# Chapter 1 : Numerical Series.

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## 1 Definition and first properties

**Definition 1.1.** Given a sequence of real or complex numbers $a = (a_n)_{n\geq 1}$, we define the sequence $(s_n(a))$ of *partial sums* by

$$s_n(a) = \sum_{k=1}^{n} a_k.$$

The series associated to $a$ is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n,\ \sum_{n\geq 1} a_n \text{ or just } \sum a_n,$$

and is said *convergent* if the sequence of partial sums converges to a limit $S$ called the *sum* of the series. In this case, it will be usefull to note $(r_n(a))$ the sequence of the *remainders* associated to the convergent series $\sum a_n$, defined by

$$r_n(a) = S - s_n(a) = \sum_{k=n+1}^{\infty} a_k.$$

The series $\sum a_n$ is said *divergent* if the sequence $(s_n(a))$ diverges.

**Examples 1.2.**

- *Telescoping series :* a sequence $(a_n)$ and the telescoping series $\sum(a_{n+1} - a_n)$ have the same behaviour.
- *Geometric series :* for a fixed real number $x \neq \pm 1$, we have
$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \Rightarrow \Big(\sum x^n \text{ converges } \Leftrightarrow |x| < 1\Big).$$

**Theorem 1.3.** *[Cauchy criterion] The series $\sum a_n$ converges if and only if*

$$\forall \epsilon > 0\ ,\ \exists N \in \mathbb{N}\ ,\ \forall p \geq q \geq N\ ,\ \left|\sum_{k=q}^{p} a_k\right| \leq \epsilon.$$

*Proof.* Cauchy criterion on $(s_n(a))$. □

**Corollary 1.4.** *A necessary condition for $\sum a_n$ to converge is the convergence of $a_n$ to $0$.*

*Proof.* Consider $p = q$ in the previous proof. □

**Example 1.5.** $\sum(-1)^n$ diverges.

*Remark* 1.6. The condition is not sufficient : we'll see later that $\sum \frac{1}{n}$ diverges.

*Notations* 1.7. I will often note $(\forall n \gg 0)$ instead of $(\exists N \in \mathbb{N}$ such that $\forall n \geq N)$.

**Theorem 1.8.** *Suppose* $\forall n \gg 0,\ a_n \geq 0$. *We have*

$$\sum a_n \textit{ converges} \Leftrightarrow (s_n(a)) \textit{ bounded.}$$

*Proof.* $\exists N$ such that $(s_n(a))_{n \geq N}$ is an increasing sequence. □

## 2 Comparison tests

*Notations* 2.1. [*Landau notations*] Let $(a_n)$ and $(b_n)$ be two sequences.
(Big O) We note $a_n = O(b_n)$ if $\exists K \in \mathbb{R},\ \forall n \gg 0,\ |a_n| \leq K|b_n|$.
(little o) We note $a_n = o(b_n)$ if $\forall \epsilon > 0,\ \forall n \gg 0,\ |a_n| \leq \epsilon |b_n|$.
(equivalence of sequences) We note $a_n \sim b_n$ if $a_n - b_n = o(a_n)$.

*Exercise* 2.2. Check that $a_n - b_n = o(a_n) \Leftrightarrow a_n - b_n = o(b_n)$.

*Remark* 2.3.
- Suppose $\forall n \gg 0,\ b_n > 0$. Then we have
$a_n = O(b_n) \Leftrightarrow \dfrac{a_n}{b_n}$ bounded, $a_n = o(b_n) \Leftrightarrow \dfrac{a_n}{b_n} \to 0,\ a_n \sim b_n \Leftrightarrow \dfrac{a_n}{b_n} \to 1.$
- Be careful with the implication

$$(a_n \sim \alpha_n,\ b_n \sim \beta_n) \Rightarrow a_n + b_n \sim \alpha_n + \beta_n\ ,$$

it's false if $\forall n \gg 0,\ \alpha_n + \beta_n = 0$ : it would mean that $\forall n \gg 0,\ a_n + b_n = 0$, which is obviously not necessarily true. Consider $a_n = 1/(n+1)$, $b_n = -1/(n+2)$ and $\alpha_n = -\beta_n = 1/n$ : in fact we have $a_n + b_n = 1/((n+1)(n+2)) \sim 1/n^2$. In such cases, it's more safe to use equalities instead of equivalences, for example with the $o$ and $O$ notations.

