Chapter 1 : Numerical Series.

1 Definition and first properties

**Definition 1.1.** Given a sequence of real or complex numbers \(a = (a_n)_{n \geq 1}\), we define the sequence \((s_n(a))\) of partial sums by

\[ s_n(a) = \sum_{k=1}^{n} a_k. \]

The series associated to \(a\) is denoted by the symbol

\[ \sum_{n=1}^{\infty} a_n, \]

and is said convergent if the sequence of partial sums converges to a limit \(S\) called the sum of the series. In this case, it will be useful to note \((r_n(a))\) the sequence of the remainders associated to the convergent series \(\sum a_n\), defined by

\[ r_n(a) = S - s_n(a) = \sum_{k=n+1}^{\infty} a_k. \]

The series \(\sum a_n\) is said divergent if the sequence \((s_n(a))\) diverges.

**Examples 1.2.**

- **Telescoping series** : a sequence \((a_n)\) and the telescoping series \(\sum(a_{n+1} - a_n)\) have the same behaviour.
- **Geometric series** : for a fixed real number \(x \neq \pm 1\), we have

\[ \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \Rightarrow \left( \sum x^n \text{ converges } \iff |x| < 1 \right). \]

**Theorem 1.3.** [Cauchy criterion] The series \(\sum a_n\) converges if and only if

\[ \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall p \geq q \geq N, \left| \sum_{k=q}^{p} a_k \right| \leq \epsilon. \]

**Proof.** Cauchy criterion on \((s_n(a))\).

**Corollary 1.4.** A necessary condition for \(\sum a_n\) to converge is the convergence of \(a_n\) to 0.

**Proof.** Consider \(p = q\) in the previous proof.
Example 1.5. $\sum (-1)^n$ diverges.

Remark 1.6. The condition is not sufficient: we’ll see later that $\frac{1}{n}$ diverges.

Notations 1.7. I will often note $(\forall n \gg 0)$ instead of $(\exists N \in \mathbb{N} \text{ such that } \forall n \geq N)$.

Theorem 1.8. Suppose $\forall n \gg 0$, $a_n \geq 0$. We have

$$\sum a_n \text{ converges } \Leftrightarrow (s_n(a)) \text{ bounded.}$$

Proof. $\exists N$ such that $(s_n(a))_{n\geq N}$ is an increasing sequence.

2 Comparison tests

Notations 2.1. [Landau notations] Let $(a_n)$ and $(b_n)$ be two sequences.

(Big O) We note $a_n = O(b_n)$ if $\exists K \in \mathbb{R}$, $\forall n \gg 0$, $|a_n| \leq K|b_n|$.

(little o) We note $a_n = o(b_n)$ if $\forall \epsilon > 0$, $\forall n \gg 0$, $|a_n| \leq \epsilon|b_n|$.

(equivalence of sequences) We note $a_n \sim b_n$ if $a_n - b_n = o(a_n)$.

Exercise 2.2. Check that $a_n - b_n = o(a_n) \Leftrightarrow a_n - b_n = o(b_n)$.

Remark 2.3. – Suppose $\forall n \gg 0$, $b_n > 0$. Then we have $a_n = O(b_n) \Leftrightarrow \frac{a_n}{b_n}$ bounded, $a_n = o(b_n) \Leftrightarrow \frac{a_n}{b_n} \rightarrow 0$, $a_n \sim b_n \Leftrightarrow \frac{a_n}{b_n} \rightarrow 1$.

– Be careful with the implication

$$(a_n \sim a_n, b_n \sim b_n) \Rightarrow a_n + b_n \sim a_n + b_n,$$

it’s false if $\forall n \gg 0$, $a_n + b_n = 0$ : it would mean that $\forall n \gg 0$, $a_n + b_n = 0$, which is obviously not necessarily true. Consider $a_n = 1/(n+1)$, $b_n = -1/(n+2)$ and $a_n = -b_n = 1/n$ : in fact we have $a_n + b_n = 1/((n+1)(n+2)) \sim 1/n^2$. In such cases, it’s more safe to use equalities instead of equivalences, for example with the $o$ and $O$ notations.

