.3: Illustration of the filled-in peeling algorithm used in the case of face percolation. The finite parts in green have not been explored (they are filled-in during the process). Notice that the boundary condition white-black-white is preserved and that by induction all the faces adjacent to the black boundary are part of the percolation cluster of the origin black face.

expectation of the above increment is $\delta = p\,\mathrm{e_q} - 1 - \mathrm{g_q}$. Clearly if $\delta > 0$ then by the law of large numbers, the walk drifts towards infinity, and there is a positive probability that B stay positive forever. On the contrary if $\delta \leq 0$ then the walk almost surely visits $\mathbb{Z}_{\leq 0}$ and so B is eventually 0. Combining this with the relation $\mathrm{e_q} = 1 + 2\mathrm{g_q}$ gives our proposition.
If $\mathrm{g_q}$ or $\mathrm{e_q}$ is infinite then by Remark 11.2 we have $\mathrm{g_q} = \infty$. In this case even if $p = 1$, by Proposition 6.7 the random walk with increments $-1 - \mathcal{G}_r + \mathcal{E}$ will reach negative values almost surely and so the cluster $\mathcal{C}$ is almost surely finite. □

Let us use the previous exploration to prove that there cannot be two disjoint infinite black clusters in the map:

Proposition 11.6 (Uniqueness of the infinite cluster)**.** *Suppose that* $\mathbf{q}$ *is admissible and that* $\bar{\mathbf{e}}_0$ *has a white-black-white-black-white boundary condition as depicted below:*



Then almost surely for $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{face}}(\mathrm{d}\eta)$ *the two clusters of the two black boundaries cannot form two disjoint infinite clusters.*

Proof. Starting from the above configuration with $0 < b^{(1)}, w, b^{(2)} < \infty$ we perform the $\mathcal{A}_{\mathrm{face}}$ peeling along the face percolation interface at the middle white-black interface and denote respectively $B_n^{(1)}, W_n$ and $B_n^{(2)}$ the length of the black, white and black segments they form. As long as W_n and $B_n^{(2)}$ stay positive, the sum $W_n + B_n^{(2)}$ evolves as a random walk with step distribution ν. We know from Theorem 5.4 and Remark 5.5 that (S) either drifts towards $-\infty$ (in the subcritical case) or oscillates (in the critical case), in both cases $\liminf_{n\to\infty} S_n = -\infty$. We deduce

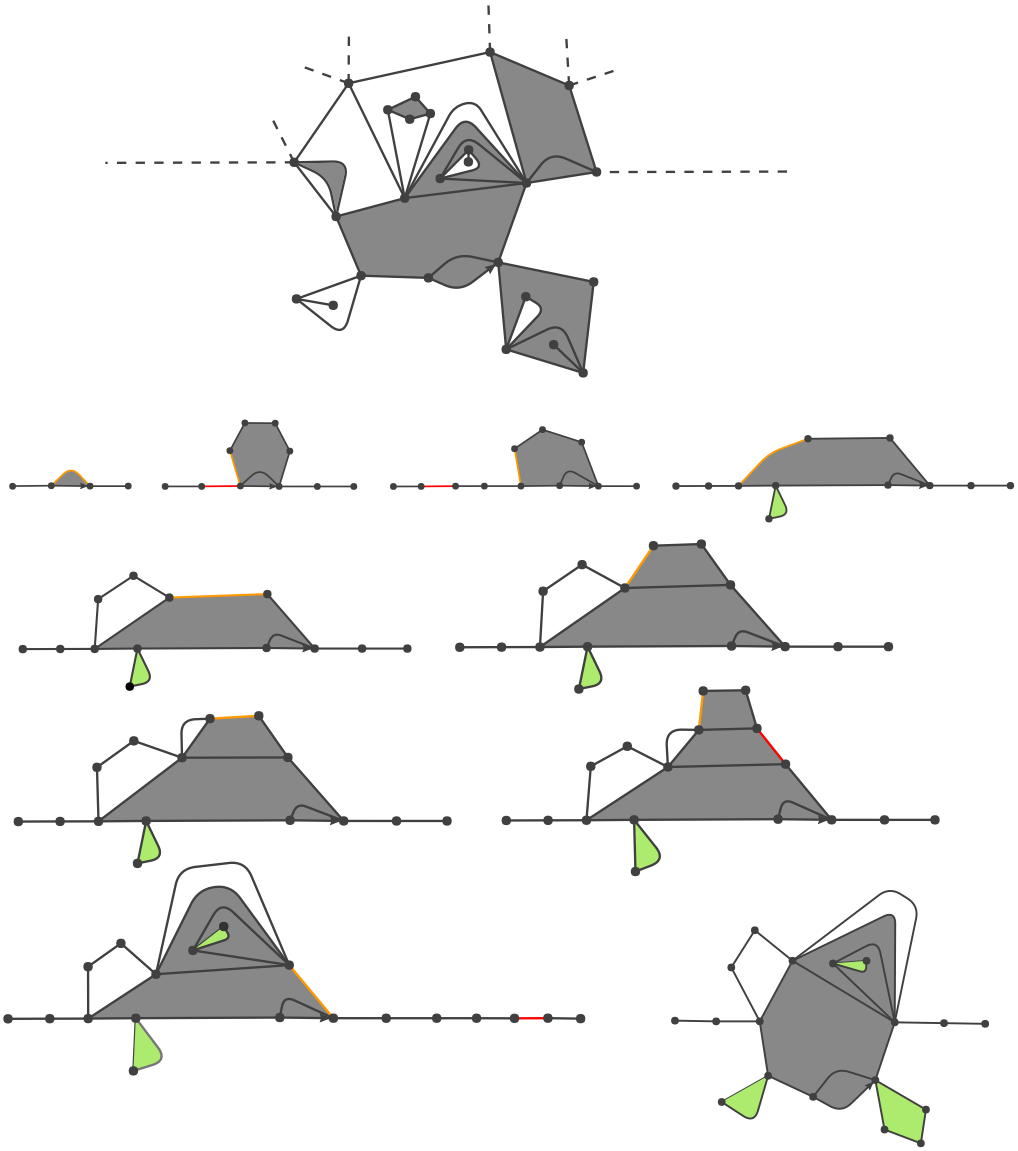


Figure 11.4: The full exploration of the black face percolation cluster of the origin in a percolated map of the half-plane. The edge to peel is in orange and is case of event of type $G_{\cdot,\cdot}$ it is identified with the edge in red on the boundary. The green regions are filled-in with finite Boltzmann maps. The exploration stops after 9 steps since we are sure that $\mathcal{C}$ is finite.

that eventually either W or $B^{(2)}$ dies out and this implies that we cannot create two infinite black clusters from the initial black boundaries:

- if $B^{(2)}$ dies out, then the cluster of second black segment is finite,
- if W dies out, then the clusters of the two black segments merge and form a single cluster

(to be precise, for the white region to disappear and connect the two black clusters, we actually require that the process W reach strictly negative values since two touching black faces may not be adjacent in the percolation).

We remark that the above argument does not rely at all on the value of $p \in [0,1]$. □

11.2.2 Proof of Theorem 11.3

Proof of Theorem 11.3. Theorem 11.3 is an easy corollary of Propositions 11.4 and 11.6 using ergodicity of the underlying maps (Proposition 6.6). Let us proceed: For any map $\mathfrak{m}$ of the half-plane, denote by

$$\begin{aligned}\tilde{p}_{c,\text{face}}(\mathfrak{m}) &= \inf\left\{p \in (0,1) : \mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\exists \text{ an infinite black face percolation cluster}) > 0\right\}\\ &= \inf\left\{p \in (0,1) : \mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\exists \text{ an infinite black face percolation cluster touching } \partial\mathfrak{m}) > 0\right\},\end{aligned}$$

where the second equality follows by changing the colors of a few faces in the percolation. Clearly, $\tilde{p}_{c,\text{face}}(\mathfrak{m})$ is invariant under local modifications near the origin of $\mathfrak{m}$ and so by Proposition 6.6 the random variable $\tilde{p}_{c,\text{face}}(\mathfrak{M}^{(\infty)})$ is almost surely constant and equals $p_{c,\text{face}}$ by Proposition 11.4. By ergodicity we deduce that there is indeed no infinite cluster $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta)$-almost surely for $p \le p_{c,\text{face}}$ and that there is (at least one) infinite cluster $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta)$-almost surely for $p > p_{c,\text{face}}$. To prove uniqueness of the infinite cluster, notice that if there are two disjoint infinite black face percolation clusters, then by modifying locally the colors of the faces we can ensure that they touch the boundary in a way described in Proposition 11.6 and conclude from there. □

Remark 11.7 ($a = 2$ and $p = \frac{1}{2}$). The case when $\mathbf{q}$ is of type $a = 2$ deserves a special attention. Going through the proof of Theorem 11.3, the killed random walk B_n has increments given by

$$\begin{aligned}&\Delta B_n = -2k \quad \text{with proba } \tfrac{1}{2}\nu(-k), \quad k \le -1\\ &\Delta B_n = 2k \quad \text{with proba } p \times \nu(k), \quad k \ge 0\\ &\Delta B_n = -1 \quad \text{otherwise.}\end{aligned}$$

Recalling Proposition 5.10 for $a = 2$ and looking at the above transitions, we see that the tails are unbalanced if $p \neq 1/2$ and balanced if $p = 1/2$. Although the walk B will reach negative values whatever $p \in (0,1)$, there is a subtle "phase transition" at $p = 1/2$ for the tail of the length of the percolation interfaces, see [57, Proposition 7].

11.2.3 Dual exploration

Face percolation is not self-dual: If two white faces have only one common vertex but no common edge, they need not be part of the same white cluster, but two such faces do form a local barrier for connection of black faces. Hence the dual of face percolation is the percolation where two faces are part of the same cluster if they have the same color and if they share a vertex. We call

it face'-percolation in the sequel. We let the reader deduce the analog of Theorem 11.3 in the case of face'-percolation with

$$p_{c,\text{face}'} = 1 - p_{c,\text{face}} = \frac{g_q}{2g_q + 1},$$

if $g_q < \infty$ and $p_{c,\text{face}'} = 1$ otherwise. To do this, we use the same algorithm as for face percolation, except that we peel the edge which is immediately on the left of the left-most edge of the black boundary of $\bar{e}_n$. See Figure 11.5.

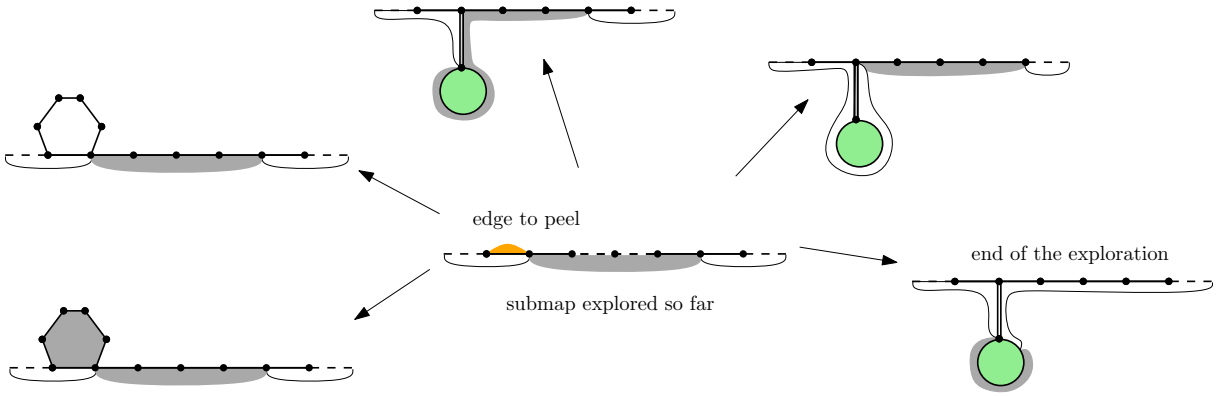


Figure 11.5: Illustration of the peeling algorithm used in the case of face'-percolation.

11.2.4 Degree percolation

The above analysis can also be carried out in the case of a **degree percolation**: in this percolation there is no additional randomness on the map, and two faces are in the same cluster if they share an edge (or a vertex in the case of face' percolation) and their degrees belong to some allowed set of degrees

$$\mathbb{D} \subset \{2, 4, 6, \ldots\}.$$

Using the same algorithm $\mathcal{A}_{\text{face}}$ (or $\mathcal{A}_{\text{face}'}$ in the case of face' percolation), the length of the black boundary now evolves as a random walk with i.i.d. increments of law

$$-1 - \mathcal{G}_r + \mathcal{E}\mathbf{1}_{\mathcal{E}+1\in\mathbb{D}},$$

(resp. $\mathcal{G}_r + \mathcal{E}\mathbf{1}_{\mathcal{E}+1\in\mathbb{D}}$). As in Theorem 11.3, there is an infinite cluster in the percolation if and only if the above increment has positive mean. Studying degree percolation for bond or site percolation seems much more challenging! *Added in proof: Tanguy Lions has proved (2022) that when* $e_q, g_q < \infty$ *the site degree-percolation has an infinite cluster under* $\mathbb{P}^{(\infty)}$ *for degrees smaller than some large constant.*

11.3 Bond percolation

We now move to Bernoulli **bond percolation** where instead of coloring the faces, we color the edges of the map independently (conditionally on the map), black with probability $p \in [0, 1]$ and

white otherwise. The clusters of the percolation are the connected components of black edges. We denote by $\mathrm{P}_{\mathfrak{m}}^{p,\mathrm{bond}}$ the resulting probability measure. We shall use the same strategy as in the previous section to prove:

Theorem 11.8 (*bond percolation threshold on the half-plane*)

Suppose $\mathbf{q}$ *is an admissible weight sequence and put*

$$p_{c,\mathrm{bond}} = 1 - \frac{1}{\mathrm{g}_{\mathbf{q}} + 1}$$

if both $\mathrm{g}_{\mathbf{q}}$ *and* $\mathrm{e}_{\mathbf{q}}$ *are finite, otherwise* $p_{c,\mathrm{bond}} = 1$. *Then almost surely for* $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{bond}}(\mathrm{d}\eta)$ *there is no infinite black cluster for the* p*-Bernoulli bond percolation if* $p \leq p_{c,\mathrm{bond}}$ *whereas there is a unique infinite black cluster if* $p > p_{c,\mathrm{bond}}$.

Remark 11.9. It is remarkable that the bond and face percolation thresholds are deterministic functions of one another. For example, in the case of uniform critical non-bipartite maps, the bond percolation should be self dual and so we expect $p_{c,\mathrm{bond}} = 1/2$ and automatically conjecture $\mathrm{g}_{\mathbf{q}} = 1$ and so $p_{c,\mathrm{face}} = 2/3$. This is indeed proved in [157].

The exploration algorithm in this case is a bit more subtle than for face percolation and may seem strange at first glance. We thus start with a recreational section where we heuristically compute the bond percolation threshold using the face percolation process on an augmented map. This might help the reader digest the next section.

11.3.1 A heuristic before the proof: adding faces of degree 2

Fix a map $\mathfrak{m}$. The idea is to relate a bond percolation on $\mathfrak{m}$ to a face'-percolation on an augmented map with a very small percolation parameter. Indeed, let us suppose that we split all the edges of $\mathfrak{m}$ into a "watermelon" of $N \gg 1$ parallel edges thus creating $N-1$ digons, see Figure 11.6. If we now sample a face'-percolation on $\mathfrak{m}$ with parameter β/N, then as $N \to \infty$, any initial face of $\mathfrak{m}$ is colored white with high probability. We see that in this limit, the clusters of black faces follow bond percolation clusters where an edge is open if its corresponding watermelon contains at least one black digon, see Figure 11.3.1, and this happens with probability

$$1 - \left(1 - \frac{\beta}{N}\right)^{N-1} \to \mathrm{e}^{-\beta}.$$

We can summarize the above discussion into the heuristic statement:

> "Bond percolation on $\mathfrak{m}$ with parameter $p = \mathrm{e}^{-\beta}$ corresponds to the limit $N \to \infty$ of face'-percolation with parameter β/N on the map obtained from $\mathfrak{m}$ by replacing each edge with a watermelon of N parallel edges."

In the context of random Boltzmann maps it is hard to control the operation of transforming each edge into a watermelon of N parallel edges for $N \gg 1$ fixed. However, there is an easy probabilistic description when we transform each edge into independent watermelons carrying a

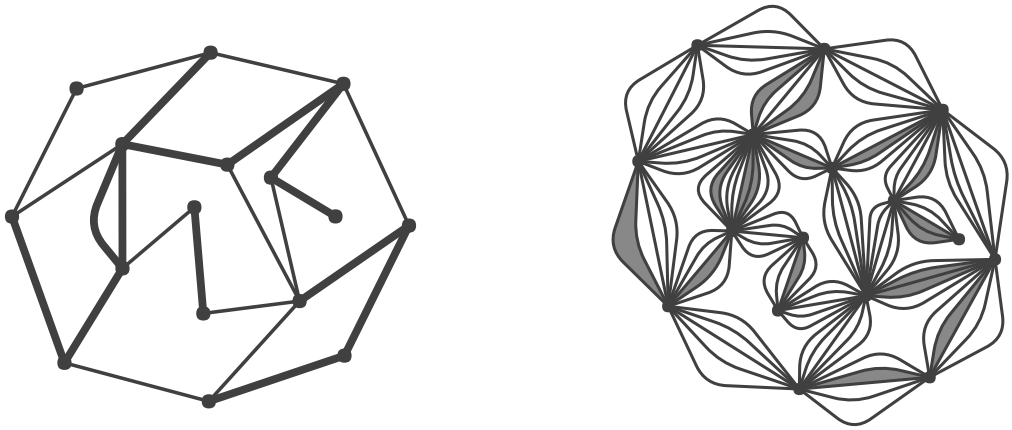


Figure 11.6: Comparing a bond percolation (left) with a face'-percolation on a watermelon-augmented map.

geometric number of edges. More precisely, let $\mathbf{q}$ be an admissible weight sequence and notice that from the admissibility criterion of Theorem 3.12 we must have $q_1 < 1$. We now create a new Boltzmann map model by increasing the number of faces of degree 2. Specifically, for $\alpha \in (0, 1]$ we define the weight sequence $\tilde{\mathbf{q}}$ as

$$\begin{cases} \tilde{q}_k = \alpha^k q_k & \text{for } k \geq 2, \\ \tilde{q}_1 = \alpha \cdot q_1 + (1 - \alpha). \end{cases}$$

It is a straightforward calculation to see that this new weight sequence is again admissible (and critical if the former was critical) and that the corresponding (smallest) solution to $f_{\tilde{\mathbf{q}}}(x) = x$ is $Z_{\tilde{\mathbf{q}}} = Z_{\mathbf{q}}/\alpha$, see Theorem 3.12. Here is the probabilist link between half-planar Boltzmann maps with weight sequences $\mathbf{q}$ and $\tilde{\mathbf{q}}$:

Proposition 11.10 (Adding faces of degree 2). *Let $\mathfrak{M}^{(\infty)}$ be distributed according to $\mathbb{P}^{(\infty)}(\mathbf{q})$. Conditionally on $\mathfrak{M}^{(\infty)}$, split independently each edge of $\mathfrak{M}^{(\infty)}$ into a watermelon of G digons (i.e. $G+1$ parallel edges) where G has law*

$$\mathbb{P}(G = k) = \alpha(1 - \alpha)^k, \qquad \text{for } k \geq 0.$$

Then the resulting map $\widetilde{\mathfrak{M}}^{(\infty)}$ is distributed according to $\mathbb{P}^{(\infty)}(\tilde{\mathbf{q}})$.

Sketch of proof. One first prove the analog statement under the measures $\mathbb{P}^{(\ell)}(\mathbf{q})$ and $\mathbb{P}^{(\ell)}(\tilde{\mathbf{q}})$ and then pass to the limit using Theorem 6.1. For this, one convient way is to decompose the maps through their skeletons: If $\mathfrak{m}$ is a (bipartite rooted) planar map with an external face of degree $2\ell > 2$ we write $< \mathfrak{m} >$ for the **skeleton** of the map, which is obtained by contracting all faces of degree 2 of the map $\mathfrak{m}$. It is easy to check that for any possible skeleton $\mathfrak{s}$ we have

$$\mathbb{P}^{(\ell)}_{\mathbf{q}}(< \mathfrak{m} >= \mathfrak{s}) \quad \propto \quad w_{\mathbf{q}}(\mathfrak{s}) \cdot \left(\frac{1}{1 - q_1}\right)^{|\mathfrak{s}|}.$$

In particular, we read in this formula that we can sample from $\mathbb{P}^{(\ell)}(\mathbf{q})$ by first sampling the skeleton and then independently replacing each edge by $\mathcal{G}(1-q_1)$ parallel edges where $\mathbb{P}(\mathcal{G}(x)=k)=(1-x)^{k-1}x$ for $k\geq 1$. Since a random sum of $\mathcal{G}(x)$ i.i.d. variables of law $\mathcal{G}(y)$ has the same distribution as $\mathcal{G}(xy)$, our result follows because $1-\tilde{q}_1=\alpha(1-q_1)$. □

Using the previous considerations we deduce (heuristically at least) that the p-Bernoulli bond percolation should be equivalent in the limit $\alpha\to 0$ to first splitting all the edges of $\mathfrak{M}^{(\infty)}$ independently into geometric(α) watermelons and then considering face'-percolation on the resulting map with parameter $q(\alpha)\to 0$ so that

$$\sum_{k=0}^{\infty}(1-\alpha)^k\alpha(1-q(\alpha))^k\to 1-p.$$

Since the resulting map has law $\mathbb{P}^{(\infty)}(\tilde{q})$ we can use Section 11.2.3 to compute the face'-percolation threshold under $\mathbb{P}^{(\infty)}(\tilde{q})$ and an easy calculation shows

$$p_{c,\text{face}'}(\tilde{\mathbf{q}}) \quad \xrightarrow[\alpha\to 0]{} \quad 1-\frac{1}{g_{\mathbf{q}}+1} \quad \overset{?}{=} \quad p_{c,\text{bond}}(\mathbf{q}).$$

We will now turn this heuristic into a proof.

11.3.2 The true proof: adding crosses!

According to the previous section, in order to study bond percolation on a map it is useful to see each edge as a geometric number of digons colored black or white. Taking the limit, this is equivalent to seeing each edge as having a "widths' given by independent exponential variables; and that those "random width" carry a Poisson point process of marks. Since the number of points of a Poisson process during an exponential time is simply a geometric random variable; this boils down to considering *maps where edges carry independent geometric number of marks.*

Marked maps and their explorations. More precisely, a **marked** (rooted bipartite finite or infinite) **map** $\mathfrak{m}$, is just a map $\mathfrak{m}$ where every edge e of $\mathfrak{m}$ carries a nonnegative integer number n_e of "marks" (represented by red crosses on our figures). We turn these marks into a coloring of the edges by declaring the edge e black if and only if $n_e>0$. An exploration of $\mathfrak{m}$ is a sequence of marked submaps $\mathfrak{e}_0\subset\cdots\subset\mathfrak{e}_n\subset\cdots\subset\mathfrak{m}$, i.e. submaps carrying marks on their edges (edges on the boundaries of holes may carry marks as well). Here, $\mathfrak{e}_i$ is a submap of $\mathfrak{m}$ means that we can recover $\mathfrak{m}$ by gluing inside the holes of $\mathfrak{e}_i$ the proper maps with a general boundary, those maps carrying themselves marks on the edges, and we simply add-up the marks in the gluing operation, see Figure 11.7.

The main difference with un-marked explorations is that we can now move from $\mathfrak{e}_i$ *to* $\mathfrak{e}_{i+1}$ *by adding crosses on the active boundary of* $\mathfrak{e}_i$ *without discovering any additional edge or face of* $\mathfrak{m}$. We call *boundary condition* of $\mathfrak{e}_i$ the marks on the boundary of its holes. In the case of filled-in exploration, we will speak of **free-black-free** boundary condition if the boundary of the

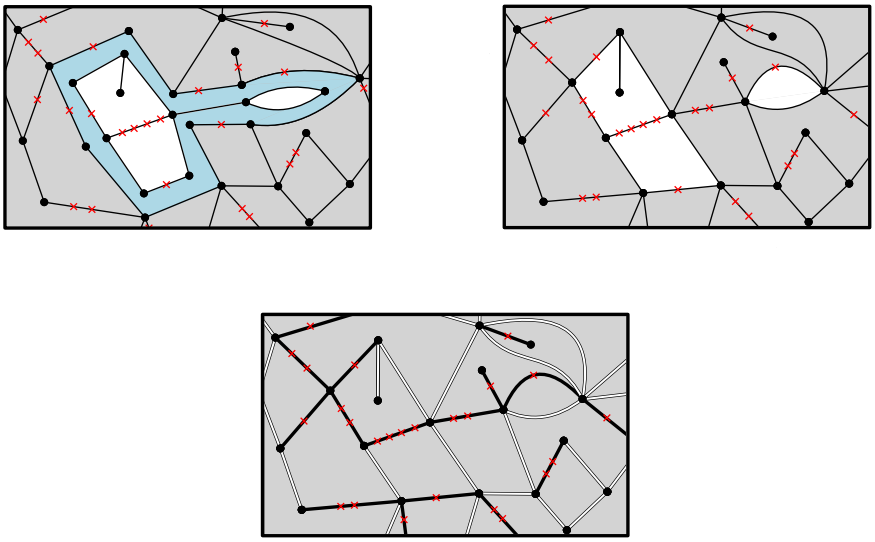


Figure 11.7: Illustration of the gluing operation in the presence of marks. Notice in particular that the marks carried by the half-edges on both sides of the "isthmus-edge" of the inside map add up in the gluing process. The bottom picture shows the interpretation of marks in terms of percolation.

unique hole is made of a single finite connected segment of edges carrying at least one mark (black edges), whereas the other ones carry no mark (free edges).

In what follows, we will consider the map $\mathfrak{M}^{(\infty)}$ as being marked by conditionally independent number of crosses $(n_e : e \in \mathsf{Edges}(\mathfrak{M}^{(\infty)}))$ distributed according to the geometric distribution, i.e. whose law is

$$\mathbb{P}(n_e = 0) = 1 - p, \quad \text{and} \quad \mathbb{P}(n_e = k) = p^k(1-p), \quad \text{for } k \geq 1$$

so that the induced bond percolation model on $\mathfrak{M}^{(\infty)}$ is the p-Bernoulli bond percolation. As before, we denote by $\mathcal{C}$ the black cluster of the origin of the root edge. We can now state the analog of Proposition 11.4 in the context of bond percolation:

Proposition 11.11 (Annealed bond percolation threshold)**.** *With the notation of Theorem 11.8 we have*

$$\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{bond}}(\mathrm{d}\eta)\mathbf{1}_{\{|\mathcal{C}|=\infty\}} = \begin{cases} 0 & \text{if } p \leq p_{c,\mathrm{bond}}, \\ 1 & \text{if } p > p_{c,\mathrm{bond}}. \end{cases}$$

Here is the filled-in marked peeling algorithm of $\mathfrak{M}^{(\infty)}$ used to prove the above proposition:

ALGORITHM $\mathcal{A}_{\text{bond}}$

Suppose that $\bar{\mathfrak{e}}_n \subset \mathfrak{m}$ is such that along the unique hole of infinite perimeter, the boundary condition is "free-black-free". If so, the next edge to peel $\mathcal{A}_{\text{bond}}(\bar{\mathfrak{e}}_n)$ is the edge immediately on the left of the black boundary. More precisely, before triggering a peeling step, we first check whether the edge belonging to the unexplored part carries a mark or not. If it does, we just discover a *single cross (mark)* and do not trigger a peeling step. Otherwise the edge carries no mark and we trigger a standard peeling step. If the black boundary is empty, the exploration stops. See Figure 11.8.

It is easy to check that the free-black-free boundary condition is preserved under filled-in marked explorations with this algorithm, see Figure 11.8. Let us make an important remark: if during the exploration, an event of type $\mathsf{G}_{\infty,k}$ identifies the peeled edge with the right-most black edge on the boundary, the convention is that the endpoint of the peeled edge belonging to the hole with free boundary serves as a *black boundary of length zero* to continue the exploration. Actually, the starting configuration $\bar{\mathfrak{e}}_0$ is of this type: a totally free boundary condition with a black boundary of length 0 being the origin of the root edge. As in the case of face-percolation we have the (almost) deterministic lemma:

Lemma 11.12. *Almost surely, the root cluster $\mathcal{C}$ is infinite if and only if the filled-in marked exploration with Algorithm $\mathcal{A}_{\text{bond}}$ goes on forever.*

Proof. Compared to the case of face percolation, this is only an almost sure statement because we need to exclude degenerate case where e.g. we discover at each step a new face and never identify edges on the active boundary... Otherwise the proof follows from the properties of our exploration (see Figures 11.8 and 11.9) and in particular :

- the (half)-edges on the black boundary are all part of the origin cluster,
- on the event "Stop" of Figure 11.8 the origin cluster is separated from infinity by a white edge and must be finite.

□

The probabilistic key, is to notice that during exploration of marked maps using algorithm $\mathcal{A}_{\text{bond}}$ we keep a certain spatial Markov property. More precisely, under $\mathbb{P}^{(\infty)}$ and if the markings are i.i.d. geometric variables, then for every $i \geq 0$, conditionally on the past exploration up to time i, the map filling in the hole of $\bar{\mathfrak{e}}_i$ is a copy of $\mathfrak{M}^{(\infty)}$ where each edge carries an independent geometric number of marks. This is clear as far as the map structure is concerned by Proposition 6.3 and extends to marked maps by the lack-of-memory property of geometric random variables:

$$\mathbb{P}(\text{Geo} = k+1 \mid \text{Geo} \geq 1) = \mathbb{P}(\text{Geo} = k).$$

As a result, if $(B_n : n \geq 0)$ denotes the length of the black component during the exploration of the map using $\mathcal{A}_{\text{bond}}$, then as long as $B_n \geq 0$ (notice here that B_n can touch 0 without being

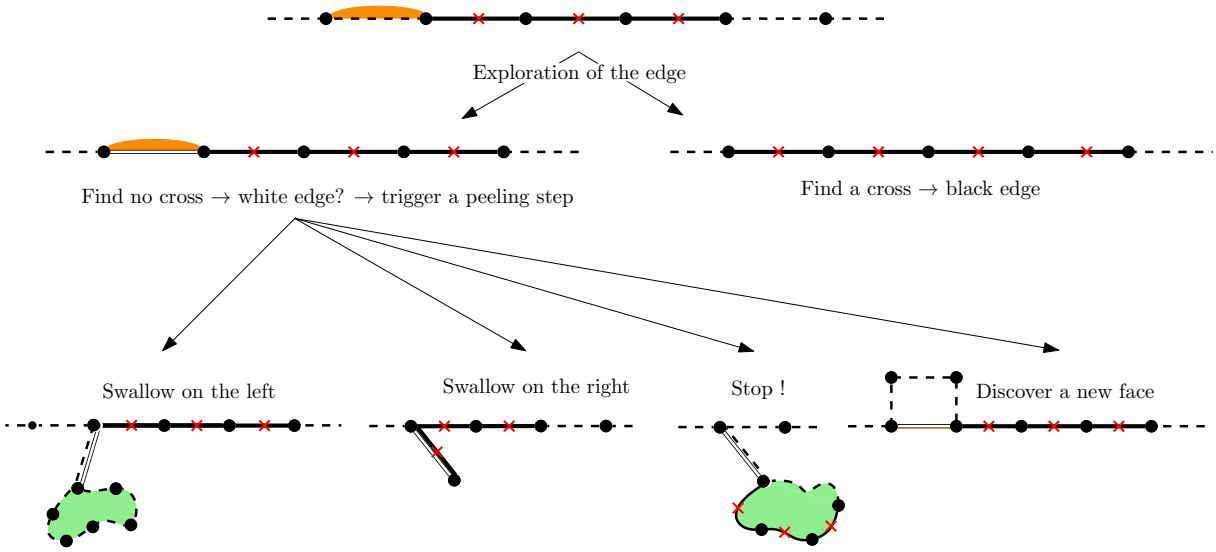


Figure 11.8: Illustration of the filled-in marked peeling process used in the case of bond percolation. We do not always trigger a peeling step but sometimes just explore "the interior" of an edge to discover its it carries a mark or not. In the case we do not discover a mark, we trigger a standard peeling step to reveal a true edge of the map. Notice that in the cases "Swallow on the left" and "Stop", this edge is white. In the case "Swallow on the right", if the peeled edge is paired with a black edge of the boundary (carrying a mark) then it becomes a black edge.

killed) its increments are distributed as

$$\epsilon - (1-\epsilon)\mathcal{G}_r,$$

where ϵ is independent of the gulp $\mathcal{G}_r$ (see Definition 11.1) and $\epsilon = 1$ if we discover a cross, otherwise $\epsilon = 0$, i.e. ϵ is a Bernoulli variable of parameter p.

Proof of Proposition 11.11. Recall Lemma 11.12. The above discussion shows that $(B_n : n \geq 0)$ is a random walk with i.i.d. increments of law $\epsilon - (1-\epsilon)\mathcal{G}_r$ killed upon touching $\mathbb{Z}_{<0}$. When $\mathfrak{g}_\mathfrak{q}$ or $\mathfrak{e}_\mathfrak{q}$ are infinite, by Remark 11.2 we have $\mathfrak{g}_\mathfrak{q} = \infty$, consequently for any $p \in (0,1)$ the above random walk drifts towards $-\infty$ and B is eventually killed. If both $\mathfrak{g}_\mathfrak{q}$ and $\mathfrak{e}_\mathfrak{q}$ are finite then the mean drift of the above walk is $\delta = p - (1-p)\mathfrak{g}_\mathfrak{q}$. We then conclude as in the above proofs that the black boundary is almost surely eventually absorbed if and only if $\delta \leq 0 \iff p \leq (\mathfrak{e}_\mathfrak{q} - 1)/(\mathfrak{e}_\mathfrak{q} + 1) = 1 - \frac{1}{\mathfrak{g}_\mathfrak{q}+1}$. □

Proof of Theorem 11.8. The rest of the proof of Theorem 11.8 is mutatis mutandis the same as for Theorem 11.3. We leave it as an exercise for the reader.

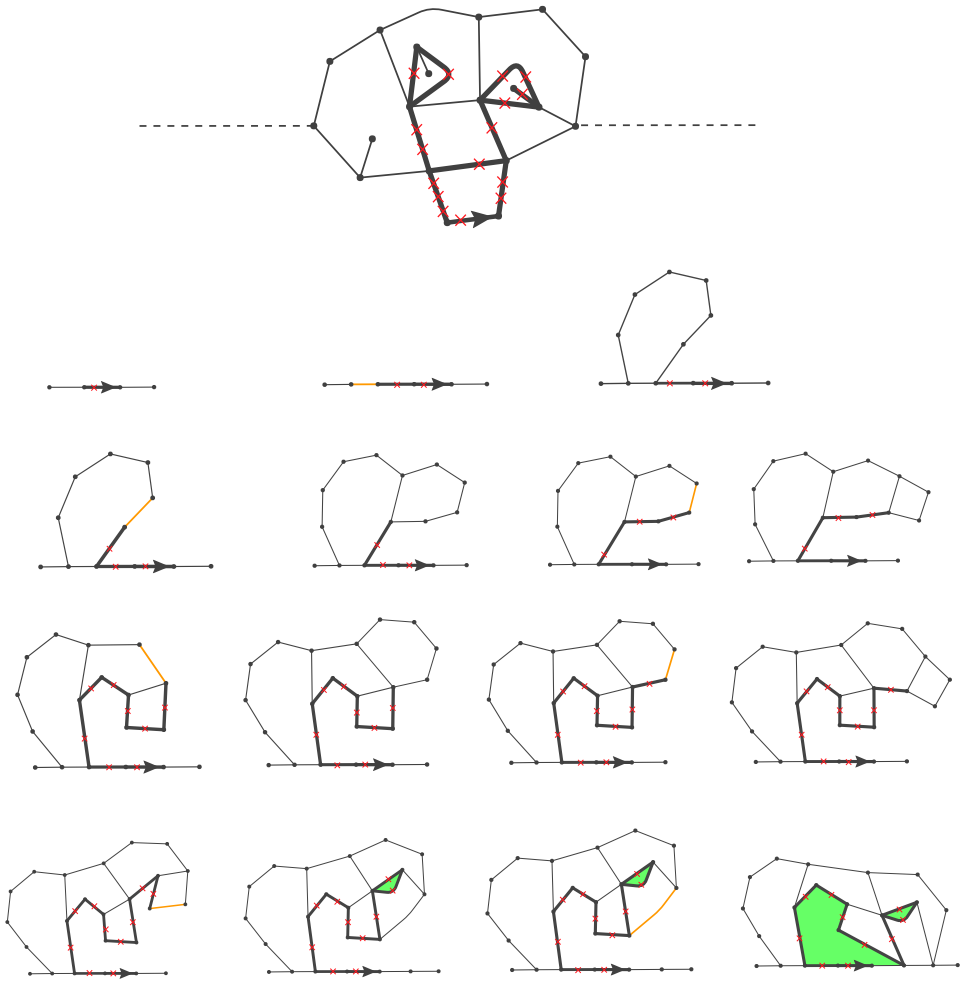


Figure 11.9: A step-by-step exploration of a finite bond percolation cluster using algorithm $\mathcal{A}_{\text{bond}}$. Notice that during a few exploration steps, we only discover a new mark on the boundary, whereas some moves are made by peeling an edge (in orange). The green regions are filled-in with finite marked Boltzmann maps.

11.4 Site percolation and the simple peeling

We now turn our attention to Bernoulli site percolation. Unfortunately, we do not know any tractable algorithm using the peeling process which explores the site percolation interface. It might not be impossible to find one, but the fact that the status of vertices on the boundary is unclear seems to be an obstacle for site percolation. To circumvent this problem, we shall rather use the *simple* peeling process defined in Section 4.3.3. Before using it for site percolation, we

shall first recast our results on face and bond percolation via the simple peeling process. We shall focus on the dilute case and leave the dense case open:

Open Question 11.13. *Study Bernoulli percolation on $\hat{\mathfrak{M}}^{(\infty)}$ for critical weight sequences $\mathbf{q}$ when $g_{\mathbf{q}} = \infty$ (we conjecture that the percolation thresholds are all trivial in this regime).*

In this section we suppose that $g_{\mathbf{q}} < \infty$ (in particular $\mathbf{q}$ is critical).

Thanks to the above hypothesis, we know from Chapter 9 that the $\mathbf{q}$-Boltzmann map $\hat{\mathfrak{M}}^{(\infty)}$ of the half-plane with a simple boundary can be defined as the infinite core with a simple boundary of $\mathfrak{M}^{(\infty)}$. In particular, the almost sure percolation thresholds for face, bond or site percolations are the same for $\mathbb{P}^{(\infty)}$ and for $\hat{\mathbb{P}}^{(\infty)}$ since $\mathfrak{M}^{(\infty)} \backslash \hat{\mathfrak{M}}^{(\infty)}$ is made of countably many finite components separated of the infinite core by a vertex. As for the standard peeling, the notion of simple gulp and exposure will play a crucial role in the determination of the critical percolation thresholds:

Definition 11.14 (Simple gulps and exposure)**.** *The simple gulps and exposure $(\hat{\mathcal{E}}, \hat{\mathcal{G}}_r, \hat{\mathcal{G}}_l)$ in the simple peeling is the law of the outer perimeter and of the part of the inner perimeter on the right and left of the root edge in a one-step simple peeling transition under $\hat{\mathbb{P}}^{(\infty)}$. The mean gulp and exposure are denoted by $\hat{g}_{\mathbf{q}} = \mathbb{E}[\hat{\mathcal{G}}_r] = \mathbb{E}[\hat{\mathcal{G}}_l]$ and $\hat{e}_{\mathbf{q}} = \mathbb{E}[\hat{\mathcal{E}}]$.*

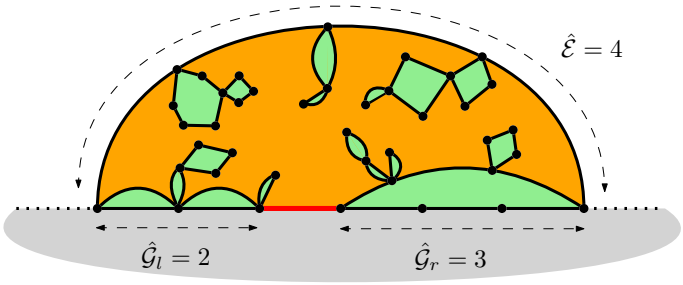


Figure 11.10: Example of a simple peeling step with $\hat{\mathcal{E}} = 4, \hat{\mathcal{G}}_r = 3$ and $\hat{\mathcal{G}}_l = 2$. The light green regions contain arbitrary bipartite maps with a simple boundary of the proper perimeter.

It is clear that in the dilute phase $g_{\mathbf{q}} < \infty$, the mean exposure is finite, since conditionally on the root edge being on the core, the exposure in the first simple peeling step is bounded above by the exposure in the standard peeling step and $e_{\mathbf{q}} < \infty$ by Remark 11.2. But it is not clear *a priori* that the mean gulp is finite. By recasting our previous results on percolation, we will derive the rather surprising fact:

Proposition 11.15. *Suppose $g_{\mathbf{q}} < \infty$. The mean gulp and exposure coincide for the standard and simple peeling process, i.e. $g_{\mathbf{q}} = \hat{g}_{\mathbf{q}}$ and $e_{\mathbf{q}} = \hat{e}_{\mathbf{q}}$. In particular $\hat{g}_{\mathbf{q}}, \hat{e}_{\mathbf{q}}$ are finite and $\hat{e}_{\mathbf{q}} = 2\hat{g}_{\mathbf{q}} + 1$.*

11.4.1 Back to bond and face percolations

In this section we prove Proposition 11.15 by revisiting our results on face and bond percolation using the simple peeling process.