**Example 2.4.** If $a_n \to 0$ we have $|a_n| < \frac{1}{2}$ for $n$ big enough, and we can write (integration by parts)

$$\int_1^{1+a_n} \frac{1 + a_n - t}{t^2} dt = a_n - \int_1^{1+a_n} \frac{dt}{t} = a_n - \ln(1 + a_n),$$

thus

$$\begin{aligned} |\ln(1+a_n) - a_n| &\leq \int_1^{1+a_n} \frac{|1 + a_n - t|}{t^2} dt \\ &\leq |a_n| \int_1^{1+a_n} \frac{dt}{t^2} = |a_n| \frac{|a_n|}{1 + a_n} \leq 2|a_n|^2, \end{aligned}$$

hence

$$\boxed{\ln(1 + a_n) = a_n + O(a_n^2)}$$

**Theorem 2.5.** *Let* $(a_n)$ *and* $(b_n)$ *be two sequences with* $\forall n \gg 0,\ b_n \geq 0$.

1. *If* $a_n = O(b_n)$,
   *(i)* $\sum b_n$ *converges* $\Rightarrow$ $\sum a_n$ *converges and* $r_n(a) = O(r_n(b))$,
   *(ii)* $\sum b_n$ *diverges* $\Rightarrow$ $s_n(a) = O(s_n(b))$.
2. *Same statements with* $o$.
3. *If* $a_n \sim b_n$, $\sum a_n$ *and* $\sum b_n$ *have the same behaviour and*

*(i)* $r_n(a) \sim r_n(b)$ *in case of convergence,*
*(ii)* $s_n(a) \sim s_n(b)$ *in case of divergence.*

*Proof.* (Partial) First, 3 directly follows from 1 and 2. Let's prove 1(i) :

$$(\exists K,\ \forall n \gg 0, |a_n| \leq K b_n) \Rightarrow \left(\exists K,\ \forall n \gg 0, \forall p, \left|\sum_{k=n+1}^{n+p} a_k\right| \leq K \sum_{k=n+1}^{n+p} b_k\right).$$

By Cauchy criterion, $\sum a_n$ converges and we can make $p \to \infty$ to obtain the result.

Let's suppose $a_n = o(b_n)$ and $\sum b_n$ divergent to prove 2(ii). We fix $\epsilon > 0$ and $N$ such that for all $n \geq N$, $|a_n| \leq \epsilon b_n$. Then

$$|s_n(a)| \leq \underbrace{|\sum_{k=1}^{N-1} a_k|}_{\text{constant } K \geq 0} + \epsilon \sum_{k=N}^{n} b_k \leq \left(\sum_{k=0}^{n} b_k\right)\left(\epsilon + \frac{K}{\sum_{k=0}^{n} b_k}\right).$$

But $\sum b_n$ diverges and $b_n \geq 0$ for $n$ big enough, so $\sum_{k=0}^{n} b_k \to \infty$ and there exists $N' \geq N$ such that $|s_n(a)| \leq 2\epsilon |s_n(b)|$, which gives the expected result. □

*Remark* 2.6. We can use the contraposition of these statements, for example

$$\sum a_n \text{ diverges and } a_n = O(b_n) \ \Rightarrow\ \sum b_n \text{ diverges.}$$