Example 2.4. If $a_n \rightarrow 0$ we have $|a_n| < \frac{1}{n}$ for $n$ big enough, and we can write (integration by parts)

$$\int_1^{1+a_n} \frac{1+a_n - t}{t^2} dt = a_n - \int_1^{1+a_n} \frac{dt}{t} = a_n - \ln(1+a_n),$$

thus

$$|\ln(1+a_n) - a_n| \leq \int_1^{1+a_n} \frac{|1+a_n - t|}{t^2} dt \leq |a_n| \int_1^{1+a_n} \frac{dt}{t^2} = |a_n| \frac{|a_n|}{1+a_n} \leq 2|a_n|^2,$$

hence

$$\ln(1+a_n) = a_n + O(a_n^2).$$

Theorem 2.5. Let $(a_n)$ and $(b_n)$ be two sequences with $\forall n \gg 0$, $b_n \geq 0$.

1. If $a_n = O(b_n)$,
   (i) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges and $r_n(a) = O(r_n(b))$, 
   (ii) $\sum b_n$ diverges $\Rightarrow$ $s_n(a) = O(s_n(b))$.

2. Same statements with $o$.

3. If $a_n \sim b_n$, $\sum a_n$ and $\sum b_n$ have the same behaviour and
(i) \( r_n(a) \sim r_n(b) \) in case of convergence,  
(ii) \( s_n(a) \sim s_n(b) \) in case of divergence.

Proof. (Partial) First, 3 directly follows from 1 and 2. Let’s prove 1(i):

\[
(\exists K, \forall n \gg 0, |a_n| \leq K b_n) \Rightarrow \left( \exists K, \forall n \gg 0, \forall p, \left| \sum_{k=n+1}^{n+p} a_k \right| \leq K \sum_{k=n+1}^{n+p} b_k \right).
\]

By Cauchy criterion, \( \sum a_n \) converges and we can make \( p \to \infty \) to obtain the result.

Let’s suppose \( a_n = o(b_n) \) and \( \sum b_n \) divergent to prove 2(ii). We fix \( \epsilon > 0 \) and \( N \) such that for all \( n \geq N \), \( |a_n| \leq \epsilon b_n \). Then

\[
|s_n(a)| \leq \left| \sum_{k=1}^{N-1} a_k \right| + \epsilon \sum_{k=N}^{n} b_k \leq \left( \sum_{k=0}^{n} b_k \right) (\epsilon + \frac{K}{\sum_{k=0}^{n} b_k}).
\]

But \( \sum b_n \) diverges and \( b_n \geq 0 \) for \( n \) big enough, so \( \sum_{k=0}^{n} b_k \to \infty \) and there exists \( N' \geq N \) such that \( |s_{n'}(a)| \leq 2\epsilon |s_n(b)| \), which gives the expected result. \( \square \)

Remark 2.6. We can use the contraposition of these statements, for example

\( \sum a_n \) diverges and \( a_n = O(b_n) \) \( \Rightarrow \) \( \sum b_n \) diverges.

Examples 2.7.

- \( a_n = \sqrt{1 + n^2} - \sqrt{n^2} - 1 = \frac{2}{\sqrt{1 + n^2} + \sqrt{n^2}} \sim \frac{1}{n^2} \) and we’ll see in the next section that \( \sum n^{-\alpha} \) converges iff \( \alpha > 1 \), so \( \sum a_n \) converges.