Face percolation. Let us begin by revisiting the exploration of the p-Bernoulli face percolation on $\hat{\mathfrak{M}}^{(\infty)}$. Imagine that under $\hat{\mathbb{P}}^{(\infty)}$ we start with a white-black-white boundary condition. We will define a peeling exploration exactly as in Section 11.2.1: Suppose that $\overline{\hat{\mathbf{e}}_n}$ is such that along the unique hole with infinite perimeter we have a white-black-white boundary condition. Then we peel the left-most black edge using a simple peeling step. We let the reader check that the boundary conditions are preserved during such step and that the length of the black boundary evolves as a random walk killed on $\{\dots, -3, -2, -1, 0\}$ with i.id increments given by

$$-\hat{\mathcal{G}}_r - 1 + \epsilon\hat{\mathcal{E}}, \tag{11.2}$$

where ϵ is independent of $(\hat{\mathcal{G}}_r, \hat{\mathcal{E}})$ and equals 1 with probability p (if the face revealed is black) and 0 otherwise.

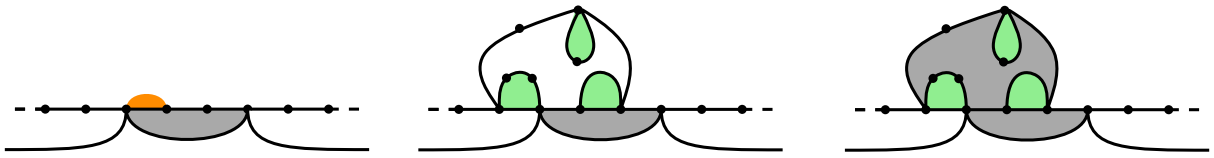


Figure 11.11: Illustration of the simple peeling algorithm to explore face percolation. As usual, the light green regions are filled-in with maps with a simple boundary having the propre perimeter. The previous two figures show the two possible outcomes of a peeling step depending on the color of the face discovered. Notice that the boundary condition black-white-black is preserved.

Bond percolation. Similarly, one can revisit the bond percolation. Imagine now that under $\hat{\mathbb{P}}^{(\infty)}$ we start with a free-black-free boundary condition. Our peeling algorithm first reveals the status of the free edge (i.e. whose color is still unrevealed) adjacent on the left on the black boundary. If this edge turns out to be white, then we trigger a simple peeling step on it. The boundary condition is again preserved and the length $(B_n : n \geq 0)$ of the black boundary evolves as a random walk with i.id increments given by

$$\epsilon - (1-\epsilon)\mathcal{G}_r, \tag{11.3}$$

with the same notation as in (11.2). Contrary to the previous cases, there is no clear stopping time for the exploration process. Indeed, even when B_n becomes strictly negative, i.e. when the cluster we are exploring is completely swallowed by a peeling step, the exploration may or may not continue depending whether or not the origin cluster is linked to the vertex at the junction

free-free, see Figure 11.12. If it is the case then B_{n+1} is set to 0 and we go on, otherwise the exploration stops. Notice that as soon as $p < 1$, when $B_n = 0$ there is a positive probability that the rightmost unknown edge turns out to be white and that the revealed face and its enclosed region block the black cluster that is, the new junction free-free vertex is not part of the black origin cluster.

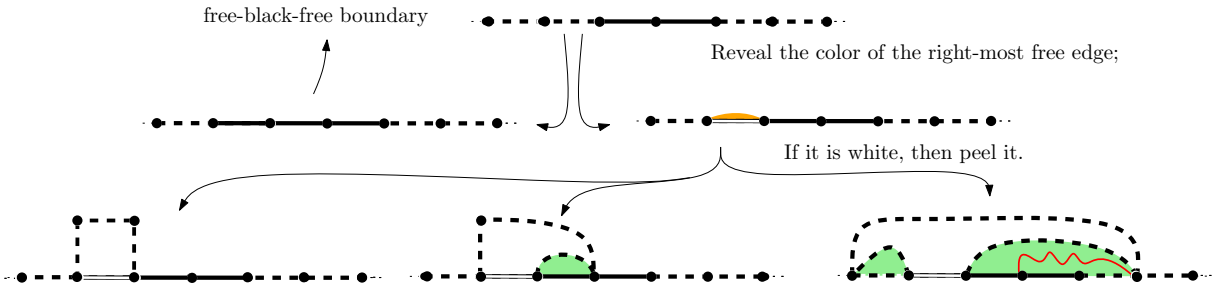


Figure 11.12: Simple peeling exploration of bond percolation. The dashed edges are free (i.e. i.i.d. black with probability p and white with probability $1-p$). When the black boundary is completely swallowed (bottom right case), then the vertex at the free-free junction may still be connected to the origin cluster (red path) and in this case we set $B_{n+1} = 0$ and continue the exploration.

Proof of Proposition 11.15. Recall from the discussion at the beginning of this section that we know from Theorems 11.3 and 11.8 the almost sure value of the face/bond percolation on $\hat{\mathfrak{M}}^{(\infty)}$. Recast in our context, it follows from (11.2) that for $p \le p_{c,\text{face}}$ the random walk with increments of law

$$-\hat{\mathcal{G}}_r - 1 + \epsilon\hat{\mathcal{E}},$$

does not drift towards $+\infty$, and does so if $p > p_{c,\text{face}}$. From the discussion just above the statement of Proposition 11.15 that we have $\mathbb{E}[\hat{\mathcal{E}}] < \infty$. Combining this with the previous remark, we easily deduce that $\hat{g}_{\mathfrak{q}}, \hat{e}_{\mathfrak{q}} < \infty$ and that

$$-\hat{g}_{\mathfrak{q}} - 1 + p_{c,\text{face}} \cdot \hat{e}_{\mathfrak{q}} = 0.$$

The same argument applied with the bond percolation and (11.3) shows that

$$p_{c,\text{bond}} - (1 - p_{c,\text{bond}})\hat{g}_{\mathfrak{q}} = 0.$$

Combining the previous two displays with the exact values of $p_{c,\text{bond}}$ and $p_{c,\text{face}}$ derived in Theorems 11.3 and 11.8 together with the equality $e_{\mathfrak{q}} = 2g_{\mathfrak{q}} + 1$ valid in the dilute phase, we (miraculously) deduce that $\hat{e}_{\mathfrak{q}} = e_{\mathfrak{q}}$ and $\hat{g}_{\mathfrak{q}} = g_{\mathfrak{q}}$ as desired. □

11.4.2 Site percolation

We now move on to site percolation. We color the vertices of the map independently in black with probability $p \in (0,1)$ and white otherwise and denote the law by $\mathrm{P}_{\mathfrak{m}}^{p,\text{site}}$ on a general map

$\mathfrak{m}$. As above, we work on $\hat{\mathfrak{M}}^{(\infty)}$ in the dilute regime $\mathrm{g_q} < \infty$ and denote as usual $\mathcal{C}$ for the (site percolation) cluster of the origin of the root edge.

Theorem 11.16 (*Site percolation threshold on the half-plane*)

Suppose $\mathbf{q}$ *is a critical weight sequence with* $\mathrm{g_q} < \infty$ *and put*

$$p_{c,\mathrm{site}} = 1 - \frac{(\sum_{k=1}^{\infty} \nu(-k))^2}{2\nu(-1)\mathrm{g_q}}$$

if both $\mathrm{g_q}$ *and* $\mathrm{e_q}$ *are finite, otherwise* $p_{c,\mathrm{site}} = 1$. *Then almost surely for* $\int \hat{\mathbb{P}}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{site}}(\mathrm{d}\eta)$ *(also for* $\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{site}}(\mathrm{d}\eta)$*) there is no infinite black cluster for the* p*-Bernoulli site percolation if* $p \leq p_{c,\mathrm{site}}$ *whereas there is a unique infinite black cluster if* $p > p_{c,\mathrm{site}}$.

The reader has already guessed that the proof of this theorem relies on a convenient (simple filled-in) peeling algorithm to explore $\mathcal{C}$. Here it is:

SITE PERCOLATION (PEELING VERTICES)

Suppose $\bar{\hat{\mathbf{e}}}_n$ has a free-black-free boundary condition. Using simple peeling steps we reveal the color of the first free vertex on the left of the black boundary. If it is black then we move on to the next step. If it is white, then we peel this vertex. To do so, we iteratively peel (via simple peeling steps) the edge immediately on the left of this white vertex until we encounter a simple peeling step that swallows it, i.e. such that $\hat{\mathcal{G}}_r > 0$. See figure below.

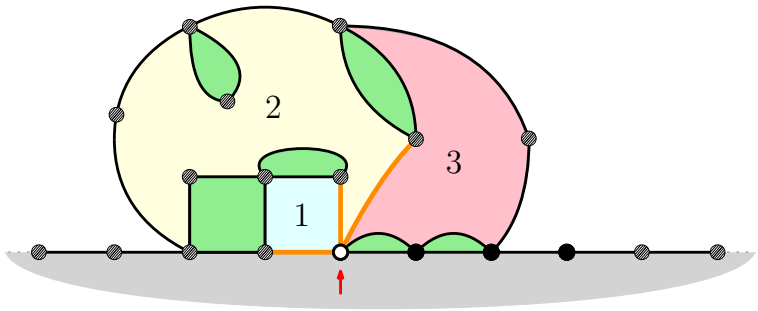


Figure: When a white vertex is discovered, we peel it by revealing the faces (3 in our example) clockwise around it until the vertex is swallowed. The light green parts are filled-in with proper maps with a simple boundary. The striped vertices are free, i.e. independently black with probability p or white with probability $1-p$.

As usual, it is easy to check that the free-black-free boundary condition is preserved along such exploration and that the length of the black boundary (i.e. number of black vertices) evolves as a random walk with i.i.d. increments distributed as

$$\epsilon - (1-\epsilon) \cdot \Big((\hat{\mathcal{G}}_r \mid \hat{\mathcal{G}}_r > 0) - 1 \Big), \tag{11.4}$$

where ϵ is a Bernoulli random variable with success parameter p independent of $(\hat{\mathcal{G}}_r \mid \hat{\mathcal{G}}_r > 0)$ which has the law of the simple gulp conditioned to be strictly positive. As in the case of

bond percolation, there is no stopping time for the exploration in terms of the length of the black boundary: when the black boundary disappears the exploration may or may not continue through the vertex at the junction depending on its color. By our standard reasoning, the threshold $p_{c,\text{site}}$ is determined by

$$p_{c,\text{site}} - (1 - p_{c,\text{site}})\mathbb{E}[\hat{\mathcal{G}}_r - 1 \mid \hat{\mathcal{G}}_r > 0] = 0. \tag{11.5}$$

Proof of Theorem 11.16. Using the previous display, we deduce that the annealed site percolation threshold is

$$p_{c,\text{site}} \underset{(11.5)}{=} 1 - \frac{\mathbb{P}(\hat{\mathcal{G}}_r > 0)}{\mathbb{E}[\hat{\mathcal{G}}_r]} \underset{\text{Prop.11.15}}{=} 1 - \frac{\mathbb{P}(\hat{\mathcal{G}}_r > 0)}{g_{\mathbf{q}}}.$$

The next lemma computes $\mathbb{P}(\hat{\mathcal{G}}_r > 0)$ in terms of the measure ν. The rest of the proof of the theorem is similar to that of Theorem 11.3 and is left as an exercise to the reader.

Lemma 11.17. *We have*

$$\mathbb{P}(\hat{\mathcal{G}}_r > 0) = \frac{(\sum_{k=1}^{\infty} \nu(-k))^2}{2\nu(-1)} \underset{(9.1)}{=} \hat{c}_{\mathbf{q}}^{-1}$$

Proof. Using the same strategy and notation as in the proof of Proposition 9.13, we see that the (standard) one-step peeling in $\mathfrak{M}^{(\infty)}$ yields a simple peeling step in $\hat{\mathfrak{M}}^{(\infty)}$ so that $\mathcal{G}_r = 0$ if the red edge on the right of Figure 9.4 is attached to the vertex linking the component of the root edge to the core. Hence we have

$$\mathbb{P}(\mathcal{G}_r = 0) = \frac{1}{\mathbb{P}(\vec{e} \text{ is on the Core})} \frac{W_c}{\overrightarrow{W}_c} \sum_{k\geq 1} q_k c_{\mathbf{q}}^{k-1} \sum_{\ell=0}^{k-1} W^{(\ell)} c_{\mathbf{q}}^{-\ell}.$$

Expressed in terms of ν this becomes

$$\mathbb{P}(\mathcal{G}_r = 0) = \frac{1}{\nu(-1)} \sum_{k=1}^{\infty} \nu(k-1) \sum_{\ell=0}^{k-1} \nu(-\ell-1).$$

The sum on the right-hand side can be further simplified using that ν is normalized,

$$1 = \left(\sum_{k=-\infty}^{\infty} \nu(k)\right)^2 = \left(\sum_{k=0}^{\infty} \nu(k)\right)^2 + \sum_{\ell=1}^{\infty} \sum_{k=-\infty}^{\infty} \nu(k)\nu(-\ell-k-1) + 2\sum_{k=1}^{\infty} \nu(k-1) \sum_{\ell=0}^{k-1} \nu(-\ell-1),$$

and satisfies Tutte's equation (3.4) rewritten as

$$\nu(-\ell-1) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \nu(k)\nu(-\ell-k-1).$$

Together these identities imply that

$$\mathbb{P}(\mathcal{G}_r = 0) = \frac{1}{\nu(-1)} \left[\frac{1}{2} - \frac{1}{2}\left(1 - \sum_{\ell=1}^{\infty} \nu(-\ell)\right)^2 - \sum_{\ell=1}^{\infty} \nu(-\ell-1)\right] = 1 - \frac{\left(\sum_{\ell=1}^{\infty} \nu(-\ell)\right)^2}{2\nu(-1)}.$$

The last fraction is seen to be equal to $\hat{c}_{\mathbf{q}}^{-1}$ defined in (9.1) after a simple calculation using the definition of ν. □

Remark 11.18 ($p_{c,\text{site}} \geq p_{c,\text{bond}}$). This remark and its proof is due to Tanguy Lions. Suppose that $\mathrm{g}_{\mathbf{q}}, \mathrm{e}_{\mathbf{q}} < \infty$ then we have

$$p_{c,\text{site}} \geq p_{c,\text{bond}}.$$

To see this, work under the law $\hat{\mathbb{P}}^{(\infty)}$ of the simple half-plane and adapt the above algorithm to the exploration of a bond percolation:

APPROXIMATE BOND PERCOLATION

Suppose $\bar{\hat{\mathbf{e}}}_n$ has a free-black-free boundary condition. Then reveal the color of the first edge on the left of the black boundary. If this edge is white then we peel the edge on the left of the white edge until the white edge is completely swallowed. We highlight the fact that this peeling algorithm *does not* follow the left-most boundary of cluster of black edges. However if it goes on forever then there exists an infinite cluster of black edges. See figure below.

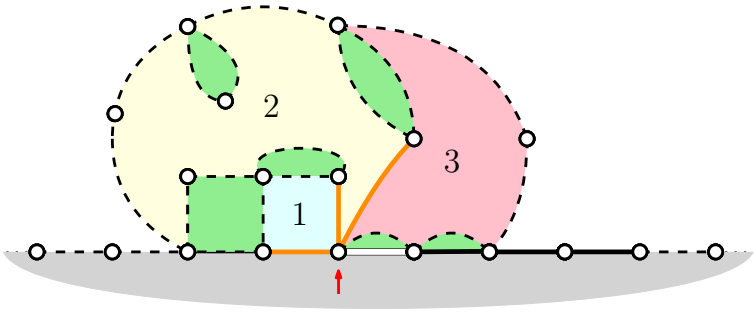


Figure: If the above algorithm goes on forever, then the black cluster is infinite (but this is not a necessary and sufficient condition since the black cluster may sneak inside the peeled region and continue on its left).

The evolution of the number of black edges on the boundary during such an exploration is a killed random walk with i.i.d. increments of law

$$\epsilon - (1-\epsilon)(\hat{\mathcal{G}}_r - 1 \mid \hat{\mathcal{G}}_r > 0),$$

where ϵ is a Bernoulli variable of parameter p. Comparing with (11.4), we deuce that if $p > p_{c,\text{site}}$ then $p > p_{c,\text{bond}}$ so that $p_{c,\text{site}}$ is indeed larger or equal to $p_{c,\text{bond}}$ as wanted.

Bibliographical notes. To our knowledge, the first study of percolation on random planar maps is due to Kazakov [129] who studied (non-rigorously) bond percolation on random triangulations as the limit $q \to 1$ of the q-states Potts models via matrix integrals (following the idea of de Gennes [100])! See also [132, 131] for some numerical studies. The first mathematical work is the pioneer paper of Angel [13] where he proved that the critical threshold for site percolation on the UIPT is $\frac{1}{2}$. These results have then been extended to the case of bond and face percolations on triangulations and quadrangulations in [14], to bond percolation on uniform maps in [157], and later to site percolation on quadrangulations by Richier [172] (see also partial results in

	quadrangulations	6-angulations	8-angulations	$2p$-angulations
$p_{c,\text{site}}$	$\frac{5}{9}$	$\frac{76}{125}$	$\frac{5197}{8085}$	$1-\frac{2p}{p-1}\frac{\left(\frac{e_q}{1-2p}+1\right)^2}{e_q-1}$
$p_{c,\text{bond}}$	$\frac{1}{3}$	$\frac{5}{11}$	$\frac{11}{21}$	$\frac{e_q-1}{e_q+1}$
$p_{c,\text{face}}$	$\frac{3}{4}$	$\frac{11}{16}$	$\frac{21}{32}$	$\frac{1}{2}(1+\frac{1}{e_q})$

Table 11.1: Explicit values of the percolation thresholds for $2p$-angulations. The *exposure* $e_q = 2g_q + 1$ in the last column is given by $e_q = 4^{p-1}/\binom{2p-2}{p-1}$.

[44]). The results of this chapter can be seen as a "Boltzmann generalization" of those results. The exact form of algorithms $\mathcal{A}_{\text{face}}$ and $\mathcal{A}_{\text{bond}}$ are taken respectively from [57] and [99]. The algorithm for site percolation is due to Richier [172] and Theorem 11.16 in this generality is proved in [56].

Chapter XII : More on bond percolation

In this chapter, we gather a few more results and open problems concerning percolations on random maps. In particular, we study percolation on random maps of the plane (as opposed to the half-plane in the previous chapter), on finite Boltzmann maps and on hyperbolic random maps. Since the geometry of those maps is substantially different from that of $\mathfrak{M}^{(\infty)}$, a careful analysis is needed and indeed displays a few differences, notably in the hyperbolic case where the percolation model witnesses another phase transition regarding the *uniqueness* phase. We also focus on the existence of (annealed) critical exponents describing the tail behavior of geometric events at criticality (or near criticality).

In this chapter *we focus on the case of bond percolation* for which the current set of results is the most satisfying and leave it as an open problem to extend the results to the case of face or site percolations which seem more difficult to study.

Open Question 12.1. *Extend the results of this chapter (percolation thresholds under $\mathbb{P}_\infty$, finite Boltzmann approach to percolation, annealed critical exponents, percolation on hyperbolic random maps...) to Bernoulli face and site percolations.*

In the rest of this chapter we focus on Bernoulli bond percolation of parameter $p \in (0,1)$ and denote by $\mathcal{C}$ the cluster of the origin. As in the previous chapter, this process is obtained by decorating independently each edge of the underlying map $\mathfrak{m}$ with a geometric number of marks: the black edges are those having at least one mark. Recall also the expression

$$p_{c,\text{bond}} = 1 - \frac{1}{g_q + 1}.$$

12.1 Critical exponents in the half-plane

In this section we establish finer properties of the peeling exploration of bond percolation under $\mathbb{P}^{(\infty)}$. We have seen in the previous chapter that the peeling along the percolation interface is intimately connected with a random walk. On top of establishing a clear phase transition for percolation, this link enables us to compute critical and near critical exponents. Let us give a few examples concerning the perimeter of $\mathcal{C}$.

Recall from Section 11.3.2 that the peeling exploration with Algorithm $\mathcal{A}_{\text{bond}}$ follows the left boundary of the cluster $\mathcal{C}$ until the later touches the right boundary of the map. During this

exploration we write $(B_i)_{i\geq 0}$ for the length of the black boundary which evolves as a random walk with increments

$$\epsilon - (1-\epsilon)\mathcal{G}_r,$$

killed when it touches $\mathbb{Z}_{<0}$. Appart from the increment 0 we have

$$\mathbb{P}(\Delta B_i = +1) = p \quad \text{and} \quad \mathbb{P}(\Delta B_i = -2k-1) = \frac{1}{2}(1-p)\nu(-k),$$

and so the walk has a negative drift if and only if $p < p_{c,\text{bond}} = \mathrm{g_q}/(1+\mathrm{g_q})$. We put $(B^*_i : i \geq 0)$ for the process where we erased all zero steps of B i.e. when $\Delta B = 0$: those are the times where we find no mark and trigger a peeling step which results in an event of type $\mathrm{G}_{k,\infty}$ or C_k. These steps do not correspond to following an edge of $\partial^*\mathcal{C}$. It should be clear from the exploration that, as long as $B^* > 0$, each step of B^* corresponds to the exploration of exactly one *half-edge* on the boundary of $\mathcal{C}$, see Figure 12.1 below.

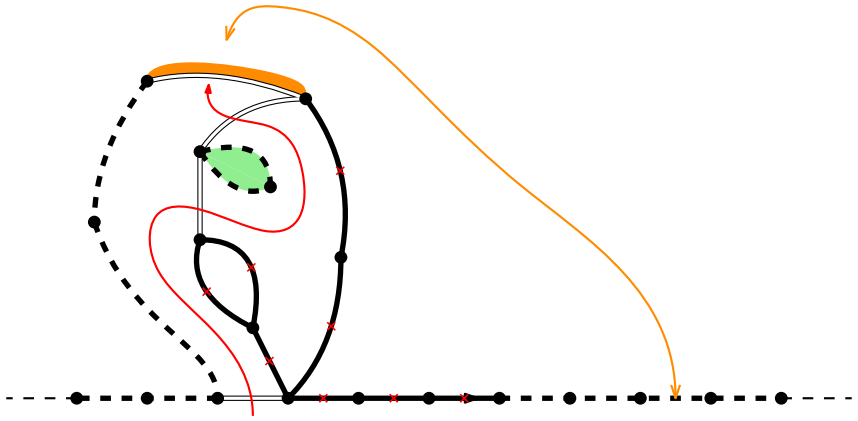


Figure 12.1: The peeling process along the percolation interface just before the exploration stops. Notice that each time B^* moves correspond to step along the percolation interface (except maybe the last step).

12.1.1 Length of exploration

Let us denote by τ the number of steps needed for B^* to reach $\mathbb{Z}_{<0}$ which we can interpret using the above discussion as the length of the percolation interface, or as the external perimeter of the cluster $\mathcal{C}$ following it from the left of the root edge until it is disconnected from infinity. Using standard results on random walk in the domain of attraction of stable Lévy processes we can prove:

Proposition 12.2. *We have the following*

- *if* $p < p_{c,\text{bond}}$ *then* $\mathbb{P}(\tau = n) \leq \mathrm{e}^{-cn}$ *for some* $c > 0$,
- *if* $p > p_{c,\text{bond}}$ *then* $\mathbb{P}(\tau = \infty) = p - (1-p)\mathrm{g_q} > 0$,

- *if furthermore* $\mathbf{q}$ *is of type* $a \in (2;5/2]$ *then at* $p = p_{c,\mathrm{bond}}$ *we have for some* $c > 0$

$$\mathbb{P}(\tau = n) \sim c \cdot n^{-2-\frac{1}{a-1}}, \quad \text{as } n \to \infty.$$

The first item shows that the length of the percolation interface has an exponential tail in the subcritical regime. For the second item, the function $\mathbb{P}(\tau = \infty)$ is usually written $\theta(p) = \int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{bond}}(\mathrm{d}\eta)\mathbf{1}_{\{|\mathcal{C}|=\infty\}}$ in percolation theory. In particular, we deduce the near-critical exponent $\theta(p_{c,\mathrm{bond}} + \varepsilon) = \varepsilon^{1+o(1)}$.

Proof of Proposition 12.2. Notice that B^* is a skip-free ascending random walk (see Section A.2.2), meaning that its positive jumps are only $+1$. In the subcritical case $p < p_{c,\mathrm{bond}}$, then B^* has a negative drift and so by an easy large deviation estimate we deduce that the probability it stays positive until time n decays exponentially. The second point follows from a variation of the ballot theorem [5, Theorem 3] which says that the probability that a skip-free ascending random walk (starting from 1) stays positive for all time is equal to its drift if the latter is positive. The last item follows from estimates on the hitting time of $\mathbb{Z}_{<0}$ for random walks in the domain of attraction of spectrally negative Lévy processes: see [187, Theorem 8], and [103, Theorem 1] for integrated results. □

The last item of the previous proposition has a geometric corollary: Suppose that $\mathbf{q}$ is a weight sequence of type $a \in (2;5/2]$, then there exists two constants $0 < c_1 < c_2 < \infty$ (depending on $\mathbf{q}$) so that at $p = p_{c,\mathrm{bond}}$ for large enough n's we have

$$\frac{c_1}{n^{\frac{a-2}{a-1}}} \leq \mathbb{P}\left(|\partial^* \mathcal{C}| \geq n\right) \leq \frac{c_2}{n^{\frac{a-2}{a-1}}},$$

where $|\partial^* \mathcal{C}|$ is the length of the external perimeter of $\mathcal{C}$. To prove this, recall that τ describes a part of the boundary $\partial^* \mathcal{C}$ namely, following the cluster from the left of the root edge until it is disconnected from ∞. We can similarly define the *right peeling exploration*, symmetric to the previous one, which always explores the free edge incident to the right of the black segment on the boundary of the hole. If similarly τ' is the length of this interface then we clearly have

$$\tau \vee \tau' \leq |\partial^* \mathcal{C}| \leq \tau + \tau' + 1.$$

Hence $\mathbb{P}(\tau + \tau' \geq 2n) \leq \mathbb{P}(\tau \geq n) + \mathbb{P}(\tau' \geq n) \sim 2cn^{-\frac{1}{a-1}-1}$ by the last point of the previous proposition. Notice that the perimeter of the region explored during the filled-in peeling of $\mathcal{C}$ actually as a perimeter of order $|\partial^* \mathcal{C}|^{1/(a-1)}$. This is due to the fact that the cluster $\mathcal{C}$ has many narrow "fjords" which once filled-in diminish drastically the perimeter. This is a well-know phenomena for $2d$-critical percolation, see [105].

12.1.2 More open questions

In the case of face percolation, the length of the black interface is not anymore a skip-free ascending random walk (except in the case of quadrangulation where we can take the half-length of the boundary) and so proving the analogs of (the last two points of) the previous

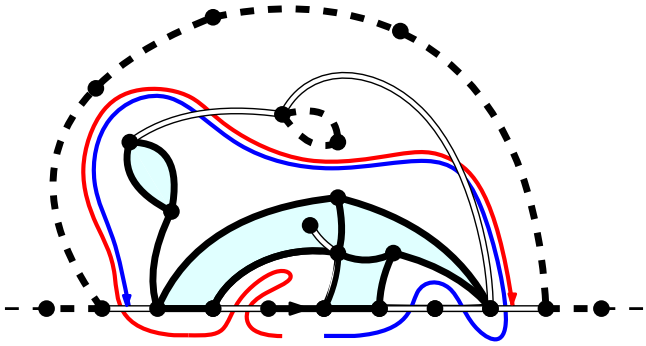


Figure 12.2: The left percolation interface of the cluster $\mathcal{C}$ (in red), and its right percolation interface (in blue).

proposition is more complicated, see Open question 12.1. Other open questions may include: existence (and computation) of other (annealed) critical exponents (e.g. for volume of the hull of the cluster, volume of the cluster, the diameter of the cluster) or the existence of quenched critical exponents (i.e. by conditioning on the map first). One could also try to extend the result of [173] and define the (annealed) incipient infinite critical cluster.

12.2 A Boltzmann approach to bond percolation

In this section, we focus on bond percolation on finite $\mathbf{q}$-Boltzmann maps. We will see that that under $\mathbb{P}^{(1)}$ the origin cluster $\mathcal{C}$ is itself a Boltzmann planar map with weight sequence

$$\tilde{\mathbf{q}}_p.$$

This can be seen using a decomposition of the map along the percolation clusters and the "islands" that it isolates, see Figure 12.3. More precisely, under $\mathbb{P}^{(1)}$ the probability that the cluster of the origin is a given map $\mathfrak{c}$ is equal to

$$\mathbb{P}(\mathcal{C} = \mathfrak{c}) = p^{\#\mathsf{Edges}(\mathfrak{c})} \prod_{f \in \mathsf{F}(\mathfrak{c})} I^{(p)}_{\deg(f)},$$

where $I^{(p)}_{2k}$ is equal to the $w_{\mathbf{q}}$-weight of so-called **islands**, which are maps with a simple boundary of perimeter $2k$ in which the boundary edges are black (counted with weight 1), and all internal edges adjacent to the boundary are white (and counted with weight $1-p$). Using Euler's formula $\#\mathsf{Edges}(\mathfrak{c}) = \frac{1}{2}\sum_{f\in\mathsf{F}(\mathfrak{c})} \deg(f)$, we can indeed write the previous equation as a product over Boltzmann weights $\tilde{\mathbf{q}}_p$. However this expression is not useful to deduce properties of $\tilde{\mathbf{q}}_p$ (apart from the trivial fact that $\tilde{\mathbf{q}}_p$ is admissible!).

12.2.1 Duality of stable maps via percolation

When furthermore $\mathbf{q}$ is of type $a \in (2; 5/2]$ then $\tilde{\mathbf{q}}_p$ is also regular and its type depends whether $p < p_{c,\text{bond}}$ or $p = p_{c,\text{bond}}$ or $p > p_{c,\text{bond}}$. The following result is proved in [99]:

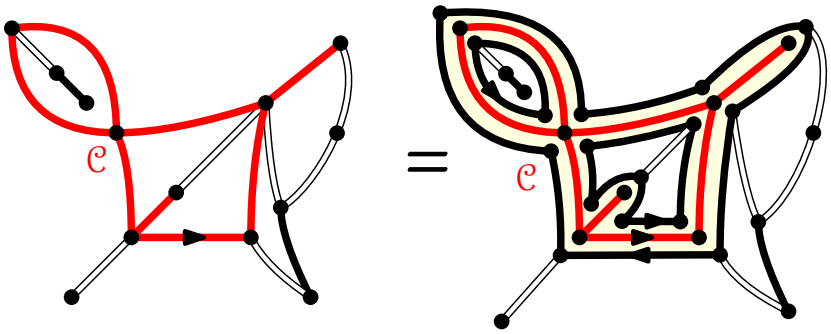


Figure 12.3: The decomposition of a map with perimeter 12 into its percolation cluster $\mathcal{C}$ with perimeter 8 (in red) and three islands.

Theorem 12.3 (***Duality via percolation***)

Let $\mathbf{q}$ *be a critical weight sequence of type* $a \in (2; 5/2]$. *Denote by* $\mathcal{C}$ *the* p*-Bernoulli bond percolation cluster of the origin in a Boltzmann map of law* $\mathbb{P}^{(1)}$. *Then* $\mathcal{C}$ *is again a Boltzmann map with weight sequence* $\tilde{\mathbf{q}}_p$ *where*

- $\tilde{\mathbf{q}}_p$ *is subcritical (of type* $a = 3/2$*) when* $p < p_{c,\mathrm{bond}}$,
- $\tilde{\mathbf{q}}_{p_{c,\mathrm{bond}}}$ *is of type* $\frac{a}{a-1} \in [\frac{5}{3}, 2)$,
- $\tilde{\mathbf{q}}_p$ *is of type* a *when* $p > p_{c,\mathrm{bond}}$.

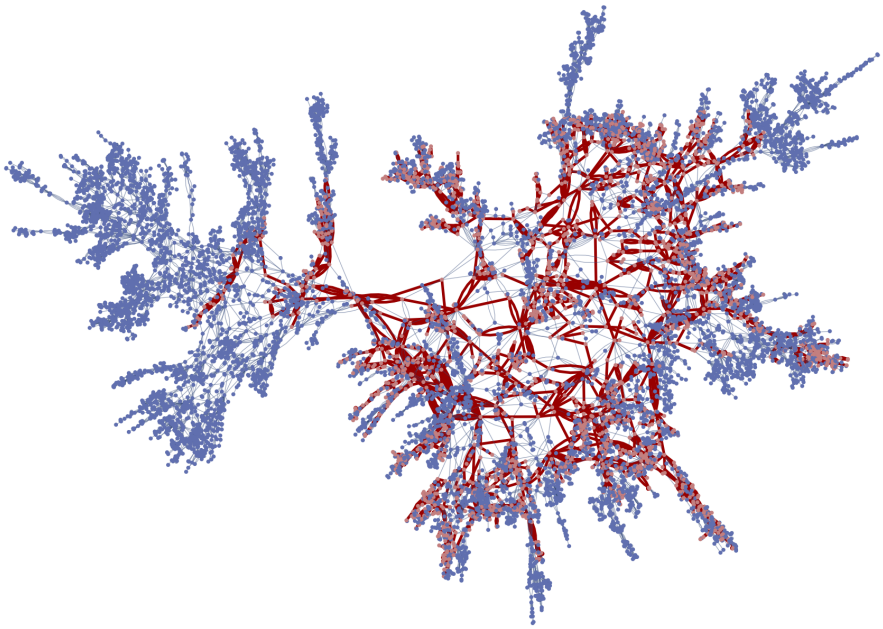


Figure 12.4: A large Boltzmann map with a large bond percolation cluster (in red).

We interpret this result as a duality property for Boltzmann maps through critical percolation. This has also been observed in the so-called *Schramm–Loewner Evolutions* (SLE) and *Conformal Loop Ensembles* (CLE) by Miller, Sheffield and Werner [163, 164]. Indeed, recall from Section 5.3.4 that discrete stable maps are known to be related to planar maps equipped with a critical $O(n)$ loop model through the so-called *gasket decomposition*. Given the (still conjectural) link between critical $O(n)$ models and SLE processes, there is a speculative relation between the parameter $a \in (3/2; 5/2]$ of Boltzmann maps and the parameter κ of CLEs, given by the formula

$$a = 1 + \frac{4}{\kappa}.$$

We can thus check that the relation of Theorem 12.3 corresponds to the duality relation $\kappa \leftrightarrow \kappa/16$ for SLEs and CLEs. Notice in particular that if $a \in (2; 5/2]$ is in the dilute phase, then $a/(a-1) < 2$ is in the dense phase. This is because percolation clusters have many "pinch points" and so large faces of the map $\mathcal{C}$ are self-intersecting in the scaling limit, see Figure 12.5.

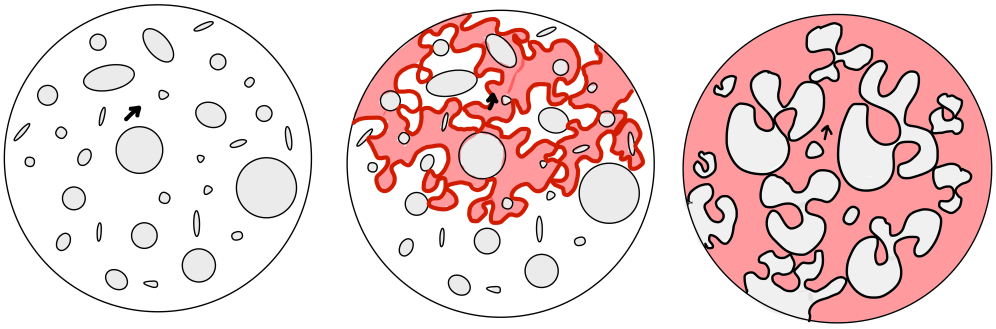


Figure 12.5: Illustration of Theorem 12.3. On the left, a Boltzmann map in the dilute phase (large faces non intersecting), on the middle its origin bond percolation cluster (which is large for the purpose of the illustration). On the right, the map formed by the percolation cluster.

The proof of Theorem 12.3 is ultimately derived from a variation on the third item of Proposition 12.2 and follows from the fine study of the exploration along bond percolation interfaces in $\mathfrak{M}^{(\infty)}$ using algorithm $\mathcal{A}_{\text{bond}}$. The complete argument can be found in [99] and we shall not include it here (in [99] the result is stated for regular weight sequences which could have an additional slowly varying function in their definitions, but it is straightforward to adapt the arguments to our definition of weight sequences of type $a \in (3/2; 5/2]$).

12.2.2 Critical exponents and open questions

Using the work [141] on scaling limits of bipartite discrete stable maps conditioned to be large, as well as the description of the geometry of their faces in [174], one can describe several critical

exponents related to the bond percolation cluster on finite $\mathbf{q}$-Boltzmann maps when $\mathbf{q}$ is of type $a \in (2; 5/2]$. Since $\mathcal{C}$ is a Boltzmann map of type $a' = \frac{a}{a-1}$, by Proposition 10.7, with a probability of order $n^{-\frac{2a'}{a'-1/2}}$ the cluster $\mathcal{C}$ has total size (number of edges) equal to n. On this event, by [141], the largest face in the cluster has a perimeter of order $n^{1/(a'-1/2)}$ and the diameter of the cluster (for the graph distance restricted on the cluster) is of order $n^{\frac{1}{2(a'-1/2)}}$, see Figure 12.6 (left).

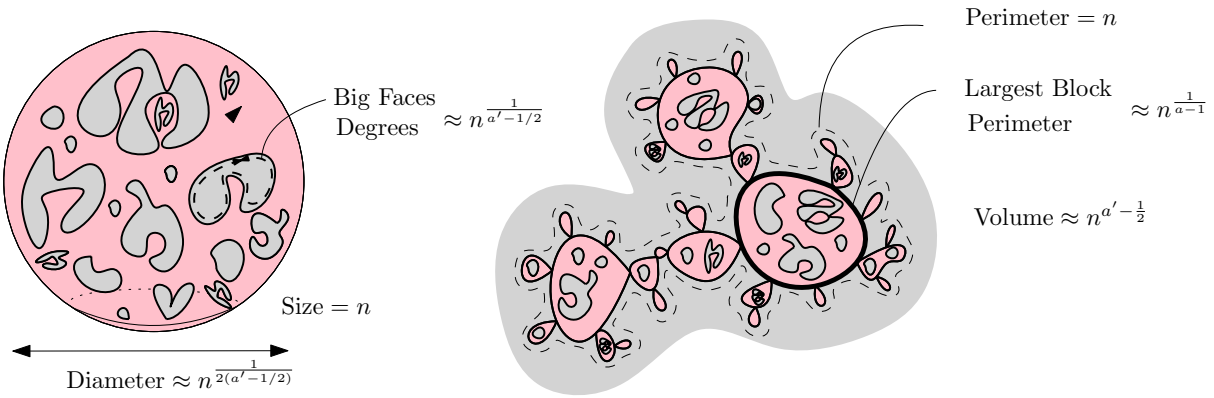


Figure 12.6: Anatomy of a large critical percolation cluster on the (unlikely) event that it has size n (left). On the right, the geometry of a large critical percolation with a boundary of perimeter n.

One can also wonder about the geometry of large critical cluster when we condition this cluster to have a root face of degree n (by the above discussion and [141] the cluster has size of order $n^{a'-\frac{1}{2}}$ in this case). As in any $\mathbf{q}$-Boltzmann map with type a', this event decays like $n^{-2a'}$, and on this event the external face is not at all a simple face but is folded on itself in a fractal fashion. This has been studied in depth in [174] and if one decomposes the cluster into blocks with simple boundary, then the tree structure of those blocks is described in the discrete setting by a multi-type critical random tree with offspring distribution in the domain of attraction of the $a-1$-stable law. In particular, the largest of these blocks has a perimeter of order $n^{1/(a-1)}$. It is tempting to believe that the total size of the cluster is already comparable to the total size of the cluster which is of order $n^{a'-\frac{1}{2}}$.

Let us now examine heuristically, in each of the above pictures, the structure of the underlying *Boltzmann map* in which those large critical clusters are found. Let us condition again on the origin cluster $\mathcal{C}$ having size n (as in the left of Figure 12.6). Of course, the initial random map can be recovered by filling-in all the faces of the cluster $\mathcal{C}$ with the appropriate percolated maps with a boundary, see the island decomposition in Figure 12.3. As we already noticed, a face of the cluster of degree d is typically folded on itself and made of a tree of simple faces whose largest one is of degree $d^{\frac{1}{a-2}}$. In each of these simple holes, one must fill-in a Boltzmann map with appropriate perimeter. Since a Boltzmann map with perimeter 2ℓ typically has size $\ell^{a-1/2}$ when the weight sequence is of type a, we deduce that the size of map inserted in a face of large degree d of the cluster is expected to be of order $(d^{1/(a-1)})^{a-1/2} = d^{\frac{a-1/2}{a-1}}$. Recalling that the

maximal degree of the faces of $\mathcal{C}$ is of order $n^{1/(a'-1/2)}$, we expect that the total size of the map containing the large cluster $\mathcal{C}$ of size n has size of order

$$(n^{1/(a'-1/2)})^{\frac{a-1/2}{a-1}} = n^{\frac{2a-1}{a+1}}.$$

This is because the maximal size of a map filling in a hole should be comparable to the total size of all those maps. We also conjecture that after proceeding to this filling operation, the initial cluster $\mathcal{C}$ has a positive chance to be the largest cluster in the obtained percolated map. The above heuristic leads us to propose the following open problem:

Open Question 12.4. *Fix an admissible weight sequence* $\mathbf{q}$ *of type* $a \in (2; 5/2]$. *Recall the function* $\theta(p)$ *for the probability that the* p*-Bernoulli bond percolation cluster* $\mathcal{C}$ *in* $\mathfrak{M}^{(\infty)}$ *is infinite. Let* $\mathcal{C}_{\max}[n]$ *be the largest cluster in a bond percolated* $\mathbf{q}$*-Boltzmann map conditioned on having size* n. *Show that*

$$\frac{\mathcal{C}_{\max}[n]}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} \theta(p).$$

In the critical case $p = p_{c,\text{bond}}$ *show that* $\mathcal{C}_{\max}[n] = n^{\frac{a+1}{2a-1}+o_{\mathbb{P}}(1)}$, *or better, study the scaling limit (in Gromov–Hausdorff-Prokhorov topology) of the sequences of first, second, third... largest clusters in the map.*

12.3 Percolations on $\mathfrak{M}_\infty$

In this section, we suppose that $\mathbf{q}$ *is critical and we work on* $\mathfrak{M}_\infty$.