**Examples 2.7.**

- $a_n = \sqrt{1+n^4} - \sqrt{n^4-1} = \dfrac{2}{\sqrt{1+n^4} + \sqrt{n^4-1}} \sim \dfrac{1}{n^2}$ and we'll see in the next section that $\sum n^{-\alpha}$ converges iff $\alpha > 1$, so $\sum a_n$ converges.
- $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n} \leq \dfrac{\sup_{[n,n+1]} |f'|}{n}$ where $f(x) = \sqrt{x}$, hence $0 \leq a_n \leq \dfrac{1}{2n\sqrt{n}}$, so $\sum a_n$ converges because $a_n = O(n^{-3/2})$. Another way to treat this kind of sequence where it appears something like $f(a) - f(b)$, with $f$ differentiable, is to write (here $f = \sqrt{\star}$)
$$a_n = \frac{1}{n\sqrt{n}} \frac{\sqrt{1+(1/n)} - \sqrt{n}}{1/n} \sim \frac{1}{n\sqrt{n}} f'(1) \sim \frac{1}{2n\sqrt{n}}$$
and we have a more precise result.
- By 2.4, we have $\sum \left(\dfrac{1}{n} - \ln\left(1 + \dfrac{1}{n}\right)\right)$ convergent, and we can note $\gamma$ its sum (the *Euler-Mascheroni constant*). We can rewrite it as
$$\gamma = \lim_{n\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - (\ln(k+1) - \ln(k)\right) = \lim_{n\infty} \left\{\left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln(n+1)\right\}$$
and finally (cf. $\ln(n+1) - \ln n = o(1)$)
$$\boxed{\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)}$$
Note that it implies $\sum_{k=1}^{n} \frac{1}{k} \sim \ln n$, which is a direct consequence of $\ln(1 + (1/n)) \sim 1/n$ and 2.5.3(ii).

## 3 Integral test

**Theorem 3.1.** *Let $f : [a, +\infty[ \to \mathbb{R}^+$ be a continuous decreasing function. Then for all $N \geq a$ we have*

$$\exists \lim_{x\to+\infty} \int_a^x f(t)dt \;\Leftrightarrow\; \sum_{n\geq N} f(n) \text{ converges.}$$

*Proof.* We write

$$\forall n \geq N,\; f(n+1) \leq \int_n^{n+1} f(t)dt \leq f(n). \tag{1}$$

Hence if $\exists \lim_{x\to+\infty} \int_a^x f(t)dt = S$,

$$\sum_{k=N}^{n} f(k) \leq f(N) + \int_N^n f(t)dt \leq S$$

and $f(k) \geq 0$ so we can use 1.8 to obtain the convergence of $\sum f(n)$. Conversely, if $\sum_{n\geq N} f(n)$ converges to $S$, because $F : x \mapsto \int_a^x f(t)dt$ is an increasing function, we just have to prove that $F$ is bounded : but for all $x \geq a$, there exists $N' \geq \max\{x, N+1\}$ and, using (1),

$$F(x) \leq F(N) + \int_N^x f(t)dt \leq F(N) + \int_N^{N'} f(t)dt \leq F(N) + \sum_{k=N}^{N'-1} f(k) \leq F(N) + S.$$

which gives the result. □

**Examples 3.2.** 1. For $\alpha > 0$, $f_\alpha : x \mapsto x^{-\alpha}$ is continuous and decreasing on $[1, +\infty[$ and $F_\alpha(x) = \int_1^x t^{-\alpha}dt = \begin{cases} \dfrac{x^{1-\alpha}-1}{1-\alpha} & \text{if } \alpha \neq 1 \\ \ln x & \text{if } \alpha = 1 \end{cases}$, which implies that $\sum n^{-\alpha}$ converges iff $\alpha > 1$ (cf. for $\alpha \leq 0$, $a_n \nrightarrow 0$, which is a necessary condition). Let's use 2.5 to find an equivalent of $r_{n,\alpha} = \sum_{k=n+1}^{\infty} n^{-\alpha}$ for $\alpha > 1$ : first we have to find an interesting equivalent for $n^{-\alpha}$, typically something telescoping to obtain a nice remainder. We rewrite the formula (1) for $f = f_\alpha$, which gives :

$$\frac{1}{(n+1)^\alpha} \leq \frac{1}{\alpha-1}\left(\frac{1}{n^{\alpha-1}} - \frac{1}{(n+1)^{\alpha-1}}\right) \leq \frac{1}{n^\alpha}$$