- \( a_n = \sqrt{n + 1} - \sqrt{n} = \frac{n}{n + 1} |f'\| \leq sup_{[n,n+1]} |f'| \) where \( f(x) = \sqrt{x} \), hence \( 0 \leq a_n \leq \frac{1}{2n\sqrt{n}} \) so \( \sum a_n \) converges because \( a_n = O(n^{-3/2}) \). Another way to treat this kind of sequence where it appears something like \( f(a) - f(b) \), with \( f \) differentiable, is to write (here \( f = \sqrt{x} \))

\[
a_n = \frac{1}{n\sqrt{n}} \left( 1 + (1/n) - \sqrt{n} \right) \sim \frac{1}{n\sqrt{n}} f'(1) \sim \frac{1}{2n\sqrt{n}}
\]

and we have a more precise result.

- By 2.4, we have \( \sum \left( \frac{1}{n} - \ln \left( 1 + \frac{1}{n} \right) \right) \) convergent, and we can note \( \gamma \) its sum (the Euler-Mascheroni constant). We can rewrite it as

\[
\gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{1}{k} - (\ln(k+1) - \ln(k)) \right) = \lim_{n \to \infty} \left\{ \left( \sum_{k=1}^{n} \frac{1}{k} \right) - \ln(n+1) \right\}
\]

and finally (cf. \( \ln(n+1) - \ln(n) = o(1) \))

\[
\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)
\]

Note that it implies \( \sum_{k=1}^{n} \frac{1}{k} \sim \ln n \), which is a direct consequence of \( \ln(1 + (1/n)) \sim 1/n \) and 2.5.3(ii).
3 Integral test

Theorem 3.1. Let \( f : [a, +\infty[ \to \mathbb{R}^+ \) be a continuous decreasing function. Then for all \( N \geq a \) we have

\[
\exists \lim_{x \to +\infty} \int_a^x f(t) \, dt \Leftrightarrow \sum_{n \geq N} f(n) \text{ converges.}
\]

Proof. We write \( \forall n \geq N, f(n+1) \leq \int_n^{n+1} f(t) \, dt \leq f(n) \).

Hence if \( \exists \lim_{x \to +\infty} \int_a^x f(t) \, dt = S \),

\[
\sum_{k=N}^{n} f(k) \leq f(N) + \int_N^n f(t) \, dt \leq S
\]

and \( f(k) \geq 0 \) so we can use 1.8 to obtain the convergence of \( \sum f(n) \). Conversely, if \( \sum_{n \geq N} f(n) \) converges to \( S \), because \( F : x \mapsto \int_a^x f(t) \, dt \) is an increasing function, we just have to prove that \( F \) is bounded : but for all \( x \geq a \), there exists \( N' \geq \max\{x, N+1\} \) and, using (1),

\[
F(x) \leq F(N) + \int_N^x f(t) \, dt \leq F(N) + \int_N^{N'} f(t) \, dt \leq F(N) + \sum_{k=N'}^{N'-1} f(k) \leq F(N) + S.
\]

which gives the result.

Examples 3.2. 1. For \( \alpha > 0 \), \( f_{\alpha} : x \mapsto x^{-\alpha} \) is continuous and decreasing on \( [1, +\infty[ \) and \( F_{\alpha}(x) = \int_1^x t^{-\alpha} \, dt = \begin{cases} \frac{x^{1-\alpha} - 1}{1-\alpha} & \text{if } \alpha \neq 1 \\ \ln x & \text{if } \alpha = 1 \end{cases} \), which implies that \( \sum n^{-\alpha} \) converges if and only if \( \alpha > 1 \) (cf. for \( \alpha \leq 0 \), \( a_n \to 0 \), which is a necessary condition). Let’s use 2.5 to find an equivalent of \( r_{n,\alpha} = \sum_{k=n+1}^{\infty} k^{-\alpha} \) for \( \alpha > 1 \) : first we have to find an interesting equivalent for \( n^{-\alpha} \), typically something telescoping to obtain a nice remainder. We rewrite the formula (1) for \( f = f_{\alpha} \), which gives :

\[
(\alpha-1)(n+1)^{\alpha} - \frac{1}{n^{\alpha-1}} - \frac{1}{(n+1)^{\alpha-1}} \leq 1
\]