We now turn to the analysis of Bernoulli percolations on the full-plane topology of $\mathfrak{M}_\infty$. Ideally, one first would like to prove Theorems 11.3, 11.8 and 11.16 after replacing $\mathbb{P}^{(\infty)}$ by $\mathbb{P}_\infty$ with the same definition of the percolation thresholds. Although we believe this is true, we cannot yet prove it in this generality:

Open Question 12.5. *Suppose* $\mathbf{q}$ *is critical. Extend Theorems 11.3, 11.8 and 11.16 with the same thresholds as in the case of the* $\mathbf{q}$*-Boltzmann maps of the plane under law* $\mathbb{P}_\infty$.

12.3.1 Do plane and half-plane bond percolation thresholds coincide?

Towards Conjecture 12.5, we prove a more modest result by focusing on bond percolation and on regular weight sequences:

Theorem 12.6 (*bond percolation threshold on* $\mathbf{M}_\infty$)

Let $\mathbf{q}$ *be a critical weight sequence of type* $a \in (3/2; 5/2]$. *Then the almost sure bond percolation threshold under* $\mathbb{P}_\infty$ *is equal to* $p_{c,\text{bond}}$. *Furthermore, there is no percolation at criticality.*

It will follow from Proposition 13.12 that if $\mathbf{q}$ is critical and of type $a \in (3/2; 2]$ then $\mathfrak{M}_\infty$ almost surely has infinitely many cut-edges, hence the critical threshold for bond percolation is 1. We can safely restrict ourselves to $a \in (2; 5/2]$ in the proof below. The most obvious path

to prove the previous theorem is to adapt the exploration of the percolation interface from the half-plane to the plane setting. We will see that this works nicely in the subcritical regime, but the supercritical regime is trickier and we shall rather rely on Theorem 12.3.

Proof of Theorem 12.6. By ergodicity, the percolation threshold on $\mathfrak{M}_\infty$ is constant. To explore the Bernoulli bond percolation on $\mathfrak{M}_\infty$ we can again use an algorithm very similar to $\mathcal{A}_{\text{bond}}$, i.e. we also peel the interface free-black, and if the boundary of the explored part is monochromatic (i.e. totally free or totally black) then we may peel any edge until we find again a bi-chromatic boundary condition in which case we follow $\mathcal{A}_{\text{bond}}$ again. We still denote by $(B_n : n \geq 0)$ the length of the black boundary and introduce $(F_n : n \geq 0)$ the length of the free boundary in this peeling exploration so that

$$\forall n \geq 0, \qquad B_n + F_n = 2P_n,$$

where $2P_n$ is the perimeter of $\bar{\mathfrak{e}}_n$. Obviously, the process (F_n, B_n) is a Markov chain whose variations are ruled by the peeling steps as follows: if $F_n > 0$ and $B_n > 0$ we explore the free edge immediately on the left of the black boundary: if it turns out to be black (i.e. if it carries a mark) then we have $\Delta F_n = -1$ and $\Delta B_n = +1$. Otherwise it carries no mark and we trigger a peeling step:

$$\begin{aligned}
(\Delta F_n = 2k-2, \Delta B_n = 0) &\quad \text{event of type } \mathsf{C}_k \\
(\Delta F_n = -1, \Delta B_n = -2k-1) &\quad \text{event of type } \mathsf{G}_{*,k} \text{ with } 2k+1 \leq B_n, \\
(\Delta F_n = B_n - 2k - 2, \Delta B_n = -B_n) &\quad \text{event of type } \mathsf{G}_{*,k} \text{ with } 2k+1 > B_n, \\
(\Delta F_n = -2k-2, \Delta B_n = 0) &\quad \text{event of type } \mathsf{G}_{k,*} \text{ with } 2k+1 \leq F_n - 1, \\
(\Delta F_n = -F_n, \Delta B_n = F_n - 2k - 2) &\quad \text{event of type } \mathsf{G}_{k,*} \text{ with } 2k+1 > F_n - 1.
\end{aligned} \tag{12.1}$$

If $F_n = 0$ or $B_n = 0$ then we explore any edge until we find a new bi-chromatic boundary condition. By a variation on Lemma 11.5, knowing whether $\mathcal{C}$ is infinite is a.s. equivalent to determine whether $B_n > 0$ for all $n \geq 0$. The basic idea is of course to compare the evolution of B_n with that of the random walk of Section 11.3.2 with i.i.d. increments $\epsilon - (1-\epsilon)\mathcal{G}_r$.

Subcritical phase: Suppose first that $p < p_{c,\text{bond}}$. From the exact transitions of the chain B and the fact $\mathbb{E}[\epsilon - (1-\epsilon)\mathcal{G}_r] < 0$, one can find b_0 so that if $B_n > b_0$, then conditionally on $\mathcal{F}_n$, the increment ΔB_n is stochastically dominated by a law ξ supported by $\mathbb{Z}_{\leq 1}$ and with strictly negative mean. It easy follows that B_n cannot stay positive forever with positive probability and then $\mathcal{C}$ is finite almost surely.

Critical phase (Sketch): In the critical case $p = p_{c,\text{bond}}$ the comparison with the half-planar case is more subtle and will rely on (a variation of) Proposition 10.3. More precisely, we will show that B has a positive probability to die at every scale (and so by an easy exercise, B almost surely dies out):

Lemma 12.7. *Suppose that* $\mathbf{q}$ *is of type* $a \in (2; 5/2]$ *and take* $p = p_{c,\mathrm{bond}}$. *There exists* $c > 0$ *such that for any* $n \geq 0$ *if* $\mathcal{F}_n$ *is the* σ*-field of the exploration up to time* n *then*

$$\mathbb{P}(\exists 0 \leq i \leq (P_n)^{a-1} : B_{n+i} = 0 \mid \mathcal{F}_n) > c.$$

Proof. Let us condition on $\mathcal{F}_n$ and write $P_n = m$. Without loss of generality we can suppose we are in the best position for the black cluster, that is we have a full black boundary of perimeter $2m$. We need to prove that within m^{a-1} exploration steps with the above rules, with a positive probability independent of m the black boundary can disappear. For this, it will be convenient to introduce a slight modification of the processes (F_n, B_n) defined as follows: $(\tilde{F}_0 = 0, \tilde{B}_0 = 2m)$ and as long as $B_{n+i} > 0$ we put

$$\begin{aligned}
(\Delta\tilde{F}_i = -1, \Delta\tilde{B}_i = 1) &\quad \text{discovery of a black edge} \\
(\Delta\tilde{F}_i = 2k-2, \Delta\tilde{B}_i = 0) &\quad \text{discovery of a white edge and event of type } \mathsf{C}_k \\
(\Delta\tilde{F}_i = -1, \Delta\tilde{B}_i = -2k-1) &\quad \text{discovery of a white edge and event of type } \mathsf{G}_{*,k} \\
(\Delta\tilde{F}_i = -2k-2, \Delta\tilde{B}_i = 0) &\quad \text{discovery of a white edge and event of type } \mathsf{C}_{k,*}.
\end{aligned} \tag{12.2}$$

Let $\theta = \inf\{i \geq 0 : B_{n+i} = 0\}$ be the first time after n when B touches 0 with the usual convention that $\inf \varnothing = +\infty$. Consider also $\underline{\tilde{F}}$ the running infimum of $\tilde{F}$ i.e. $\underline{\tilde{F}}_k = \inf\{\tilde{F}_i : 0 \leq i \leq k\}$. Then it is easy so see that $\theta = \inf\{k \geq 0 : \tilde{B}_k + \underline{\tilde{F}}_k \leq 0\}$ and that we can recover B and P as follows:

$$\forall 0 \leq i < \theta, \quad B_{n+i} = \tilde{B}_i + \underline{\tilde{F}}_i, \quad \text{and} \quad \forall 0 \leq i \leq \theta, \quad 2P_{n+i} = \tilde{B}_i + \tilde{F}_i.$$

We introduce yet another process $(\hat{F}, \hat{B})$ which under $\mathbf{P}^{(m)}$ evolves as $(\tilde{F}, \tilde{B})$ but in the half-plane setting, i.e. where the transitions of (12.2) take place in the half-plane. In other words, $(\hat{F}, \hat{B})$ has i.i.d. increments and $(\tilde{F}_i, \tilde{B}_i)_{0 \leq i \leq \theta}$ is a h-transform of the process $(\hat{F}, \hat{B})_{0 \leq i \leq \theta}$ for the function $h(f, b) = h^{\uparrow}((f+b)/2)$. Using these considerations we have

$$\mathbf{P}^{(m)}(\exists k \leq m^{a-1} : B_k = 0) = \mathbf{P}^{(m)}(\theta \leq m-1) \quad = \quad \mathbf{E}^{(m)}\left[\mathbf{1}_{\theta \leq m^{a-1}} \frac{h^{\uparrow}(\frac{1}{2}\hat{B}_\theta + \frac{1}{2}\hat{F}_\theta)}{h^{\uparrow}(m)}\right]. \tag{12.3}$$

We leave the reader adapt the proof of Proposition 10.3 (see [88, Proposition 2] for similar arguments) to see that at $p = p_{c,\mathrm{bond}}$, the process $(\hat{F}, \hat{B})$ has both components centered and converges after scaling towards a pair of independent $(a-1)$-stable processes:

$$\text{under } \mathbf{P}^{(m)}, \qquad \left(\frac{\hat{F}_{[m^{a-1}t]}}{2m}, \frac{\hat{B}_{[m^{a-1}t]}}{2m}\right)_{t \geq 0} \xrightarrow[m \to \infty]{} (\mathfrak{S}_t, \mathfrak{B}_t)_{t \geq 0},$$

where $\mathfrak{S}$ is an $(a-1)$-stable Lévy process started from 0 and independent of $\mathfrak{B}$ which is another $(a-1)$-stable Lévy process (spectrally negative this time) started from 2. Using this and (12.3), we leave the reader check that $\mathbf{P}^{(m)}(\theta \leq m^{a-1})$ is larger than some constant $c > 0$ for all m large enough. This implies our Lemma 12.7 and concludes the critical case. □

Supercritical phase: Following the same strategy as in the previous cases, when $p > p_{c,\text{bond}}$ one would like to prove that B_n survives with positive probability. This is not so easy and let us explain the main difficulties. First of all, the comparison with a random walk with positive drift is hard to perform because $B_n \le 2P_n$ and in the regime of interest we have $P_n = o(n)$. We should then rather focus on the length of the free interface F_n which, roughly speaking, has a negative drift when positive. This should entail that F_n stays small (of order $\log n$) and so $B_n = 2P_n - F_n$ tends to infinity by Proposition 10.9. This argument works well when ν is supported by $\{\dots, -3, -2, -1, 0, 1, 2, \dots, k_0\}$ for some k_0, because ΔF_n has an exponential tail on the positive side and (eventually) a negative drift, see [13, 157] for similar arguments. However, when $\mathbf{q}$ is of type $a \in (2; 5/2)$, then sometimes F_n becomes large, due to the discovery of a face of degree of order $n^{1/(a-1)}$: at those times the length of the free interface is of the same order as the total perimeter. Obviously, when such a face is discovered, the process F is then "pushed down" by a negative drift and quickly reaches small values. In the mean time, it is very unlikely that the black boundary got swallowed... but this requires a proof!

To circumvent those problems, we shall use Theorem 12.3: for $p > p_{c,\text{bond}}$ we know that the origin cluster $\mathcal{C}$ in a supercritical bond percolation inside the *finite* Boltzmann map $\mathfrak{M}$ is again a Boltzmann map with a weight sequence of the same type as $\mathbf{q}$. In particular, using Proposition 10.7 we deduce that for some $c_1, c_2 > 0$ we have

$$\mathbb{P}(|\mathcal{C}| \ge n) \sim c_1 n^{-1-\frac{1}{a-1/2}} \quad \text{and} \quad \mathbb{P}(|\mathfrak{M}| \ge n) \sim c_2 n^{-1-\frac{1}{a-1/2}}, \quad \text{as } n \to \infty.$$

Hence for every N_0 fixed we have

$$\mathbb{P}(|\mathcal{C}| \ge N_0 \mid |\mathfrak{M}| \ge n) \ge \mathbb{P}(|\mathcal{C}| \ge n \mid |\mathfrak{M}| \ge n) = \frac{\mathbb{P}(|\mathcal{C}| \ge n)}{\mathbb{P}(|\mathfrak{M}| \ge n)} \xrightarrow[n\to\infty]{} \frac{c_1}{c_2} > 0.$$

Since $\{|\mathcal{C}| \ge N_0\}$ is a local event and since $\mathbb{P}^{(1)}(\cdot \mid |\mathfrak{m}| \ge n) \to \mathbb{P}_\infty$ by Theorem 7.1, we deduce that for any N_0 we have $\mathbb{P}_\infty(|\mathcal{C}| \ge N_0) \ge \frac{c_1}{c_2}$. Letting $N_0 \to \infty$ yields $\mathbb{P}_\infty(|\mathcal{C}| = \infty) \ge \frac{c_1}{c_2}$ and this completes the proof. □

12.3.2 Open questions

Along with Open Question 12.5, a natural quest concerning bond percolation on $\mathfrak{M}_\infty$ would be to study (annealed) critical and near-critical exponents in the spirit of Section 12.1. In particular, the recent work [113] computed the critical exponent for the length of the site-percolation exploration in the UIPT and showed it is different from its analog in the half-plane setting. See also [156] for other critical exponents and a computation of $\theta(p)$ in the case of site-percolation on the UIPT. Extending this work to other classes of maps and to maps with weight sequence of type $a \in (2; 5/2]$ would certainly be of great interest. Other questions may include the construction and the study of the Infinite Incipient Cluster as in [173].

12.4 Percolation on hyperbolic random maps

In this section, we suppose that $\mathbf{q}$ *is subcritical and satisfies Cramér's condition.*

In this section, we fix a subcritical weight sequence $\mathbf{q}$ satisfying Cramér's condition (see Chapter 8) and we study the Bernoulli bond percolation under $\mathbb{H}^{(\infty)}$. Contrary to the previous cases, we are now facing a double phase transition: a first almost sure threshold for the existence of infinite clusters, and a second one for the **uniqueness of the infinite cluster**. Exactly as for the previous cases, the analysis will crucially rely on the notion of hyperbolic gulp and exposure:

Definition 12.8 (Hyperbolic gulp and exposure)**.** *Let* $(\mathcal{E}^{\mathrm{hyp}}, \mathcal{G}_l^{\mathrm{hyp}}, \mathcal{G}_r^{\mathrm{hyp}})$ *be the number of exposed edges, edges swallowed on the left and on the right of the peeling edge during a one-step peeling under* $\mathbb{H}^{(\infty)}$ *and denote by*

$$\mathrm{e}_{\mathbf{q}}^{\mathrm{hyp}} = \mathbb{E}[\mathcal{E}^{\mathrm{hyp}}] \quad \text{and} \quad \mathrm{g}_{\mathbf{q}}^{\mathrm{hyp}} = \mathbb{E}[\mathcal{G}_l^{\mathrm{hyp}}].$$

Contrary the non-hyperbolic case, we *do not have* $\mathrm{e}_{\mathbf{q}}^{\mathrm{hyp}} = 2\mathrm{g}_{\mathbf{q}}^{\mathrm{hyp}} + 1$ but rather

$$\mathrm{e}_{\mathbf{q}}^{\mathrm{hyp}} - 2\mathrm{g}_{\mathbf{q}}^{\mathrm{hyp}} - 1 = \mathbf{E}[\tilde{S}_1] > 0$$

where $\tilde{S}$ is the random walk associated with S as in Section A.3.5. Notice also that we always have $\mathrm{g}_{\mathbf{q}}^{\mathrm{hyp}} < \infty$ but $\mathrm{e}_{\mathbf{q}}^{\mathrm{hyp}}$ might be infinite (see Open Question 8.7).

12.4.1 Critical and uniqueness thresholds

Theorem 12.9 (***Phase transitions for percolation on hyperbolic half-planes***)

Suppose $\mathbf{q}$ *is subcritical and satisfies Cramér's condition. We put*

$$p_{c,\mathrm{bond}}^{\mathrm{hyp}} = 1 - \frac{1}{1 + \mathrm{g}_{\mathbf{q}}^{\mathrm{hyp}}} \quad \text{and} \quad p_{u,\mathrm{bond}}^{\mathrm{hyp}} = 1 - \frac{1}{\mathrm{e}_{\mathbf{q}}^{\mathrm{hyp}} - \mathrm{g}_{\mathbf{q}}^{\mathrm{hyp}}}.$$

Then almost surely for $\int \mathbb{H}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{bond}}(\mathrm{d}\eta)$ *there is no infinite black cluster for the* p*-Bernoulli bond percolation if* $p \le p_{c,\mathrm{bond}}^{\mathrm{hyp}}$*, there are infinitely many infinite black clusters if* $p_{c,\mathrm{bond}}^{\mathrm{hyp}} < p < p_{u,\mathrm{bond}}^{\mathrm{hyp}}$ *and a unique infinite black cluster if* $p \ge p_{u,\mathrm{bond}}^{\mathrm{hyp}}$.

Remark 12.10. The similarity in the expression of $p_{c,\mathrm{bond}}^{\mathrm{hyp}}$ and $p_{c,\mathrm{bond}}$ should not come as a surprise. Remark that since $\mathrm{e}_{\mathbf{q}}^{\mathrm{hyp}} - 2\mathrm{g}_{\mathbf{q}}^{\mathrm{hyp}} - 1 > 0$ we indeed have $p_{u,\mathrm{bond}}^{\mathrm{hyp}} > p_{c,\mathrm{bond}}^{\mathrm{hyp}}$.

Proof. Let us start with the value of the percolation threshold $p_{c,\mathrm{bond}}^{\mathrm{hyp}}$. For this we adapt the proof of Theorem 11.8 and peel the bond percolation interface using algorithm $\mathcal{A}_{\mathrm{bond}}$ of Proposition 11.11 (when seeing the bond percolation as induced by geometric number of marks on every edge). When doing so, the length B_n of the black boundary evolves as a random walk killed on $\mathbb{Z}_{<0}$ with i.i.d. increments of law

$$\epsilon + (1-\epsilon)\mathcal{G}_r^{\mathrm{hyp}},$$

where $\epsilon \in \{0,1\}$ is a Bernoulli variable independent of $\mathcal{G}_r^{\mathrm{hyp}}$ with law $\mathbb{P}(\epsilon = 1) = p$. Hence, B_n has a non-zero chance to stay positive if and only if its mean

$$p - (1-p)\mathrm{g}_{\mathbf{q}}^{\mathrm{hyp}}$$

is positive. This is indeed equivalent to $p > p_{c,\mathrm{bond}}^{\mathrm{hyp}}$. We then conclude by ergodicity that there is almost surely an infinite cluster in $\mathfrak{H}^{(\infty)}$ if and only if we are above the percolation threshold. To study the uniqueness of infinite clusters, we adapt the argument of Proposition 11.6: the evolution of the length of a free boundary part in-between two black boundaries, is a killed random walk with i.i.d. increments of law

$$-1 + (1-\epsilon)(\mathcal{E}^{\mathrm{hyp}} - \mathcal{G}_l^{\mathrm{hyp}}),$$

where ϵ is independent of $(\mathcal{E}^{\mathrm{hyp}}, \mathcal{G}_l^{\mathrm{hyp}})$ and equals 1 with probability p and 0 otherwise. Again, this region has a non-zero chance to survive forever if the mean of the above increment is positive which is equivalent to $p < p_{u,\mathrm{bond}}^{\mathrm{hyp}}$. It follows that if $p \geq p_{u,\mathrm{bond}}^{\mathrm{hyp}}$ there cannot be two distinct infinite black clusters. We leave the details to the reader. □

The following open question should not be particularly hard to settle:

Open Question 12.11. *Extend the above theorem to the hyperbolic random maps of the plane.*

12.4.2 Open questions

Let us close this chapter with a few open questions about percolations on hyperbolic maps. First of all, one can try to extend the above results to face or site percolation. The next item as usual concerns critical exponents where we expect a very different behavior compared to non-hyperbolic random maps. Those critical or near-critical exponents should describe the rough geometry of critical clusters, but those can be studied directly from the geometric point of view:

Open Question 12.12. *Study the geometry of critical clusters on hyperbolic random maps conditioned to be large and show they converge after rescaling towards Brownian continuum random trees (at least when* $\mathbf{q}$ *is finitely supported).*

Bibliographical notes. The first annealed critical exponents for percolation on random maps were computed in [14] in the case of site percolation on half-planar triangulations. The Boltzmann approach to percolation on random maps was initiated in [30] for Bernoulli site and bond percolations on triangulations using analytic combinatorics, the result presented here is taken from [99] and is rather based on peeling techniques. Section 12.2 is largely adapted from the previous two references. A nice corollary (omitted in these lecture notes) of those techniques is the "sharpness of the phase transition", namely the fact that in the subcritical regime, the size of the origin cluster has an exponential tail. Finer geometric properties of percolation cluster such as the geometry of its outer face were studied in [90], see also [174] for a more general treatment. The transfer of the Bernoulli percolation thresholds from the half-plane to the full-plane

(Theorem 12.6) is new in this generality, the only case known so far were site percolation on the UIPT [13], face and bond percolations on quadrangulations, and bond percolation on uniform random planar maps [157]. In particular, the last part of the proof of Theorem 12.6 is adapted from [30, Proof of Theorem 1.2]. As mentioned above, the critical exponent for the length of the exploration was computed in the case of site percolation on the UIPT in [113]. Recently, Ménard [156] managed to combine the decomposition of [30] and analytic combinatorics techniques to compute exactly the percolation probability as well as the critical exponent of the volume. Percolation on hyperbolic maps was first studied in [170] in the case of site percolation on hyperbolic random triangulations and Theorem 12.9 is adapted from those results. We conclude this part on percolation by mentioning the far reaching results of Gwynne and Miller [118] relating critical percolation (in their case, face percolation on random quadrangulations) to SLE_6-decorated Liouville Quantum Gravity surfaces with parameter $\sqrt{8/3}$. En route they compute the scaling of the percolation interface in finite Boltzmann quadrangulations. Building on this idea, Holden & Sun [123] managed to prove the convergence of random triangulations under the so-called Cardy embedding towards Liouville Quantum Gravity surfaces with parameter $\sqrt{8/3}$. Part of this program contains an especially useful bijective encoding of critical percolated triangulations [31] which can be thought of as a "space-filling" version of the (simple) peeling process. They can compute the scaling limit of all interfaces simultaneously using this approach. Extending those ideas to the more general case (say, of bond percolation on bipartite maps) would certainly yield a better understanding and might cut short Open Question 12.4.

Part V:
Geometry

In this part, we study the volume growth of our random infinite planar maps, for the primal distance, the dual distance and the Eden distance (first passage percolation with exponentially distributed weights). We also connect the scaling limits of these distances to growth-fragmentations processes which can, in a sense, be seen as branching Lévy processes.

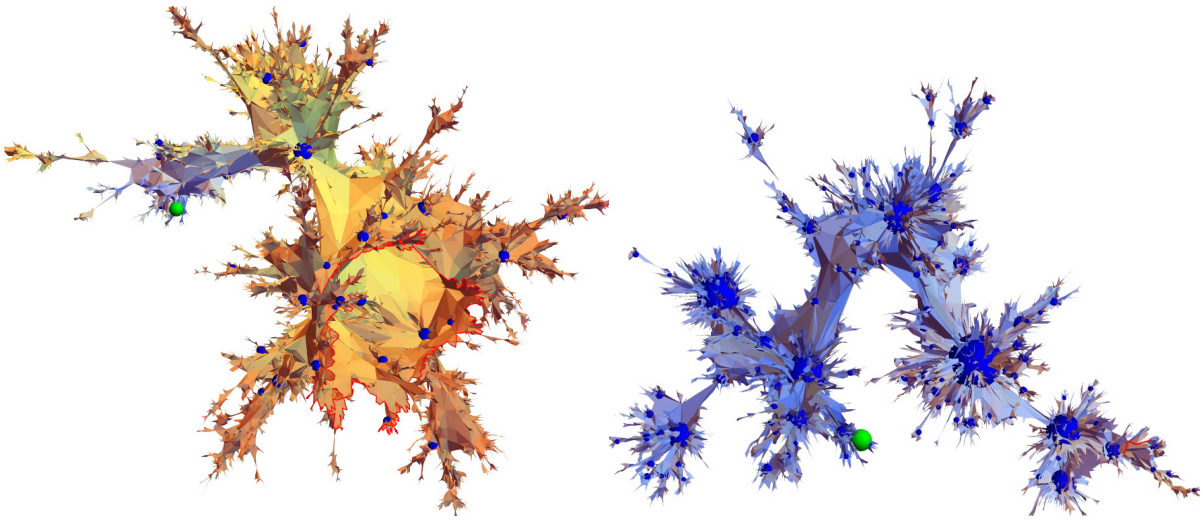


Figure 12.7: Two representations of the neighborhood of the root in an infinite Boltzmann maps with large degree vertices in the dilute case (left) and in the dense case (right). The root is represented by a green ball, while the high degree vertices are represented by blue balls of size proportional to the degree. The boundary is colored in red.

Chapter XIII : Metric growths

In this chapter, we only deal with random maps of the plane.

In this chapter, we study the geometry of the random maps sampled according to the distribution $\mathbb{P}_\infty$ in the case when $\mathbf{q}$ is a weight sequence of type $a \in (3/2; 5/2]$ (in particular critical) or according to $\mathbb{H}_\infty$ when $\mathbf{q}$ is subcritical and satisfies Cramér's condition. We will measure the distances according to three metrics:

- the **graph distance** on $\mathsf{Vertices}(\mathfrak{m})$ denoted by d_{gr},
- the **dual-graph distance** on $\mathsf{Vertices}(\mathfrak{m}^\dagger) = \mathsf{Faces}(\mathfrak{m})$ denoted by $\mathrm{d}^\dagger_{\mathrm{gr}}$,
- the **Eden model** on $\mathsf{Vertices}(\mathfrak{m}^\dagger)$ denoted by $\mathrm{d}_{\mathrm{fpp}}$ which corresponds to First-Passage Percolation distance with independent exponential edge weights.

We will see that the geometry of those maps is different and depends on the metric we choose and on $a \in (3/2; 5/2]$. We will start with the Eden distance which is technically the simplest one and then move on to the dual distance. As of today, the primal distance is tractable via the peeling process only in the case of triangulations or quadrangulations. In each case, the strategy is the same: devise a filled-in peeling algorithm $(\bar{\mathfrak{e}}_n : n \geq 0)$ which roughly explores hulls of balls in $\mathfrak{M}_\infty$ or $\mathfrak{H}_\infty$ for the required distance with a random increasing radius R_n. Once the growth of R_n is understood, one can use the general results on the volume and perimeter during a peeling process (Theorem 10.8 and Proposition 10.12) to recover information about volume and perimeter of hulls of balls in $\mathfrak{M}_\infty$ or $\mathfrak{H}_\infty$.

13.1 Eden model: exponential FPP distances on the dual

We first define the Eden model on a general map and link it to the uniform peeling. As in Section 11.3.2, the properties of the exponential law will be crucial in this section.

13.1.1 Definition of the Eden distance

Let $\mathfrak{m}$ be a (finite or infinite) planar map. Under the probability measure $\mathrm{P}^{\mathrm{fpp}}_{\mathfrak{m}}$, we sample independent weights x_e for each edge $e \in \mathsf{Edges}(\mathfrak{m}^\dagger)$ (equivalently for each edges of $\mathfrak{m}$) distributed according to the exponential law $\mathcal{E}(1) = \mathrm{e}^{-x}\mathrm{d}x\mathbf{1}_{x>0}$. These weights can be used to modify the

usual dual graph metric on $\mathfrak{m}^\dagger$ by considering the first-passage percolation (abbreviate below by fpp) distance: for $f_1, f_2 \in \mathsf{Faces}(\mathfrak{m})$

$$\mathrm{d}_{\mathrm{fpp}}(f_1, f_2) = \inf \sum_{e \in \gamma} x_e,$$

where the infimum is taken over all paths $\gamma : f_1 \to f_2$ in the dual map $\mathfrak{m}^\dagger$. This model (first-passage percolation with exponential edge weights on the dual graph) is often referred to as the Eden model[1]. It is convenient in this section to view the edges of the map $\mathfrak{m}^\dagger$ as real segments of length x_e for $e \in \mathsf{Edges}(\mathfrak{m}^\dagger)$ glued together according to the incidence relations of the map. This operation turns $\mathfrak{m}^\dagger$ into a continuous length space (but we keep the same notation) and the distance $\mathrm{d}_{\mathrm{fpp}}$ extends easily to all points of this space. As usual, if $\mathfrak{m}$ is a priori a map without boundary we see it as a map with a boundary of perimeter 2 by Figure 3.2.

Definition 13.1 (Fpp ball). *For $t > 0$ we denote by*

$$\mathrm{Ball}_t^{\mathrm{fpp}}(\mathfrak{m})$$

the submap of $\mathfrak{m}$ whose associate connected subset of dual edges $\left(\mathrm{Ball}_t^{\mathrm{fpp}}(\mathfrak{m})\right)^\circ$ in $\mathfrak{m}^\dagger$ is the set of all dual edges which have been fully-explored by time $t > 0$, i.e. whose points are all at FPP*-distance less than t from the origin of $\mathfrak{m}^\dagger$ (the root-face of $\mathfrak{m}$). We put $\mathrm{Ball}_0^{\mathrm{fpp}}(\mathfrak{m}) = f_{\mathrm{r}}$.*

If $\mathfrak{m}$ is given with a distinguished point or is infinite and one-ended, we can consider the hull $\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{m})$ of $\mathrm{Ball}_t^{\mathrm{fpp}}(\mathfrak{m})$ obtained by filling-in the finite components of its complement in $\mathfrak{m}$ except for the hole containing the distinguished point/infinity.

13.1.2 Uniform peeling

Since the exponential lengths on the edges are independent and have a density, it is easy to see that, as t increases, there are jump times $0 = T_0 < T_1 < T_2 < \cdots$ for the process $\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{m})$ and that almost surely (depending on the randomness of the x_e) the map $\overline{\mathrm{Ball}}_{T_{i+1}}^{\mathrm{fpp}}(\mathfrak{m})$ is obtained from $\overline{\mathrm{Ball}}_{T_i}^{\mathrm{fpp}}(\mathfrak{m})$ by the peeling of one edge and by filling-in the finite component possibly created. Let us recall the most obvious (randomized in the terminology of Section 11.1.1) peeling algorithm (we focus on filled-in peeling for simplicity):

UNIFORM PEELING

Define a random algorithm $\mathcal{A}_{\mathrm{uniform}}$ as follows: For all submap $\bar{\mathfrak{e}}$ with one hole, pick independently $\mathcal{A}_{\mathrm{uniform}}(\bar{\mathfrak{e}}) \in \partial^* \bar{\mathfrak{e}}$ uniformly at random. If $\mathfrak{m}$ is a random map independent of those choices, then the peeling of $\mathfrak{m}$ using $\mathcal{A}_{\mathrm{uniform}}$ is called the uniform peeling of $\mathfrak{m}$.

The following proposition only relies the randomness of the weights, the map $\mathfrak{m}$ being fixed.

Proposition 13.2 (Eden model and uniform peeling). *If $\mathfrak{m}$ is an infinite map of the plane then under $\mathrm{P}_{\mathfrak{m}}^{\mathrm{fpp}}$ we have:*

[1] Conceived first by Eden Murray in the late 50's: `https://ethw.org/Oral-History:Murray_Eden`

- *the law of* $(\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m}))_{i\geq 0}$ *is that of a filled-in peeling* $\bar{\mathfrak{e}}_0 \subset \cdots \subset \bar{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{m}$ *performed using* $\mathcal{A}_{\mathrm{uniform}}$.
- *conditionally on* $(\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m}))_{i\geq 0}$ *the variables* $\Delta T_i = T_{i+1} - T_i$ *are independent and distributed as exponential variables of parameter (the inverse of the mean) given by the perimeter of its unique hole.*

Proof. Fix $\mathfrak{m}$ and let us imagine the situation at time T_i for $i \geq 0$. We condition on the σ-field $\mathcal{F}_i$ generated by all the exploration up to time T_i. Let us examine the edges in $\mathfrak{m}^\dagger$ which are dual to the boundary of $\bar{\mathfrak{e}}_i = \overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m})$. These come in two types: *type-1* edges that are adjacent to a new face in the unexplored part (that is, if we peel one of those edges we have an event C), and *type-2* edges that link two faces adjacent to the boundary of the explored part (that is, if we peel one of these edges we have an event G_{k_1,k_2}). See Figure 13.1.

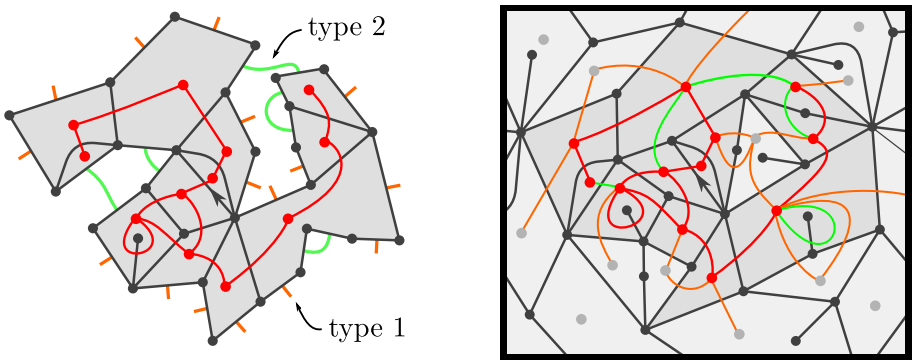


Figure 13.1: Illustration of the proof of Proposition 13.2. The edges of the first type are in orange and those of the second type are in green. Regardless of their number and locations, the next edge to peel can be taken uniformly on the boundary and the increase of time is given by an exponential variable of parameter given by the perimeter.

Let us consider an edge $e^{(1)}$ of the first type and denote by $e^{(1)}_-$ its extremity in the explored region. Since this edge has not been fully explored at time T_i, its weight $x_{e^{(1)}}$ must satisfy $x_{e^{(1)}} > T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(1)}_-, f_{\mathrm{r}})$ and furthermore by properties of exponential variables conditionally on $\mathcal{F}_i$, the variable

$$y_{e^{(1)}} := x_{e^{(1)}} - \big(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(1)}_-, f_{\mathrm{r}})\big)$$

has the law $\mathcal{E}(1)$ of an exponential variable of parameter 1. Let us now examine the situation for an edge $e^{(2)}$ of the second type. We denote by $e^{(2)}_-$ and $e^{(2)}_+$ its endpoints. Since $e^{(2)}$ is being explored from both sides but has not been fully explored by time T_i, we must have $x_{e^{(2)}} > \big(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(2)}_-, f_{\mathrm{r}})\big) + \big(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(2)}_+, f_{\mathrm{r}})\big)$ and by the same argument as above, conditionally on $\mathcal{F}_i$, the variable

$$y_{e^{(2)}} := x_{e^{(2)}} - \big(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(2)}_-, f_{\mathrm{r}})\big) - \big(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(2)}_+, f_{\mathrm{r}})\big)$$

is again exponentially distributed with parameter 1. Of course, an edge of the second type is dual to two edges of the boundary of $\overline{\mathfrak{e}}_i$. Apart from this trivial identification, the variables y_e where e runs over the edges dual to the boundary of $\overline{\mathfrak{e}}_i$ are, conditionally on $\mathcal{F}_i$, independent of each other. Now, the time it takes until a new edge is fully explored is equal to

$$T_{i+1} - T_i = \inf\{y_e : e \text{ of the first type}\} \wedge \frac{1}{2}\inf\{y_e : e \text{ of the second type}\},$$

where the factor $1/2$ again comes from the fact that edges of the second type are explored from both sides. By the above independence property and standard properties of exponential laws, $T_{i+1} - T_i$ is thus distributed as an exponential variable of parameter (the inverse of its mean)

$$T_{i+1} - T_i \overset{(d)}{=} \mathcal{E}(\#\{\text{edges of the first type}\} + 2\#\{\text{edges of the second type}\}) = \mathcal{E}(2\ell)$$

where 2ℓ is the perimeter of the hole of $\overline{\mathfrak{e}}_i$. That proves the second part of the proposition. To see that conditionally on $\mathcal{F}_i$ the next edge to peel is uniform on the boundary, we may replace for each edge $e^{(2)}$ of the second type the variable $\frac{1}{2}y_{e^{(2)}}$ of law $\mathcal{E}(2)$ by the minimum of two independent exponential variables $\tilde{y}_{e_1^{(2)}}$ and $\tilde{y}_{e_2^{(2)}}$ of law $\mathcal{E}(1)$ which we attach on the two edges on the boundary of $\overline{\mathfrak{e}}_i$ traversed by $e^{(2)}$. Finally, everything boils down to assigning to each edge of the boundary of the explored map an independent exponential variable of parameter 1; the next edge to peel is the one carrying the minimal weight which is then uniform as desired. This completes the proof. □

13.1.3 Fpp growth on $\mathfrak{M}_\infty$

We can now apply our results on the scaling limit of filled-in peeling processes (Theorem 10.8 and Proposition 10.12) to study the geometry of the balls of increasing radius for the fpp-distance on infinite Boltzmann maps of the plane. Let $(\overline{\mathfrak{e}}_n)_{n\geq 0}$ be the filled-in peeling exploration associated to the fpp-distance on $\mathfrak{m}^\dagger$ by Proposition 13.2 and recall the notation T_n for the fpp-time to $\partial^*\overline{\mathfrak{e}}_n$. To simplify notation we again write $P_n = |\partial^*\overline{\mathfrak{e}}_n|$ and $V_n = |\overline{\mathfrak{e}}_n|$. The evolution of (P_n, V_n) is understood in Chapter 14. The crux is that by Proposition 13.2 we known that conditionally on $(P_i, V_i)_{i\geq 0}$ we have

$$T_n \quad = \quad \sum_{i=0}^{n-1} \frac{\mathbf{e}_i}{2P_i}, \qquad \text{for all } n \geq 0, \tag{13.1}$$

where $\mathbf{e}_i$ are independent exponential variables of expectation 1. Hence, the scaling limit of the perimeter process will drive the scaling limit of the fpp-distance process and different regime will appear depending whether $\sum_i \frac{1}{P_i}$ is convergent, divergent or barely divergent.

Theorem 13.3 (*Growth of Fpp on* $\mathbf{M}_\infty$)

Fix a weight sequence $\mathbf{q}$ *of type* $a \in (3/2; 5/2]$ *and recall the notation of Theorem 10.8:*

- *If $a \in (2;5/2]$ we have the convergence in distribution in the sense of Skorokhod*

$$\left(\frac{|\partial^*\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{tn}(\mathfrak{M}_\infty)|}{n^{\frac{1}{a-2}}}, \frac{|\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{tn}(\mathfrak{M}_\infty)|}{n^{\frac{a-1/2}{a-2}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^{\uparrow}, \mathfrak{b}_{\mathfrak{q}} \cdot \mathcal{V}_a(\Upsilon_a^{\uparrow})\right)(\vartheta_{2\mathfrak{p}_{\mathfrak{q}}} t)\right)_{t\geq 0}, \quad (13.2)$$

  where for $t \geq 0$ we have put $\vartheta_t = \inf\{s \geq 0 : \int_0^s \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)} \geq t\}$.

- *If $a = 2$ we have the convergences in probability*

$$\left(\frac{\log|\partial^*\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{tn}(\mathfrak{M}_\infty)|}{n}, \frac{\log|\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{tn}(\mathfrak{M}_\infty)|}{n}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(\mathbb{P})} \pi^2\mathfrak{p}_{\mathfrak{q}}\left(t, \frac{3}{2}t\right)_{t\geq 0}.$$

- *If $a \in (3/2;2)$ we have*

$$\mathbb{E}\left[\mathrm{d}_{\mathrm{fpp}}(f_{\mathrm{r}}, \infty)\right] = \mathbf{E}^{(1)}[N_0] < \infty,$$

  where $\mathrm{d}_{\mathrm{fpp}}(f_{\mathrm{r}}, \infty)$ is the infimum of the FPP*-length of all infinite paths in $\mathfrak{m}^\dagger$, and N_0 is the number of times the random walk $(S_i)_{i\geq 0}$ started at 1 visits 0.*

Remark 13.4. The transformation $(X_t)_{t\geq 0} \mapsto (X_{\vartheta_t})_{t\geq 0}$ where $\vartheta_t = \inf\{s \geq 0 : \int_0^s X_u^{-1}\mathrm{d}u \geq t\}$ is called the **Lamperti transformation**, see [136]. It is ubiquitous in the study of positive self-similar Markov processes. In the case when $a = \frac{5}{2}$ since $\Upsilon_{5/2}$ has no positive jumps, the Lamperti transform $\Upsilon_a^{\uparrow}(\vartheta_t)$ has the same distribution as the time-reversal of a branching process with branching mechanism $\phi(\lambda) = c\lambda^{3/2}$ started from $-\infty$ and conditioned to die at 0, for some $c > 0$. See [91, Section 4.4]. This is not true for other values of a since the process Υ_a has positive and negative jumps.

We thus see a different behavior depending on a: polynomial volume growth if $a \in (2;5/2]$, exponential volume growth at $a = 2$ and even explosiveness when $a \in (3/2;2)$. The proof of the above theorem relies on Theorem 10.8 and on the understanding of the scaling limit of $\sum_{0\leq i\leq n}\frac{1}{P_i}$. Recall that we write $\mathbf{P}^{(\ell)}$ and $\mathbf{E}^{(\ell)}$ for the probability and expectation under which S,$S^{\uparrow}$ and $S^{\downarrow}$ are started from $\ell \geq 1$.