Multiplying this line by $n^\alpha$, we remark

$$\frac{1}{\alpha-1}\left(\frac{1}{n^{\alpha-1}} - \frac{1}{(n+1)^{\alpha-1}}\right) \sim \frac{1}{n^\alpha},$$

and using

$$\sum_{k=n+1}^{\infty}\left(\frac{1}{k^{\alpha-1}} - \frac{1}{(k+1)^{\alpha-1}}\right) = \frac{1}{(n+1)^{\alpha-1}} \sim \frac{1}{n^{\alpha-1}},$$

we obtain (cf. 2.5.3(i))

$$\boxed{r_{n,\alpha} \sim \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}}$$

2. [*Bertrand series*] Let $a_n = n^{-\alpha} \ln^{-\beta} n$.
   - if $\alpha < 1$, $\exists \alpha' \in ]\alpha, 1[$, and $n^{-\alpha'} = o(a_n)$. But $\sum n^{-\alpha'}$ diverges so by 2.5, $\sum a_n$ diverges.
   - if $\alpha > 1$, $\exists \alpha' \in ]1, \alpha[$, and $a_n = o(n^{-\alpha'})$. Hence, this time, $\sum a_n$ converges.
   - if $\alpha = 1, \beta \leq 0$, $n^{-1} = O(a_n)$, so $\sum a_n$ diverges.
   - if $\alpha = 1, \beta > 0$, $f \left| \begin{array}{rcl} [2, +\infty[ & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x \ln^\beta x} \end{array} \right.$ is continuous, decreasing, and

$$\int_2^x f(t)dt \underset{u=\ln t}{=} \int_{\ln 2}^{\ln x} \frac{du}{u^\beta} = F_\beta(\ln x) - F_\beta(\ln 2),$$

which has a finite limite iff $\beta > 1$ (cf. first example).

## 4 Ratio tests

**Proposition 4.1.** *Let $(a_n)$ be a sequence such that $\forall n \gg 0$, $|a_n| > 0$.*

*(i) If $\exists \alpha < 1$ such that $\forall n \gg 0$, $\frac{|a_{n+1}|}{|a_n|} < \alpha$, then $\sum a_n$ converges.*

*(ii) If $\forall n \gg 0$, $\frac{|a_{n+1}|}{|a_n|} \geq 1$, then $\sum a_n$ diverges.*

*Proof.* For (i), $\exists N$ such that $\forall n \geq N$, $|a_{n+1}| \leq \alpha' |a_n|$ with $\alpha' \in ]\alpha, 1[$. Thus $\forall n \geq N$ we have $|a_n| \leq \alpha'^{n-N} |a_N|$ which implies $a_n = O(\alpha'^n)$ and so the result. For (ii), $a_n \nrightarrow 0$. □

**Corollary 4.2.** *[De D'Alembert rule] With the same hypothesis,*

*(i) If $\exists \lim\limits_{n\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum a_n$ converges.*

*(ii) If $\exists \lim\limits_{n\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then $\sum a_n$ diverges.*

*Remark* 4.3. This test is very exigent ! In most cases it will fail to solve your problem. For example you can't apply it to the Riemann series $\sum n^{-\alpha}$.