Multiplying this line by \( n^{\alpha} \), we remark

\[
\frac{1}{\alpha-1} \left( \frac{1}{n^{\alpha-1}} - \frac{1}{(n+1)^{\alpha-1}} \right) \sim \frac{1}{n^{\alpha}}
\]

and using

\[
\sum_{k=n+1}^{\infty} \left( \frac{1}{k^{\alpha-1}} - \frac{1}{(k+1)^{\alpha-1}} \right) = \frac{1}{(n+1)^{\alpha-1}} \sim \frac{1}{n^{\alpha-1}},
\]

we obtain (cf. 2.5.3(i))

\[
r_{n,\alpha} \sim \frac{1}{\alpha-1} \cdot \frac{1}{n^{\alpha-1}}
\]
2.  [Bertrand series] Let \( a_n = n^{-\alpha} \ln^{-\beta} n \).
- If \( \alpha < 1 \), \( \exists \alpha' \in \]0, 1[\), and \( n^{-\alpha'} = o(a_n) \). But \( \sum n^{-\alpha'} \) diverges so by 2.5, \( \sum a_n \) diverges.
- If \( \alpha > 1 \), \( \exists \alpha' \in \]1, \alpha[\), and \( a_n = o(n^{-\alpha'}) \). Hence, this time, \( \sum a_n \) converges.
- If \( \alpha = 1, \beta \leq 0 \), \( n^{-1} = O(a_n) \), so \( \sum a_n \) diverges.
- If \( \alpha = 1, \beta > 0 \), \( f \begin{array}{ccc} [2, +\infty[ & \rightarrow & \mathbb{R} \\ x & \mapsto & \frac{1}{x \ln^{\beta} x} \end{array} \) is continuous, decreasing, and

\[
\int_2^x f(t)dt = \int_{\ln 2}^{\ln x} \frac{du}{u^{\beta}} = F_{\beta}(\ln x) - F_{\beta}(\ln 2),
\]

which has a finite limite iff \( \beta > 1 \) (cf. first example).

4 Ratio tests

Proposition 4.1. Let \((a_n)\) be a sequence such that \( \forall n \gg 0, |a_n| > 0 \).

(i) If \( \exists \alpha < 1 \) such that \( \forall n \gg 0, \frac{|a_{n+1}|}{|a_n|} < \alpha \), then \( \sum a_n \) converges.

(ii) If \( \forall n \gg 0, \frac{|a_{n+1}|}{|a_n|} \geq 1 \), then \( \sum a_n \) diverges.

Proof. For (i), \( \exists N \) such that \( \forall n \geq N, |a_{n+1}| \leq \alpha'|a_n| \) with \( \alpha' \in \]0, 1[\). Thus \( \forall n \geq N \) we have \( |a_n| \leq \alpha'^n|a_N| \) which implies \( a_n = O(\alpha'^n) \) and so the result. For (ii), \( a_n \to 0 \).

Corollary 4.2. [De D’Alembert rule] With the same hypothesis,

(i) If \( \exists \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} < 1 \), then \( \sum a_n \) converges.

(ii) If \( \exists \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \), then \( \sum a_n \) diverges.

Remark 4.3. This test is very exigent! In most cases it will fail to solve your problem. For example you can’t apply it to the Riemann series \( \sum n^{-\alpha} \).

Theorem 4.4. [Raabe-Duhamel test] We suppose \( \forall n \gg 0, a_n > 0 \).

1. If

\[ \exists \alpha \in \mathbb{R}, \frac{a_{n+1}}{a_n} = 1 - \frac{\alpha}{n} + o \left( \frac{1}{n} \right), \]

then

(i) \( \alpha > 1 \) \( \Rightarrow \) \( \sum a_n \) converges;

(ii) \( \alpha < 1 \) \( \Rightarrow \) \( \sum a_n \) diverges.