Lemma 13.5. *Suppose that $\mathbf{q}$ is critical. We have*

$$\mathbf{E}^{(1)}\left[\frac{1}{S_n^{\uparrow}}\right] = \mathbf{P}^{(1)}(S_n^{\downarrow} > 0) = 2\sum_{k=n+1}^{\infty}\frac{1}{k}\mathbf{P}^{(1)}(S_k = 0). \quad (13.3)$$

Proof. The first equality in (13.3) follows directly from the definition of the $h^{\uparrow}$-transform and the exact forms of $h^{\uparrow}$ and $h^{\downarrow}$:

$$\begin{aligned}\mathbf{E}^{(1)}\left[\frac{1}{S_n^{\uparrow}}\right] &= \sum_{k=1}^{\infty}\frac{1}{k}\mathbf{P}^{(1)}(S_n^{\uparrow} = k) = \sum_{k=1}^{\infty}\frac{1}{k}\frac{h^{\uparrow}(k)}{h^{\uparrow}(1)}\mathbf{P}^{(1)}(S_i > 0 \text{ for } 1 \leq i < n,\, S_n = k)\\ &= \frac{h^{\downarrow}(1)}{h^{\uparrow}(1)}\sum_{k=1}^{\infty}\frac{h^{\uparrow}(k)}{kh^{\downarrow}(1)}\mathbf{P}^{(1)}(S_n^{\downarrow} = k) = \sum_{k=1}^{\infty}\mathbf{P}^{(1)}(S_n^{\downarrow} = k) = \mathbf{P}^{(1)}(S_n^{\downarrow} > 0),\end{aligned}$$

which gives the first claim. Since $\inf\{i \geq 0 : S_i^{\downarrow} = 0\}$ is a.s. finite, we may identify

$$\mathbf{E}^{(1)}\left[\frac{1}{S_n^{\uparrow}}\right] = \mathbf{P}^{(1)}(S_n^{\downarrow} > 0) = \frac{1}{h^{\downarrow}(1)} \sum_{k=n+1}^{\infty} \mathbf{P}^{(1)}(S_i > 0 \text{ for } 1 \leq i < k,\, S_k = 0). \tag{13.4}$$

We now use the cyclic lemma (Lemma A.4) to re-interpret the probabilities in the sum and get

$$\mathbf{P}^{(1)}(S_i > 0 \text{ for } 1 \leq i < k,\, S_k = 0) = \frac{1}{k}\mathbf{P}^{(1)}(S_k = 0).$$

Together with (13.4) and $h^{\downarrow}(1) = 1/2$ this implies the last equality in (13.3). □

Elements of proof of Theorem 13.3. Dense phase $a \in (3/2; 2)$. We start with the dense phase $a < 2$ which is the easiest. This boils down to computing the expectation of $T_\infty = \lim_{i\to\infty} T_i$. Using (13.1) we have

$$\mathbb{E}[T_\infty] = \sum_{i=0}^{\infty} \mathbb{E}\left[\frac{1}{2P_i}\right] \underset{\text{Lem. 13.5}}{=} \sum_{k=1}^{\infty} \mathbf{P}^{(1)}(S_k = 0) = \mathbf{E}^{(1)}[N_0]. \tag{13.5}$$

From the local limit theorem (Theorem A.21) we have $\mathbf{P}^{(1)}(S_k = 0) \sim C_0\, k^{-1/(a-1)}$ as $k \to \infty$ for some constant $C_0 > 0$ and so when $a \in (3/2; 2)$ we have $\mathbf{E}^{(1)}[N_0] < \infty$ (in other words, the walk $(S_i)_{i\geq 0}$ is transient whenever $a < 2$).

Critical phase $a = 2$. We only sketch the proof and refer the reader to [57] for details. In the case $a = 2$ the perimeter P_i is of order i and actually one can prove $\mathbb{E}[i/P_i] \to \mathbb{E}[(\mathfrak{p}_{\mathfrak{q}}\Upsilon_1^{\uparrow}(1))^{-1}] = 2/(\pi^2\mathfrak{p}_{\mathfrak{q}})$ as $i \to \infty$ (see [57, Remark 3]), so

$$\frac{1}{\log n}\sum_{i=1}^{n} \mathbb{E}\left[\frac{1}{2P_i}\right] \to \frac{1}{\pi^2\mathfrak{p}_{\mathfrak{q}}}. \tag{13.6}$$

It turns out that the process $(P_i)_{i\geq 0}$ decorrelates over scales, and since each scale contributes to roughly a constant in $\sum 1/P_i$, we can prove concentration of $\sum_{i=1}^{n} \frac{1}{2P_i} \approx T_n$ around its mean

$$\mathbb{E}\left[\left|\frac{T_n}{\log n} - \frac{1}{\pi^2\mathfrak{p}_{\mathfrak{q}}}\right|\right] \to 0. \tag{13.7}$$

From this, the proof of the second item of the theorem is easy to finish: Since the process T is increasing, if we set $U_t = \inf\{n \geq 1 : T_n > t\}$ then we have $n^{-1}\log U_{nt} \to t \cdot \pi^2\mathfrak{p}_{\mathfrak{q}}$ in probability (in the Skorokhod topology) and this convergence holds jointly with that of Lemma 10.9. Recalling that $\bar{\mathfrak{e}}_i = \overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m})$ we easily deduce the second statement of the theorem.

Dilute case $a \in (2; 5/2]$. We now focus on the case when $a \in (2; 5/2]$. As above, since we have $\bar{\mathfrak{e}}_i = \overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m})$ the first item of the theorem is easily deduced once we have established the following convergence in distribution (in the Skorokhod sense) of perimeters, volume and fpp-times in the uniform exploration of $\mathfrak{M}_\infty$:

$$\left(\frac{P_{\lfloor nt\rfloor}}{n^{\frac{1}{a-1}}}, \frac{V_{\lfloor nt\rfloor}}{n^{\frac{a-1/2}{a-1}}}, \frac{T_{\lfloor nt\rfloor}}{n^{\frac{a-2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^{\uparrow}, \mathfrak{b}_{\mathfrak{q}}\mathcal{V}_a(\Upsilon_a^{\uparrow}), \frac{1}{2\mathfrak{p}_{\mathfrak{q}}}\int_0^{\cdot} \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)}\right)(\mathfrak{p}_{\mathfrak{q}}t)\right)_{t\geq 0}.$$

The joint convergence of the first two components is given by Theorem 10.8. We now prove the convergence of the third component jointly with the first two using (13.1). By the convergence of the first component in the proposition, for every $0 < s < t$ we have

$$n^{-\frac{a-2}{a-1}} \sum_{i=[ns]+1}^{[nt]} \frac{1}{P_i} = \int_{n^{-1}([ns]+1)}^{n^{-1}([nt]+1)} \frac{\mathrm{d}u}{n^{\frac{-1}{a-1}} P_{[nu]}} \xrightarrow[n\to\infty]{(d)} \int_s^t \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(\mathfrak{p}_{\mathfrak{q}} u)},$$

and, on the other hand,

$$\begin{aligned}\mathbb{E}\left[\left(n^{-\frac{a-2}{a-1}} \sum_{i=[ns]+1}^{[nt]} \frac{\mathbf{e}_i}{P_i} - n^{-\frac{a-2}{a-1}} \sum_{i=[ns]+1}^{[nt]} \frac{1}{P_i}\right)^2 \middle| (P_k)_{k\geq 0}\right] \\ = n^{-\frac{2a-4}{a-1}} \sum_{i=[ns]+1}^{[nt]} \frac{1}{(P_i)^2} = \frac{1}{n} \int_{n^{-1}([ns]+1)}^{n^{-1}([nt]+1)} \frac{\mathrm{d}u}{(n^{\frac{-1}{a-1}} P_{[nu]})^2}\end{aligned}$$

converges to 0 in probability as $n \to \infty$. It easily follows that

$$\left(n^{-\frac{a-2}{a-1}} \left(T_{[nt]} - T_{[n\varepsilon]}\right)\right)_{t\geq\varepsilon} \xrightarrow[n\to\infty]{(d)} \left(\frac{1}{2\mathfrak{p}_{\mathfrak{q}}} \int_{\mathfrak{p}_{\mathfrak{q}}\varepsilon}^{\mathfrak{p}_{\mathfrak{q}} t} \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)}\right)_{t\geq\varepsilon}, \tag{13.8}$$

and this convergence holds jointly with the first two components considered in the proposition. Hence, to finish the proof of the proposition, it suffices to see that for any $\delta > 0$ we have

$$\lim_{\varepsilon\to 0} \sup_{n\geq 1} \mathbb{P}\left(n^{-\frac{a-2}{a-1}} \cdot T_{[n\varepsilon]} > \delta\right) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \mathbb{P}\left(\int_0^{\varepsilon} \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)} > \delta\right) = 0, \tag{13.9}$$

so that we could harmlessly let $\varepsilon \to 0$ in (13.8). To prove the first claim, notice that from Lemma 13.5 together with the local limit theorem we deduce that for some constant $C > 0$ we have for any $n \geq 1$

$$\mathbb{E}^{(1)}[1/S_n^{\uparrow}] \leq C n^{-1/(a-1)}.$$

We use this estimate to get

$$\mathbb{E}[T_{[n\varepsilon]}] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=0}^{[n\varepsilon]-1} \frac{\mathbf{e}_i}{2P_i} \middle| (P_i)_{i\geq 0}\right]\right] = \sum_{i=0}^{[n\varepsilon]-1} \mathbb{E}\left[\frac{1}{2P_i}\right] \quad \leq \quad C(\varepsilon n)^{\frac{a-2}{a-1}},$$

for some constant $C > 0$. The first point of (13.9) thus follows from an application of Markov's inequality. The second statement just follows from the fact that $(\Upsilon_a^{\uparrow}(t))^{-1}$ is almost surely integrable around 0^+ since $a > 2$, and this follows from the previous estimate: take expectations in (13.8) and use Fatou's lemma to get

$$\frac{1}{2\mathfrak{p}_{\mathfrak{q}}} \mathbb{E}\left[\int_{\varepsilon}^{1} \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)}\right] \leq C(1 - \varepsilon^{\frac{a-2}{a-1}}), \qquad \text{for some } C > 0.$$

Sending $\varepsilon \to 0$ we deduce that indeed $(\Upsilon_a^{\uparrow}(u))^{-1}$ is almost surely integrable around 0. □

Open Question 13.6 (Eden height of a cylinder)**.** *In conjunction with Open Question 10.2, consider $\mathcal{L}(p,q)$ be the Eden distance between the root and target face of a finite Boltzmann map $\mathfrak{M}_q^{(p)}$ of the "cylinder", i.e. under the law $\mathbb{P}_q^{(p)}$. Is there an exact expression for $\mathbb{E}[\mathcal{L}(p,q)]$ (in the same fashion as in (13.5))? Study the scaling limit of $\mathcal{L}(p,q)$ for regular weight sequences of type $a \in (3/2;5/2]$.*

13.1.4 Fpp growth on $\mathfrak{H}_\infty$

We now state the analogs of the previous results for hyperbolic random maps. As in Section 8.2.2, we restrict to finitely supported subcritical weight sequences $\mathbf{q}$ for which we can prove a very precise exponential volume growth in $\mathfrak{H}_\infty$:

Proposition 13.7 (Sharp exponential volume growth)**.** *Suppose that $\mathbf{q}$ is subcritical and finitely supported. With the notation of Proposition 10.12 we have*

$$\exp\left(-2\mathrm{p}_{\mathbf{q}}^{\mathrm{hyp}}\cdot t\right)\cdot\left(|\partial^*\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{H}_\infty)|,|\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{H}_\infty)|\right)\xrightarrow[t\to\infty]{a.s.}\mathcal{X}_\infty\cdot(\mathrm{p}_{\mathbf{q}}^{\mathrm{hyp}},\mathrm{b}_{\mathbf{q}}^{\mathrm{hyp}}),$$

where $\mathcal{X}_\infty$ is a positive random variable.

Proof. The proof is very similar to the previous ones: given Proposition 10.12 we just have to verify that if (P_n) is the process of the half-perimeters during the uniform peeling of $\mathfrak{H}_\infty$ then the fpp time $T_n=\sum_{i=0}^{n-1}\frac{\mathbf{e}_i}{2P_i}$ satisfies

$$T_n=\frac{1}{2\mathrm{p}_{\mathbf{q}}^{\mathrm{hyp}}}\left(\log n+Z_\infty+\delta_n\right),\tag{13.10}$$

where Z_∞ is a real random variable and $\delta_n\to 0$ almost surely. The proposition follows from the previous display and Proposition 10.12 with $\mathcal{X}_\infty=\exp(-Z_\infty)$. To prove (13.10), notice that if $\tilde{S}_n$ is a $\tilde{\nu}$-random walk, since $\tilde{\nu}$ has finite variance (it has finite support) and positive mean $\mathrm{p}_{\mathbf{q}}^{\mathrm{hyp}}$, the law of the iterated logarithm shows that

$$\frac{\tilde{S}_n-\mathrm{p}_{\mathbf{q}}^{\mathrm{hyp}}\cdot n}{n^{1/2+\varepsilon}}\xrightarrow[n\to\infty]{a.s.}0.$$

This result is easily extended to P_n instead of $\tilde{S}_n$ by conditioning the walk to stay positive (an event of positive probability). Hence we can write

$$\begin{aligned}T_n&=\sum_{i=0}^{n-1}\frac{\mathbf{e}_i}{2P_i}=\sum_{i=1}^{n-1}\frac{1}{2\mathrm{p}_{\mathbf{q}}^{\mathrm{hyp}}\cdot i}+\left(\sum_{i=0}^{n-1}\frac{1}{2P_i}-\sum_{i=1}^{n-1}\frac{1}{2\mathrm{p}_{\mathbf{q}}^{\mathrm{hyp}}\cdot i}\right)+\left(\sum_{i=0}^{n-1}\frac{\mathbf{e}_i}{2P_i}-\sum_{i=0}^{n-1}\frac{1}{2P_i}\right)\\&=\frac{\log n+\gamma+o(1)}{2\mathrm{p}_{\mathbf{q}}^{\mathrm{hyp}}}+\sum_{i=1}^{n-1}\frac{O(i^{1/2+\varepsilon})}{i^2}+\sum_{i=0}^{n}\frac{\mathbf{e}_i-1}{2P_i},\end{aligned}$$

where γ is Euler's constant and $O(\cdot)$ is almost surely bounded by the previous display. The first series on the previous line converges almost surely, as well as the second one by an easy application of the three-series theorem. This completes the proof of (13.10) and of the proposition. □

13.2 Dual graph distances

In this section we use the peeling process to study the graph metric on the *dual* map $\mathfrak{m}^\dagger$. The strategy is the same as in the previous section: we first design a peeling algorithm that explores balls of increasing radii for the dual distance and then study the growth of the radius during the exploration.

13.2.1 Exploration of dual metric

Let $\mathfrak{m}$ be a finite or infinite (planar bipartite rooted) map, we denote by $\mathrm{d}_{\mathrm{gr}}^\dagger$ the graph distance on the dual map $\mathfrak{m}^\dagger$. If $f \in \mathsf{Faces}(\mathfrak{m})$ the dual distance to the root face $\mathrm{d}_{\mathrm{gr}}^\dagger(f, f_{\mathrm{r}})$ is called the *height* of f in $\mathfrak{m}$ (as usual if the map has no boundary, we may and will consider that is has a boundary of perimeter 2).

Definition 13.8 (Ball for the dual distance). *We define the dual ball of radius r*

$$\mathrm{Ball}_r^\dagger(\mathfrak{m})$$

as the map made by keeping only the faces of $\mathfrak{m}$ that are at height less than or equal to r and cutting along all the edges which are adjacent on both sides to faces at height r. Equivalently, the corresponding connected subset

$$\left(\mathrm{Ball}_r^\dagger(\mathfrak{m})\right)^\circ$$

of dual edges in $\mathfrak{m}^\dagger$ is given by those edges of $\mathfrak{m}^\dagger$ which contain at least one endpoint at height strictly less than r. By convention we also put $\mathrm{Ball}_0^\dagger(\mathfrak{m})$ to be the root face of $\mathfrak{m}$.

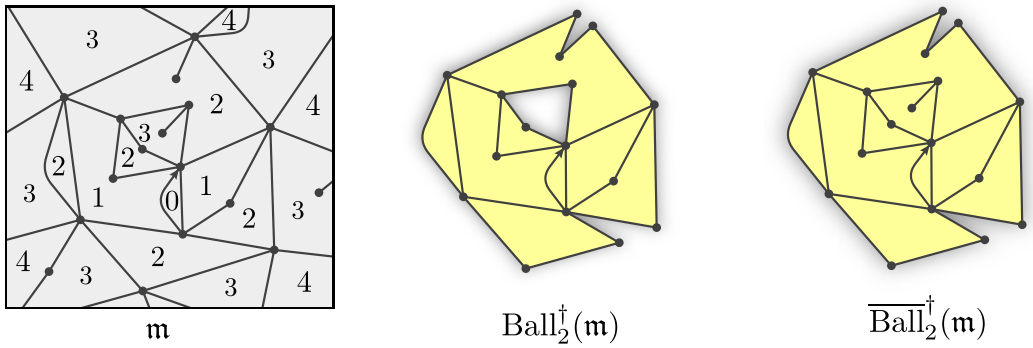


Figure 13.2: The dual ball of radius 2 and its hull in some underlying planar map $\mathfrak{m}$.

Notice that with this definition, $\mathrm{Ball}_r^\dagger(\mathfrak{m})$ is indeed a submap of $\mathfrak{m}$ in the sense of Section 4.1. Also, when $\mathfrak{m}$ is pointed or one-ended we write $\overline{\mathrm{Ball}}_r^\dagger(\mathfrak{m})$ for the hull of these balls, which are obtained by filling-in all the holes of $\mathrm{Ball}_r^\dagger(\mathfrak{m})$ inside $\mathfrak{m}$ which do not contain the distinguished point (or infinity). We will now devise a peeling algorithm that discovers those balls for increasing radii. We concentrate on filled-in peeling for simplicity.

PEELING BY LAYERS ON THE DUAL

Suppose that $\bar{\mathfrak{e}} \subset \mathfrak{m}$ verifies the following property: **(H)** there exists an integer $h \geq 0$ such that all the edges on the boundary of the hole are at height h or $h+1$, and that the set of edges at height h forms a connected segment. Then if all edges on the boundary of the hole of $\bar{\mathfrak{e}}$ are at the same height h then $\mathcal{A}_{\mathrm{dual}}(\bar{\mathfrak{e}})$ is any (deterministically chosen) edge on the boundary; otherwise $\mathcal{A}_{\mathrm{dual}}(\bar{\mathfrak{e}})$ is the unique edge at height h such that the edge immediately on its left is at height $h+1$.

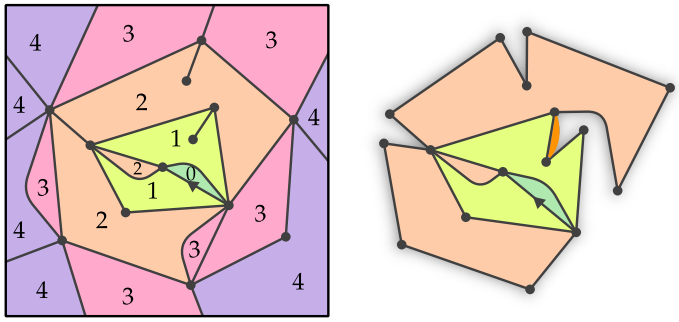


Figure: The left figure shows a portion of an infinite planar map with faces coloured according to (dual) graph distance to the root-face. The submap on the right depicts a possible state of the peeling by layers. Its hole has three edges at height $h = 1$ and eleven edges at height $h+1 = 2$. The next edge to peel is indicated in orange.

The first step $\bar{\mathfrak{e}}_0$ satisfies **(H)** and one easily checks inductively that if we use this algorithm, then for any $i \geq 0$, the i-th step $\bar{\mathfrak{e}}_i$ satisfies it as well, see Figure 13.3. The sequence $(\bar{\mathfrak{e}}_i)_{i\geq 0}$ thus forms a well-defined filled-in peeling exploration which we call the *peeling by layers* of $\mathfrak{m}^\dagger$. The name is justified by the following observation (which we leave as an exercise for the reader): for every $i \geq 0$, let H_i be the minimal height of an edge on the boundary of $\bar{\mathfrak{e}}_i$, so that $H_0 = 0$ and for each $i \geq 0$, we have $\Delta H_i := H_{i+1} - H_i \in \{0, 1\}$. For each $r \geq 0$, if we denote by $\theta_r = \inf\{i \geq 0 : H_i \geq r\}$ then

$$\bar{\mathfrak{e}}_{\theta_r} = \overline{\mathrm{Ball}}_r^\dagger(\mathfrak{m}) \qquad \text{for every} \qquad r \geq 0. \tag{13.11}$$

13.2.2 Growth of the dual metric in $\mathfrak{M}_\infty$

Here is the analog of Theorem 13.3 for the dual distance.

Theorem 13.9 (*Growth of dual metric in* $\mathbf{M}_\infty$)

Fix a weight sequence $\mathbf{q}$ *of type* $a \in (3/2; 5/2]$. *With the same notation as in Theorem 10.8 and recalling Definition 11.1, we have the convergences in distribution in the sense of Skorokhod*

- *if* $a \in (2; 5/2]$

$$\left(\frac{|\partial^* \overline{\mathrm{Ball}}^\dagger_{\lfloor tn \rfloor}(\mathfrak{M}_\infty)|}{n^{\frac{1}{a-2}}}, \frac{|\overline{\mathrm{Ball}}^\dagger_{\lfloor tn \rfloor}(\mathfrak{M}_\infty)|}{n^{\frac{a-1/2}{a-2}}} \right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^\uparrow, \mathfrak{b}_{\mathbf{q}} \cdot \mathcal{V}_a(\Upsilon_a^\uparrow) \right) (\vartheta_{2\mathfrak{p}_{\mathbf{q}} t/(1+\mathfrak{g}_{\mathbf{q}})}) \right)_{t\geq 0},$$

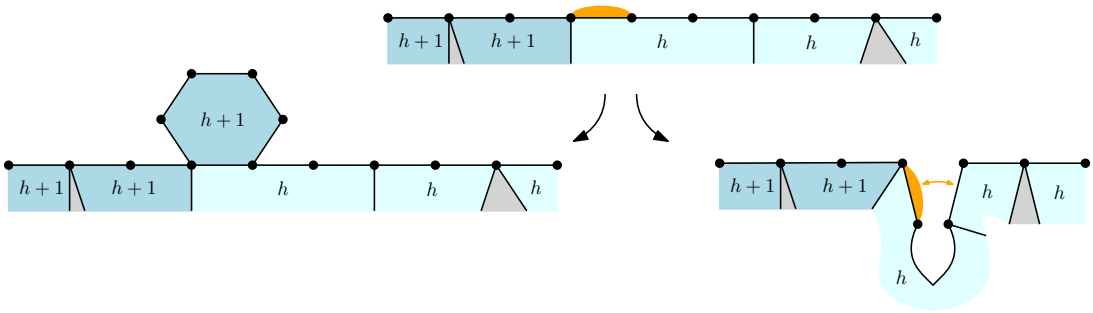


Figure 13.3: Illustration of two possible outcomes of one step of the peeling by layers for the dual distance. The edge to peel is in orange. Notice that the boundary condition (**H**) is preserved.

where for $t \geq 0$ we have put $\vartheta_t = \inf\{s \geq 0 : \int_0^s \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)} \geq t\}$.

- *if $a = 2$*

$$\left(\frac{\log|\partial^*\overline{\mathrm{Ball}}^{\dagger}_{[tn]}(\mathfrak{M}_\infty)|}{\sqrt{n}}, \frac{\log|\overline{\mathrm{Ball}}^{\dagger}_{[tn]}(\mathfrak{M}_\infty)|}{\sqrt{n}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \pi\sqrt{2}\cdot\left(\sqrt{t}, \frac{3}{2}\cdot\sqrt{t}\right)_{t\geq 0},$$

- *if $a \in (3/2; 2)$ there exists a constant $\mathsf{c}_a \in (0,\infty)$ such that*

$$\left(\frac{\log|\partial^*\overline{\mathrm{Ball}}^{\dagger}_{[tn]}(\mathfrak{M}_\infty)|}{n}, \frac{\log|\overline{\mathrm{Ball}}^{\dagger}_{[tn]}(\mathfrak{M}_\infty)|}{n}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \mathsf{c}_a\cdot(t, (a-1/2)\cdot t)_{t\geq 0}.$$

Notice the similarity between the first item of the previous theorem and the first item of Theorem 13.3. Indeed, the only difference in the limiting process is the linear time change $t \mapsto \frac{t}{(1+\mathsf{g_q})}$. This obviously motivates the following conjecture:

Open Question 13.10. *When $a \in (2; 5/2]$ show that $\overline{\mathrm{Ball}}^{\dagger}_{[r(1+\mathsf{g_q})]}(\mathfrak{M}_\infty) \approx \overline{\mathrm{Ball}}^{\mathrm{fpp}}_{r}(\mathfrak{M}_\infty)$ in the sense that for any $\varepsilon > 0$, almost surely, we have for all sufficiently large r*

$$\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{[r(1-\varepsilon)]}(\mathfrak{M}_\infty) \subset \overline{\mathrm{Ball}}^{\dagger}_{[r(1+\mathsf{g_q})]}(\mathfrak{M}_\infty) \subset \overline{\mathrm{Ball}}^{\mathrm{fpp}}_{[r(1+\varepsilon)]}(\mathfrak{M}_\infty).$$

This is proved [92] in the case of random triangulations.

Remark also that in the dense phase $a \in (3/2; 2)$, the volume growth of dual balls is now exponential whereas it was explosive in the fpp metric. In the critical case $a = 2$ we find a rather exotic intermediate volume growth $\exp(\Theta(\sqrt{n}))$.

Rough idea of the proof. The proof of the above result is more technical than for the fpp metric but relies on the same set of ideas. We will present the main heuristic behind the proof and

refer the reader to [55] and [57] for details. Let $(\overline{\mathfrak{e}}_n)_{n\geq 0}$ be the filled-in peeling exploration with algorithm $\mathcal{A}_{\text{dual}}$ and denote by P_n, V_n and H_n respectively the half-perimeter, volume processes as well as the minimal height of an edge on the boundary of $\partial^*\overline{\mathfrak{e}}_n$. Recall from Theorem 10.8 that we have the convergence

$$\left(\frac{P_{[nt]}}{n^{\frac{1}{a-1}}}, \frac{V_{[nt]}}{n^{\frac{a-1/2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^{\uparrow}, \mathfrak{b}_{\mathfrak{q}}\mathcal{V}_a(\Upsilon_a^{\uparrow})\right)(\mathfrak{p}_{\mathfrak{q}}t)\right)_{t\geq 0}.$$

Using the same arguments as in the proof of Theorem 13.3, the theorem will be proved provided that we can show that jointly with the previous display we have

$$\left(\frac{H_{[nt]}}{n^{\frac{a-2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\frac{1+\mathfrak{g}_{\mathfrak{q}}}{2\mathfrak{p}_{\mathfrak{q}}}\int_0^{\mathfrak{p}_{\mathfrak{q}}t} \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)}\right)_{t\geq 0} \qquad a\in(2;5/2] \tag{13.12}$$

$$\frac{H_n}{\log^2 n} \xrightarrow[n\to\infty]{(\mathbb{P})} \frac{1}{2\pi^2} \qquad a=2 \tag{13.13}$$

$$\frac{H_n}{\log n} \xrightarrow[n\to\infty]{(\mathbb{P})} \frac{1}{c_a} \qquad a\in(3/2;2). \tag{13.14}$$

Contrary to the case of the first-passage percolation where we had $T_{n+1}-T_n = \frac{\mathfrak{e}_n}{2P_n}$, to understand how H evolves, one needs to understand at which "speed" the peeling with algorithm $\mathcal{A}_{\text{dual}}$ "turns" around the current boundary. More precisely, imagine that at time θ_h we are facing a boundary of perimeter $2p = 2P_{\theta_h}$ with all edges at height h: how many peeling steps $\Delta\theta_h$ with algorithm $\mathcal{A}_{\text{dual}}$ do we need to turn around this boundary so that all edges of the explored part are now at height $h+1$? The answer turns out to be

$$\underbrace{\Delta\theta_h \approx \frac{2p}{1+\mathfrak{g}_{\mathfrak{q}}}}_{\text{if } a\in(2;5/2]}, \qquad \underbrace{\Delta\theta_h \approx \frac{1}{\mathfrak{p}_{\mathfrak{q}}}\cdot\frac{2p}{\log p}}_{\text{if } a=2}, \qquad \underbrace{\Delta\theta_h \lesssim p^{a-1}}_{\text{if } a\in(3/2;2)}. \tag{13.15}$$

To see why those assertions are believable, consider a "typical" peeling step that occur between height h and $h+1$. When performing such a step, we will typically have a large boundary component of edges at height $h+1$ on one's left and a large boundary component of edges at height h on one's right, see Figure 13.3. Hence the change in the number of edges at height h can be approximated by

$$-1-\tilde{\mathcal{G}}_r$$

where $\tilde{\mathcal{G}}_r$ is the number of edges swallowed on the right of the peeled edge in the plane model (we neglect the fact that a peeling step may actually swallow the entire set of edges at height h). When the perimeter is large, we have the half-planar approximation $\tilde{\mathcal{G}}_r \approx \mathcal{G}_r$. When $a\in(2;5/2]$ the number of edges at height h has an approximate mean drift of

$$1+\mathfrak{g}_{\mathfrak{q}},$$

hence the guess $\Delta\theta_h \approx \frac{2p}{1+\mathfrak{g}_{\mathfrak{q}}}$. When $a=2$ the speed at which the peeling by layer turns around $\overline{\mathfrak{e}}_n$ is given by an *unbalanced* random walk with Cauchy-type tails. By standard results, the rate of

growth of such random walks is of order $\mathsf{p_q} n \log n$. This justifies the heuristic $\Delta\theta_h \approx \frac{1}{\mathsf{p_q}} \cdot \frac{2p}{\log p}$. In the dense setting, the measure ν has a heavy tail and S_n grows like $n^{1/(a-1)}$ hence it is logical to expect that we need of order p^{a-1} peeling steps to complete a full layer of perimeter p. With a leap of faith one may thus believe in the rough differential picture:

$$\Delta H_n \approx \frac{1+\mathsf{g_q}}{2}\frac{1}{P_n}, \quad \Delta H_n \approx \mathsf{p_q} \cdot \frac{\log P_n}{P_n} \quad \text{and} \quad \Delta H_n \approx \frac{\mathrm{cst}}{(P_n)^{a-1}} \approx \frac{\mathrm{cst}}{n}, \tag{13.16}$$

respectively for $a \in (2;5/2]$, $a = 2$ and $a \in (3/2;2)$. Integrating (formally) those approximations and using Theorem 10.8 one can understand why (13.12) (13.13) and (13.14) are reasonable to expect. □

13.2.3 Growth of the dual metric on $\mathfrak{H}_\infty$

We now state the analog of Proposition 13.7 for the dual metric on hyperbolic random maps. Notice again the change $2\mathsf{p_q^{hyp}} \to \frac{2\mathsf{p_q^{hyp}}}{1+\mathsf{g_q^{hyp}}}$ which now appears in the exponential volume growth:

Proposition 13.11 (Growth of dual distances on $\mathfrak{H}_\infty$). *Suppose that* $\mathbf{q}$ *is subcritical and finitely supported. With the notation Proposition 10.12 and Definition 12.8 we have for* $r \in \{0,1,2,3,\dots\}$

$$\exp\left(-\frac{2^{\mathsf{hyp}}_{\mathbf{q}}}{1+\}^{\mathsf{hyp}}_{\mathbf{q}}} \cdot r\right) \cdot \left(|\partial^* \overline{\mathrm{Ball}}^\dagger_r(\mathfrak{H}_\infty)|, |\overline{\mathrm{Ball}}^\dagger_r(\mathfrak{H}_\infty)|\right) \xrightarrow[r\to\infty]{a.s.} \mathcal{Y}_\infty \cdot (\mathsf{p_q^{hyp}}, \mathsf{b_q^{hyp}}),$$

where $\mathcal{Y}_\infty$ *is a positive random variable.*

Proof. We leave the proof as an exercise for the reader by adapting the argument of [89, Section 2.2] in this case. □

13.2.4 Cut-points in the dense phase

We conclude this section by proving the existence of cut-points in $\mathfrak{M}_\infty^\dagger$ if $\mathbf{q}$ is of type $a \in (3/2;2]$. For this, we again use the peeling algorithm of Section 13.2.1.

Proposition 13.12. *Let* $\mathbf{q}$ *be of type* $a \in (3/2;2]$. *Then the random map* $\mathfrak{M}_\infty^\dagger$ *contains infinitely many cut-points separating the origin from infinity almost surely.*

Proof (Sketch). Suppose first that $a < 2$. Perform any filled-in Markovian exploration of the map $\mathfrak{M}_\infty$ until the following event $\mathcal{B}_n$ happens: at time n we have an event C_p with $p \geq |\partial^* \bar{\mathfrak{e}}_n|$. We claim that conditionally on $\mathcal{B}_n$ and on the past exploration, the face f of degree $2p$ that the nth peeling step just revealed is a cut-point in $\mathfrak{M}_\infty^\dagger$ with probability at least c for some $c > 0$ independent of n. To see this, imagine that after $\mathcal{B}_n$ happens, we peel along the edges of the face f, from left to right using the peeling by layers on the dual algorithm $\mathcal{A}_{\mathrm{dual}}$. Since $a < 2$, by (13.15), it takes roughly p^{a-1} peeling steps to entirely swallow the exposed edges of f and so it may happen that in the mean time we performed a peeling step of type $\mathsf{G}_{k,*}$ with $k \approx p$

identifying two edges of f in such a way that the origin and infinity are separated in $\mathfrak{M}_\infty^\dagger$ by f, see Figure 13.4. A moment's though using Theorem 10.8 shows that the above scenario happens with a probability bounded away from 0 independently of $|\partial^* \bar{\mathfrak{e}}_n|$ and $p \geq |\partial^* \bar{\mathfrak{e}}_n|$ indeed. Since by Theorem 10.8 there are almost surely infinitely many n's so that $\mathcal{B}_n$ happens, we conclude by a conditional version of Borel–Cantelli lemma that there are indeed infinitely many cut-points in $\mathfrak{M}_\infty^\dagger$ almost surely. The argument in the case $a = 2$ is roughly the same, but this time the

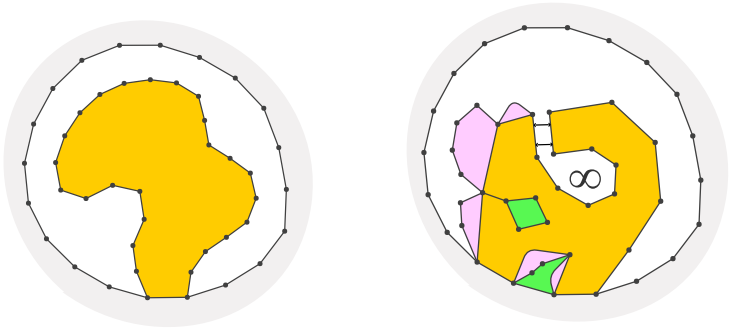


Figure 13.4: Creation of a cut-point in the dual map from a large face: On the left the situation just after an event $\mathcal{B}_n$, on the right, the exploration the exposed edges of the large face yields the disconnection of the origin from ∞ in the dual map.

probability that a large face as above is a cut face decreases to 0 but not so quickly...: Consider a peeling of $\mathfrak{M}_\infty$ and assume that between times 2^k and 2^{k+1} we discover a new face of degree proportional to 2^k. By Theorem 10.8 and Proposition 5.10, this event happens with a probability bounded away from 0 (uniformly in k). When this happens, we explore the exposed edges of this face as before. This time, there is a probability of order $1/k$ that two edges of this face are identified and form a cut-edge of $\mathfrak{M}_\infty$. This can for example occur if, when performing a peeling by layers along this face, we discover an unusually large jump of size 2^k within the first $2^k/\log 2^k$ steps and that this jump creates a cut-edge as in the above figure. Summarizing, with a probability of order $1/k$ a cut-point in $\mathfrak{M}_\infty^\dagger$ occurs between the peeling times 2^k and 2^{k+1}. Since the harmonic series diverges and these events are "roughly" independent, this implies the existence of infinitely many cut-edges in $\mathfrak{M}_\infty$. We leave to the reader turning the above sketch into a rigorous proof. □

Contrary the half-plane case (Theorem 6.7) we do not know of a simple and universal criterion for the existence of cut-points in the plane topology:

Open Question 13.13. *Is $\sum_{k\geq 0} \nu(-k)k = \infty$ a necessary and sufficient condition for the existence of infinitely many cut edges in $\mathfrak{M}_\infty$?*

13.3 Primal graph distances

The sane readers may be puzzled by the fact that so far we considered the Eden distance, the dual distance, but not the primal graph distance on our maps! Alas, it does not seem easy to use the peeling process to systematically study the graph metric on the primal $\mathfrak{m}$ (this is because the degree of the faces are not bounded and so when discovering a new large face, one cannot *a priori* know what are the distances to the root of all of its incident vertices). However when the faces of the map are small this can be done:

13.3.1 Triangulations

In this section we study the peeling process on triangulations making an exception to our rule since triangulations are not bipartite. However, as soon as the enumerative formulas are at hand (which is the case for critical triangulations, see Section 3.5), one can develop the peeling machinery exactly as in the bipartite case. Let $\mathfrak{t}$ be a triangulation. The following filled-in peeling algorithm is a small variation of the algorithm $\mathcal{A}_{\text{metric}}$ which we use to study primal graph distances on triangulations:

PEELING BY LAYERS ON THE PRIMAL FOR TRIANGULATIONS

Suppose $\bar{\mathfrak{e}}_n \subset \mathfrak{t}$ verifies the following property: **(H)** there exists an integer $r \geq 0$ so that the distances, *inside* $\bar{\mathfrak{e}}_n$, between ρ and the vertices of $\partial^*\bar{\mathfrak{e}}_n$ are in $\{r, r+1\}$. Furthermore those vertices at distance r form a connected interval on $\partial^*\bar{\mathfrak{e}}_n$. If so, the next edge to peel is the only edge whose extremities are at distance $r+1$ and r from left to right along the boundary of the hole. If all vertices are at the same distance from the origin, then we peel any edge. See figure below.

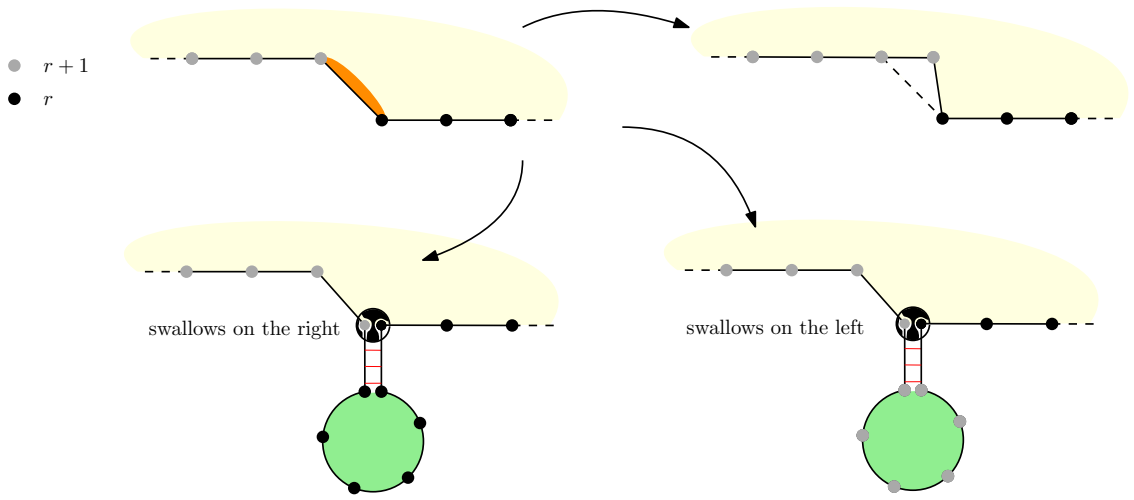


Figure: Illustration of the peeling by layers for the primal distance in the case of triangulations. When the peeled edge is identified with another edge of the boundary, two vertices are identified to form a new vertex on the boundary and the distances of this vertex to ρ is the minimum of the previous two distances.

It is easy to check that the boundary condition (**H**) is preserved when exploring a triangulation using this algorithm (we crucially use the fact that the face degrees are bounded by 3). Also, if θ_r is the first time when all the vertices of $\partial^* \overline{\mathfrak{e}}_n$ are at distance at least $r+1$ from the origin (inside the explored map) then it is easy to see that $\overline{\mathfrak{e}}_{\theta_r}$ is related to the hull of the ball of radius r for the primal distance in $\mathfrak{t}$ in the sense that $\overline{\mathfrak{e}}^{\circ}_{\theta_r}$ is the hull of all the dual edges having one extremity at distance less than r from the origin.

Let us denote by P_n the perimeter (and not the half-perimeter) of the unique hole during a filled-in exploration under (the triangular equivalent to) $\mathbb{P}_\infty$. We also write $D_n^{\mathrm{fpp}}, D_n^{\mathrm{dual}}$ and D_n^{prim} respectively for the minimal Eden/dual/primal distance to the boundary of $\overline{\mathfrak{e}}_n$ using respectively the uniform peeling, the peeling by layers on the dual and the peeling by layers on the primal. Then we claim that we have the formal heuristic growth mechanism:

$$\text{“}\Delta D_n^{\mathrm{fpp}} = \frac{1}{\mathrm{P}_n}, \quad \Delta D_n^{\mathrm{dual}} = \frac{1+\mathrm{g_q}}{\mathrm{P}_n}, \quad \Delta D_n^{\mathrm{prim}} = \frac{\mathrm{g_q}}{\mathrm{P}_n}.\text{”} \tag{13.17}$$

The first equation is a heuristic version of (13.1), the second one is explained in (13.16) and the third one follows from the same arguments (in a limiting half-plane setting the number of vertices at distance r has a negative drift equal to $-\mathrm{g_q}$). This is of course not rigorous, but it explains why in large random triangulations the dual and Eden distance are roughly proportional to the primal graph distance with proportionally factor equal to $\frac{1+\mathrm{g_q}}{\mathrm{g_q}} = 1+2\sqrt{3}$ and $\frac{1}{\mathrm{g_q}} = 2\sqrt{3}$ respectively. See [91] and [92] for details and proper statements.