**Theorem 4.4.** *[Raabe-Duhamel test] We suppose $\forall n \gg 0$, $a_n > 0$.*

1. *If*

$$\exists \alpha \in \mathbb{R}, \quad \frac{a_{n+1}}{a_n} = 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right),$$

*then*
*(i) $\alpha > 1 \Rightarrow \sum a_n$ converges ;*
*(ii) $\alpha < 1 \Rightarrow \sum a_n$ diverges.*

2. *Same conclusions if*

$$\exists \alpha \in \mathbb{R}, \quad \frac{a_{n+1}}{a_n} = 1 - \frac{1}{n} - \frac{\alpha}{n \ln n} + o\left(\frac{1}{n \ln n}\right).$$

*Proof.* For 1. : if $\alpha > 1$ (resp. $< 1$), consider $\alpha' \in ]1, \alpha[$ (resp. $]\alpha, 1[$). To exploit the hypothesis, it's relevant to consider the sequence $b_n = \ln(n^{\alpha'} a_n)$. One way to study such a sequence, considering the $\ln$ and the ratio hypothesis, is to consider the associated telescoping series $u_n = b_{n+1} - b_n$ :

$$
\begin{aligned}
u_n &= \alpha' \ln\left(1+\frac{1}{n}\right) + \ln\frac{a_{n+1}}{a_n} \\
&= \alpha' \ln\left(1+\frac{1}{n}\right) + \ln\left(1-\frac{\alpha}{n}+o\left(\frac{1}{n}\right)\right) \\
&\underset{2.4}{=} \frac{\alpha'}{n} + O\left(\frac{1}{n^2}\right) - \frac{\alpha}{n} + o\left(\frac{1}{n}\right) + O\left(\left\{-\frac{\alpha}{n}+o\left(\frac{1}{n}\right)\right\}^2\right)
\end{aligned}
$$

but (cf. definition of the Landau notations), $o(1/n)^2 = o(1/n^2)$, $(1/n)o(1/n) = o(1/n^2)$ and of course $o(1/n^2) = O(1/n^2)$, so (we also use the Minkowski inequality)

$$\left\{-\frac{\alpha}{n}+o\left(\frac{1}{n}\right)\right\}^2 = O\left(\frac{1}{n^2}\right).$$

As we also have $\forall a_n$, $O(O(a_n)) = O(a_n)$ and $O(a_n/n) = o(a_n)$ (cf. $1/n \to 0$), we finally obtain

$$u_n = \frac{\alpha'-\alpha}{n} + o\left(\frac{1}{n}\right) \sim \frac{\alpha'-\alpha}{n}.$$

Thus for (i), $\alpha' - \alpha < 0$ implies $\sum u_n \to -\infty$, which means $b_n \to -\infty$, which means $n^{\alpha'} a_n \to 0$, which means $a_n = o(n^{-\alpha'})$ which gives the result ($\alpha' > 1$). For (ii), $\alpha' - \alpha > 0$ gives us $n^{\alpha'} a_n \to +\infty$, so $n^{-\alpha} = O(a_n)$, which leads to the result ($\alpha' < 1$).

For 2. : same proof, using this time $b_n = \ln(n \ln^{\alpha'}(n) a_n)$. ☐

*Exercise* 4.5. Considering $a_n = \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2$, prove that the first Raabe test fails ($\alpha = 1$ in the hyporhesis of 1.), but not the second ($\alpha = 0$ in the hypothesis of 2.).

## 5 Further results

**Theorem 5.1.** *[Leibniz criterion] Suppose $a_n = (-1)^n b_n$ with $(b_n)_{n\geq 1}$ a decreasing sequence which tends to zero. Then*

1. *$\sum_{n\geq 1} a_n$ converges ;*
2. *if we note $S$ its sum, $S \leq 0$ ;*
3. *$\forall n$, $|r_n(a)| \leq |a_{n+1}| = b_{n+1}$.*

*Proof.* $(s_{2n}(a))$ is decreasing, $(s_{2n+1}(a))$ is increasing and $s_{2n+1}(a) - s_{2n}(a) \to 0$. Hence there exists $S$ such that $s_{2n+1}(a) \overset{\leq}{\to} S \overset{\leq}{\leftarrow} s_{2n}(a)$. As a consequence of these inequalities, we have $|r_n(a)| = |S - s_n(a)| \leq |s_n(a) - s_{n+1}(a)| = b_{n+1}$. For 2., just use $S \leq s_2(a)$. ☐