2. Same conclusions if

\[ \exists \alpha \in \mathbb{R}, \frac{a_{n+1}}{a_n} = 1 - \frac{1}{n} - \frac{\alpha}{n \ln n} + o \left( \frac{1}{n \ln n} \right). \]

Proof. For 1. : if \( \alpha > 1 \) (resp. \( < 1 \)), consider \( a' \in \]1, \alpha[\) (resp. \( \]0, 1[\)). To exploit the hypothesis, it’s relevant to consider the sequence \( b_n = \ln(n^\alpha a_n) \). One way to study such a sequence, considering the \( \ln \) and the ratio hypothesis, is to consider the associated telescoping series \( u_n = b_{n+1} - b_n \):
\[
\begin{align*}
    u_n &= \alpha' \ln \left(1 + \frac{1}{n}\right) + \ln \frac{a_{n+1}}{a_n} \\
    &= \alpha' \ln \left(1 + \frac{1}{n}\right) + \ln \left(1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right)\right) \\
    &= \frac{\alpha'}{n} + O\left(\frac{1}{n^2}\right) - \frac{\alpha}{n} + o\left(\frac{1}{n}\right) + O\left\{\left(-\frac{\alpha}{n} + o\left(\frac{1}{n}\right)\right)^2\right\}
\end{align*}
\]

but (cf. definition of the Landau notations), \(o(1/n)^2 = o(1/n^2), (1/n)o(1/n) = o(1/n^2)\) and of course \(o(1/n^2) = O(1/n^2)\), so (we also use the Minkowski inequality)

\[
    \left\{-\frac{\alpha}{n} + o\left(\frac{1}{n}\right)\right\}^2 = O\left(\frac{1}{n^2}\right).
\]

As we also have \(\forall a_n, O(O(a_n)) = O(a_n)\) and \(O(a_n/n) = o(a_n)\) (cf. 1/\(n\) → 0), we finally obtain

\[
u_n = \frac{\alpha' - \alpha}{n} + o\left(\frac{1}{n}\right) \sim \frac{\alpha' - \alpha}{n}.
\]

Thus for (i), \(\alpha' - \alpha < 0\) implies \(\sum u_n \to -\infty\), which means \(b_n \to -\infty\), which means \(n^{\alpha} a_n \to 0\), which means \(a_n = o(n^{-\alpha'})\) which gives the result \((\alpha' > 1)\). For (ii), \(\alpha' - \alpha > 0\) gives us \(n^{\alpha} a_n \to +\infty\), so \(n^{-\alpha} = O(a_n)\), which leads to the result \((\alpha' < 1)\).

For 2.: same proof, using this time \(b_n = \ln(n \ln n')(n)a_n\).

\[\Box\]

Exercise 4.5. Considering \(a_n = \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2\), prove that the first Raabe test fails \((\alpha = 1\) in the hypothesis of 1.), but not the second \((\alpha = 0\) in the hypothesis of 2.).

5 Further results

Theorem 5.1. [Leibniz criterion] Suppose \(a_n = (-1)^n b_n\) with \((b_n)_{n \geq 1}\) a decreasing sequence which tends to zero. Then

1. \(\sum_{n \geq 1} a_n\) converges;
2. if we note \(S\) its sum, \(S \leq 0\);
3. \(\forall n, |r_n(a)| \leq |a_{n+1}| = b_{n+1}\).

Proof. \((s_{2n}(a))\) is decreasing, \((s_{2n+1}(a))\) is increasing and \(s_{2n+1}(a) - s_{2n}(a) \to 0\).