13.3.2 Quadrangulations

Let us now adapt the former algorithm to the case of quadrangulations. An important remark is that since any quadrangulation $\mathfrak{q}$ with one hole is bipartite, the distance of the vertices along the boundary of the hole must vary at each step by $+1$ or -1.

PEELING BY LAYERS ON THE PRIMAL FOR QUADRANGULATIONS

Suppose $\bar{\mathfrak{e}}_n \subset \mathfrak{q}$ verifies the following property: **(H)** there exists an integer $r \geq 0$ so that the distances, *inside* $\bar{\mathfrak{e}}_n$, between ρ and the vertices of $\partial^* \bar{\mathfrak{e}}_n$ are in $\{r-1, r, r+1\}$. Furthermore, the edges having a vertex at distance $r-1$ form a connected interval on $\partial^* \bar{\mathfrak{e}}_n$. If so, the next edge to peel is the left-most edge along the boundary of the hole having an extremity at distance $r-1$ from the origin. If all vertices along the boundary are at distance r or $r+1$ then we may peel any edge. See figure below.

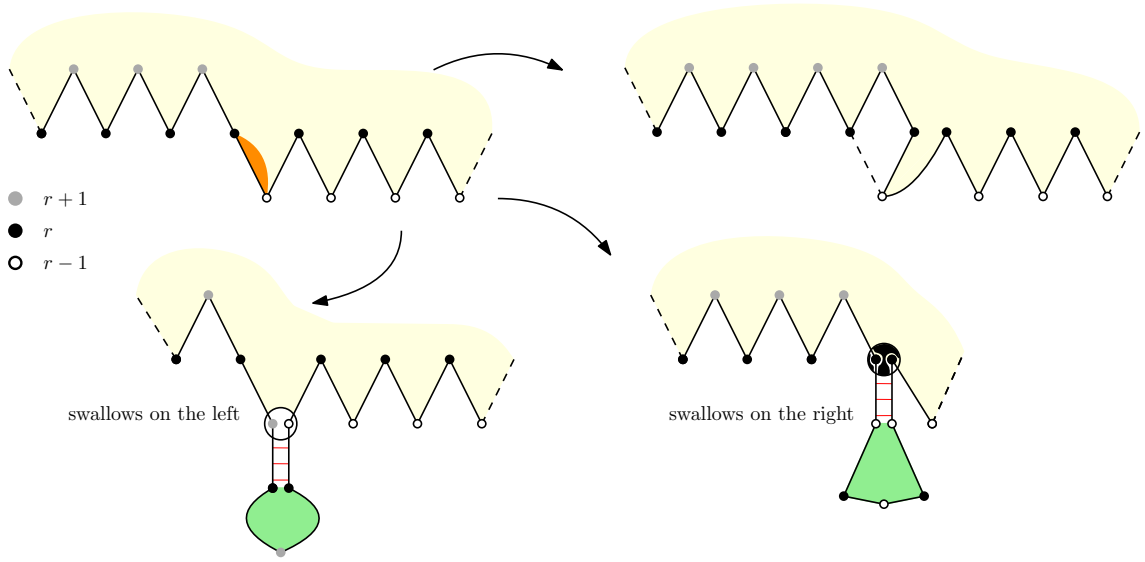


Figure: Illustration of the peeling by layers for the primal distance in the case of quadrangulations. When the peeled edge is identified with another edge of the boundary, two vertices are identified to form a new vertex on the boundary and the distances of this vertex to ρ is the minimum of the previous two distances.

Again, it is easy to check that the boundary condition **(H)** is preserved when exploring a quadrangulation using this algorithm and as above one can relate the hull of the ball of radius r in $\mathfrak{q}$ to the map explored when all vertices on the boundary are at distance larger than $r+1$. It is again possible to analyse the speed at which this peeling turns around primal layers and deduce that if P_n is the half-perimeter during filled-in exploration of the UIPQ then the analog of (13.17) becomes here

$$\text{“}\Delta D_n^{\text{fpp}} = \frac{1}{2P_n}, \quad \Delta D_n^{\text{dual}} = \frac{1+\mathrm{g}_{\mathfrak{q}}}{2P_n}, \quad \Delta D_n^{\text{prim}} = \frac{\frac{1+\mathrm{g}_{\mathfrak{q}}}{2}}{2P_n}.\text{”}$$

13.3.3 General case

In the general (bipartite) case, it is still possible to design a peeling algorithm that “turns around” primal layers by peeling on vertices at minimal distances from the origin of the map (inside the explored map), see Algorithm $\mathcal{A}_{\text{metric}}$.

However, we do not know yet how to control the speed at which this exploration turns

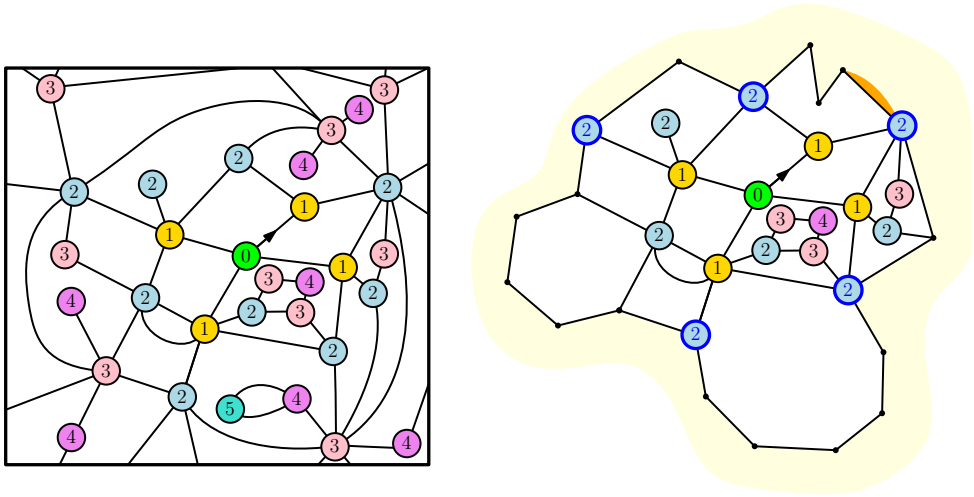


Figure 13.5: Illustration of the filled-in version of algorithm $\mathcal{A}_{\text{metric}}$ in the general bipartite case. The labelling represents distances of the vertices to the origin. On the right a current state $\bar{\mathfrak{e}}_i$ of the exploration. Notice that the vertices on the boundary with minimal distances to the origin have the same labels in $\bar{\mathfrak{e}}_i$ and in the underlying map. We always peel at an edge whose right end point minimises this distance (inside $\bar{\mathfrak{e}}_i$).

around our layers using only peeling techniques[2]. Surprisingly, using bijective constructions (see Appendix B) it is possible to guess the correct speed constant:

Open Question 13.14 (Timothy Budd, [52]). *Suppose* $\mathbf{q}$ *is of type* $a = 5/2$. *If* D_n^{prim} *is the minimal primal distance to the hole of half-perimeter* P_n *in a filled-in exploration using algorithm* $\mathcal{A}_{\text{metric}}$ *then we should have*

$$\Delta D_n^{\text{prim}} \approx \frac{\frac{4\mathrm{p_q}\sqrt{\pi}}{3}}{2P_n},$$

or more precisely, the time needed to complete a layer of perimeter $2p$ *at given height is roughly of order* $\frac{3p}{2\mathrm{p_q}\sqrt{\pi}}$ *when* p *is large.*

13.4 ... and for the half-plane ?

We leave open the task of extending the results of this chapter to the half-plane models $\mathfrak{M}^{(\infty)}$ and $\mathfrak{H}^{(\infty)}$. This is merely an exercise for the case of $\mathfrak{H}^{(\infty)}$ but a substantial difficulty arises for $\mathfrak{M}^{(\infty)}$ in the case of critical weight sequence: In the Eden case, Equation (13.1) has to be replaced by $T_n = \sum_{i=0}^{n-1} \frac{\mathrm{e}_i}{\mathrm{E}_i}$ where $(E_i : i \geq 0)$ is the perimeter of the *exposed* boundary during the uniform peeling process. Establishing functional scaling limits for this process is an interesting question on its own. Similar problems pop-up when dealing with the dual graph distance and boil down to understanding the length of the exposed boundary as well.

[2]This would require to understand the repartition of vertices of minimal distances along the boundary of the hole during such a peeling exploration.

Open Question 13.15. *Extend the result of this chapter to the half-plane models. In passing, establish functional scaling limit of the length of the exposed boundary during a peeling of* $\mathfrak{M}^{(\infty)}$ *with algorithm* $\mathcal{A}_{\text{uniform}}$ *or* $\mathcal{A}_{\text{dual}}$.

Some geometric information may already be deduced from the works [65], [67] and [92, Section 4].

Bibliographical notes. The study of the volume growth of random triangulations using the peeling process can be traced back to Watabiki and Ambjørn [10] (non-rigorously at the time but proved to be exact for the Eden distance later by Ambjørn and Budd [8]). Angel [13] derived the growth exponent of the UIPT (up to logarithmic fluctuations) using the simple peeling. Convergence in distribution was later established in [91] in the case of triangulations and quadrangulations using the simple peeling and then extended to Boltzmann maps of type $a = 5/2$ by Budd [53]. Proposition 13.12 and Theorems 13.3 and 13.9 are taken from [55] except for the case $a = 2$ which was treated separately in [57]. The volume growth of hyperbolic Boltzmann maps (Propositions 13.7 and 13.11) is new in this form but largely inspired from the triangular case [89, 170]. The fact that several distances should be proportional at large scales was observed first by Timothy Budd [52]. It is proved in [92] in the case of triangulations, in [145] for quadrangulations but is still open in the generality of Conjecture 13.10. The peeling by layer on the primal lattices for triangulations and quadrangulations is adapted from [91, Section 6]. Conjecture 13.14 is taken from the enlightening talk [52] of Budd.

Chapter XIV : A taste of scaling limit

In this chapter, we overfly the scaling limit theory for random planar maps. From this point of view, we shall consider our random maps as random discrete metric spaces. We stay at a rather high and unprecise level since this is not the main focus of these lecture notes. The reader eager to learn more about this subject is referred to [142, 158]. This chapter can be seen as a pretext to include a bunch of nice simulations in these lecture notes (as well as speaking about Gromov-Hausdorff convergence and growth-fragmentation processes!).

14.1 Gromov–Hausdorff topology

In order to speak about "continuous limit" of large graphs, or more generally about convergence of metric spaces, one should first introduce the proper notion of space and convergence. The most common is the so-called Gromov–Hausdorff topology, introduced among others by Gromov in order to study large scale geometric properties of Cayley graphs of groups. We refer the interested reader to [114] and [66] for background on metric geometry.

14.1.1 Space of metric spaces

How can we speak of convergence of metric spaces?

One naïve way to start would be to consider the set of all possible metric spaces. Of course, as far as geometry is concerned, two isometric metric spaces are indistinguishable and one should rather consider the set of *isometry classes* of metric spaces. The reader will be rapidly convinced that this space is huge and one descent simplification is to consider the set Pol of all isometry classes of Polish space (where we recall that a Polish space is a separable complete metric space). If (E, d) is a Polish space, by taking a dense sequence of points $(x_i : i \geq 0)$ one can identify E with an infinite matrix $(d(x_i, x_j) : 0 \leq i, j)$. Thus Pol can be seen as a quotient of $\mathbb{R}_+^{\mathbb{Z}^2_{\geq 0}}$. Unfortunately this quotient space is nasty (it possesses not good Borel structure nor topology) and the problem of classification of Polish spaces up to isometry is, quoting A. Vershik [188], "an enormous task". One should then further restrict our class of metric spaces down to essentially finite ones. There are two common ways to proceed: either considering *compact metric spaces*, or considering Polish metric spaces endowed *with a Borel probability measure*. Those approaches are similar in spirit since it is well-known that a probability measure on a Polish space is tight, i.e. puts essentially all its mass on a compact subset. In this lecture note we shall take the first approach.

14.1.2 Gromov–Hausdorff topology

Following the above discussion, we restrict to *compact metric spaces* and consider

$$\mathbb{K} = \{\text{isometry classes of compact metric spaces}\}.$$

This space is not too big and has a natural notion of distance. Indeed, two compact metric spaces are identified in $\mathbb{K}$ if they are isometric, and relaxing the constraint of exact isometry yields the definition

Definition 14.1 (Gromov–Hausdorff distance)**.** *Let (E, d) and (E', d') be two compact metric spaces. The Gromov–Hausdorff distance $\mathrm{d_{GH}}$ between $\mathbf{E}$ and $\mathbf{E'}$ is less than $\varepsilon > 0$ if there exists two ε-isometries $\phi : E \to E'$ and $\phi' : E' \to E$ i.e. if*

$$|d'(\phi(x), \phi(y)) - d(x, y)| \leq \varepsilon \quad \text{and} \quad |d(\phi'(x'), \phi'(y')) - d'(x', y')| \leq \varepsilon,$$

for all $x, y \in E$ and $x', y' \in E'$.

The Gromov-Hausdorff distance is indeed a metric on $\mathbb{K}$. An alternative definition of this distance (although less convenient in practice) uses the Hausdorff distances (hence the name) of isometric copies of E and E' into the same metric space. It is then not hard to show that the space

$$(\mathbb{K}, \mathrm{d_{GH}})$$

is complete and that the set of all finite metric spaces is dense in $\mathbb{K}$. Furthermore, there is nice characterization of relative compactness in $\mathbb{K}$ which relies on existence of ε-net of bounded cardinality. In a nutshell, $(\mathbb{K}, \mathrm{d_{GH}})$ is a nice state space for random variables. The following exercise shows that it is mandatory to consider compact metric spaces:

Exercise 14.2. Find X and Y two (non-compact) metric spaces such that $\mathrm{d_{GH}}(X, Y) = 0$ without X and Y being isometric.

In the following, we will be interested in convergence of random graph towards random compact metric spaces. If G_n is a random graph on n vertices, one associates with it the random compact (finite!) metric space $(G_n, \mathrm{d_{gr}})$ and its scaled version $(G_n, \lambda_n \cdot \mathrm{d_{gr}})$ where λ_n is a deterministic sequence such that λ_n^{-1} is roughly of the order of the diameter of G_n. We speak of a scaling limit result when we have

$$(G_n, \lambda_n \cdot \mathrm{d_{gr}}) \xrightarrow[n\to\infty]{(d)} (\mathcal{G}, \Delta),$$

where $(\mathcal{G}, \Delta)$ is a random variable taking values in $\mathbb{K}$ and the above convergence in distribution is the usual weak-convergence for probability measures on the Polish space $\mathbb{K}$. Proving convergence of the above type requires to understand the behavior of all mutual distances in the graph G_n and not only the distances to the origin vertex (which is what we get when we study the volume growth).

14.1.3 Properties

Many metric observables (such as the diameter, the circum-radius, the distance set...) are continuous for the Gromov–Hausdorff topology. However, finer quantities such as the Hausdorff dimension is easily seen *not* to be continuous for this topology. Even the topology itself can undergo dramatic change during a Gromov–Hausdorff convergence. Notice however that in the case of two-dimensional manifold, if $X_n \to X$ for $\mathrm{d_{GH}}$ and X_n is homeomorphic to a fixed genus g torus, then the limit X "may look no worse than the following bubble (or branching) space":

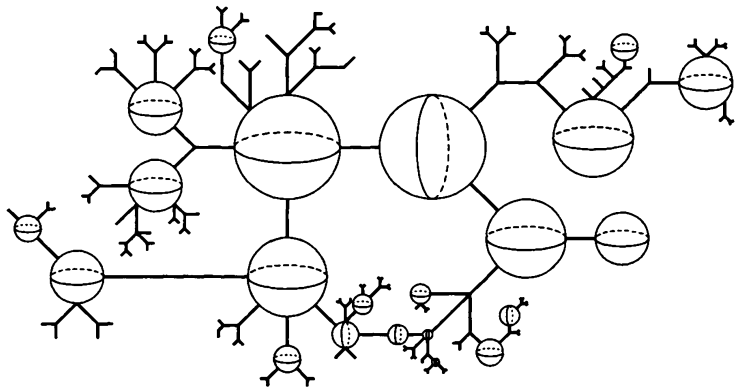


Figure 14.1: A possible Gromov–Hausdorff limit of topological torus of fixed genus. Figure taken from [114, Section 3.32].

In our context of scaling limit of graphs, it is easy to see that any Gromov–Hausdorff limits of scaled connected finite graphs $(\mathfrak{g}_n, \lambda_n \cdot \mathrm{d_{gr}})$ are automatically geodesic spaces when $\lambda_n \to 0$. Recall that a compact metric space (X, d) is a geodesic space if between any two points $x, y \in X$ there exists a geodesic $\gamma : x \to y$ (i.e. an isometric image of $[0, \mathrm{d}(x, y)]$ in X going from x to y).

Remark 14.3. There are extensions of the Gromov–Hausdorff topology to the locally compact (totally bounded) case, in particular in the case of length spaces. We refer to [66] or [3] for details.

14.2 Scaling limits for large Boltzmann maps

In this section, we present the scaling limits results for random bipartite maps endowed with the (primal) graph distance. We have seen in the previous chapter that primal graph distances are hard to control via the peeling process (except in the case of triangulations and quadrangulations) and most of the results concerning the large scale geometry for the primal graph distance are based on constructions of maps via labeled trees. Such constructions are described in Appendix B, see [158] or [142] for more details.

14.2.1 The Brownian sphere

The most satisfying results hold when $\mathbf{q}$ is critical of type $a = \frac{5}{2}$, in which case large random Boltzmann planar maps converge towards the so-called *Brownian sphere* which is a random compact metric space $(\mathcal{M}, \Delta)$. The following result has been established by Marzouk following the strategy developed by Le Gall (see also Miermont in the case of quadrangulations):

Theorem 14.4 (*Convergence towards the Brownian sphere*)

Let $\mathbf{q}$ be a critical weight sequence of type $a = \frac{5}{2}$. If $\mathfrak{M}_n$ is sampled according to $\mathbb{P}_{\mathbf{q}}^{(1)}(\cdot | \#\mathsf{Vertices}(\mathfrak{m}) = n)$ then along the values of n for which this makes sense we have the following convergence in distribution for the Gromov–Hausdorff topology

$$(\mathfrak{M}_n, n^{-1/4} \cdot \mathrm{d}_{\mathrm{gr}}) \xrightarrow[n\to\infty]{} \left(\mathcal{M}, \mathrm{D}_{\mathbf{q}}^{\mathrm{vertices}} \cdot \Delta\right),$$

where $\mathrm{D}_{\mathbf{q}}^{\mathrm{vertices}} = \frac{1}{9}(2\sqrt{\pi}\mathrm{p}_{\mathbf{q}})c_{\mathbf{q}}$. The former convergence holds true if instead of vertices, we condition on having n faces or n edges (along the values of n for which this is possible) as long as the scaling constant $\mathrm{D}_{\mathbf{q}}^{\mathrm{vertices}}$ is changed for

$$\mathrm{D}_{\mathbf{q}}^{\mathrm{faces}} = \mathrm{D}_{\mathbf{q}}^{\mathrm{vertices}} \frac{\mathrm{b}_{\mathbf{q}}^{\mathrm{vertices}}}{\mathrm{b}_{\mathbf{q}}^{\mathrm{faces}}} = \frac{4}{9}(2\sqrt{\pi}\mathrm{p}_{\mathbf{q}}) \frac{c_{\mathbf{q}}}{c_{\mathbf{q}} - 4}, \quad \text{and} \quad \mathrm{D}_{\mathbf{q}}^{\mathrm{edges}} = \mathrm{D}_{\mathbf{q}}^{\mathrm{vertices}} \frac{\mathrm{b}_{\mathbf{q}}^{\mathrm{vertices}}}{\mathrm{b}_{\mathbf{q}}^{\mathrm{edges}}} = \frac{4}{9}(2\sqrt{\pi}\mathrm{p}_{\mathbf{q}}).$$

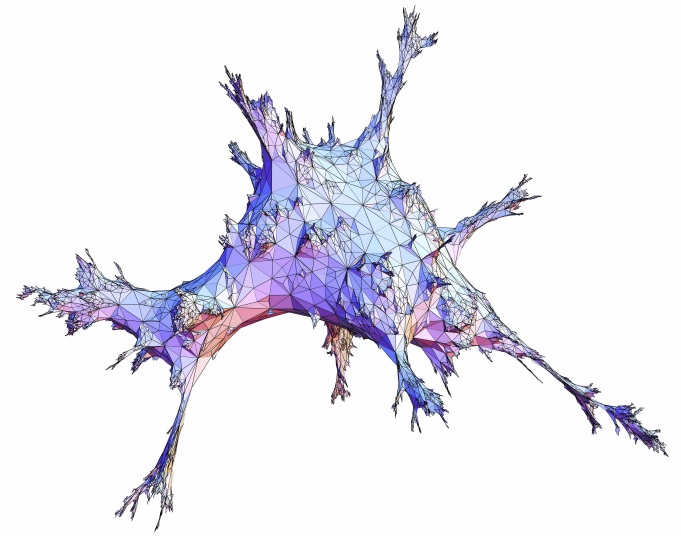


Figure 14.2: A simulation of a large random map with critical weight sequence of type $a = 5/2$ (in this case, a quadrangulation). The map is embedded in $\mathbb{R}^3$ using the GraphPlot function of Mathematica. Although this embedding is not an isometry, it represents well the overall geometry and the fractal structure of the Brownian sphere.

The Brownian sphere[1] is a random compact metric sphere which is almost surely homeomorphic to the 2-sphere and of Hausdorff dimension 4 (echoing the volume growth in $(\text{radius})^4$ in $\mathfrak{M}_\infty$ when $\mathbf{q}$ is of type $a = 5/2$, see Theorem 13.3). The construction of the Brownian sphere is rather involved and uses the Brownian motion indexed by the continuum random tree called the Brownian snake. Roughly speaking, it is a continuum version of the discrete "Schaeffer-type" construction of maps from labeled trees. We refer to the lecture notes of Miermont [158] for a complete account on this subject. Let us mention a nice open problem related to the universality of the Brownian sphere:

Open Question 14.5. *Let G_n be a (rooted) uniform planar **graph** with n edges (i.e. chosen uniformly at random among the finitely many (rooted) planar graphs with n edges). Show that for some $c > 0$ (computable?) there is the Gromov–Hausdorff convergence*

$$\left(G_n, n^{-1/4} \cdot \mathrm{d}_{\mathrm{gr}}\right) \xrightarrow[n\to\infty]{(d)} c \cdot (\mathcal{M}, D^*).$$

14.2.2 The stable maps

If $\mathbf{q}$ is a weight sequence of type $a \in (3/2; 5/2)$ then the convergence towards the Brownian sphere does not hold since large faces persist in the limit and create holes in the scaling limits. The state-of-the-art is given by the result of Le Gall & Miermont:

Theorem 14.6 (*Towards stable gaskets and carpets*)

Let $\mathbf{q}$ be a critical weight sequence of type $a \in (3/2; 5/2)$. If $\mathfrak{M}_n$ is sampled according to $\mathbb{P}_{\mathbf{q}}^{(1)}(\cdot | \#\mathsf{Vertices}(\mathfrak{m}) = n)$ then

$$\left(\mathfrak{M}_n, n^{-1/(2a-1)} \cdot \mathrm{d}_{\mathrm{gr}}\right)$$

is tight for the Gromov–Hausdorff topology.

By Prokhorov theorem, the above result shows that, at least along some subsequence $(n_k : k \geq 0)$ the random metric spaces $(\mathfrak{M}_{n_k}, n_k^{-1/(2a-1)} \mathrm{d}_{\mathrm{gr}})$ converge as $k \to \infty$. It is believed that we have an actual convergence in distribution without passing to a subsequence, and Le Gall and Miermont introduced the candidate scaling limit $(\mathcal{S}_a, D_a^*)$ which is a random metric space of Hausdorff dimension $2a - 1$ a.s. Contrary to the case $a = 5/2$, the topology of $\mathcal{S}_a$ is not that of the sphere since it has "holes". It is conjectured that $\mathcal{S}_a$ has the topology of the Sierpinski carpet if $a \in [2; 5/2)$ and that of a gasket if $a \in (3/2; 2)$.

The scaling factor $n^{-1/(2a-1)}$ indicates that the volume growth in $\mathfrak{M}_\infty$ for the primal distance is of order r^{2a-1}. This is not a result that can be proved using peeling techniques as of today, see the related Open Question 13.14.

[1] It is called the "Brownian map" in the literature.

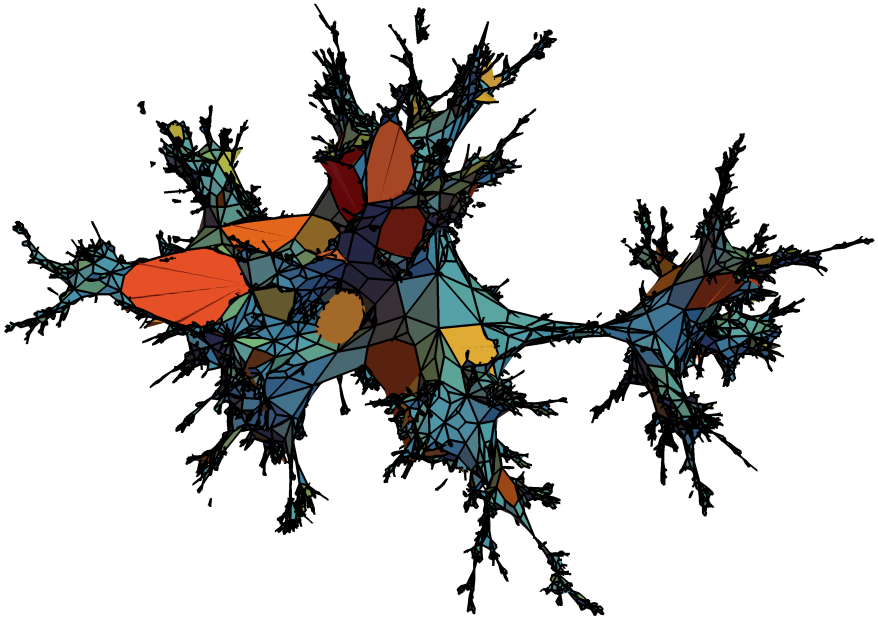


Figure 14.3: A simulation of large Boltzmann map with weight sequence of type $a = 1.8$. We clearly see large faces persisting in the limit.

14.3 Scaling limit for dual maps and Growth-Fragmentation trees

If one considers our maps endowed with their (Eden or) dual graph distance, then there is no tractable Schaeffer-type constructions of those maps via labeled trees where dual distances are encoded in an exploitable form on the tree. As a result, the understanding of scaling limit of random maps with large degree vertices is much less advanced than for large degree faces maps. However, a road map is to use the peeling process with algorithm $\mathcal{A}_{\text{dual}}$ to control those distances. In these lecture notes, we mainly focused on filled-in peeling exploration, but in order to understand the whole geometric structure of a map one must control the behavior of a (branching) peeling process.

14.3.1 Genealogy on holes

If $\mathfrak{m}$ is a map and $\mathcal{A}$ a peeling algorithm, recall that $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m}$ is the **non-filled-in** peeling exploration of $\mathfrak{m}$ using algorithm $\mathcal{A}$. We will suppose in what follows that every hole of $\mathfrak{e}_i$ is eventually peeled, that is, $\cup \mathfrak{e}_i = \mathfrak{m}$. There is an intuitive genealogy on the set $\mathcal{H}$ of all holes of $\mathfrak{e}_i$ which we can encode via the infinite binary tree $\text{Bin} = \{0,1\}^{\mathbb{Z}_{\geq 0}}$:

- the ancestor hole $h_\varnothing$ is the only hole of $\mathfrak{e}_0$,
- for $u \in \text{Bin}$, if at step $i \geq 0$ the ith peeling step does not occur on the hole $h_u \in \partial^* \mathfrak{e}_i$ then h_u is naturally identified with the hole of $\mathfrak{e}_{i+1}$ descending from it,

- when a peeling step occur on h_u it may split into 1 or 2 holes (depending whether we have an event of type C or G) which we denote by h_{u0} and h_{u1}, so that h_{u0} has the largest perimeter (with ties broken deterministically).

In filled-in peeling exploration we focused on the behavior of the boundary of distinguished hole, and we shall now encode the behavior of the boundary of all holes simultaneously. More precisely, for $u \in \mathrm{Bin}$ we denote by $\mathfrak{b}_u$ the half-perimeter of h_u and we put $\mathfrak{b}_u = 0$ if u does not show up as a hole's label. It is easy to see that together with the peeling algorithm $\mathcal{A}$ the labeled tree

$$\mathfrak{T} = (\mathfrak{b}_u : u \in \mathrm{Bin}),$$

completely characterize the map $\mathfrak{m}$.

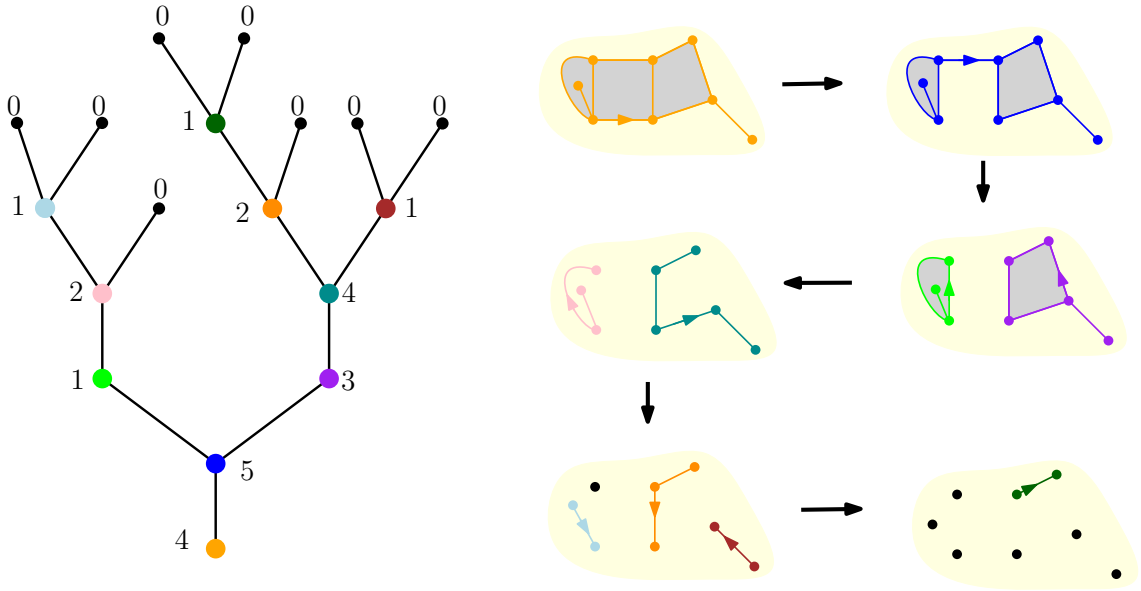


Figure 14.4: An example of a peeling exploration of a quadrangulation of size 3 and boundary of perimeter 10 and the resulting labeled plane tree (the component at the origin of the oriented edge peeled is placed on the left in the tree). The choice of the oriented edge to peel in each component is driven by the peeling algorithm $\mathcal{A}$. In particular, once $\mathcal{A}$ is fixed, the encoding of the map into its labeled tree is a bijection.

As in Proposition 10.3 it is tempting to ask for a scaling limit of this labeled tree and this has been done in recent works:

Proposition 14.7 (Scaling limits for peeling trees). *Recall the notation of the previous subsection. Let* $\mathbf{q}$ *be a weight sequence of type* $a \in [3/2; 5/2]$. *Then under* $\mathbb{P}^{(\ell)}$ *for any peeling algorithm, the labeled tree obtained as in Figure 14.4 converges after renormalizing the distances by* $\ell^{\frac{1}{a-1}}$ *and the labels by* ℓ *towards a "continuum labeled tree" heuristically described as a branching self-similar Markov process related to the* $a-1$*-stable Lévy process.*

This result follows from the techniques developed in [35] and later applied in [34] to the case of the peeling process of Boltzmann bipartite planar maps. In particular, Proposition 6.6 in [34] proves the convergence of the half-perimeters of the "locally largest" cycle in the exploration. The desired convergence follows from an iteration of this convergence using the Markov property together with cut-off estimates obtained using martingale arguments. The proper topology on the space of labeled trees as well as the convergence under various conditionings are proved in the monograph [36].

14.3.2 Slicing at heights

Combining the techniques developed in Chapter 13 with the previous result on branching peeling exploration it is also possible to establish scaling limits for the labeled peeling tree obtained with the Eden algorithm, and where the length of the branches in the tree are now measured in terms of the Eden distance to the origin face. See [34, Theorem 6.8]. We restricted to $a \in (2; 5/2]$ because the Lamperti time-change used to the Eden distance from the process of the perimeters blows up when $a \leq 2$ and indeed, the dual volume growth inside large bipartite maps of type $a \in (3/2; 2]$ is not polynomial and one expects no scaling limit result. The previous result is a first step towards understanding the whole geometry of those maps and should serve a the stepping stone to construct their potential scaling limits.

Open Question 14.8. *Construct the scaling limit "Stable spheres" of large* $\mathbf{q}$*-Botlzmann maps of type* $a \in (2; 5/2)$ *endowed with the dual graph distance and prove convergence of the discrete models towards the continuous limit.*

Bibliographical notes. The Gromov–Hausdorff topology has been popularized and investigated by Gromov in the 80's, see [114]. Similar ideas and earlier developments can be traced back to Edwards [107]. It has since then been advertised in probability by Evans [108]. We refer to [66] and [114] for details about this topology. One can fairly say that the theory of scaling limits of random planar map has been launched by the work of Chassaing and Schaeffer [75], followed by Le Gall [139]. The construction of the possible scaling limit of random quadrangulations (initially called the Brownian map) is first due to Marckert and Mokkadem [151] but the convergence in the Gromov–Hausdorff topology has only been proved in the breakthrough of Miermont [159] and independently of Le Gall [140]. In particular Le Gall [140] proposed a method to prove convergence towards the Brownian sphere which was then applied to many models see e.g. [4, 1, 40] and Theorem 14.4 proved in [153]. We refer the reader to [158] or [112] for details about construction of the Brownian sphere. The Brownian sphere is believed to be *universal* in the sense that it should be the scaling limit of many models of (critical) random planar graphs, see e.g. the Conjecture 14.5 and [92] for partial results concerning the universality with respect to the choice of the discrete metric on random triangulations. Let us also mention the deep result of Miller and Sheffield [161, 162] which identifies the Brownian sphere as Liouville Quantum Gravity with parameter $\gamma = \sqrt{8/3}$. Results concerning the scaling limit of maps when $\mathbf{q}$ is of

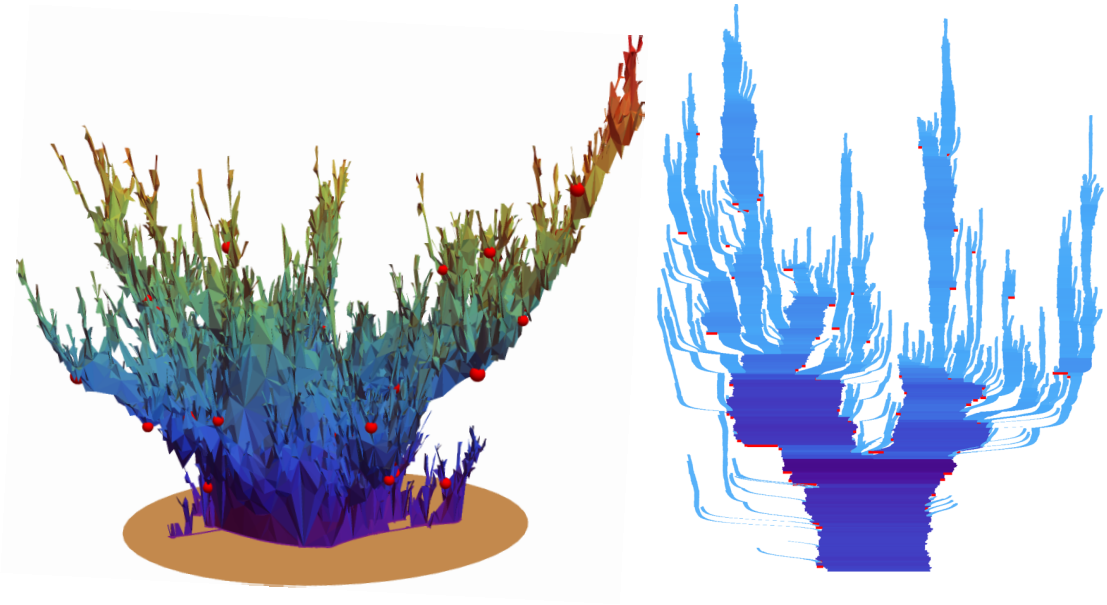


Figure 14.5: Left: A cactus representation of a random planar map with certain vertices of high degrees (which are the red dots), where the height of a vertex is its distance to the orange boundary. Right: A (different) simulation of the growth-fragmentation process describing the scaling limit of its perimeters at heights (the red part corresponds to positive jumps of the process).

type $a \in (3/2; 5/2)$ are much sparser. The case of the primal distance in finite Botlzmann maps is studied in [141] (a work in progress [98] address the uniqueness of the stable maps) and the dual distance on infinite maps is addressed in [55, 57]. The case of maps with prescribed degrees is studied in [152]. Proposition 14.7 establishing the link between random planar maps and Bertoin's growth-fragmentation processes [33] was proved in [35] for simple peeling exploration on triangulations and in [34] for **q**-Boltzmann maps of type $a \in (3/2; 5/2]$.

Part VI:
Simple random walk

In this part, we study the behavior of the simple random walk (recurrence, transience, intersection properties, subdiffusive behavior) on $\mathfrak{M}_\infty$ or $\mathfrak{H}_\infty$ and their duals. The reader will see that our understanding of the spectral properties of random planar maps is far from being complete!

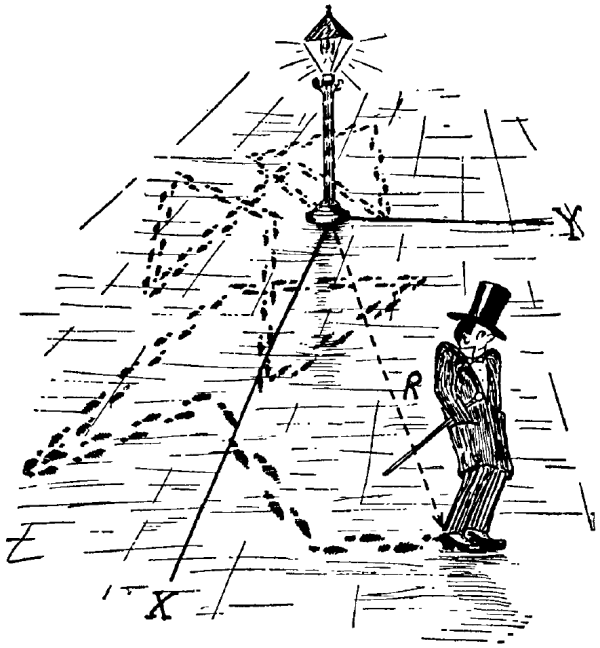


Figure 14.6: A drunkard walking on the plane (from Georges Gamow's book "One, two, three... infinity").

Chapter XV : Recurrence, Transience, Liouville and speed

In this chapter, we study the behavior of the simple random walk (SRW) on infinite Boltzmann planar maps of the plane under $\mathbb{P}_\infty$ or $\mathbb{H}_\infty$. We first recall the classical notions of recurrence/transience, intersection property, Liouville property and positive speed as well as their implications in the case of planar graph (with or without bounded degree). We then present the application of the circle packing theory to the Benjamini & Schramm theorem on recurrence of local limits of uniformly pointed maps. This theorem, and its extension by Gurel-Gurevich & Nachmias, shows that for any critical weight sequence $\mathbf{q}$, the random infinite Boltzmann map of the plane $\mathfrak{M}_\infty$ is almost surely recurrent. However, the case of its dual is largely open and we present some partial results using the peeling process. The hyperbolic regime is, of course, very different and we show there that the simple random walk has a positive speed and that $\mathfrak{H}_\infty$ and $\mathfrak{H}_\infty^\dagger$ are non-Liouville.

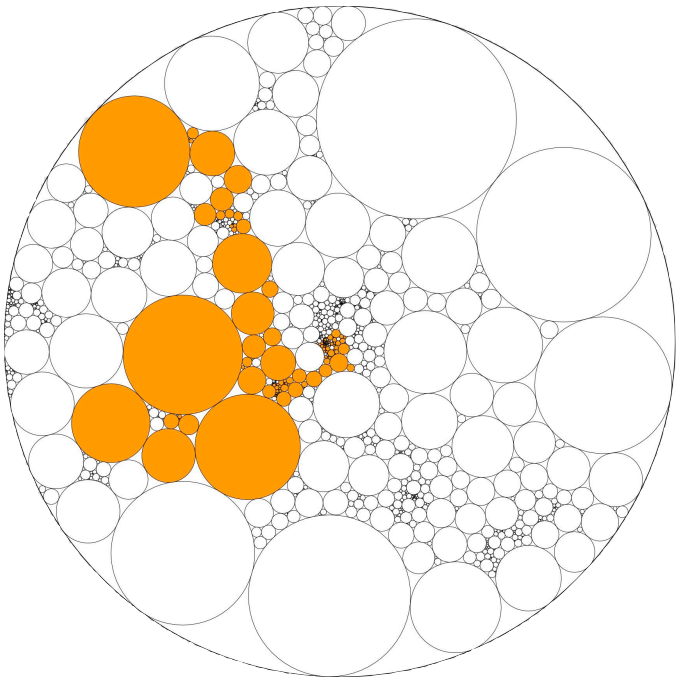


Figure 15.1: The circle packing representation of a large ball in the UIPT and the trace of the simple random walk until it reaches its boundary. Courtesy of Thomas Budzinski.