**Example 5.2.** $\sum \frac{(-1)^n}{n}$ converges. Lets's calculate its limit : we write

$$\sum_{k=1}^{n} \frac{(-1)^k}{k} = \sum_{k=0}^{n-1} \int_0^1 (-t)^k dt = \int_0^1 \left(\sum_{k=0}^{n-1} (-t)^k\right) dt = \int_0^1 \frac{1-(-t)^n}{1+t} dt = \ln 2 - \alpha_n$$

with

$$|\alpha_n| = |\int_0^1 \frac{(-t)^n}{1+t} dt| \leq \int_0^1 t^n dt = \frac{1}{n+1} \to 0.$$

Finally

$$\boxed{\sum_{n\geq 1}\frac{(-1)^n}{n}=\ln 2}$$

*Exercise* 5.3. Prove that we can apply the Leibniz criterion to $\sum r_n(a)$ with $a_n = \dfrac{(-1)^n}{\ln n}$.

**Definition 5.4.** Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ two sequences. The *Cauchy product* of $\sum a_n$ and $\sum b_n$, noted $(\sum a_n)\star(\sum b_n)$, is the series $\sum c_n$, with

$$c_n=\sum_{k=0}^{n} a_k b_{n-k}.$$

**Theorem 5.5.** *Suppose $\sum |a_n|$ and $\sum b_n$ converge and note $A, B$ the sums of $\sum a_n, \sum b_n$. Then $\sum c_n$ converge and its sum is $AB$.*

*Proof.* We write

$$\begin{aligned} s_n(c) &= \sum_{k=0}^{n}\sum_{i=0}^{k} a_i b_{k-i} = \sum_{i=0}^{n}\sum_{k=i}^{n} a_i b_{k-i} \\ &\underset{h=k-i}{=} \sum_{i=0}^{n} a_i \sum_{h=0}^{n-i} b_h = \sum_{i=0}^{n} a_i s_{n-i}(b) \\ &= \sum_{i=0}^{n} a_i (B - r_{n-i}(b)) = \underbrace{s_n(a)B}_{\to AB} - \alpha_n \end{aligned}.$$

Let's prove $\alpha_n = \sum_{i=0}^{n} a_i r_{n-i}(b) \to 0$. For $\epsilon > 0$, $\exists N$ sucht that $\forall n \geq N$, $|r_n(b)| \leq 0$. We note $\mathfrak{A}$ the sum of $\sum |a_n|$. Then

$$|\alpha_n| \leq |\sum_{i=0}^{N} a_i r_{\underbrace{n-i}_{\geq n-N}}(b)| + \mathfrak{A}\epsilon.$$

But $a_n \to 0$, so $\exists N'$ sucht that $\forall n \geq N'$, $|a_n| \leq \epsilon$. Hence

$$\forall n \geq N + N',\quad n - N \geq N' \Rightarrow |\alpha_n| \leq (K + \mathfrak{A})\epsilon$$

with $K = \sum_{i=0}^{N} |r_i(b)|$. □

**Proposition 5.6.** *[Abel's summation by parts formula] Given to sequences $(a_n)$ and $(b_n)$, we have the following formulas $\forall p, q$ :*

*(i)* $\sum_{n=p+1}^{q} a_n(b_n - b_{n-1}) = \sum_{n=p+1}^{q} (a_n - a_{n+1})b_n + a_{q+1}b_q - a_{p+1}b_p$

*(ii)* $\sum_{n=p+1}^{q} a_n b_n = \sum_{n=p+1}^{q} (a_n - a_{n+1})s_n(b) + a_{q+1}s_q(b) - a_{p+1}s_p(b)$