Hence there exists \(S\) such that \(s_{2n+1}(a) \leq S < s_{2n}(a)\). As a consequence of these inequalities, we have \(|r_n(a)| = |S - s_n(a)| \leq |s_n(a) - s_{n+1}(a)| = b_{n+1}\). For 2., just use \(S \leq s_2(a)\).

\[\Box\]

Example 5.2. \(\sum_{-\infty} \frac{(-1)^n}{n}\) converges. Let's calculate its limit: we write

\[
    \sum_{k=1}^{n} \frac{(-1)^k}{k} = \sum_{k=0}^{n-1} \int_{0}^{1} (-t)^k dt = \int_{0}^{1} \left(\sum_{k=0}^{n-1} (-t)^k\right) dt = \int_{0}^{1} \frac{1 - (-t)^n}{1 + t} dt = \ln 2 - \alpha_n
\]

with

\[
    |\alpha_n| = \left| \int_{0}^{1} \frac{(-t)^n}{1 + t} dt \right| \leq \int_{0}^{1} t^n dt = \frac{1}{n + 1} \to 0.
\]
Finally

\[
\sum_{n \geq 1} \frac{(-1)^n}{n} = \ln 2
\]

Exercise 5.3. Prove that we can apply the Leibniz criterion to \( \sum r_n(a) \) with \( a_n = \frac{(-1)^n}{\ln n} \).

Definition 5.4. Let \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) two sequences. The Cauchy product of \( \sum a_n \) and \( \sum b_n \), noted \((\sum a_n) \star (\sum b_n)\), is the series \( \sum c_n \), with

\[
c_n = \sum_{k=0}^{n} a_k b_{n-k}.
\]

Theorem 5.5. Suppose \( \sum |a_n| \) and \( \sum b_n \) converge and note \( A,B \) the sums of \( \sum a_n, \sum b_n \). Then \( \sum c_n \) converge and its sum is \( AB \).

Proof. We write

\[
s_n(c) = \sum_{i=0}^{n-k} \sum_{i+k}^{n} a_i b_{n-i} = \sum_{i=0}^{n-k} \sum_{k=0}^{n-i} a_i b_{n-i} = \sum_{k=0}^{n} \sum_{i=0}^{n-k} a_i b_{n-i} = a_n(B - r_{n-i}(b)) = s_n(a)B - a_n \to AB
\]

Let’s prove \( a_n = \sum_{i=0}^{n} a_i r_{n-i}(b) \to 0 \). For \( \epsilon > 0 \), \( \exists N \) sucht that \( \forall n \geq N, |r_n(b)| \leq 0 \). We note \( A \) the sum of \( \sum |a_n| \). Then

\[
|a_n| \leq \left| \sum_{i=0}^{N} a_i r_{n-i}(b) \right| + A \epsilon.
\]

But \( a_n \to 0 \), so \( \exists N' \) sucht that \( \forall n \geq N', |a_n| \leq \epsilon \). Hence

\[
\forall n \geq N + N', \ n - N \geq N' \Rightarrow |a_n| \leq (K + A) \epsilon
\]

with \( K = \sum_{i=0}^{N} |r_i(b)| \).

Proposition 5.6. [Abel’s summation by parts formula] Given to sequences \( (a_n) \) and \( (b_n) \), we have the following formulas \( \forall p,q \):

\[
(i) \quad \sum_{n=p+1}^{q} a_n (b_n - b_{n-1}) = \sum_{n=p+1}^{q} (a_n - a_{n+1}) b_n + a_{q+1} b_q - a_{p+1} b_p
\]

\[
(ii) \quad \sum_{n=p+1}^{q} a_n b_n = \sum_{n=p+1}^{q} (a_n - a_{n+1}) s_n(b) + a_{q+1} s_q(b) - a_{p+1} s_p(b)
\]