Notions. Recall from Section 7.2.1 the definition of the simple random walk on a connected (infinite) multi-graph $\mathfrak{g}$. Formally the process takes values in the set of oriented edges of the graph, but we will usually consider the process induced on the vertices which we denote by $X_0, X_1, \ldots$. We denote by $\mathbf{SRW}_{\mathfrak{g}}^x$ the law of the simple random walk on $\mathfrak{g}$ starting from $x \in \mathsf{V}(\mathfrak{g})$. We remind the reader of the following classical definitions:

- The graph $\mathfrak{g}$ is **recurrent** if the random walk almost surely comes back to its starting point, i.e. $X_n = X_0$ for infinitely many n's almost surely. Otherwise the graph is **transient**.
- The graph $\mathfrak{g}$ has the **intersection property**, if for any two $x, y \in \mathfrak{g}$, the trajectories (i.e. the set of vertices visited by the walks) of two independent simple random walks started respectively from x and y, intersect almost surely.
- The graph $\mathfrak{g}$ has the **Liouville** property (in short, is Liouville) if there are no non-constant function $h : \mathsf{V}(\mathfrak{g}) \to [0, 1]$ harmonic for the simple random walk.
- The graph $\mathfrak{g}$ has **positive speed** for the simple random walk if almost surely

$$\liminf_{n\to\infty} n^{-1} \cdot \mathrm{d}_{\mathrm{gr}}(X_0, X_n) > 0.$$

We first recall the well-known sequence of implications as well as the $\mathbb{Z}^d, d \geq 1$ lattices that satisfy them:

$$\begin{array}{ccccc} \text{recurrence} & \Rightarrow & \text{intersection} & \Rightarrow & \text{Liouville} \\ \mathbb{Z}^d, d = 1, 2 & & \mathbb{Z}^d, d = 1, 2, 3, 4 & & \mathbb{Z}^d, d \geq 1. \end{array}$$

The first implication is easy. For the second one, notice that if h is a non-constant bounded harmonic function then $(h(X_n) : n \geq 0)$ is a bounded martingale and so converges almost surely. Since h is non-constant $\lim_{n\to\infty} h(X_n)$ is non constant and so with positive probability two independent random walk paths do not intersect after some time. This proves that the intersection property cannot hold if $\mathfrak{g}$ is non Liouville. The positive speed property of course implies transience, but is in general is not implied by nor implies the Liouville property.

15.1 $\mathfrak{M}_\infty$ is recurrent

In this section we study the SRW on $\mathfrak{M}_\infty$ for a critical weight sequence $\mathbf{q}$. After reviewing the connections between recurrence, transience, intersection and Liouville property in the case of PLANAR graphs, we shall recall the Benjamini–Schramm–Gurel-Gurevich–Nachmias theorem. The interested reader is referred to Nachmias' lecture notes [167] for proofs and much more about this theory. In our context this theorem implies the main result of this section:

Theorem 15.1 (*Recurrence of the primal map* $\mathbf{M}_\infty$)

If $\mathbf{q}$ *is a critical weight sequence then* $\mathfrak{M}_\infty$ *is almost surely recurrent.*

In the rest of this section we shall explain the general strategy underlying the proof of this result, and for this we need to discuss discrete uniformization of planar graphs via circle packings.

15.1.1 Discrete uniformization of infinite planar graphs

Although in general the Liouville property seems much weaker than recurrence, surprisingly, in the case of bounded-degree planar graph, those properties are equivalent:

Theorem 15.2 (*Dichotomy for bounded-degree planar graphs*)

If $\mathfrak{g}$ is a planar connected multi-graph then the Liouville property is equivalent to the intersection property. If furthermore $\sup\{\deg(x) : x \in V(\mathfrak{g})\} < \infty$ *then recurrence, intersection and Liouville property are equivalent.*

The first part of the theorem is due to [26] and is rather elementary. The second part was proved by Benjamini & Schramm [28] using the theory of circle packing: We saw in Section 2.3.2 that every finite simple planar graph can be represented as a circle packing (the Koebe–Andreev–Thurston theorem). In the case of infinite maps of the plane (recall Definition 2.8), He & Schramm [122] proved the following dichotomy:

Theorem 15.3 (*Infinite circle packing theorem*)

Let $\mathfrak{m}$ be an infinite simple map of the plane. Then we have one of the mutually excluding alternatives:

- *Parabolic case: either there is a circle packing whose carrier is $\mathbb{R}^2$ representing $\mathfrak{m}$,*
- *Hyperbolic case: or there is a circle packing whose carrier is $\mathbb{D}$ representing $\mathfrak{m}$,*

where the **carrier** *of a packing $\mathcal{P}$ is the subset of the plane made of the union of all the circles as well as the interstices between them. Furthermore, if the vertex degrees of $\mathfrak{m}$ are bounded, the above dichotomy corresponds to the case when $\mathfrak{m}$ is recurrent (packing in $\mathbb{R}^2$) or transient and non-Liouville (packing in $\mathbb{D}$).*

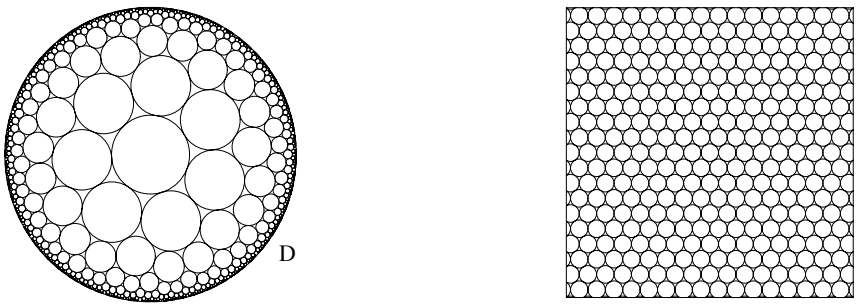


Figure 15.2: Illustration of Theorem 15.3: the 7-regular infinite triangulation is circle-packed in the disk (and is transient and non-Liouville) whereas the 6-regular infinite triangulation is circle-packed in the plane (and is recurrent). Images of Kenneth Stephenson.

The above theorem can be seen as a discrete counter-part to the dichotomy for simple connected Riemann surfaces homeomorphic to the disk: either such a surface is conformally

equivalent to the disk (and Brownian motion on the surface is transient) or it is conformally equivalent to the plane (and Brownian motion on the surface is recurrent).

15.1.2 Benjamini–Schramm limits

One key observation of Benjamini & Schramm [29] was that if $\mathfrak{M}$ is an infinite random map obtained as a local limit of uniformly rooted planar maps, then $\mathfrak{M}$ must be almost surely parabolic. Hence in the bounded degree case, those maps are recurrent. This can roughly be stated as follows:

"A very large finite planar graph seen from a typical point is recurrent".

More precisely,

Definition 15.4 (Benjamini–Schramm limit)**.** *A random infinite rooted map $\mathfrak{M}$ of the plane is called a Benjamini–Schramm limit of planar maps if there exists a sequence $\mathfrak{M}_n$ of finite stationary random rooted planar maps (equivalently the root edge can be chosen uniformly among all oriented edges of $\mathfrak{M}_n$) satisfying*

$$\mathfrak{M}_n \xrightarrow[n\to\infty]{(d)} \mathfrak{M},$$

in distribution for the local topology.

Theorem 15.5 (*Benjamini–Schramm limits with controlled degrees are recurrent*)

If $\mathfrak{M}$ is a Benjamini–Schramm limit of finite planar *maps and if furthermore we have*

$$\mathbb{P}(\deg(\rho) \geq k) \leq c^k, \qquad \text{for some } c \in (0,1),$$

where ρ is the origin of the root edge in $\mathfrak{M}$; then $\mathfrak{M}$ is almost surely recurrent.

The theorem was first proved by Benjamini & Schramm [29] under a bounded degree assumption, which was then released by Gurel-Gurevich & Nachmias [115] enabling its application to many models of random planar maps. (To be accurate, they considered random pointed graphs rather than random rooted graphs but the translation between the two concepts is easy to do after biasing by the inverse of the degree of the origin vertex).

Proof of Theorem 15.1. Clearly by Theorem 7.1 our infinite $\mathbf{q}$-Boltzmann map of the plane $\mathfrak{M}_\infty$ are Benjamini–Schramm limits of finite planar maps. The theorem thus follows from Theorem 15.5 and the following control on the distribution of the origin-degree in $\mathfrak{M}_\infty$:

Lemma 15.6. *Let $\mathbf{q}$ be a critical weight sequence. There exists $c \in (0,1)$ such that if ρ denotes the origin vertex of $\mathfrak{M}_\infty$ we have*

$$\mathbb{P}_\infty(\deg(\rho) \geq k) \leq c^{k-1}.$$

Proof. To prove this we use a *non filled-in* exploration that is a variation of the algorithm $\mathcal{A}_{\text{metric}}$, which we first used in Section 6.1.3:

Peeling the root vertex

As long as the root vertex belongs to $\partial^* \mathfrak{e}_i$, we peel the edge of the boundary adjacent on the left of the origin vertex until it becomes an interior vertex after an event of type $\mathrm{G}_{k,0}$ for some $k \geq 0$.

Using the explicit transition of the peeling process (distinguishing whether the origin vertex is on the boundary of a hole containing the infinite part of $\mathfrak{M}_\infty$ or not) we have for some constant $c > 0$,

$$\inf_{\ell \geq 2} b_\infty^{(\ell)}(*, 0) \wedge \inf_{\ell \geq 2} b^{(\ell)}(\ell - 1, 0) > c,$$

and so, conditionally on the past exploration, if $\ell \geq 2$ the exploration with the above algorithm has a probability at least $c > 0$ of ending at the next step (a similar statement holds for the case $\ell = 1$ by considering two steps). The total number of step of such exploration then has an exponential tail, and since each step adds two units to the degree of the origin the lemma is proved. $\square_{\text{lemma}}$ $\square_{\text{theorem}}$

Although the previous result settles the case of $\mathfrak{M}_\infty$, it leaves wide open the question of recurrence/transience of $\mathfrak{M}^{(\infty)}$ since the later is *not* a Benjamini–Schramm limit. Of course if the mean gulp is infinite then by Proposition 6.7 the map $\mathfrak{M}^{(\infty)}$ has infinitely many cut-edges so must be recurrent by Nash–Williams criterion. The triangular case has been solved by Angel & Ray [19], and the case of critical (non bipartite) general maps is recently tackled in [62], and both half-plane lattices are recurrent. This pushes us to conjecture:

Open Question 15.7. *Let* $\mathbf{q}$ *be an admissible weight sequence. Is* $\mathfrak{M}^{(\infty)}$ *almost surely recurrent? What about* $\hat{\mathfrak{M}}^{(\infty)}$ *(in particular in the dense phase)?*

Theorem 15.5 does not cover the case of the dual maps $\mathfrak{M}_\infty^\dagger$ in general. Indeed, although $\mathfrak{M}_\infty^\dagger$ is again a Benjamini–Schramm limit of finite planar graphs, the root degree distribution is

$$\mathbb{P}_\infty^\dagger(\deg(\rho) = 2k) = b_\infty^{(1)}(k) = \frac{h^\uparrow(k)}{h^\uparrow(1)} \nu(k-1), \tag{15.1}$$

and as soon as $\nu(k)$ does not decay exponentially fast when $k \to \infty$ (in particular if $\mathbf{q}$ is of type $a \in (3/2; 5/2)$) then the origin degree of $\mathfrak{M}_\infty^\dagger$ does not have an exponential tail. This case is investigated in the next section.

15.2 Simple random walk on $\mathfrak{M}_\infty^\dagger$

We now focus on the properties of the simple random walk on the dual maps $\mathfrak{M}_\infty^\dagger$ and in particular when $\mathbf{q}$ is of type $a \in (3/2; 5/2)$. The presence of the large degree vertices in $\mathfrak{M}_\infty^\dagger$ decreases substantially the resistance of those lattices and in fact we show that these dual maps can actually be transient.

15.2.1 Transience of $\mathfrak{M}_\infty^\dagger$ in the dense case

Recalling Proposition 13.12, the reader may wrongly think that $\mathfrak{M}_\infty^\dagger$ is recurrent when $\mathbf{q}$ is of type $a \in (3/2; 2)$: actually those separating vertices have a huge degree and so it is not possible to turn these cut-points into reasonably small cut-set of edges as required by the Nash–Williams criterion. We will actually see as a corollary of our study of the fpp-distance on $\mathfrak{m}^\dagger$ that those graphs are almost surely transient. The proof uses the method of **random paths**, see [148, Section 3.1].

Corollary 15.8. *Let $\mathbf{q}$ be a critical weight sequence such that the random walk (S) with step distribution ν is transient (e.g. if $\mathbf{q}$ is of type $a \in (3/2; 2)$). Then the random lattice $\mathfrak{M}_\infty^\dagger$ is almost surely transient.*

Proof. Recall the setup of Section 13.1.1 and the third item of Theorem 13.3. The exponential fpp-model on $\mathfrak{M}_\infty^\dagger$ enables us to distinguish an infinite oriented path $\vec{\Gamma} : f_r \to \infty$ in $\mathfrak{M}_\infty^\dagger$ which is the shortest infinite path starting from the origin for the fpp-distance (almost sure uniqueness of this path is easy to prove under $\int \mathbb{P}_\infty(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(\mathrm{d}(x_e))$). From this path $\vec{\Gamma}$ one constructs a unit flow θ on the directed edges with source at f_r by putting for any oriented edge $\vec{e}$ of $\mathfrak{m}$

$$\theta(\vec{e}) = \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(\vec{e} \in \vec{\Gamma}) - \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(\overleftarrow{e} \in \vec{\Gamma}).$$

To show that the energy of this flow is finite, we compare it to the expected fpp-length of $\vec{\Gamma}$ which is almost surely finite under $\int \mathbb{P}_\infty(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(\mathrm{d}(x_e))$ by the third item of Theorem 13.3 (the proof was done in the case of regular weight sequence of type $a \in (3/2; 2)$ but extends readily as soon as (S) is transient). More precisely, if x_{e_0} denotes the exponential weight of a given edge e_0, and if $\mathrm{E}_\mathfrak{m}^{\mathrm{fpp}}$ denotes the expectation under $\mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}$, we just remark that there exists a constant $C > 0$ such that for any event A we have

$$\mathrm{E}_\mathfrak{m}^{\mathrm{fpp}}[x_{e_0} \mathbf{1}_A] \geq C\, \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(A)^2.$$

Indeed, if $\delta = \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(A)$ we have $\mathrm{E}_\mathfrak{m}^{\mathrm{fpp}}[x_e \mathbf{1}_A] \geq \mathrm{E}_\mathfrak{m}^{\mathrm{fpp}}[x_e \mathbf{1}_A \mathbf{1}_{x_e \geq \delta/2}] \geq \delta/2\, \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(A \cap \{x_e \geq \frac{\delta}{2}\})$ and use the fact that $\mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(A \cap \{x_e \geq \frac{\delta}{2}\}) \geq \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(A) + \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(x_e \geq \frac{\delta}{2}) - 1 = \delta + \mathrm{e}^{-\delta/2} - 1 \geq \delta/2$. Using this we can write for a fixed infinite map $\mathfrak{m}$ of the plane

$$\begin{aligned} \sum_{\vec{e} \in \overrightarrow{\mathrm{Edges}}(\mathfrak{m}^\dagger)} \theta(\vec{e})^2 \leq 4 \sum_{e \in \mathrm{Edges}(\mathfrak{m}^\dagger)} \mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}(e \in \Gamma)^2 &\leq \frac{4}{C} \sum_{e \in \mathrm{Edges}(\mathfrak{m}^\dagger)} \mathrm{E}_\mathfrak{m}^{\mathrm{fpp}}\left[\mathbf{1}_{e \in \Gamma} x_e\right] \\ &= \frac{4}{C} \mathrm{E}_\mathfrak{m}^{\mathrm{fpp}}[\mathrm{Length}_{\mathrm{fpp}}(\Gamma)]. \end{aligned}$$

But in our case $\mathfrak{m} = \mathfrak{M}_\infty$ and it follows from the third item of Theorem 13.3 that

$$\mathbb{E}_\infty\left[\mathrm{E}_\mathfrak{m}^{\mathrm{fpp}}[\mathrm{Length}_{\mathrm{fpp}}(\Gamma)]\right] < \infty,$$

and so that $\mathrm{E}_\mathfrak{m}^{\mathrm{fpp}}[\mathrm{Length}_{\mathrm{fpp}}(\Gamma)]$ is almost surely finite under $\mathbb{P}_\infty$. This proves almost sure transience of the lattice as desired. □

15.2.2 Intersection and recurrence

On the other hand, we show in this section that the random maps $\mathfrak{M}_\infty^\dagger$ always possess the intersection property. Unfortunately, we have to restrict to regular weight sequences, but we believe the result holds for any critical weight sequence:

Proposition 15.9 (Intersection property)**.** *Let* $\mathbf{q}$ *be a weight sequence of type* $a \in (3/2; 5/2]$*, then* $\mathfrak{M}^\dagger$ *almost surely has the intersection property. In particular it is almost surely Liouville.*

To prove this proposition we shall explore the random map $\mathfrak{M}_\infty$ along the simple random walk in $\mathfrak{M}_\infty^\dagger$: we let the simple random walk move freely and when it is necessary, we trigger a new peeling step to discover the edge it wants to go through (it is a randomized algorithm in the sense of Section 11.1.1).

PEELING ALONG THE DUAL RANDOM WALK

Let $(\vec{E}_n^\dagger : n \geq 0)$ be the dual oriented edges traversed by a random walk on $\mathfrak{m}^\dagger$ started from the root face. If $\mathfrak{e}$ is the map explored so far, the next edge to peel is the first dual edge of $\partial^*\mathfrak{e}$ through which the dual simple random walk goes.

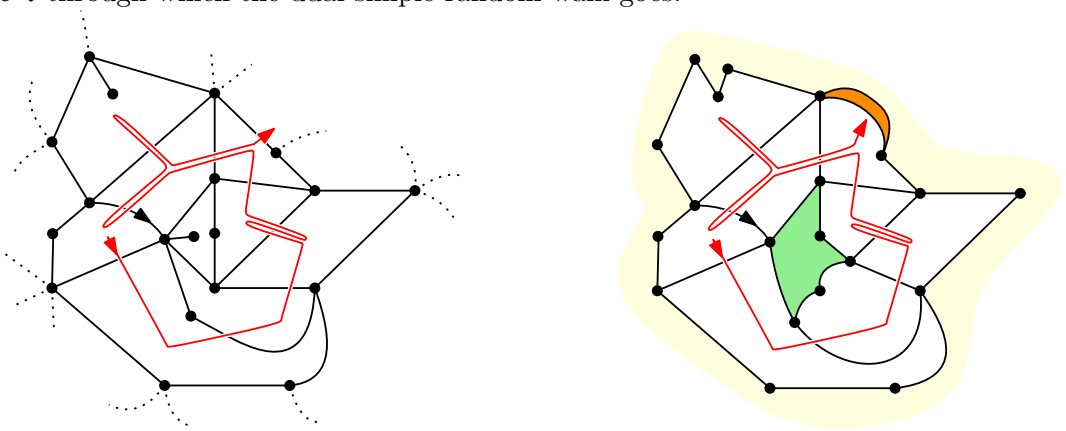


Figure: On the left the first few steps of the random walk on the dual graph. On the right the peeling step that it triggers (in orange).

Equivalently, the associated growing connected subset of the dual $\mathfrak{e}_n^\circ$ can be defined as $\mathfrak{e}_n^\circ = \{\vec{E}_k^\dagger : 0 \leq k \leq n\}$. This is not exactly a peeling exploration since it might be that $\bar{\mathfrak{e}}_n = \bar{\mathfrak{e}}_{n+1}$ if the edge $\vec{E}_n$ has previously been visited by the walk. However, after erasing the repetitions, this defines a (non filled-in) peeling exploration, and since the choice of the next edge to peel is independent of the undiscovered part, the transitions follow that of Proposition 7.4. With this algorithm in hands, the proof of the previous proposition is straightforward:

Proof of Proposition 15.9. We in fact prove the stronger statement: almost surely the trace of a simple random walk on $\mathfrak{M}_\infty^\dagger$ disconnects the root face from ∞. Let $(\bar{\mathfrak{e}}_n : n \geq 0)$ be the filled-in exploration of $\mathfrak{M}_\infty$ with the above algorithm. Since $\mathbf{q}$ is of type $a \in (3/2; 5/2]$, by the second point of Theorem 10.13 we know that

$$\bigcup_{n\geq 0} \bar{\mathfrak{e}}_n = \mathfrak{M}_\infty \quad \text{almost surely.}$$

With the geometric interpretation of $\bar{\mathfrak{e}}_n$ in term of the dual random walk $(\vec{E}_n^\dagger)_{n\geq 0}$ this means that the hull of the trace $\{\vec{E}_n^\dagger : n \geq 0\}$ is the full map a.s. In particular, the complement of the vertices in $\mathfrak{M}_\infty^\dagger$ visited by the walk is made of finite components only and so any other random walk must intersect the vertex-trajectory of $(\vec{E}_n^\dagger)_{n\geq 0}$ almost surely. □

We are ashamed to leave the following question open:

Open Question 15.10. *If* $\mathfrak{q}$ *be of type* $a \in [2; 5/2)$*, is* $\mathfrak{M}_\infty^\dagger$ *transient?*

15.3 Hyperbolic maps and positive speed

We now turn our attention to simple random walks on the hyperbolic random maps $\mathfrak{H}^{(\infty)}$ and $\mathfrak{H}_\infty$. As expected, a "hyperbolic" behavior appears, i.e. the simple random walk has a positive speed and $\mathfrak{H}_\infty$ is non-Liouville a.s. As for Theorem 8.11 or Proposition 13.11 we shall restrict our attention to subcritical weight sequences with bounded support.

15.3.1 Anchored expansion, speed and stationarity

Recall from Definition 8.10 the definition of anchored expansion. It is well-known that isoperimetric inequalities help understanding the behavior of the random walk. In that direction, let us recall the fundamental result of Virag [189] which says that if a connected infinite graph $\mathfrak{g}$ with bounded degree is such that $\mathrm{Cheeger}^\bullet(\mathfrak{g}) > 0$, then the simple random walk on $\mathfrak{g}$ has positive speed. Combined with Theorem 8.11, the previous result immediately implies that the random walk on the duals of hyperbolic maps with subcritical weight sequences of bounded support have positive speed. For the walk on the primal graph however, although our random hyperbolic maps $\mathfrak{H}_\infty$ and $\mathfrak{H}^{(\infty)}$ are anchored non-amenable, the lack of a bound on their degrees prevent us from applying Virag's result. As in the first section of this chapter, we shall rely on an extension which trades off the bounded degree assumption in exchange of a certain homogeneity of the graph:

Theorem 15.11 (*Stationary non-amenable random graphs have positive speed*)

Let $(G, \vec{E})$ *be an ergodic stationary random graph such that*

$$\mathrm{Cheeger}^\bullet(G) > 0 \quad a.s., \qquad \limsup_{r\to\infty} (\#[G]_r)^{1/r} < \infty \quad a.s., \qquad \text{and} \qquad \mathbb{E}[\log \deg(\rho)] < \infty.$$

Then G *has positive speed and is non-Liouville*

Proof: The positive speed part of the theorem is proved by Benjamini, Paquette and Pfeffer [27] using the island/ocean decomposition of Virag and ergodic techniques, see also [17] for similar arguments. In general, the Liouville property does not prevent positive speed, except when the graph is somehow "homogeneous" (Cayley graphs or more generally stationary random graphs). More precisely, since G has positive speed, one can apply [71, Lemma 5.1] (see also [24, Proposition 3.6]) and get under our assumptions that the entropy of G is almost surely positive, which implies it is non-Liouville a.s. □

We thus have:

Corollary 15.12 (Hyperbolic random maps of the plane have positive speed.)**.** *If* $\mathbf{q}$ *is a subcritical weight sequence with a finite support, then* $\mathfrak{H}_\infty$ *and* $\mathfrak{H}_\infty^\dagger$ *have positive speed and are non-Liouville.*

Remark 15.13. In particular, we deduce from the above result and Theorem 15.5 that $\mathfrak{H}_\infty^\dagger$ is not a Benjamini–Schramm limit of finite *planar* graphs, recall indeed from Open Question 8.2 that we conjecture that it is a limit of maps whose genus is proportional to their sizes.

Proof. Under the hypotheses of the corollary, $\mathfrak{H}_\infty$ almost surely has a positive anchored expansion constant by Theorem 8.11 and exponential growth by Proposition 13.11. We leave the reader check the easy estimate $\mathbb{E}_\infty[\log \deg(\rho)] < \infty$. Since it is a stationary and ergodic random graph by Proposition 8.9, we can conclude from the previous theorem that it must have positive speed (almost surely constant) and be non-Liouville. □

Exercise 15.14. If $\mathbf{q}$ is a subcritical weight sequence with a finite support, prove that $\mathfrak{H}^{(\infty)}$ and its dual also have positive speed.

Poisson boundary. In the hyperbolic setup, or more generally when a graph is non-Liouville, it is interesting to study its **Poisson boundary**. In heuristic terms, the Poisson boundary contains the information of the tail of the path $(X_i : i \geq 0)$ and in turn enables us to describe all bounded harmonic functions on the graph, see [147, Proposition 14.12]. For random hyperbolic maps, this boundary coincides with the geometric boundary of the circle packing representation. See [15, 16, 124].

Bibliographical notes. The study of the behavior of random walks on planar graphs and in particular of their Poisson boundaries was triggered in [28] and since then witnessed many developments. We refer to Nachmias's lecture notes [167] and the paper [16] for a beautiful treatment of this subject. The study of the recurrence/transience of random walk on infinite random planar maps obtained as local limit of finite planar maps started with the wonderful work of Benjamini & Schramm [29]. Their result did not cover the case of the Uniform Infinite Planar Triangulation and the recurrence of the later was open until the breakthrough of Gurel-Gurevich & Nachmias [115]. This theorem was since then widely applied to many cases, in particular Theorem 15.1 is due to Bjornberg & Stefansson [43]. The behavior of random walk on $\mathfrak{M}_\infty^\dagger$ is still widely open. Proposition 15.8 is taken from [55] whereas Proposition 15.9 seems to be new. Positive speed on the hyperbolic random maps (and the associated non-Liouville property) was first proved in [89] in the triangular case, see also [17, 27]. Recently, an alternative proof of the recurrence of the UIPT (via estimation of resistance) has been found using a connection with Liouville Quantum Gravity, see [119].

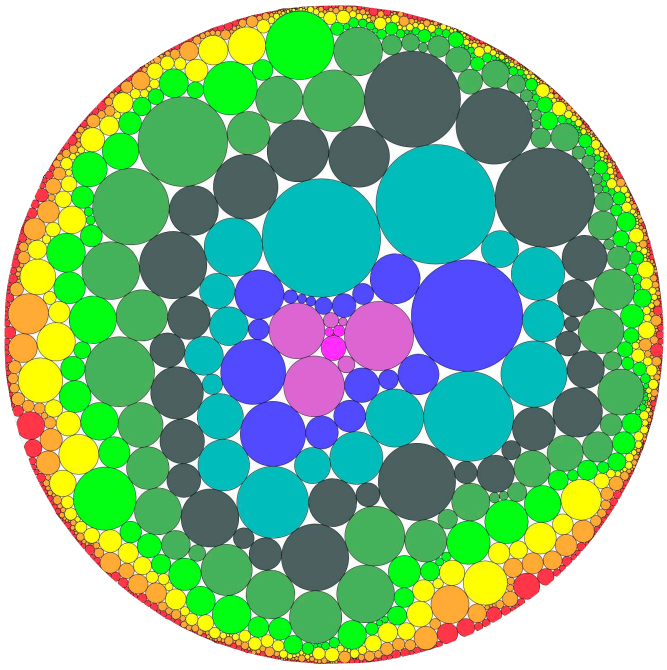


Figure 15.3: A circle packing representation of (a simple version) of a hyperbolic triangulation. The colors correspond to distances to the origin vertex. The boundary of the circle packing coincides with the Poisson boundary. Courtesy of Thomas Budzinski.

Chapter XVI : Subdiffusivity and pioneer points

Appart from recurrence/transience, one can study many other properties of the simple random walk on a graph such as resistance growth, spectral dimension or rate of escape. In this section, we will focus on the rate of escape of simple random walk $(X_0, X_1, ...)$ on $\mathfrak{M}_\infty$ and $\mathfrak{M}_\infty^\dagger$ and show a *subdiffusive phenomenon*: the walk displaces much more slowly than on a regular lattice.

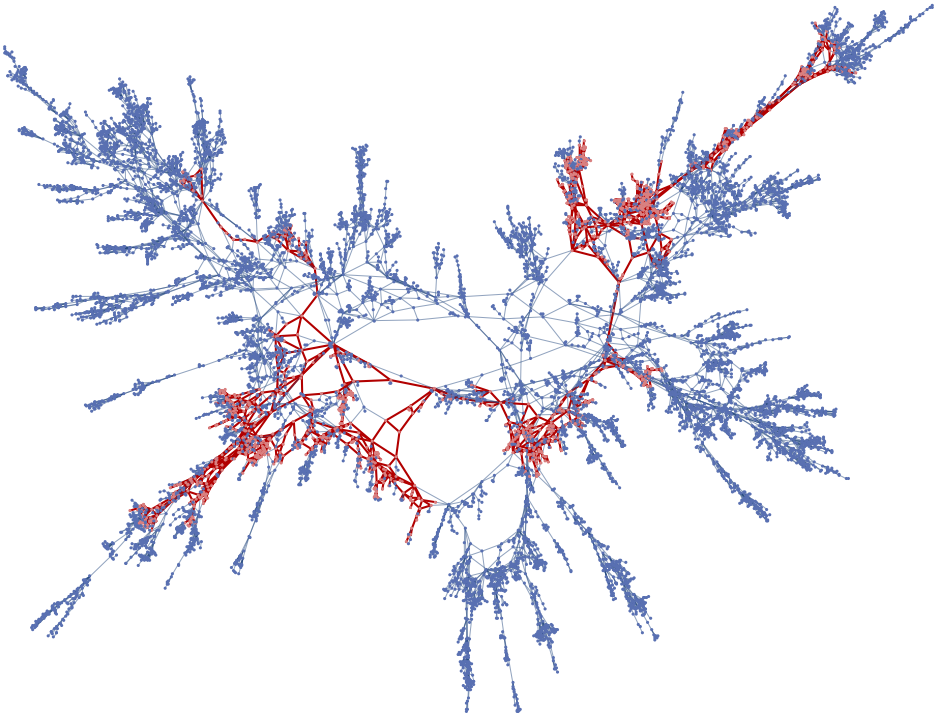


Figure 16.1: A (non-isometric) embedding in $\mathbb{R}^3$ of a random Boltzmann map with 20000 edges with weight sequence of type $a = 2.3$ in blue and the trace of the first 5000 steps of a random walk in red.

Subdiffusivity and the ant in the labyrinth Let us first describe heuristically the phenomenology of sub/super diffusive behaviors. We stay at a high and unprecise level in order not to dive into unnecessary technicalities at this point. In many situations, the random walker on a (infinite

connected) graph $\mathfrak{g}$ has a *diffusive* behavior, meaning that

$$\mathrm{d}_{\mathrm{gr}}(X_0, X_n) \approx n^{1/2}$$

for large n's. This is for example the case for the simple random walk on $\mathbb{Z}^d$ for any $d \geq 1$, or more generally on any transitive graph $\mathfrak{g}$ of polynomial growth (see [144]). If the walk displaces faster (resp. slower) than $n^{1/2+\varepsilon}$ (resp. $n^{1/2-\varepsilon}$) for some $\varepsilon > 0$, we speak of super-diffusive (resp. sub-diffusive) behavior. A particular case of super-diffusion behavior is the ballistic case when the simple random walk has positive speed. One can also reach other super-diffusivity exponents on transitive graphs [11].

We shall now focus on subdiffusive situations, also called "anomalous diffusions", which we cannot appear if the graph is too regular. Indeed, the subdiffusivity phenomenon is caused by "impurities" in the graph which slows down the random walker. This has been popularized in a wonderful article by P.G. De Gennes[1] [100] where he proposed to study the random walk on critical percolation clusters: "the ant in the labyrinth". On such random graphs, the tortuosity of the media trap the walker and delay it at every scale, forcing it to be subdiffusive. This was rigorously proved for two-dimensional percolation critical clusters and critical random trees (conditioned to survive) by Kesten [130], see also [22]. The main result of this chapter is to establish a subdiffusive behavior for simple random walk on $\mathfrak{M}_\infty$ and $\mathfrak{M}_\infty^\dagger$:

Theorem 16.1 (*Stable Boltzmann infinite maps are always subdiffusive*)

Let $\mathbf{q}$ *be a weight sequence of type* $a \in (3/2; 5/2]$. *Let* $(X_n : n \geq 0)$ *(resp.* $(X_n^\dagger : n \geq 0)$*) be the vertices visited by a simple random walk on* $\mathfrak{M}_\infty$ *(resp.* $\mathfrak{M}_\infty^\dagger$*), then for every* $\varepsilon > 0$ *we have*

$$\lim_{r\to\infty} \mathbb{P}\left(X_i \in \overline{\mathrm{Ball}}_r(\mathfrak{M}_\infty) \text{ for all } 0 \leq i \leq r^{3-\varepsilon}\right) = 1,$$
$$\lim_{r\to\infty} \mathbb{P}\left(X_i^\dagger \in \overline{\mathrm{Ball}}_r^\dagger(\mathfrak{M}_\infty) \text{ for all } 0 \leq i \leq r^{3-\varepsilon})\right) = 1.$$

In words, the above result shows that the time needed for the random walk to exit the hull of the ball of radius r is of order at least r^3 (both on the primal of dual lattice). If we were able to replace the hull of the ball with the standard ball, this would prove a subdiffusive behavior with exponent less than $1/3$. We will however not enter those geometric subtleties now, and refer to Section 16.1.3 for a discussion. We will describe two very different techniques to prove subdiffusive behavior of random walk on $\mathfrak{M}_\infty$ and $\mathfrak{M}_\infty^\dagger$: The first one will be based on the study of pioneer points via the peeling process, the second one will exploit stationarity of the graph and horofunctions measuring distances from infinity. The following diagram encapsulate all the known bounds on the subdiffusivity exponents so far:

[1] In French, and written initially for a broad audience, De Gennes surveys in this paper the phase transition in percolation, mean-field approximation, Fortuin-Kesteleyn model, anomalous diffusions and Anderson localisation problem as well as practical applications !!!

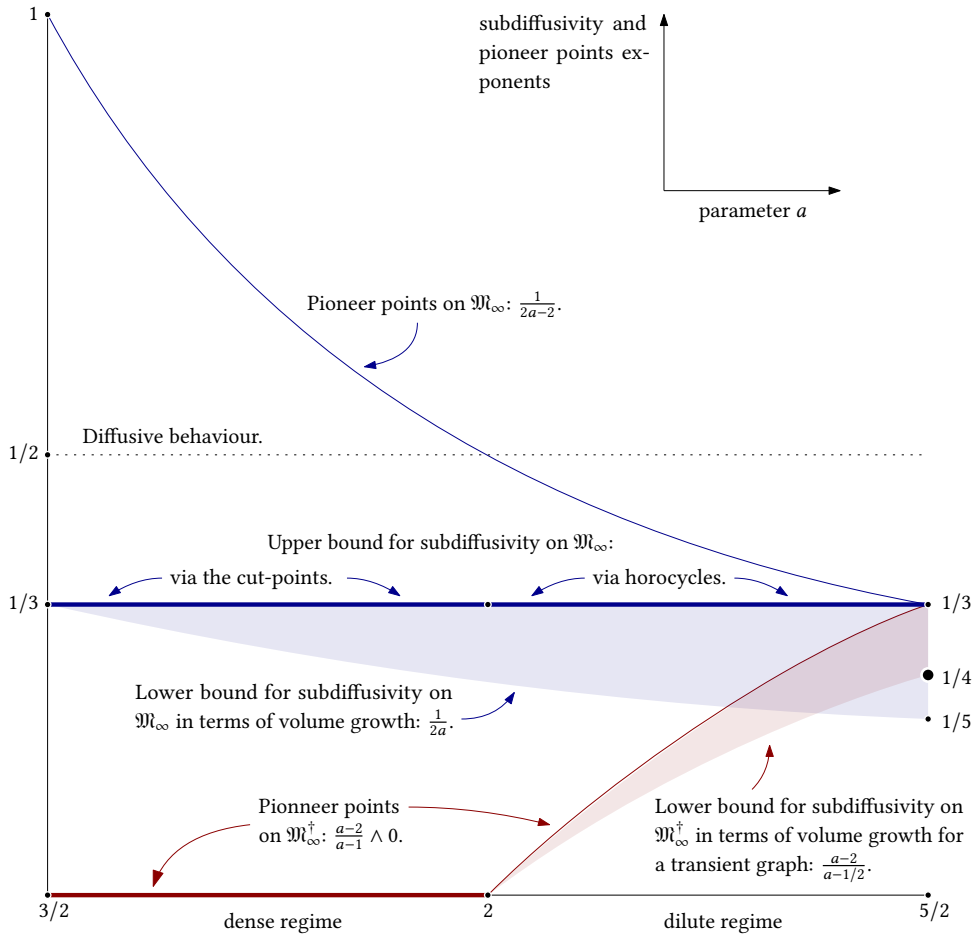


Figure 16.2: A schematic representation of the bounds on the subdiffusivity exponents for the random walk on $\mathfrak{M}_\infty$ in blue (top) and the one on $\mathfrak{M}_\infty^\dagger$ in red (bottom). Theorem 16.1 provides an upper bound for $s_a \leq 1/3$ valid on the primal lattice, in all regimes (thick blue horizontal line on Figure 16.2). On the other hand, a general result in terms of volume growth suggests that $s_a \geq \frac{1}{2a}$. When $a = \frac{5}{2}$ we expect $s_{5/2} = \frac{1}{4}$ to hold in a broad generality (see Gwynne–Miller and Gwynne–Hutchcroft for the UIPT [119, 117]). In the case of the dual map, in the dilute regime $a > 2$, we believe that the bound obtained via pioneer points remains valid with balls instead of their hulls (thin red curve in Figure 16.2); for a lower bound, if these lattices were transient (see Open Problem 15.10) then the preceding lower bound in terms of the volume growth could be sharpened to $\frac{a-2}{a-1/2}$. Indeed, in a transient stationary random graph, the range of the walk grows linearly and so the displacement exponent is at least the inverse of the volume growth exponent. This lower bound should be exact at $a = 5/2$, although in this case the lattices are expected to be recurrent (see Theorem 15.1).

Open Question 16.2. *Improve the above diagram by computing (or narrowing the estimations about) the subdiffusivity exponent of $\mathfrak{M}_\infty$ for a weight sequence $\mathbf{q}$ of type $a \in (3/2; 5/2]$.*

16.1 Pioneer points and subdiffusivity

In this section, we explain how the peeling algorithm we used in the previous chapter to deduce the intersection property for simple random walk on $\mathfrak{M}_\infty^\dagger$ can actually yield a control on subdiffusivity when combined with Theorem 10.13.

16.1.1 Pioneer points

Recall the algorithm described in Section 15.2.2 where we peel along a random walk on $\mathfrak{M}_\infty^\dagger$ and trigger a peeling step each time it is necessary to allow the random walker to perform one more step. When we consider the filled-in version of this exploration, those times where we trigger a peeling step are called the *pioneer steps* for the random walk. Equivalently, those steps are characterized by the following property:

Definition 16.3 (Pioneer step). *A pioneer step for the random walk $(X_i^\dagger : i \geq 0)$ on $\mathfrak{M}_\infty^\dagger$ is a time $i \geq 0$ such that $X_i^\dagger$ is a fresh point for the random walk (i.e. a face of $\mathfrak{M}_\infty$ that had not visited before time i) and such that $X_i^\dagger$ is not disconnected from ∞ by the trace $(X_0^\dagger, ..., X_{i-1}^\dagger)$ of the walk up to time i.*

Proof of Theorem 16.1 in the case of the walk on $\mathfrak{M}_\infty^\dagger$. Let $\mathbf{q}$ be a weight sequence of type $a \in (3/2; 5/2]$ and let us explore $\bar{\mathfrak{e}}_0^{\mathcal{A}} \subset \bar{\mathfrak{e}}_1^{\mathcal{A}} \subset \cdots \subset \bar{\mathfrak{e}}_n^{\mathcal{A}} \subset \cdots \subset \mathfrak{M}_\infty$ using the filled-peeling along the random walk on $\mathfrak{M}_\infty^\dagger$ as explained in Section 15.2.2 which we denote here by algorithm $\mathcal{A}$. For all $k \geq 0$, let us denote by $\mathcal{P}_k^\dagger$ the k-th pioneer step. Clearly, the first n steps of the walk on $\mathfrak{M}_\infty^\dagger$ take place inside $\bar{\mathfrak{e}}_n^{\mathcal{A}}$. We now use Theorem 10.13 and argue that for $\varepsilon > 0$ fixed, one can find $C_\varepsilon > 0$ such that for any other peeling algorithm $\mathcal{B}$ we have for n large enough

$$\mathbb{P}_\infty\left(\bar{\mathfrak{e}}_n^{\mathcal{A}} \subset \bar{\mathfrak{e}}_{C_\varepsilon n}^{\mathcal{B}}\right) \geq 1 - \varepsilon.$$

We apply the above inequality where the algorithm $\mathcal{B}$ is the peeling by layers for the dual distance as explained in Section 13.2.1: Using Theorem 13.9 (more precisely, the equations (13.12), (13.13) and (13.14)), it is easy to see that for some other constant C'_ε we have

$$\begin{aligned}
\bar{\mathfrak{e}}_n^{\mathcal{B}} &\subset \overline{\mathrm{Ball}}^\dagger_{C'_\varepsilon n^{\frac{a-2}{a-1}}}(\mathfrak{M}_\infty) && \text{when} \quad a \in (2, 5/2],\\
\bar{\mathfrak{e}}_n^{\mathcal{B}} &\subset \overline{\mathrm{Ball}}^\dagger_{(1+\varepsilon)c_a \log n}(\mathfrak{M}_\infty) && \text{when} \quad a \in (3/2; 2),\\
\bar{\mathfrak{e}}_n^{\mathcal{B}} &\subset \overline{\mathrm{Ball}}^\dagger_{\frac{1+\varepsilon}{2\pi^2}\log^2 n}(\mathfrak{M}_\infty) && \text{when} \quad a = 2,
\end{aligned}$$

with probability at least $1 - \varepsilon$ for n large enough. This, together with the penultimate display implies Theorem 16.1 in the case of $\mathfrak{M}_\infty^\dagger$. In the statement of the theorem, we could in fact replace $r^{3-\varepsilon}$ respectively by $r^{\frac{a-1}{a-2}-\varepsilon}$ when $a \in (2; 5/2]$, by $\mathrm{e}^{c\sqrt{r}}$ when $a = 2$ and by e^{cr} when $a \in (3/2; 2)$ for some $c > 0$ depending on a. □

16.1.2 Primal distances

One can obviously try to adapt the above proof in the case of the random walk on the primal graph $\mathfrak{M}_\infty$. In this case the peeling algorithm works as follows: we let the random walk $(X_i : i \geq 0)$ move on $\mathfrak{M}_\infty$ until it is located on the boundary $\partial^* \bar{\mathfrak{e}}_n$ of the current explored part and write τ_n for this random walk time. When it is so, we peel (with a filled-in peeling) the edge on the boundary which lies immediately to the left of X_{τ_n} and continue to do so until X_{τ_n} does not belong to the boundary of the explored part: The 1-neighborhood of X_{τ_n} has then been completely explored and we may perform a new random walk step.