*Proof.* First, (ii) is just (i) applied to $s_n(b)$ instead of $b_n$. For (i) :

$$\begin{aligned}
\sum_{n=p+1}^{q} a_n(b_n - b_{n-1}) &= \sum_{n=p+1}^{q} a_n b_n - \sum_{n=p+1}^{q} a_n b_{n-1} \\
&= \sum_{n=p+1}^{q} a_n b_n - \sum_{n=p}^{q-1} a_{n+1} b_n \\
&= \sum_{n=p+1}^{q} a_n b_n - \sum_{n=p+1}^{q} a_{n+1} b_n - a_{p+1}b_p + a_{q+1}b_q \\
&= \sum_{n=p+1}^{q} (a_n - a_{n+1})b_n + a_{q+1}b_q - a_{p+1}b_p
\end{aligned}.$$

□

**Example 5.7.** Let $u_n = \dfrac{\cos(n\theta)}{n^\alpha}$.

- If $\alpha > 1$, $u_n = O(n^{-\alpha}) \Rightarrow \sum u_n$ converges.
- If $\alpha \leq 0$, $u_n \nrightarrow 0 \Rightarrow \sum u_n$ diverges.
- If $\alpha \in ]0, 1]$, we already know that $\sum u_n$ diverges if $\theta \equiv 0 \pmod{2\pi}$, so we may assume $e^{i\theta} \neq 1$. In order to apply Abel's formula (ii), we note $a_n = n^{-\alpha}$ and $b_n = \cos(n\theta)$ and we have (cf. $s_0(b) = \cos 0 = 1$)

$$\sum_{n=1}^{q} u_n = \sum_{n=1}^{q} \underbrace{(a_n - a_{n+1})s_n(b)}_{v_n} + a_{q+1}s_q(b) - 1.$$

But

$$\begin{aligned}
s_n(b) &= \Re\left(\sum_{k=0}^{n} e^{ik\theta}\right) \underset{e^{i\theta}\neq 1}{=} \Re\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right) \\
&= \Re\left(\frac{e^{i(n+1)\theta/2} 2i \sin((n+1)\theta/2)}{e^{i\theta/2} 2i \sin(\theta/2)}\right) = \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \Re(e^{in\theta/2}) \\
&= \frac{\cos(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)} \\
&\Rightarrow |s_n(b)| \leq K = \frac{1}{\sin(\theta/2)} \\
&\Rightarrow |v_n| \leq K\left(\frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha}\right) = K\frac{1}{n^\alpha}\left(1 - \left(1 + \frac{1}{n}\right)^{-\alpha}\right),
\end{aligned}$$

and (with $f_\alpha : x \mapsto x^{-\alpha}$) :

$$\frac{1 - \left(1 + \frac{1}{n}\right)^{-\alpha}}{\frac{1}{n}} \to f'_\alpha(1) = -\alpha \;\Rightarrow\; 1 - \left(1 + \frac{1}{n}\right)^{-\alpha} = O\left(\frac{1}{n}\right)$$

Finally $v_n = O(n^{-(\alpha+1)})$ and $\sum v_n$ converges (cf. $\alpha + 1 > 1$). But $a_{q+1} \to 0$, so the Abel's formula proves the convergence of $\sum u_n$.

We finish with the Fubini's theorem for double series :

**Theorem 5.8.** *Suppose $(a_{m,n}) \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ is such that for all $m$, $\sum_n |a_{m,n}|$ converges to a limit noted $\sigma_m$ and that $\sum \sigma_m$ converges to a limit noted $\Sigma$. Then*
*(i) for all $n$, $\sum_m |a_{m,n}|$ converges to a limit noted $\sigma'_n$,*
*(ii) $\sum \sigma'_n$ converges,*
*(iii) $\sum_m \sum_n a_{m,n} = \sum_n \sum_m a_{n,m}$ (noted $\sum_{m,n} a_{m,n}$).*

*Proof.* (i) : For $n_0 \in \mathbb{N}$, we have for all $M \in \mathbb{N}$

$$\sum_{m=0}^{M} |a_{n_0,m}| \leq \sum_{m=0}^{M} \sigma_m \leq \Sigma,$$

so we have the result.