Proof. First, (ii) is just (i) applied to \( s_n(b) \) instead of \( b_n \). For (i) :
\[
\sum_{n=p+1}^{q} a_n(b_n - b_{n-1}) = \sum_{n=p+1}^{q} a_nb_n - \sum_{n=p+1}^{q} a_nb_{n-1}
\]
\[
= \sum_{n=p+1}^{q} a_nb_n - \sum_{n=p+1}^{q} a_{n+1}b_n
\]
\[
= \sum_{n=p+1}^{q} a_nb_n - \sum_{n=p+1}^{q} a_{n+1}b_n - a_{p+1}b_p + a_{q+1}b_q
\]
\[
= \sum_{n=p+1}^{q} (a_n - a_{n+1})b_n + a_{q+1}b_q - a_{p+1}b_p
\]

Example 5.7. Let \( u_n = \frac{\cos(n\theta)}{n^\alpha} \).

- If \( \alpha > 1 \), \( u_n = O(n^{-\alpha}) \) \( \Rightarrow \) \( \sum u_n \) converges.
- If \( \alpha \leq 0 \), \( u_n \not\to 0 \) \( \Rightarrow \) \( \sum u_n \) diverges.
- If \( \alpha \in [0, 1] \), we already know that \( \sum u_n \) diverges if \( \theta \equiv 0 \pmod{2\pi} \), so we may assume \( e^{i\theta} \neq 1 \). In order to apply Abel’s formula (ii), we note \( a_n = n^{-\alpha} \) and \( b_n = \cos(n\theta) \) and we have (cf. \( s_0(b) = \cos 0 = 1 \))

\[
\sum_{n=1}^{q} u_n = \sum_{n=1}^{q} (a_n - a_{n+1})s_n(b) + a_{q+1}s_q(b) - 1.
\]

But

\[
s_n(b) = \Re \left( \sum_{k=0}^{n} e^{ik\theta} \right) = \Re \left( \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \right)
\]
\[
= \Re \left( \frac{e^{i(n+1)\theta}/2i \sin((n+1)\theta/2)}{e^{i\theta}/2i \sin(\theta/2)} \right) = \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \Re(e^{i\theta/2})
\]
\[
= \frac{\cos(n\theta/2) \sin((n+1)\theta/2)}{\sin(\theta/2)}
\]
\[
\Rightarrow |s_n(b)| \leq K = \frac{1}{\sin(\theta/2)}
\]
\[
\Rightarrow |v_n| \leq K \left( \frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} \right) = K \frac{1}{n^\alpha} \left( 1 - \left(1 + \frac{1}{n}\right)^{-\alpha} \right),
\]
and (with \( f_\alpha : x \mapsto x^{-\alpha} \)):

\[
1 - \left(1 + \frac{1}{n}\right)^{-\alpha} \to f_\alpha'(1) = -\alpha \Rightarrow 1 - \left(1 + \frac{1}{n}\right)^{-\alpha} = O \left( \frac{1}{n} \right)
\]

Finally \( v_n = O(n^{-(\alpha+1)}) \) and \( \sum v_n \) converges (cf. \( \alpha+1 > 1 \)). But \( a_{q+1} \to 0 \), so the Abel’s formula proves the convergence of \( \sum u_n \).
We finish with the Fubini’s theorem for double series:

**Theorem 5.8.** Suppose \((a_{m,n}) \in \mathbb{C}^{N \times N}\) is such that for all \(m\), \(\sum_n |a_{m,n}|\) converges to a limit noted \(\sigma_m\) and that \(\sum \sigma_m\) converges to a limit noted \(\Sigma\). Then

(i) for all \(n\), \(\sum_m |a_{n,m}|\) converges to a limit noted \(\sigma'_n\),

(ii) \(\sum \sigma'_n\) converges,

(iii) \(\sum_m \sum_n a_{m,n} = \sum_n \sum_m a_{n,m}\) (noted \(\sum_{n,m} a_{m,n}\)).

**Proof.** (i) : For \(n_0 \in \mathbb{N}\), we have for all \(M \in \mathbb{N}\)

\[
\sum_{m=0}^M |a_{n_0,m}| \leq \sum_{m=0}^M |a_{n,m}| \leq |a_{n,m}|,
\]

so we have the result.