With this exploration process at hand, one could try to mimic the above proof. A first difficulty is that we may trigger several peeling steps before the random walk might be able to displace, in other words we may have $\tau_n \ll n$. However, if X_{τ_n} is on the boundary of the explored part of perimeter say $2p$, there is a probability at least $\nu(-1)h^\uparrow(p-1)/h^\uparrow(p)$ that X_{τ_n} is swallowed by a peeling step of type $\mathsf{G}_{*,0}$ and is not exposed on the boundary of the explored part anymore. If $p = 1$ the point X_{τ_n} can be swallowed in two peeling steps. Since $\inf_{p\geq 2} \nu(-1)h^\uparrow(p-1)/h^\uparrow(p) > c > 0$ we see that the time it takes to discover the neighbourhood of a given pioneer point is stochastically dominated by a geometric random variable. It easily follows that $\tau_n \geq cn$ with very high probability for some constant $c \in (0,1)$. For here, using Theorem 10.13 again, one can include the piece of the map discovered after n peeling steps inside $\bar{\mathfrak{e}}^{\mathcal{B}}_{C_\varepsilon n}$ where $\mathcal{B}$ is the algorithm $\mathcal{A}_{\text{metric}}$ of Section 6.1.3. We said in Section 13.3.3 that studying primal graph distances is a hard task (except in the case of triangulations or quadrangulations), see Conjecture 13.14. However, based on the geometric information provided by Schaeffer type constructions (see Theorem 14.6) it is natural to expect that $\bar{\mathfrak{e}}^{\mathcal{B}}_n$ roughly discover the (primal) metric hull of radius $n^{\frac{1}{2(a-1)}}$. Hence, the first n steps of the walk take place in the hull of the ball of radius $n^{\frac{1}{2(a-1)}+\varepsilon}$ with high probability as $n \to \infty$. Although correct, this argument does not yield the exponent $1/3$ claimed in Theorem 16.1 and only provides some subdiffusivity information when $a \in (2; 5/2)$ where $\frac{1}{2(a-1)} < 1/2$.

16.1.3 About tentacles

As mentioned above, Theorem 16.1 does not prove a subdiffusive behavior in a strict sense since the distances in the hulls of the ball of radius r can *a priori* be much larger than r: imagine a long finite tentacle starting within the ball of radius r and reaching a level $h_r \gg r$ that would be included in the hull. It can be proved that this phenomenon does not occur in $\mathfrak{M}_\infty$ when $\mathbf{q}$ is regular, but this result is not yet available for $\mathfrak{M}^\dagger_\infty$ (and actually believed to be wrong in the dense regime).

16.2 Subdiffusivity via stationarity

We now describe the other technique used to get the universal bound $1/3$ on the subdiffusivity exponent in Theorem 16.1. It relies on a simple idea (formalized in the following general result)

which gives an upper bound on the displacement of a random walk on a random graph by "flashing it" on a certain subgraph. We only sketch the idea and refer the reader to [95] for details.

16.2.1 Subdiffusivity from diffusivity on a sparse subgraph

Let us denote by $\mathcal{G}$ a random connected (multi-)graph, either finite or infinite, but locally finite in this case, with a distinguished origin vertex ρ, and consider a simple random walk $(X_n)_{n\geq 0}$ on $\mathcal{G}$ started at $X_0 = \rho$. Denote by $\mathcal{B}_R$ the ball of radius R around the origin ρ (for the graph distance) in $\mathcal{G}$. In the following proposition if $\mathcal{G}_R$ is a finite subset of vertices of $\mathcal{G}$, we equip $\mathcal{G}_R$ with a graph structure by declaring that two vertices are adjacent if one can go from one to the other in $\mathcal{G} \setminus \mathcal{G}_R$ (beware, it is not the induced subgraph structure).

Lemma 16.4. *Let $(\beta_R)_{R\geq 1}$ and $(\gamma_R)_{R\geq 1}$ be two positive sequences and $d \geq 1$. Suppose that for any integer $R \geq 2$, we are given a subset of vertices $\mathcal{G}_R$ of the graph $\mathcal{G}$ such that:*

1. (Polynomial growth) *with high probability as $R \to \infty$, the ball $\mathcal{B}_{R+1}$ has less than R^d vertices;*
2. (Geometric Separation) *With high probability as $R \to \infty$, a simple random walk on $\mathcal{G}$ started from ρ has moved for a distance at least β_R in $\mathcal{G}_R$ (for the distance in $\mathcal{G}_R$) before exiting the ball $\mathcal{B}_R$ (for the distance in $\mathcal{G}$);*
3. (Density) *For every $n \geq 1$, we have that $\mathbb{P}(X_n \in \mathcal{G}_R) \leq \gamma_R^{-1}$.*

Then with high probability as $R \to \infty$, the random walker X_i belongs to $\mathcal{B}_R$ for every $i \leq \gamma_R \beta_R^2 \log^{-\frac{7}{4}} R$.

The idea of flashing a random walk on a certain subset to deduce subdiffusivity was already used by Kesten [130] where he considered the backbone of the critical infinite incipient percolation cluster on $\mathbb{Z}^2$.

Proof. We shall consider the walk *flashed* on $\mathcal{G}_R$, i.e. the sequence $(Y_i)_{i\geq 1}$ of successive vertices of $\mathcal{G}_R$ visited by the walk. Recall that the subset $\mathcal{G}_R$ can be equipped with a connected graph structure induced by $\mathcal{G}$ as follows: two vertices of $\mathcal{G}_R$ are linked by an edge if there exists a path in $\mathcal{G}$ going from one to the other without visiting any other vertex of $\mathcal{G}_R$. By decomposing the probability that Y moves from a vertex to another over all possible corresponding paths for X, it is straightforward to show that Y is a (possibly stopped) reversible Markov chain with respect to $\deg_{\mathcal{G}}(\cdot)$, the degree of the vertices in the *original graph* $\mathcal{G}$. The Varopoulos–Carne bound (see e.g. Lyons & Peres [147, Theorem 13.4]) then shows that for $n \geq 1$ and two vertices y and y' in $\mathcal{G}_R$ at distance d *in* $\mathcal{G}_R$, the probability that the flashed walk Y goes from y to y' in exactly n steps is at most

$$2\left(\frac{\deg_{\mathcal{G}}(y)}{\deg_{\mathcal{G}}(y')}\right)^{1/2} \exp\left(-\frac{d^2}{2n}\right).$$

Now, Assumption 2 shows that the random walker X needs to move for a distance at least β_R within the graph $\mathcal{G}_R$ in order to escape from $\mathcal{B}_R$ with high probability. Let us denote by σ_R the first instant at which the walk X has made $\beta_R^2 \log^{-3/2} R$ steps in the subset $\mathcal{G}_R$. Summing over all possible starting and ending points inside $\mathcal{G}_R \cap \mathcal{B}_R$, all times $n \leq \beta_R^2 \log^{-3/2} R$, and crudely bounding the degrees by the volume, we deduce that the probability to move distance β_R across $\mathcal{G}_R$ before time σ_R is bounded above by (dropping along the way a few factors which are less than 1 for large R's)

$$2 \max_{y \in \mathcal{G}_R} \deg_{\mathcal{G}}(y)^{1/2} \cdot \#\mathcal{B}_R^2 \cdot \beta_R^2 \cdot \exp\left(-\frac{1}{2}\log^{3/2} R\right) \leq 2R^{d/2+2d+2d} \exp\left(-\frac{1}{2}\log^{3/2} R\right) \xrightarrow[R\to\infty]{} 0,$$

on the event where 1 and 2 are satisfied.

On the other hand, by 3 and Markov's inequality

$$\begin{aligned}
\Pr \sigma_R \leq \gamma_R \beta_R^2 \log^{-7/4} R &= \Pr \sum_{k \leq \gamma_R \beta_R^2 \log^{-7/4} R} \mathbf{1}_{X_k \in \mathcal{G}_R} \geq \beta_R^2 \log^{-3/2} R \\
&\leq \beta_R^{-2} \log^{3/2} R \sum_{k \leq \gamma_R \beta_R^2 \log^{-7/4} R} \Pr X_k \in \mathcal{G}_R \\
&\leq C \log^{-1/4} R \xrightarrow[R\to\infty]{} 0.
\end{aligned}$$

We deduce that with high probability, the random walk X was not able to move across distance β_R in $\mathcal{G}_R$ within the first $\gamma_R \beta_R^2 \log^{-7/4} R$ steps. A fortiori it could not have escaped $\mathcal{B}_R$ by 2. □

The proof of Theorem 16.1 reduces to finding such $\mathcal{G}_R$'s which are big enough so β_R is large, but not too big so γ_R is also large. Indeed, a caricature consists in taking $\mathcal{G}_R$ to be the entire ball of radius R, then $\beta_R = R$ but $\gamma_R = 1$ which shows that the walk is at most diffusive; another extreme consists in taking $\mathcal{G}_R$ to be the union of the boundaries of the balls of radius R and $R/2$, which lie at distance 1 in $\mathcal{G}_R$, but now γ_R is quite large and this again would yield a diffusive upper bound in our case.

16.2.2 Heuristic for $\mathcal{G}_R$

A natural guess for $\mathcal{G}_R$ which is thinner than the entire ball $\mathcal{B}_R$ but which still necessitate about R (flashed) steps to traverse it is the set $\widetilde{\mathcal{G}}_R$ of vertices which separates $\mathcal{B}_{R/2}$ from infinity (see Figure 16.3 left). Namely those are the vertices $u \in \mathcal{G} \setminus \mathcal{B}_{R/2}$ from which there exists an infinite path along which the distance to ρ is nondecreasing. The main drawback is that estimating $\mathbb{P}(X_n \in \mathcal{G}_R)$ is a difficult task. This is due to the fact that this set strongly depends on ρ. We shall rather construct our random subsets $\mathcal{G}_R$ in a *stationary* way, i.e. such that $\mathbb{P}(X_n \in \mathcal{G}_R) = \mathbb{P}(\rho \in \mathcal{G}_R)$ for all n. Since the random graph $\mathfrak{M}_\infty$ is itself a stationary random graph (Proposition 7.9), it suffices to construct $\mathcal{G}_R$ in a way that does not depend on the origin ρ of $\mathfrak{M}_\infty$. A way to proceed is to use "distances from infinity" or horodistances rather than distances to ρ. These horodistances are defined by

$$h(u) = \lim_{z \to \infty} \mathrm{d}_{\mathrm{gr}}(z, u) - \mathrm{d}_{\mathrm{gr}}(z, \rho) \in \mathbb{Z}, \tag{16.1}$$

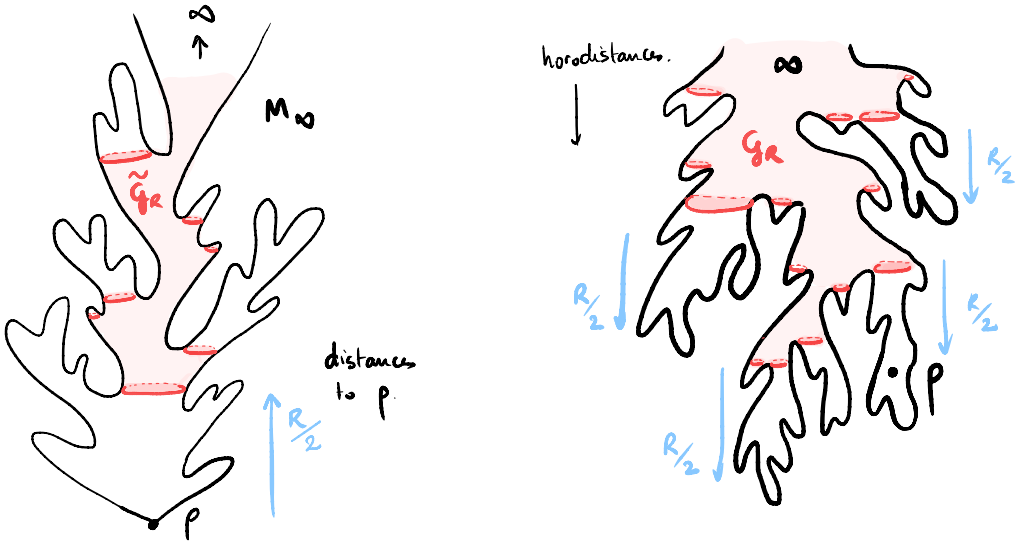


Figure 16.3: A natural try for $\mathcal{G}_R$ on the left (not stationary) and its stationary version using horodistances. The problem is that horodistances are not proved to exist in $\mathfrak{M}_\infty$ in general...

where $z \to \infty$ means that z escapes from any finite set in the map. Then *define* the set $\mathcal{G}_R$ as those vertices u such that at least R^{2a-1} (the typical volume of a ball of radius R) different vertices lie "under u", i.e. may be joined to u by a path visiting only vertices with horodistance non-greater than $h(u)$ (see Figure 16.3 right). Since the definition of $\mathcal{G}_R$ does not depend on the origin of the map, it is stationary in the sense that $\mathbb{P}(X_n \in \mathcal{G}_R)$ is constant in n. In the notation of Lemma 16.4 we expect both $\beta_R \approx R$ and $\gamma_R \approx R$ which yields an upper bound of $1/3$ on the subdiffusivity exponent by Lemma 16.4.

Open Question 16.5. *Show that horodistances (i.e. defined by (16.1)) exist on $\mathfrak{M}_\infty$ or $\mathfrak{M}_\infty^\dagger$ (an easier task adapting [155] would be to show their existence of the Eden distance on $\mathfrak{M}_\infty^\dagger$).*

Actually since horodistances (16.1) are not yet proved to exist in general Boltzmann maps (see [96, 155] for the case of the UIPQ and UIPT) the paper [95] uses a trick and emulates them on finite maps by replacing horodistances with the distances to an extra large boundary, far away from the root edge.

Cut-points. Recall from Chapter 13 that the geometry of $\mathfrak{M}_\infty$ undergoes a phase transition at $a = 2$, and the dense phase $a < 2$ is very different from the dilute phase $a > 2$; in particular, when $a < 2$, the map possesses *cut edges*: large faces touch themselves and disconnect the origin from infinity (Proposition 13.12). In this phase $a < 2$, we can actually give another version of a stationary set $\mathcal{G}_R$ as the set of all (vertices adjacent to) edges which separate from infinity a part of the map with volume at least R^{2a-1}, see Figure 16.4. It is possible to control the number of such cut edges and to prove with the preceding notation that $\beta_R \approx R^{4-2a}$, see [95]. Similarly,

it is possible to evaluate the density of $\mathcal{G}_R$ and proving $\gamma_R \approx R^{4a-5}$. This gives $\gamma_R \beta_R^2 \approx R^3$ and proves the same upper bound of $1/3$ on the subdiffusivity exponent on $\mathfrak{M}_\infty$.

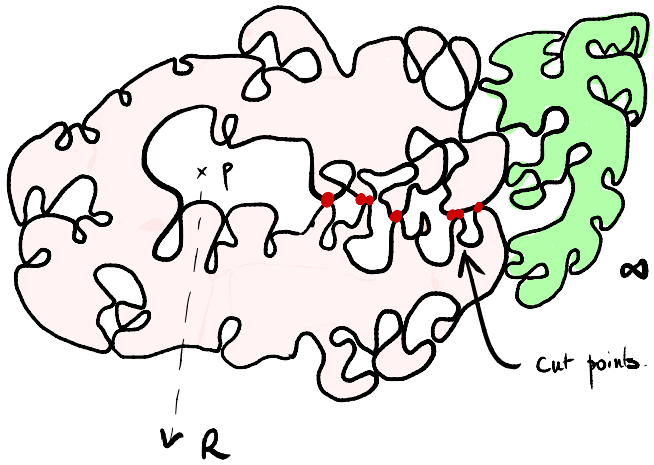


Figure 16.4: Illustration of the cut points in the dense case. We have roughly R^{4-2a} cut points separating the origin from infinity in the ball of radius R. Combining a diffusivity estimate for the random walk flashed on these cut points and the density R^{5-4a} of these cut points yields an upper bound of $1/3$ on the diffusivity exponent.

Bibliographical notes. The subdiffusivity phenomenon on random maps was first established in [25] in the case of the UIPQ using the pioneer point technique. It was then improved and generalized in [93]. Since then, a complete understanding of the subdhiffusive behavior on the UIPT/UIPQ has been reached thanks to a series of works of Gwynne, Miller and Hutchcroft [119, 117] based on Liouville Quantum Gravity. However, the case of maps with large degrees/faces remains widely open and Theorem 16.1 is taken from [95]. A powerful technique to prove subdiffusivity using conformal metrics has also been developed by Lee [143].

Chapter I : Elements of fluctuation theory

In this chapter, we gather a few general results on random walks on $\mathbb{Z}$: fluctuation theory, cyclic lemma, conditioning to stay postive, strong ratio limit theorem and local limit theorem. We suppose that $(S_i : i \geq 0)$ is an aperiodic random walk on $\mathbb{Z}$ with i.i.d. increments of law ν which under the law $\mathbf{P}^{(\ell)}$ is started from $\ell \in \mathbb{Z}$. We generically suppose that S starts from 0 and simply write $\mathbf{P}$ for $\mathbf{P}^{(0)}$ to simplify notation (beware, the random walk in the rest of the notes started from 1 which induces a shift of 1 in the notation). We will recall some elements of fluctuation theory for such walks which studies the return times and entrance heights in half-spaces.

A.1 Oscillations, duality

Recall the classification of one-dimensional random walks (whose proof can be found in many textbooks):

Definition A.1. *A (non-trivial) one-dimensional random walk* (S) *falls into exactly one of the three categories:*

(i) Either $S_n \to \infty$ *a.s. as* $n \to \infty$ *in which case* (S) *is said to* drift towards ∞,

(ii) Or $S_n \to -\infty$ *a.s. as* $n \to \infty$ *in which case* (S) *is said to* drift towards $-\infty$,

(iii) Or (S) oscillates *i.e.* $\limsup_{n\to\infty} S_n = +\infty$ *and* $\liminf_{n\to\infty} S_n = -\infty$ *almost surely.*

The above classification is equivalently expressed in terms of the so-called ladder process which we now introduce:

Definition A.2 (Ladder heights and epochs). *We define by induction* $T_0^> = T_0^< = T_0^\geq = T_0^\leq = 0$ *as well as* $H_0^> = H_0^< = H_0^\geq = H_0^\leq = 0$ *and for* $i \geq 1$ *we put*

$$\begin{aligned} T_i^> &= \inf\left\{k > T_{i-1}^> : S_k > H_{i-1}^>\right\} \quad \text{and} \quad H_i^> = S_{T_i^>}, \\ T_i^\geq &= \inf\left\{k > T_{i-1}^\geq : S_k \geq H_{i-1}^\geq\right\} \quad \text{and} \quad H_i^\geq = S_{T_i^\geq}, \\ T_i^< &= \inf\left\{k > T_{i-1}^< : S_k < H_{i-1}^<\right\} \quad \text{and} \quad H_i^< = S_{T_i^<}, \\ T_i^\leq &= \inf\left\{k > T_{i-1}^\leq : S_k \leq H_{i-1}^\leq\right\} \quad \text{and} \quad H_i^\leq = S_{T_i^\leq}. \end{aligned}$$

If T_i^* *is not defined (i.e. we take the infimum over the empty set) then we put* $T_j^* = H_j^* = \pm\infty$ *for all* $j \geq i$. *The variables* $(T^>/T^\geq)$ *(resp.* $(T^</T^\leq)$*) are called the strict/weak ascending (resp. descending) ladder epochs. The associated* H *processes are called the (strict/weak ascending/descending) ladder heights.*

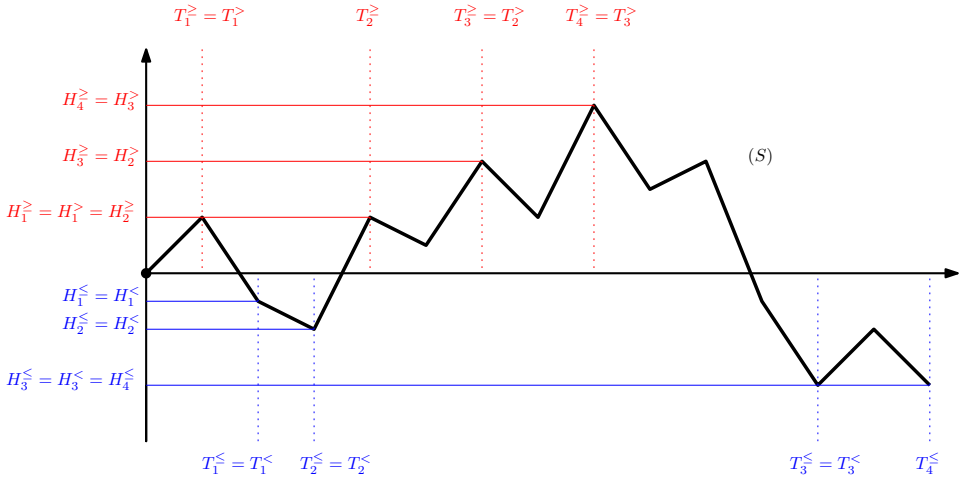


Figure A.1: Illustration of the definition of the ladder heights and epochs.

In the following we write H and T generically for one of the four couples

$$(T^\geq, H^\geq), \quad (T^>, H^>), \quad (T^<, H^<) \quad \text{or} \quad (T^\leq, H^\leq).$$

The ladder epochs are clearly stopping times for the natural filtration generated by the walk and the strong Markov property shows that N defined as $\inf\{i \geq 0 : T_i = \infty\}$ is a geometric random variable with distribution

$$\mathbf{P}(N = k) = \mathbf{P}(T_1 = \infty)\mathbf{P}(T_1 < \infty)^{k-1},$$

and that conditionally on N the random variables $((T_{i+1} - T_i), (H_{i+1} - H_i))_{0 \leq i \leq N-1}$ are i.i.d. with law (T_1, H_1) conditioned on $T_1 < \infty$. Combining these observations with Definition A.1 we deduce a characterization of drift/oscillation in terms of the ladder epochs: The walk (S) drifts towards $+\infty$ if and only if $\mathbf{P}(T_1^< = \infty) > 0$ and $\mathbf{P}(T_1^> = \infty) = 0$. The walk (S) oscillates if and only

if $\mathbf{P}(T_1^< = \infty) = \mathbf{P}(T_1^> = \infty) = 0$. The ladder variables are also connected via a simple but surprisingly important observation called *duality*.

Proposition A.3 (Duality). *For each fixed $n \geq 0$, we have the following equality in distribution*

$$(0 = S_0, S_1, \ldots, S_n) \overset{(d)}{=} (S_n - S_n, S_n - S_{n-1}, S_n - S_{n-2}, \ldots, S_n - S_1, S_n).$$

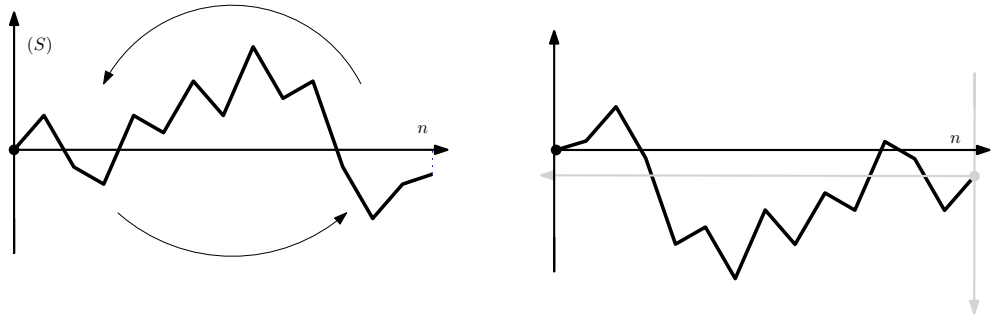


Figure A.2: Geometric interpretation of the duality: the rotation by angle π of the first n steps of the walk (S) leaves the distribution invariant.

This innocent proposition enables us to connect the strict descending ladder variables to the weak ascending ones. Indeed, notice (on a drawing) that for any $n \geq 0$

$$\begin{aligned}
\mathbf{P}(T_1^< > n) &= \mathbf{P}(S_0 = 0, S_1 \geq 0, \ldots, S_n \geq 0) \\
&\underset{\text{duality}}{=} \mathbf{P}(S_n - S_n = 0, S_n - S_{n-1} \geq 0, \ldots, S_n \geq 0) \\
&= \mathbf{P}(S_n \geq S_{n-1}, S_n \geq S_{n-2}, \ldots, S_n \geq S_0) = \mathbf{P}(n \text{ is a weak ladder epoch}).
\end{aligned}$$

Summing over $n \geq 0$ we deduce that

$$\begin{aligned}
\mathbf{E}[T_1^<] &= \sum_{n \geq 0} \mathbf{P}(T_1^< > n) \\
&= \mathbf{E}[\text{number of weak ascending finite ladder epochs}] \\
&= \frac{1}{\mathbf{P}(T_1^{\geq} = \infty)}, \qquad \text{(A.1)}
\end{aligned}$$

because the total number of weak ascending finite ladder epochs follows a geometric distribution with success parameter $\mathbf{P}(T_1^{\geq} = \infty)$. We similarly establish that $\mathbf{E}[T_1^{\leq}] = 1/\mathbf{P}(T_1^> = \infty)$.

A.2 Cyclic lemma and applications

We now present the cyclic lemma, which is a very powerful tool when analyzing random walks. It has many applications and corollaries such as the Kemperman formula. The lemma in this form is due to Feller[1].

[1]  William Feller 1906–1970, born Vilibald Srećko Feller

A.2.1 Feller's cyclic lemma

Let $x_1, x_2, \dots, x_n$ be real numbers which we consider as the increments of the walk (s) defined by

$$s_0 = 0, s_1 = x_1, s_2 = x_1 + x_2, \quad \dots \quad , s_n = x_1 + \cdots + x_n.$$

Recall that i is a strict ascending ladder epoch for (s) if $s_i > s_{i-1}, s_i > s_{i-2}, \dots, s_i > s_0$. If $\ell \in \{0, 1, 2, \dots, n-1\}$ we consider $(s^{(\ell)})$ the ℓ-th cyclic shift of the walk obtained by cyclically shifting its increments ℓ times, that is

$$s_0^{(\ell)} = 0, s_1^{(\ell)} = x_{\ell+1}, \quad \dots \quad , s_{n-\ell}^{(\ell)} = x_{\ell+1} + \cdots + x_n, \quad \dots \quad , s_n^{(\ell)} = x_{\ell+1} + \cdots + x_n + x_1 + \cdots + x_\ell.$$

Lemma A.4 (Feller). *Suppose that $s_n > 0$. We denote by $k \in \{0, 1, 2, \dots, n\}$ the number of cyclic shifts $(s^{(\ell)})$ with $\ell \in \{0, 1, 2, \dots, n-1\}$ for which n is a strict increasing ladder epoch. Then $k \geq 1$ and any of those cyclic shifts has exactly k strict ascending ladder epochs.*

Proof. Let us first prove that $k \geq 1$. For this consider the first time $\ell \in \{1, 2, \dots, n\}$ such that the walk (s) reaches its maximum. Then clearly (make a drawing) the time n is a strict ascending ladder epoch for $s^{(\ell)}$. We can thus suppose without loss of generality that n is a strict ascending ladder epoch for (s). It is now clear (see Figure A.3 below) that the only possible cyclic shifts of the walk such that the resulting walk admits a strict ascending ladder epoch at n correspond to the strict ascending ladder epochs of (s). Moreover these cyclic shifts do not change the number of strict ascending ladder epochs. □

Remark A.5. Beware, Feller's combinatorial lemma does not say that the cyclic shifts $(s^{(\ell)})$ are distinct. Indeed, in the action of $\mathbb{Z}/n\mathbb{Z}$ on $\{(s^{(\ell)} : \ell \in \{0, 1, \dots, n-1\}\}$ by cyclic shifts, the size of the orbit is equal to n/j where $j|n$ is the cardinal of any stabilizer of $(s^{(\ell)})$. In our case, it is easy to see that j must also divide k and in this case, there are only k/j distinct cyclic shifts having n as the kth strict ascending ladder time.

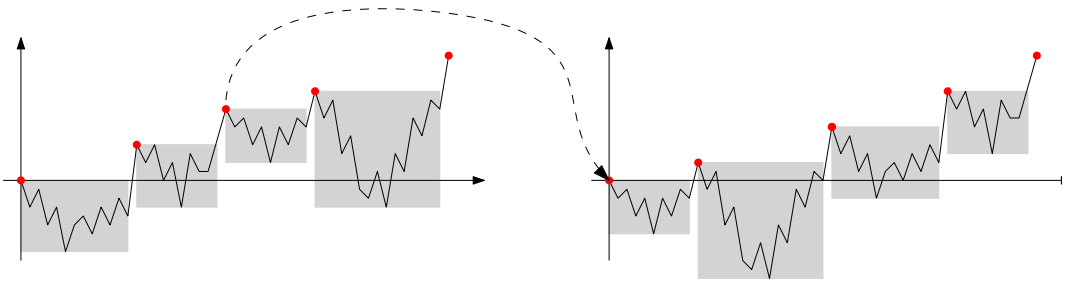


Figure A.3: Illustration of Feller's combinatorial lemma. We show a walk such that n is a strict ascending ladder epoch and the cyclic shift corresponding to the second strict ascending ladder epoch.

The above lemma also holds if we replace strict ascending ladder epochs by weak/descending ladder epochs provided that $s_n \geq 0$ or $s_n \leq 0$ or $s_n < 0$ depending on the case. Another equivalent way to reformulate the previous lemma is to write for $n \geq 1$ and any measurable subset $A \subset \mathbb{R}_+^*$ that we have

$$\mathbf{1}_{s_n \in A} = \sum_{i=0}^{n-1} \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{1}_{T_k^>(s^{(i)})=n} \mathbf{1}_{H_k^>(s^{(i)}) \in A}.$$

Indeed, if the walk (s) is such that $s_n \in A$ in particular $s_n > 0$ and there exists a unique k such that exactly k cyclic shifts do not annulate the indicator functions on the right-hand side. Since we divide by k the total sum is one. If we take expectation in the case when $(s) = (S)$ is a one-dimensional random walk with i.i.d. increments, then using the fact that for all $n \geq 0$ we have $(S_j^{(i)})_{0 \leq j \leq n} = (S_j)_{0 \leq j \leq n}$ in distribution we deduce the following corollary:

Corollary A.6. *For every $n \geq 1$ and any measurable subset $A \subset \mathbb{R}_+^*$ we have*

$$\frac{1}{n} \mathbf{P}(S_n \in A) = \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{P}(T_k^> = n, H_k^> \in A).$$

A.2.2 Applications

Let us mention a few important applications of the preceding results:

Skip-free walks

The ν-walk (S) is said to be **skip-free ascending** (resp. descending) if $\nu(\{1, 2, 3, \dots\}) = \nu_1$ (resp. $\nu(\{\dots, -3, -2, -1\}) = \nu_{-1}$); or in words when the only positive (resp. negative) jumps of S are jumps of $+1$ (resp. -1).

The best examples of skip-free walks are simple random walks where the step distribution is supported by ± 1 (they are both skip-free ascending and descending). The nice thing with skip-free walks is the fact their k-th (ascending or descending) ladder height must be equal to k when it is finite. This simple observation turns out to have many implications. E.g., Corollary A.6 reduces to the well-known Kemperman's [2] formula

Proposition A.7 (Kemperman's formula)**.** *Let (S) be a skip-free descending walk. Then for every $n \geq 1$ and every $k \geq 1$ we have*

$$\frac{1}{n} \mathbb{P}(S_n = -k) = \frac{1}{k} \mathbb{P}(T_k^< = n).$$

Plane trees

Also, skip-free descending walks are in bijection with plane trees. Let us quickly recall the basics of plane trees. Within this lecture notes, the most natural definition of a plane tree is the following:

[2] Johannes Henricus Bernardus Kemperman 1924–2011

Definition A.8. *A* plane tree τ *is a finite rooted planar map with a unique face.*

Obviously, there are many equivalent definitions of plane trees and there is a down-to-earth way to encode them, via Neveu's notation, as subset of words on $\{1, 2, ...\}$, see Figure A.4 left. More important for us, plane trees can be encoded by skip-free descending walk via their so-called Łukasiewicz walk. The Łukasiewicz walk associated to a finite plane tree τ is obtained by first listing the vertices $u_1 \prec u_2 \prec \cdots \prec u_n$ of τ in depth-first order (equivalent in lexicographical order in Neveu's notation) and consider the discrete path: $\mathcal{W}_0(\tau) = 0$ and for $0 \leq k \leq n-1$:

$$\mathcal{W}_{k+1}(\tau) = \mathcal{W}_k(\tau) + \#\mathsf{Children}(u_k \text{ in } \tau) - 1.$$

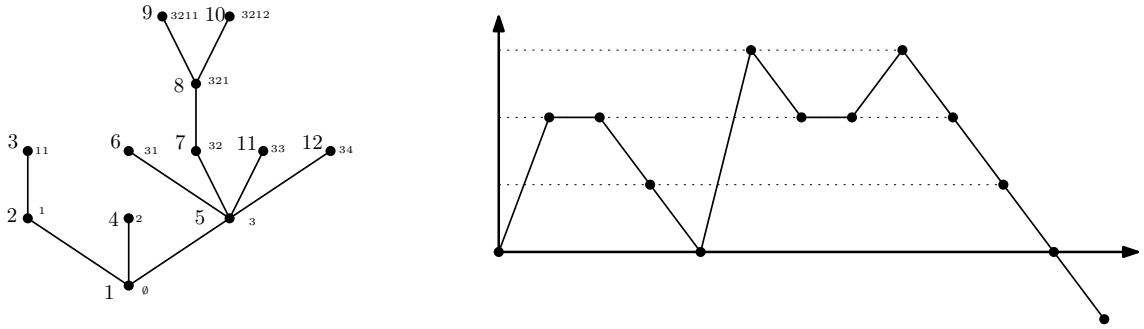


Figure A.4: Left: a finite plane tree and its vertices listed in lexicographical order. Right: its associated Łukasiewicz walk.

The above encoding is bijective and the total number of vertices of the tree τ is equal to the first hitting time of -1 by its Lukasiewciz walk (the process is killed afterwards). Here is a immediate corollary of the above encoding together with Feller's combinatorial lemma:

Theorem A.9 (*Harary & Prins & Tutte (1964)*)

The number of plane trees with d_i vertices with $i \geq 0$ children, and with $n = 1 + \sum id_i = \sum d_i$ vertices is equal to

$$\frac{(n-1)!}{d_0!d_1!\cdots d_i!\cdots}.$$

Proof. Fix d_i and n as in the theorem. Notice that we must have $n = 1 + \sum id_i = \sum d_i$. By the encoding of plane trees into their Łukasiewicz path it suffices to enumerate the number of paths starting from 0, ending at -1 at n and with d_i steps of $i-1$ and which stay non-negative until time $n-1$. Clearly, if one removes the last assumption there are

$$\binom{n}{d_0, \ldots, d_i, \ldots} = \frac{n!}{d_0!d_1!\cdots}$$

such paths. If we partition those paths according to the cyclic shift equivalence relation, then by Lemma A.4 (see Remark A.5) we know that there each equivalence class has cardinal n and

has a unique element which stays non-negative until time $n-1$. Hence the quantity we wanted to enumerate is equal to

$$\frac{1}{n}\binom{n}{d_0,\ldots,d_i,\ldots} = (n-1)!\prod_i \frac{1}{d_i!}.$$

□

If $\mu \neq \delta_1$ is an offspring distribution on $\{0,1,2,...\}$ with mean less than 1, then the Łukasiewicz path associated to a μ-Bienaymé–Galton–Watson plane tree is a skip-free descending random walk started from 0, killed at $T_1^<$ (wich is almost surely finite) and with step distribution $\nu(k) = \mu(k+1)$ for $k \geq -1$.

A.3 Random walks conditioned to stay positive

We now use the ladder processes to introduce and study the "random walk (S) conditioned to stay positive for ever". Unless in the case when (S) drifts towards $+\infty$, the previous conditioning is degenerate and one needs to work to make sense of it. Before doing this, let us recall some basics about Doob's h-transform.

A.3.1 h-transform of Markov chains

Suppose that $p(x,y)$ are probability transitions of a discrete Markov chain on a countable state space Ω (although we could proceed similarly for a continuous state space). Suppose that $h : \Omega \to \mathbb{R}_+$ is a non-negative function which is harmonic and positive on $A \subset \Omega$ i.e.

$$h(x) > 0, \quad \text{and} \quad h(x) = \sum_{y\in\Omega} p(x,y)h(y), \qquad \forall x \in A.$$

Under these circumstances, one can define a new transition kernel q on A by the formula:

$$q(x,y) = \frac{h(y)}{h(x)}p(x,y), \quad x \in A, y \in \Omega.$$

It is plain from the harmonicity of h on A that q indeed defines a transition kernel. The Markov chain obtained by starting in A, using those transitions rules until we possibly enter $\Omega\backslash A$ (where we stop) is called the **is called the Doob h-transform of p**. The law of the q-chain is equivalently characterized as follows: for any given path $x_0, x_1, \ldots, x_{n-1}$ in A and $x_n \in \Omega$ we have

$$\prod_{i=0}^{n-1} q(x_i, x_{i+1}) = \frac{h(x_n)}{h(x_0)} \prod_{i=0}^{n-1} p(x_i, x_{i+1}). \tag{A.2}$$

In particular, notice that if h is zero on $\Omega\backslash A$, the q-Markov chain never escapes A and so can be interpreted as a way to *condition the p-chain to stay in A* (see below for a justification of this name in the case of random walks).