(ii) : For all $N \in \mathbb{N}$

$$\sum_{n=0}^{N} \sigma'_n = \sum_{m=0}^{\infty} \sum_{n=0}^{N} |a_{m,n}| = \lim_{M\to\infty} \sum_{m=0}^{M} \sum_{n=0}^{N} |a_{m,n}|$$

and $\sum_{n=0}^{N} |a_{m,n}| \leq \sigma_m$, so $\sum_{m=0}^{M} \sum_{n=0}^{N} |a_{m,n}| \leq \Sigma$, and thus $\sum_{n=0}^{N} \sigma'_n \leq \Sigma$ which is enough to conclude.

(iii) : First, both members of the equality exist : we note $S_m = \sum_n a_{m,n}$ and $S'_n = \sum_m a_{m,n}$ so that $|S_m| \leq \sigma_m$ and $|S'_n| \leq \sigma'_n$ imply the convergence of $\sum S_m$ and $\sum S'_n$.

Let $\epsilon > 0$. We have for all $(M, N) \in \mathbb{N} \times \mathbb{N}$

$$\sum_{m=0}^{M} S_m = \sum_{m=0}^{M} \sum_{n\geq 0} a_{m,n} = \sum_{n\geq 0} \sum_{m=0}^{M} a_{m,n} = \\ \sum_{n=0}^{N} \sum_{m=0}^{M} a_{m,n} + \sum_{n\geq N+1} \sum_{m=0}^{M} a_{m,n}$$

where, because $\sum \sigma'_n$ converges, there exists $N_\epsilon \in \mathbb{N}$ such that for all $N \geq N_\epsilon$

$$|\sum_{n\geq N+1} \sum_{m=0}^{J} a_{m,n}| \leq \sum_{n\geq N+1} \sum_{m=0}^{M} |a_{m,n}| \leq \sum_{n\geq N+1} \sigma'_n \leq \epsilon,$$

and where, because $\sum \sigma_m$ converges, there exists $M_\epsilon \in \mathbb{N}$ such that for all $M \geq M_\epsilon$

$$|\sum_{n=0}^{N} S_n - \sum_{n=0}^{N} \sum_{m=0}^{M} a_{m,n}| \leq \sum_{n=0}^{N} \sum_{m\geq M+1} |a_{m,n}| = \\ \sum_{m\geq M+1} \sum_{n=0}^{N} |a_{m,n}| \leq \sum_{m\geq M+1} \sigma_m \leq \epsilon.$$

Hence for all $N \geq N_\epsilon$ and $M \geq M_\epsilon$ we have

$$|\sum_{n=0}^{N} S_n - \sum_{m=0}^{M} S_m| \leq 2\epsilon.$$

which leads to the result. □

*Remark* 5.9.

- In fact (iii) is a particular case of the double-limit theorem you'll see in ch2. The trick is to consider $\mathbf{E} = \{x_i\}_{i\in\mathbb{N}\cup\{\infty\}} \subset \mathbb{R}$ with $x_i \xrightarrow{n\infty} x_\infty$ and to define $f_m \in \mathbb{C}^{\mathbf{E}}$ by $f_m(x_i) = \sum_{n=0}^{i} a_{m,n}$ for all $i \in \mathbb{N} \cup \{\infty\}$. We have
  – $\forall m$, $f_m(x_i) \xrightarrow{x_i\to x_\infty} f_m(x_\infty) = \sum_n a_{m,n}$ ;
  – normal convergence : $\forall x \in \mathbf{E}$, $|f_m(x)| \leq \sigma_m$.
  Hence, setting $g = \sum_{m\geq 0} f_m \in \mathbb{C}^{\mathbf{E}}$, $\exists \lim_{x_i\to x_\infty} g(x_i) = g(x_\infty)$, which exactly says that $\sum_n S'_n$ converges, and that the limit is $\sum_m S_m$.
- This theorem can be very usefull for the theory of power series - see ch3.