(ii) : For all \(N \in \mathbb{N}\)

\[
\sum_{m=0}^N \sigma'_n = \sum_{m=0}^N \sum_{n=0}^N |a_{m,n}| = \lim_{M \to \infty} \sum_{m=0}^M \sum_{n=0}^N |a_{m,n}|
\]

and \(\sum_{n=0}^N |a_{m,n}| \leq \sigma_m\), so \(\sum_{m=0}^M \sum_{n=0}^N |a_{m,n}| \leq \Sigma\), and thus \(\sum_{n=0}^N \sigma'_n \leq \Sigma\) which is enough to conclude.

(iii) : First, both members of the equality exist : we note \(S_m = \sum_n a_{m,n}\) and \(S'_n = \sum_m a_{m,n}\) so that \(|S_m| \leq \sigma_m\) and \(|S'_n| \leq \sigma'_n\) imply the convergence of \(\sum S_m\) and \(\sum S'_n\).

Let \(\epsilon > 0\). We have for all \((M,N) \in \mathbb{N} \times \mathbb{N}\)

\[
\sum_{m=0}^M S_m = \sum_{m=0}^M \sum_{n=0}^N a_{m,n} = \sum_{n=0}^N \sum_{m=0}^M a_{m,n} + \sum_{n=0}^N \sum_{m=N+1}^M a_{m,n},
\]

where, because \(\sum \sigma'_n\) converges, there exists \(N_\epsilon \in \mathbb{N}\) such that for all \(N \geq N_\epsilon\)

\[
|\sum_{m=0}^M a_{m,n}| \leq |\sum_{m=0}^N \sum_{n=0}^N a_{m,n}| \leq |\sum_{n=0}^N \sum_{n=0}^N |a_{m,n}|| \leq \sum_{n=0}^N \sum_{n=0}^N \sigma'_n \leq \epsilon,
\]

and where, because \(\sum \sigma_m\) converges, there exists \(M_\epsilon \in \mathbb{N}\) such that for all \(M \geq M_\epsilon\)

\[
|\sum_{n=0}^N S_n = \sum_{n=0}^N \sum_{m=0}^M a_{m,n}| \leq |\sum_{m=0}^M \sum_{n=0}^N a_{m,n}| \leq \sum_{m=0}^M \sum_{n=0}^N \sigma_m \leq \epsilon.
\]

Hence for all \(N \geq N_\epsilon\) and \(M \geq M_\epsilon\) we have

\[
|\sum_{n=0}^N S_n - \sum_{m=0}^M S_m| \leq 2\epsilon.
\]

which leads to the result. \(\square\)

**Remark 5.9.**

- In fact (iii) is a particular case of the double-limit theorem you’ll see in ch2.
  The trick is to consider \(E = \{x_i\}_{i \in \mathbb{N} \cup \{\infty\}} \subset \mathbb{R}\) with \(x_i \xrightarrow{n \to \infty} x_\infty\) and to define \(f_m \in \mathbb{C}^E\) by \(f_m(x_i) = \sum_{n=0}^m a_{m,n}\) for all \(i \in \mathbb{N} \cup \{\infty\}\). We have
  - \(\forall m, f_m(x_i) \xrightarrow{j \to \infty} f_m(x_\infty) = \sum_{n=0}^m a_{m,n}\);
  - normal convergence : \(\forall x \in E, |f_m(x)| \leq \sigma_m\).
  Hence, setting \(g = \sum_{m=0}^\infty f_m \in \mathbb{C}^E\), \(\exists \lim_{x_i \to x_\infty} g(x_i) = g(x_\infty)\), which exactly says that \(\sum_{n=0}^\infty S_n\) converges, and that the limit is \(\sum_{n=0}^\infty S_n\).
- This theorem can be very useful for the theory of power series - see ch3.