A.3.2 Renewal function

Recalling the definition of the ladder process we define two functions for $\ell \geq 0$

$$h^{\downarrow}(\ell) = \sum_{i\geq 0} \mathbf{P}(H_i^{<} = -\ell) = \mathbf{P}(\tau_{\mathbb{Z}_{\leq -\ell}} = \tau_{-\ell} < \infty),$$

$$h^{\uparrow}(\ell) = h^{\downarrow}(0) + \cdots + h^{\downarrow}(\ell) = \mathbf{E}\left[\sum_{i\geq 0} \mathbf{1}_{H_i^{<} \geq -\ell}\right].$$

The function $h^{\downarrow}$ is called the *pre-renewal* function and $h^{\uparrow}$ is called the *renewal function* of the walk S. Using (A.1), it is easy to see that $h^{\uparrow}$ is bounded if and only if (S) drifts towards $+\infty$. Actually, the functions $h^{\downarrow}$ and $h^{\uparrow}$ have harmonic properties with respect to the walk (S):

Proposition A.10. *The functions $h^{\downarrow}$ and $h^{\uparrow}$ are respectively harmonic on $\mathbb{Z}_{>0}$ and super-harmonic on $\mathbb{Z}_{\geq 0}$ for the walk (S). Moreover the function $h^{\uparrow}$ is harmonic on $\mathbb{Z}_{\geq 0}$ (not only super-harmonic) if and only if (S) does not drift towards $-\infty$.*

Proof. By writing the Markov property at time 1 under $\mathbf{P}^{(x)}$ for $x \in \{1, 2, \dots\}$ we have $h^{\downarrow}(x) = \sum_{k\in\mathbb{Z}} \nu(k) h^{\downarrow}(x+k)$ which is the required harmonicity of $h^{\downarrow}$ on $\mathbb{Z}_{>0}$. By summing-up these equations for $x = 1, 2, \dots, y$ we get that

$$h^{\uparrow}(y) - 1 = \sum_{k\in\mathbb{Z}} \nu(k) h^{\uparrow}(y+k) - \sum_{k\geq 0} h^{\uparrow}(k)\nu(k).$$

Hence the (super-)harmonicity of $h^{\uparrow}$ is tied to the value of $\sum_{k\geq 0} h^{\uparrow}(k)\nu(k)$. To evaluate the latter, we use duality, at fixed n:

$$\begin{aligned}
\sum_{k\geq 0} h^{\uparrow}(k)\nu(k) &= \sum_{k\geq 0} \nu(k)\mathbf{P}\left[\sum_{i\geq 0} \mathbf{1}_{H_i^{<} \geq -k}\right] \\
&= \sum_{k\geq 0} \nu(k)\left(1 + \sum_{i\geq 1}\sum_{n\geq 1} \mathbf{E}\left[H_i^{<} \geq -k \text{ and } T_i^{<} = n\right]\right) \\
&\underset{\text{duality}}{=} \sum_{k\geq 0} \nu(k)\left(1 + \sum_{n\geq 1} \mathbf{E}\left[S_1, \dots, S_n < 0 \text{ and } S_n \in [0, -k]\right]\right) \\
&= \mathbf{P}(S_1 \geq 0) + \sum_{n\geq 1}\sum_{k\geq 0} \nu(k)\mathbf{E}\left[S_1, \dots, S_n < 0 \text{ and } S_n \in [0, -k]\right] \\
&= \mathbf{P}(S_1 \geq 0) + \sum_{n\geq 1} \mathbf{P}(T_1^{\geq} = n+1) = \mathbf{P}(T_1^{\geq} < \infty).
\end{aligned}$$

The last probability is equal to 1 if and only if (S) does not drift towards $-\infty$. □

The reader may first wonder, whether there are other harmonic functions on $\mathbb{Z}_{\geq 0}$ with which one could perform the h-transformation. The answer is no in general: it follows from [104] that for an oscillating random walk there is a unique (up to multiplicative constant) harmonic function on $\mathbb{Z}_{\geq 0}$ which is null on $\mathbb{Z}_{<0}$. As performed in the lecture notes, when $h^{\uparrow}$ is harmonic, we can consider the process $S^{\uparrow}$ obtained as the Doob h-transform of the walk (S) killed on $\mathbb{Z}_{<0}$ and we thus obtain a Markov process on $\{0, 1, 2, \dots\}$ which is transient when S oscillates:

Proposition A.11 (Transience of the $h^\uparrow$-transform). *Suppose that S does not drift towards $-\infty$. Then the Markov chain $(S_i^\uparrow)_{i\geq 0}$ is transient.*

Proof. Let us consider the Markov chain $S^\uparrow$ started from ℓ and consider the first time $\tau_{<\ell}$ it reaches a value strictly lower than ℓ. By above properties of the h-transform we can write

$$\mathbf{P}^{(\ell)}(\tau_{<\ell}(S^\uparrow) < \infty) = \frac{1}{h^\uparrow(\ell)}\mathbf{E}_\ell[h^\uparrow(S_{\tau_{<\ell}})\mathbf{1}_{\tau_\ell<\infty}] \leq \frac{\sup_{\ell'<\ell} h^\uparrow(\ell')}{h^\uparrow(\ell)},$$

unless the walk has non-negative increments (in which case the statement of the proposition is plain), the function $h^\uparrow$ is strictly increasing so that the last fraction is < 1. It follows easily that $S^\uparrow$ is transient. □

Remark A.12. In the case of a skip-free descending random walk, the condition that (S) does not drift towards $-\infty$ is equivalent to $\sum kv(k) \geq 0$. If we further suppose that S does not drift towards $+\infty$, the walk must be centered and since it oscillates we have $h^\uparrow(x) = x + 1$ because $T_k^< < \infty$ and $H_k^< = k$ automatically. In this case, $(S_n : n \geq 0)$ is a martingale (for the natural filtration) and the h-transformation of (S) coincides with the martingale change of measure by (S).

A.3.3 Oscillating case and limit of large conditionings

In this section, we suppose that S does not drift towards $-\infty$ so that $h^\uparrow$ is harmonic and $S^\uparrow$ is a well-defined Markov chain. We shall see that the process $S^\uparrow$ arises as a limit in distribution of the random walk S conditioned on staying non-negative for large time, hence justifying the terminology "random walk conditioned to stay non-negative for ever". Specifically, we consider the two events

$$\Lambda_n^{(1)} = \{(S) \text{ hits } [n, \infty) \text{ before } (-\infty, 0)\},$$
$$\Lambda_n^{(2)} = \{S_k \geq 0 \text{ for all } 0 \leq k \leq n\}.$$

Theorem A.13 (*Bertoin & Doney*)

Suppose that S does not drift towards $-\infty$. The process $(S^\uparrow)$ appears as the limit in distribution of the random walk S conditioned on $\Lambda_n^{(1)}$ or $\Lambda_n^{(2)}$ as $n \to \infty$, specifically we have

$$\mathbf{P}\left(S_0 = s_0, \dots, S_k = s_k \mid \Lambda_n^{(i)}\right) \xrightarrow[n\to\infty]{} \mathbf{1}_{s_i\geq 0, \forall 0\leq i\leq k} \frac{h^\uparrow(s_k)}{h^\uparrow(s_0)} \prod_{i=0}^{k-1} \nu(s_{i+1} - s_i) = \mathbf{P}(S_0^\uparrow = s_0, \dots, S_k^\uparrow = s_k).$$

Proof. The proof is very similar in spirit to what we did in Lemma 7.7. The technical key is:

Lemma A.14. *We have for $i \in \{1, 2\}$*

$$\liminf_{n\to\infty} \frac{\mathbf{P}^{(x)}(\Lambda_n^{(i)})}{\mathbf{P}^{(0)}(\Lambda_n^{(i)})} = \frac{h^\uparrow(x)}{h^\uparrow(0)},$$

Given the lemma, the proof of the theorem is a piece of cake. Let us treat the case of $i = 2$ for example. By the Markov property applied at time k we have for $i \in \{1, 2\}$

$$\begin{aligned}
\mathbf{P}^{(0)}\left(S_0 = s_0, \dots, S_k = s_k \mid \Lambda_n^{(2)}\right) &= \mathbf{P}^{(0)}(S_0 = s_0, \dots, S_k = s_k)\mathbf{1}_{s_i \geq 0, \forall 0 \leq i \leq k} \frac{\mathbf{P}^{(s_k)}(\Lambda_{n-k}^{(2)})}{\mathbf{P}^{(0)}(\Lambda_n^{(2)})} \\
&\geq \mathbf{1}_{s_0=0}\mathbf{1}_{s_i \geq 0, \forall 0 \leq i \leq k} \prod_{i=0}^{k-1} \nu(s_{i+1} - s_i) \frac{\mathbf{P}^{(s_k)}(\Lambda_n^{(2)})}{\mathbf{P}^{(0)}(\Lambda_n^{(2)})} \\
\liminf \underset{\text{Lem.A.14}}{\geq} & \mathbf{1}_{s_0=0}\mathbf{1}_{s_i \geq 0, \forall 0 \leq i \leq k} \prod_{i=0}^{k-1} \nu(s_{i+1} - s_i) \frac{h^{\uparrow}(s_k)}{h^{\uparrow}(s_0)} \\
&= \mathbf{P}^{(0)}(S_0^{\uparrow} = s_0, \dots, S_k^{\uparrow} = s_k).
\end{aligned}$$

But since the last expression is a probability measure on positive paths, by Fatou's lemma it follows that the inequality is saturated in the limit and we got the desired convergence (see Remark 2.5 for a similar argument). □

Let us now prove Lemma A.14 for $i = 2$ again. Start from 0 and denote by $0 = T_0^<, T_1^<, ..., T_k^<$ the strict minimal records of the walk (S) and introduce the kth excursion

$$\mathrm{Exc}_k = (S_{T_k^<} - S_{T_k^<}, S_{T_k^<+1} - S_{T_k^<}, ..., S_{T_{k+1}^<} - S_{T_k^<}).$$

Notice then that for $k \geq 0$ so that $H_k \geq -x$ if the event $\Lambda_n^{(2)}$ happens for Exc_k then it happens for $(S) + x$. Hence by the inclusion-exclusion principle we have

$$\begin{aligned}
\mathbb{P}^{(x)}(\Lambda_n^{(2)}) &\geq \sum_{k \geq 0} \mathbb{P}^{(0)}(\mathrm{Exc}_k \in \Lambda_n^{(2)} \text{ and } \{H_k^< \geq -x\}) - \mathbb{P}(\exists 0 \leq i, j \leq x, \mathrm{Exc}_i \in \Lambda_n^{(2)} \text{ and } \mathrm{Exc}_i \in \Lambda_n^{(2)}) \\
&\geq \sum_{k \geq 0} \mathbb{P}^{(0)}(\Lambda_n^{(2)})\mathbf{P}(H_k^< \geq -x) - \left(x\mathbb{P}^{(0)}(\Lambda_n^{(2)})\right)^2.
\end{aligned}$$

Since $0 < \mathbb{P}^{(0)}(\Lambda_n^{(2)}) \to 0$ as $n \to \infty$, we can divide by $\mathbb{P}^{(0)}(\Lambda_n^{(2)})$ and take $\liminf$ to deduce the result. □

A.3.4 Tanaka's construction

In this section, we give a direct and very neat construction of the random walk $(S^{\uparrow})$ conditioned to stay positive due to Tanaka. To start with, let Exc be the time and space reversal of a negative excursion of S

$$\mathrm{Exc} = (0, S_{T^>} - S_{T^>-1}, S_{T^>} - S_{T^>-2}, \dots, S_{T^>} - S_1, S_{T^>})$$

where we recall that $T^> = \inf\{k \geq 0 : S_k > 0\}$. One then considers independent copies $\mathrm{Exc}_1, \mathrm{Exc}_2, \dots$ of Exc obtained by running the walk S and looking at its strict ascending ladder process, which we glue together in the most natural way to get an infinite walk. Tanaka proved that the process obtained has the law of $S^{\uparrow}$ (but started from 0 and conditioned not to touch $\mathbb{Z}_{\leq 0}$).

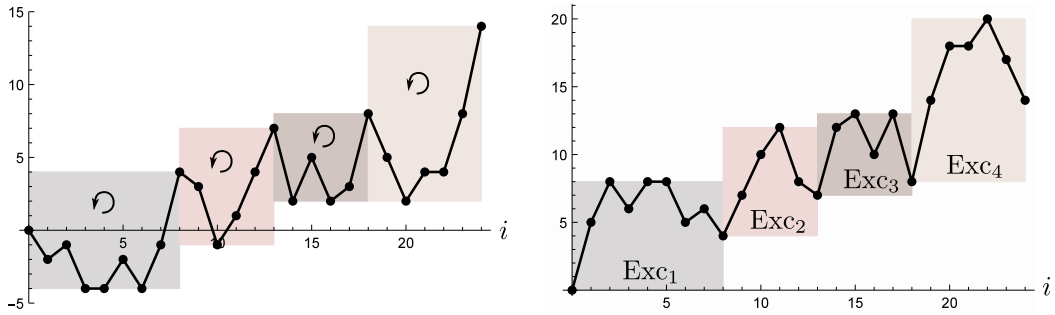


Figure A.5: Illustration of Tanaka's construction of the process $S^{\uparrow}$.

Proposition A.15 (Tanaka). *Suppose that (S) does not drift towards $-\infty$. Then the process obtained by concatenating i.i.d.time and space reversals of negative excursions of S has the same law as $S^{\uparrow}$.*

Proof. Denote by $(\mathcal{S}_n : n \geq 0)$ the process obtained by Tanaka's construction. Under our hypothesis, $\mathcal{S} \to +\infty$ and stays positive after time 1. Fix s_0 and $s_1, s_2, \ldots, s_k \in \mathbb{Z}_{>0}$ and let us try to compute directly

$$\mathbf{P}(\mathcal{S}_1 = s_1, \ldots, \mathcal{S}_k = s_k).$$

The problem is that if we are only given the path of the process up to time k, we do not know to which excursions it corresponds within the walk S. However, if we assume that the walk after time k never drops below $\mathcal{S}_k$ (large inequality), then we can reverse Tanaka's construction, in the sense that time k is a strict ascending ladder epoch for S and $S_1 = \tilde{s}_1, \ldots, S_k = \tilde{s}_k$ are obtained by reversing the excursion of the $s_1, \ldots, s_k$ when read backwards in time and space. Hence we can write

$$\begin{aligned}
\mathbf{P}(\mathcal{S}_1 = s_1, \ldots, \mathcal{S}_k = s_k) &= \sum_{s_{k+1},\ldots,s_{k+\ell} \in \mathbb{Z}_{>0}} \mathbf{P}(\mathcal{S}_1 = s_1, \ldots, \mathcal{S}_{k+\ell} = s_{k+\ell} \text{ and } \mathcal{S}_i > s_{k+\ell}, \forall i > k+\ell) \\
&= \sum_{s_{k+1},\ldots,s_{k+\ell} \in \mathbb{Z}_{>0}} \mathbf{P}(S_1 = \tilde{s}_1, \ldots, S_{k+\ell} = \tilde{s}_{k+\ell}) \\
&= \prod_{i=0}^{k-1} \nu(s_{i+1} - s_i) \underbrace{\sum_{s_{k+1},\ldots,s_{k+\ell} \in \mathbb{Z}_{>0} \text{ +condition below}} \prod_{j=0}^{\ell-1} \nu(s_{k+j+1} - s_{k+j})}_{h^{\uparrow}(s_k)},
\end{aligned}$$

simply because we are summing the probabilities over all paths starting from s_k and going to a decreasing ladder height which must be at positive height. This proves the desired result. □

Exercise A.16. Let $(S_n : n \geq 0)$ be a random walk with finite mean. We suppose that S is centered, in particular it is recurrent and thus oscillates. Show that $(S_n^{\uparrow} : n \geq 0)$ satisfies the law of large numbers:

$$\frac{S_n^{\uparrow}}{n} \xrightarrow[n\to\infty]{a.s.} 0.$$

A.3.5 Drift to $-\infty$ and Cramér's condition

Let us now suppose that we are in the annoying case when S drifts towards $-\infty$. In full generality it might be impossible to define a good notion of walk conditioned to stay non-negative since it might not exist any harmonic functions on $\mathbb{Z}_{>0}$, see Doney [104]. However, in the so-called Cramér[3] case, things are relatively under control:

Definition A.17 (Cramér's condition). *Suppose that S drifts towards $-\infty$. We say that ν satisfies Cramér's condition if there exists $\omega > 1$ so that*

$$\sum_{k\in\mathbb{Z}} \omega^k \nu(k) = 1.$$

It is easy to check that if such an ω exists, it must be unique by convexity. When Cramér's condition holds, one can define a step distribution $\tilde{\nu}(k) = \omega^k \nu(k)$ and the associated random walk $\tilde{S}$. It is easy to establish the following Radon-Nikodym derivative

$$\mathbf{E}[f(S_0, ..., S_n)\omega^{S_n}] = \mathbf{E}[f(\tilde{S}_0, ..., \tilde{S}_n)].$$

This walk $\tilde{S}$ can be seen as a first h-transformation of the walk (S) by the harmonic function $h(x) = \omega^x$. Since $\tilde{\nu}$ has an integrable left tail, its expectation is well defined and it is easy to check that $\mathbf{E}[\tilde{S}_1] > 0$ so that $\tilde{S}$ drifts towards $+\infty$. It thus admits a renewal function $\tilde{h}^{\uparrow}$ and a version $\tilde{S}^{\uparrow}$ conditioned to stay positive. By the previous display and the results of Section A.3.2 we can write

$$\tilde{h}^{\uparrow}(\ell) = \sum_{i=0}^{\ell} \tilde{h}^{\downarrow}(i) = \sum_{i=0}^{\ell} \omega^{-i} h^{\downarrow}(i).$$

In total, the process $(\tilde{S}^{\uparrow})$ can be seen as the h-transformation of the walk (S) by the function $h(x) = \omega^x \tilde{h}^{\uparrow}(x)$ which is indeed harmonic on $\mathbb{Z}_{\geq 0}$. This process will be our version of the walk (S) conditioned to stay positive as justified by the following proposition (whose proof is omitted in these lecture notes, see [37]):

Proposition A.18. *Assume that Cramér's condition holds and that $\sum_{k\in\mathbb{Z}} \tilde{\nu}(k)|k| < \infty$. Then we have*

$$\mathbf{P}\left(S_0 = s_0, \dots, S_k = s_k \mid \Lambda_n^{(1)}\right) \xrightarrow[n\to\infty]{} \mathbf{P}(\tilde{S}_0^{\uparrow} = s_0, \dots, \tilde{S}_k^{\uparrow} = s_k).$$

A.4 Ratio and local limit theorem

We end this chapter by recalling the strong ratio limit theorem valid for a large class of random walks as well as Gnedenko's local limit theorem which we will use for random walks whose increments are in the domain of attraction of a stable law.

[3] Harald Cramér (Swedish 1893 – 1985) not to confuse with Gabriel Cramer (Swiss 1704 – 1752)

A.4.1 Strong ratio limit theorem

To enjoy this theorem, let us recall the very classical ratio limit theorem for an irreducible recurrent Markov chain (X) on a countable state space

$$\frac{\sum_{k=1}^n \mathbb{P}(X_n = x)}{\sum_{k=1}^n \mathbb{P}(X_n = y)} \xrightarrow[n\to\infty]{} \frac{\mu(x)}{\mu(y)} \qquad \text{provided that } \mu(y) < \infty$$

where μ is the invariant measure of the chain (unique up to multiplicative constant). The following theorem is a considerable reinforcement of the previous convergence which holds for random walks on $\mathbb{Z}^d$ (not necessarily recurrent). As the reader will see, the proof is short and elegant:

Theorem A.19 (*Strong ratio limit theorem*)

Let ν be an aperiodic distribution on $\mathbb{Z}^d$. Let $(X_i)_{i\geq 0}$ be i.i.d. random variables with law ν and form the random walk $S_n = \sum_{k=1}^n X_i \in \mathbb{Z}^d$. Then for all $m \geq 0$ and $b \in \mathbb{Z}^d$ we have

$$\lim_{n\to\infty} \frac{\mathbf{P}(S_{n-m} = s_n - b)}{\mathbf{P}(S_n = s_n)} = 1, \qquad \text{as soon as} \qquad \mathbf{P}(S_n = s_n)^{1/n} \to 1.$$

Remark A.20. Notice that $\mathbf{P}(S_n = 0)$ is a super-multiplicative function so that $\mathbf{P}(S_n = 0)^{1/n}$ always has a limit which must of course be 1 in the case of a recurrent random walk.

Proof. Using the aperiodicity of ν it is easy to see that it is sufficient to restrict our attention to $m = 1$ and $b \in \mathbb{Z}^d$ such that $\mathbf{P}(X_1 = b) > 0$. For such m, b let us introduce $N_n = \#\{1 \leq i \leq n : X_i = b\}$ and notice that by symmetry of the increments of the walk we have

$$\mathbf{E}\left[\frac{N_n}{n}\,\middle|\, S_n = s_n\right] \underset{\text{symmetry}}{=} \mathbf{P}(X_1 = b \mid S_n = s_n) \underset{\text{Markov}}{=} \mathbf{P}(X_1 = b) \cdot \frac{\mathbf{P}(S_{n-1} = s_n - b)}{\mathbf{P}(S_n = s_n)}.$$

On the other hand, N_n has binomial $\mathrm{Bin}(\mathbf{P}(X_1 = b), n)$ distribution and in particular an easy large deviation estimate shows that for any $\varepsilon > 0$ we have $\mathbf{P}(|\frac{N_n}{n} - \mathbf{P}(X_1 = b)| \geq \varepsilon) \leq \mathrm{e}^{-c_\varepsilon n}$ for some $c_\varepsilon > 0$. Combining this with $\mathbf{P}(S_n = s_n) = \mathrm{e}^{-o(n)}$ we easily deduce that $\mathbf{E}\left[\frac{N_n}{n} \mid S_n = s_n\right] \to \mathbf{P}(X_1 = b)$ as $n \to \infty$. Thank to the previous display, the theorem is proved. □

One can give a sufficient condition on the sequence s_n so that $\mathbf{P}(S_n = s_n)^{1/n} \to 1$. Let us recall the classical definition of the log-Laplace function and its Legendre transform, for $x, \theta \in \mathbb{R}^d$ put

$$\phi(\theta) = \log \int \mathrm{d}\nu(x)\ \exp((\theta|x)) \in \mathbb{R} \cup \{\infty\} \qquad \text{and} \qquad \psi(x) = \left(\sup_{\theta\in\mathbb{R}^d} (\theta|x) - \phi(\theta)\right) \in \mathbb{R}_+ \cup \{\infty\}.$$

In [2, Section 4] it is shown that as soon as

$$\sup |s_n/n| < \infty \text{ and } \lim_{n\to\infty} \psi(s_n/n) = 0 \qquad \text{then} \qquad \mathbf{P}(S_n = s_n)^{1/n} \to 1. \tag{A.3}$$

We shall not reproduce the argument here but invite the reader to connect this with the classical large deviations Cramér theory for sums of independent random variables in $\mathbb{R}^d$.

A.4.2 Local limit theorem

The local limit theorem is a reinforcement of the central limit theorem (or its equivalent in the stable regime). Remarkably, it can be proved under the same assumption as for the central limit theorem. Let us first recall the basics of the domain of (strict) attraction of the stable law.

Suppose that ν is a law on $\mathbb{Z}$ which is in the (strict) domain of attraction of some α-stable law for some $\alpha \in (0,2]$. This means that if $(S_n : n \geq 0)$ is a ν-random walk then we have the following convergence in the Skorokhod sense:

$$\left(\frac{S_{\lfloor nt \rfloor}}{n^{1/\alpha}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\mathcal{S}_t : t \geq 0), \tag{A.4}$$

where $\mathcal{S}$ is an α-stable Lévy process, see [32, Section VIII] for background. This is in particular the case if ν has finite variance and mean zero for $\alpha = 2$, or if ν has stable tails with index $\alpha \in (0,2)$

$$\nu((-\infty,-k]) \sim c_- k^{-a+1} \quad \text{and} \quad \nu([k,\infty]) \sim c_+ k^{-a+1}, \quad \text{as } k \to \infty,$$

for some $c_+, c_- \geq 0$ non both zero, and if additionally ν is centered if $\alpha \in (1,2)$ or satisfy an approximate drift condition in the case $\alpha = 1$, see [42, Theorem 8.3.1]. The so-called positivity parameter $\varrho = \mathbb{P}(\mathcal{S}_t > 0)$ is given in terms of c_+ and c_- using Zolotarev' formula:

$$\varrho = \frac{1}{2} + \frac{1}{a\pi}\arctan\left(\frac{c_+ - c_-}{c_+ + c_-}\tan\left(\frac{a\pi}{2}\right)\right). \tag{A.5}$$

The convergence of the renormalized walk towards $\mathcal{S}$ can thus be seen as a functional version of the stable-central limit theorem (equivalently a stable version of Donsker's theorem). Actually under the sole assumption (A.4) we can give a local version of the central limit theorem:

Theorem A.21 (*Gnedenko's local limit theorem*)

Suppose that ν is aperiodic and that (A.4) holds. If $g(\cdot)$ denotes the density of $\mathcal{S}_1$ then we have

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \left| n^{1/\alpha}\mathbb{P}(S_n = k) - g(k \cdot n^{-1/\alpha}) \right| = 0.$$

Bibliographical notes. Most of this chapter is adapted from the lecture notes [87]. Standard references for fluctuation theory of random walks are [111, 80]. See [138] for more about random plane trees and their encodings. Results of Section A.3 are taken from the beautiful paper [37] of Bertoin & Doney. Tanaka's construction was first explained in [183]. The strong ratio limit theorem is due to [2, Theorem 4.7] in this form, and the proof is adapted from the beautiful argument of Neveu [168] (simplifying earlier works of Kesten). A proof of Gnedenko's local limit theorem can be found in [125, Theorem 4.2.1].

Chapter II : Coding of bipartite maps with labeled trees

In this chapter, we present the coding of (bipartite) planar maps via labeled trees, based on a variant of the construction of Schaeffer and Bouttier–Di Francesco–Guitter. This coding enables a slick proof of Theorem 3.12 which was the key enumerative input in these lecture notes. It is also instrumental in the study of scaling limits of large Boltzmann maps endowed with their primal distance as mentioned in Chapter 14.

B.1 Bouttier–Di Francesco–Guitter coding of bipartite maps

In this section we present the Bouttier–Di Francesco–Guitter bijection (rather a construction more than a bijection in our setup) between pointed bipartite planar map and labeled forests.

B.1.1 From maps to trees

Let $\mathfrak{m}_* = (\mathfrak{m}, \delta)$ be a bipartite planar map of perimeter 2ℓ having a distinguished vertex δ. Perform the following operations:

1. Draw a vertex in each face of $\mathfrak{m}$ (including the external face). The new vertices are considered black ($\bullet$) and the old ones white ($\circ$). Label each white vertex by its distance to the distinguished vertex δ. Since the map is bipartite, the labels of any two adjacent vertices differ exactly by one.

2. For a face f of $\mathfrak{m}$ and a white vertex adjacent to f, link the white vertex to the black vertex inside f if the next white vertex in the clockwise order around f has a smaller label.

3. Remove the edges of $\mathfrak{m}$ and the vertex δ. It can be shown that the resulting graph is a tree.

4. Let v_0 be the black vertex corresponding to the external face of $\mathfrak{m}$. By removing v_0 and its adjacent edges, we obtain a forest of cyclically ordered trees, rooted at the neighbors of v_0. Finally, we choose uniformly at random one of the trees to be the first one, subtract the labels by a constant so that the label of the root vertex of this first tree becomes zero.

With a moment of thought on the Step 2 of the above construction, one observes that

(i) Each internal face of degree $2k$ in $\mathfrak{m}$ gives rise to a black vertex of degree k in the forest, and the forest is composed of ℓ trees.

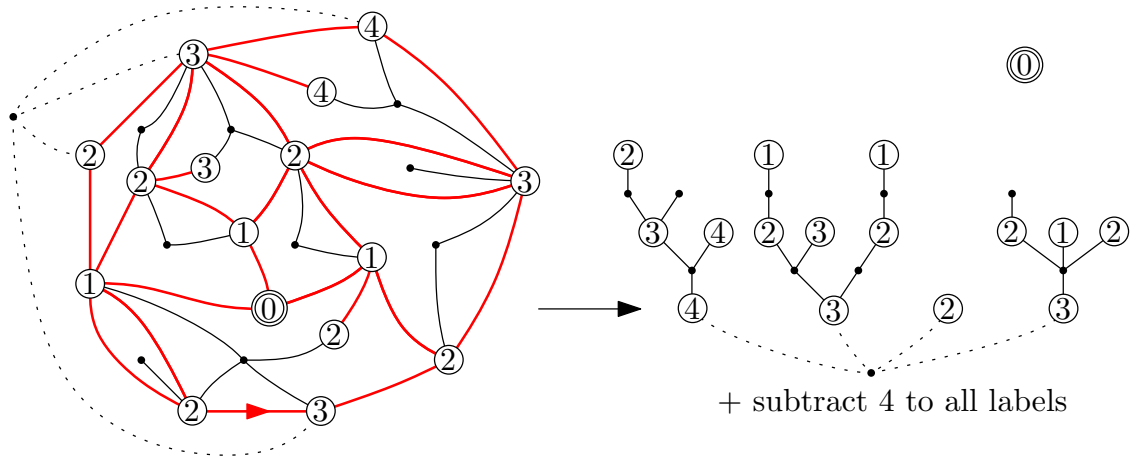


Figure B.1: Illustration of the construction of a forest of ℓ mobiles from a pointed bipartite planar map with a boundary of perimeter 2ℓ (in this example, the boundary turned out to be simple but that's a coincidence). The first mobile in the forest is not specified by the map and is chosen uniformly at random among all the mobiles.

(ii) Given a black vertex of degree k, the possible labels on its (white) neighbors are exactly those which, when read in the clockwise order around the black vertex, can decrease at most by 1.

Remark B.1. Let v be a black vertex of degree ℓ and fix the label of one of its neighbor (to 0, say). Then the number of well-labelings of the white vertices around v is given by the number of walks in $\mathbb{Z}$ which starts at 0, come back at 0 after ℓ steps, and whose steps are in $\{-1, 0, 1, 2, 3, 4, \dots\}$. Up to adding 2 to each increment, this number is given by the number of partitions of 2ℓ into ℓ positive integers which is classical to compute

$$\mathsf{N}(\ell) := \binom{2\ell - 1}{\ell}.$$

A **mobile** is a rooted plane tree whose vertices at even (resp. odd) generations are white (resp. black). We say that a forest of mobiles $(\mathfrak{t}_1, \cdots, \mathfrak{t}_p)$ is *well-labeled* if (a) the root vertex of $\mathfrak{t}_1$ has label 0, (b) the labels satisfy the constraint in the observation (ii) above, and (c) the labels of the roots of $\mathfrak{t}_1, \cdots, \mathfrak{t}_p$ satisfy the similar constraint.

The above construction thus associates $\mathfrak{m}_\bullet$ with a forest $\mathfrak{f}$ of ℓ well-labeled mobiles (modulo the addition of randomness needed to chose the first tree in the forest) and we shall denote

$$\mathfrak{f} = \mathsf{Mob}(\mathfrak{m}_\bullet).$$

B.1.2 From trees to maps

Let us now present the inverse construction. Start with the forest $\mathfrak{f}$ of ℓ mobiles $\mathfrak{t}_1, \cdots, \mathfrak{t}_\ell$ which we imagine drawn on the plane once grafted in clockwise order on a cycle of length ℓ. We then

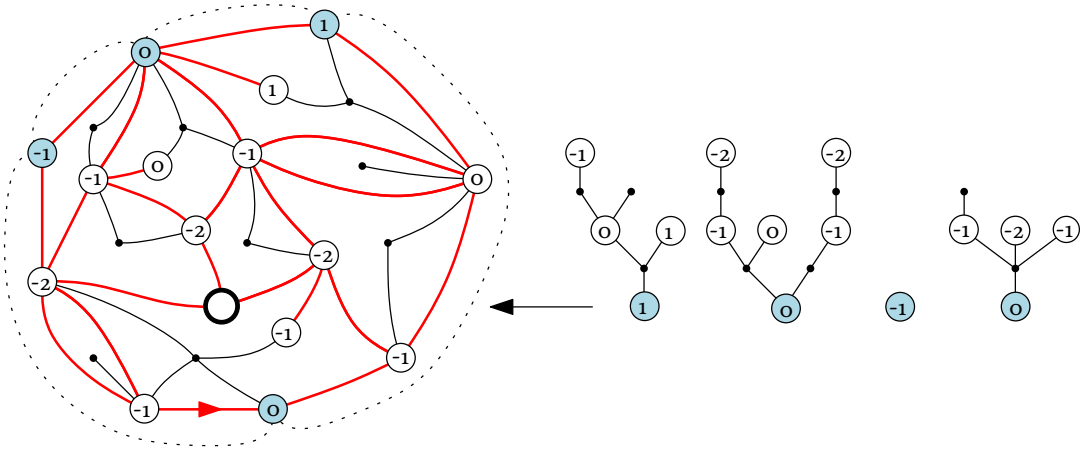


Figure B.2: Illustration of the construction of a pointed (at the white unlabelled vertex) bipartite planar map with a boundary of perimeter 8 (in red) from a forest of 4 mobiles. Note that the boundary is simple here, which may not be the case in general.

perform the usual "Schaeffer construction" by doing the contour of the mobiles and linking any corner associated to a vertex of label i to the next corner in the contour associated with a vertex of label $i-1$. If i is the minimal label then we link this corner to an additional vertex δ put in the infinite face of the embedding. The edges can be drawn in a non-crossing fashion and after erasing the embedding of the cycle and the mobiles, we are left with a bipartite map with a distinguished vertex δ. The external face of the map is the face that "encloses" the cycle on which the mobiles have been grafted. The root edge of the map is not prescribed by the forest and is taken uniformly at random on an edge of the external face of degree 2ℓ (so that the external face is on its right). We denote by $\mathsf{BDG}(\mathfrak{f})$ the resulting pointed random map. As usual in Schaeffer-type constructions, the labeling of the above forest has a geometric interpretation in terms of the map but since we do not use it we do not bother to enter the details. It should be clear on a drawing that modulo the rooting of the map we have

$$\mathfrak{m}_\bullet = \mathsf{BDG}(\mathsf{Mob}(\mathfrak{m}_\bullet)). \tag{B.1}$$

B.2 Distribution of the forest of mobiles

Let us now describe the effect of this coding on the Boltzmann measure. Recall from Section 3.2.2 the definition of the measure w_q on bipartite planar maps possibly with target. We now compute the image measure of w_q by the mapping Mob (which has an additional randomness in it for the choice of the root mobile). But before doing so we first transform a mobile into a simple plane tree by yet another mapping due to Janson & Stefansson [127].

B.2.1 Janson & Stefansson's trick

Janson & Stefánsson discovered a mapping which transforms a mobile into a rooted plane tree by keeping the same set of vertices, but changing the set of edges so that every white vertex is mapped to a leaf, and every black vertex of degree k is mapped to an internal vertex with k children. The reader may guess the rigorous definition of that transformation from the following figure: We denote by JS this transformation.

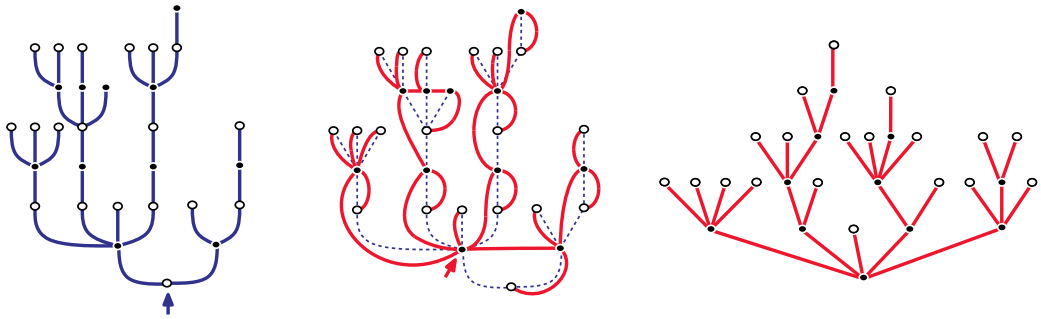


Figure B.3: Illustration of the Janson & Stefánsson transformation applied on a mobile (left) to get a plane tree (right).

B.2.2 Law of the unlabeled forest

Proposition B.2. *Let $\ell \geq 1$, then the image measure of $w_{\mathbf{q}}$ on $\mathcal{M}_0^{(\ell)}$ by the chain of mappings*

$$\mathsf{JS} \circ \mathsf{ForgetLabels} \circ \mathsf{Mob}$$

is the measure on forests of ℓ trees defined by

$$\tilde{w}_{\mathbf{q}}(\mathsf{f}) = 2 \cdot \binom{2\ell - 1}{\ell} \prod_{u \in \mathsf{f}} \tilde{q}_{k_u},$$

where $\tilde{q}_k = q_k \binom{2k-1}{k}$, $\tilde{q}_0 = 1$ and where k_u is the number of children of u in the forest f.

Proof. Let $\mathsf{f} = (\mathsf{t}_1, \mathsf{t}_2, \dots, \mathsf{t}_\ell)$ be a forest of ℓ plane trees. Let us first see which forest of ℓ mobiles give rise to such a forest of trees after applying Janson–Stefansson mapping. The unlabeled forest of mobiles $\mathfrak{f} = (\mathfrak{t}_1, \dots, \mathfrak{t}_\ell)$ can be retrieved by applying the inverse Janson–Stefansson mapping and in particular the degrees of the black vertices are given by the number of children in the forest, hence by

$$\{k_u : u \in \mathsf{f}\},$$

where k_u is the number of children of a vertex u. Now, to such a forest $\mathfrak{f}$ may correspond a lot of well-labeled forests. Specifically, using Remark B.1 around each black vertex (and around the

origins of the mobiles) since the label of the (white) origin of the first mobile is fixed to 0, there are exactly

$$\mathsf{N}(\ell) \prod_{v \in \mathrm{BlackVertices}(\mathfrak{f})} \mathsf{N}(\deg(v)) = \mathsf{N}(\ell) \prod_{u \in \mathfrak{f}} \mathsf{N}(k_u),$$

possible well-labelings of the forest where we put $\mathsf{N}(0) = 1$. Now, fix a labeling of the forest $\mathfrak{f}$ and let us see which pointed maps $\mathfrak{m}_\bullet$ can give rise to this forest by the Mob construction. By (B.1), up to the location of the root edge, the pointed map $\mathfrak{m}_\bullet$ can be recovered by applying the Bouttier–Di Francesco–Guitter construction to the forest of well-labeled mobiles. In particular since the degree of the inner faces of the map are twice those of the black vertices of the forest we deduce that the $w_{\mathbf{q}}$ weight of such a map (if rooted) is given by

$$\prod_{v \in \mathrm{BlackVertices}(\mathfrak{f})} q_{\deg(v)} = \prod_{u \in \mathfrak{f}} q_{k_u},$$

where we put $q_0 = 1$ by convention. Let us assume first that the unrooted pointed map $\mathsf{BDG}(\mathfrak{f})$ has no symmetry, i.e. that rooting the map on each of the 2ℓ edges of its boundary yield 2ℓ different rooted pointed maps. Then for each of these 2ℓ maps the Mob construction returns $\mathfrak{f}$ with probability $1/\ell$ (the probability to choose the right mobile as first one). In the case of symmetry, the fewer number of maps obtained by rooting on the boundary is exactly compensated by the larger probability to get the forest by picking the first tree (since the forest also inherits the symmetries of the map). In total, gathering the above equations we deduce that the $w_{\mathbf{q}}$ weight of all the maps $\mathfrak{m}_\bullet$ such that $\mathsf{JS} \circ \mathsf{ForgetLabels} \circ \mathsf{Mob}(\mathfrak{m}_\bullet) = \mathfrak{f}$ is

$$2\ell \cdot \frac{1}{\ell} \cdot \prod_{u \in \mathfrak{f}} q_{k_u} \cdot \mathsf{N}(\ell) \prod_{u \in \mathfrak{f}} \mathsf{N}(k_u) = 2 \cdot \binom{2\ell - 1}{\ell} \prod_{u \in \mathfrak{f}} \tilde{q}_{k_u},$$

as desired. □

B.3 Back to the enumeration results

B.3.1 Back to the admissibility criterion

Let us now use Proposition B.2 to give an alternative proof of the of Theorem 3.12 in the case $p = 0$ without using Tutte's slicing formula: By Proposition B.2, the total $w_{\mathbf{q}}$-weight $W_0^{(\ell)}$ of $\mathcal{M}_0^{(\ell)}$ is finite (for one $\ell \geq 1$ or equivalently for all $\ell \geq 1$) if and only if we have

$$\sum_{\mathfrak{t} \in \mathrm{FinitePlaneTrees}} \prod_{u \in \mathfrak{t}} \tilde{q}_{k_u} < \infty. \tag{B.2}$$

The standard way to check whether the above sum is finite is to remark, by applying a recursive decomposition at the root vertex (a tree is either equal to a single vertex or a vertex of degree k with k trees attached to them) that its value is obtained as the limit of the iterative scheme: $x_0 = 0$ and $x_{n+1} = \psi(x_n)$ where in our case we have

$$\psi(x) = 1 + \sum_{k \geq 1} \tilde{q}_k x^k = f_{\mathbf{q}}(x).$$

Hence the series (B.2) is finite if and only if $f_{\mathbf{q}}(x) = x$ has a solution, in which case the value of the series is the smallest positive such solution $Z_{\mathbf{q}} = c_{\mathbf{q}}/4$. We thus recover

$$W_0^{(\ell)} \underset{\text{Prop.B.2}}{=} 2\binom{2\ell-1}{\ell}(c_{\mathbf{q}}/4)^{\ell} = c_{\mathbf{q}}^{\ell} h^{\downarrow}(\ell),$$

as desired. □

B.3.2 Interpretation of the law J and back to criticality

With the notation of the last section, if $\mathbf{q}$ is admissible, then it is easy to see that upon normalizing $w_{\mathbf{q}}$ on $\mathcal{M}_0^{(\ell)}$ to make it a probability measure, the distribution of the ℓ plane trees obtained by pushing the previous distribution via $\mathsf{JS} \circ \mathsf{ForgetLabels} \circ \mathsf{Mob}$ is just given by ℓ independent Galton–Watson trees whose offspring distribution is given by

$$\mu_{\mathsf{JS}}(k) = \tilde{q}_k Z_{\mathbf{q}}^{k-1}, \quad k \geq 0,$$

and $\tilde{q}_0 = 1$ by convention. Indeed, since for any plane tree τ we have $\sum_{u\in\tau} k_u - 1 = -1$ the probability under the μ_{JS}-GW measure to obtain a given tree τ is equal to $1/Z_{\mathbf{q}} \prod_{u\in\tau} \tilde{q}_{k_u}$ as required by Proposition B.2. Notice that the law of μ_{JS} is nothing but the law μ introduced in Section 3.3.1 shifted by -1. We encourage the reader to use the cyclic lemma to give a new proof of Theorem 3.12 when $p = 0$.

Furthermore, it is clear from (5.6) that our definition of criticality for the weight sequence $\mathbf{q}$ is equivalent to asking that μ_{JS} be centered. Similarly, we have the following dictionary for the notions appearing in Section 5.3:

$\mathbf{q}$ subcritical	μ_{JS} of mean < 1.
$\mathbf{q}$ critical	μ_{JS} of mean 1.
$\mathbf{q}$ is of type $a \in (3/2; 5/2)$	μ_{JS} of mean 1 and has a polynomial tail of order $a - \frac{1}{2}$.

Bibliographical notes. The coding of planar maps via label trees first appeared in the seminal work of Cori & Vauquelin [84] and later made clear and popularized by Schaeffer [178]. The coding we used here is an extension of Schaeffer's initial construction due to Bouttier–Di Francesco–Guitter [48] and more precisely the variant presented in [41]. The study of the induced distribution of trees is found first in [150]. Janson & Stefansson trick is taken from [127, Section 3], we refer to [90, Section 3.2] for details of this transformation. The presentation here is partially taken from our work [77]. Notice that the use of bijections with trees is one of the most useful tool to study geometric properties of random planar maps, but in these lecture notes we preferred not to use it to focus on the peeling process itself.

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