Chapter 3 : Power Series.

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1 Definitions and first properties

Definition 1.1. A *power series* is a series of functions $\sum f_n$ where $f_n : z \mapsto a_n z^n$, (a_n) being a sequence of complex numbers. Depending on the cases, we will consider either the complex variable z, or the real variable x.

Notations 1.2. For $r \geq 0$, we will note $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$, $K_r = \{z \in \mathbb{C} \mid |z| \leq r\}$ and $C_r = \{z \in \mathbb{C} \mid |z| = r\}$.

Lemma 1.3. *[Abel's lemma] Let $\sum a_n z^n$ be a power series. We suppose that there exists $z_0 \in \mathbb{C}^*$ such that the sequence $(a_n z_0^n)$ is bounded. Then, for all $r \in]0, |z_0|[$, $\sum a_n z^n$ normally converges on the compact K_r.*

Remark 1.4.

- Note that it implies the absolute convergence on $\Delta_{|z_0|}$, ie $\forall z \in \Delta_{|z_0|}$, $\sum |a_n z^n|$ converges.
- Of course if we suppose $\sum |a_n| r^n$ convergent, we directly have the normal convergence on K_r (cf. $\forall z \in K_r$, $|a_n z^n| \leq |a_n| r^n$).

Proof. Let z be in K_r, we have

$$|a_n z^n| \leq |a_n| r^n = |a_n z_0^n| \left(\frac{r}{|z_0|}\right)^n = O\left(\left(\frac{r}{|z_0|}\right)^n\right),$$

which gives the result. □

Definition 1.5. We call the *radius of convergence* of the power series $\sum a_n z^n$ the number

$$R = \sup\{r \geq 0 \mid (a_n r^n) \text{ bounded}\} \in \overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}.$$

It will sometimes be noted $RCV(\sum a_n z^n)$.

Theorem 1.6. *Let R be the RCV of a power series $\sum a_n z^n$.*

- *For all $r < R$, $\sum a_n z^n$ normally converges on the compact K_r.*
- *For all z such that $|z| > R$, $a_n z^n \overset{n\infty}{\nrightarrow} 0$.*

Remark 1.7. It implies the absolute convergence on Δ_R.

Proof. The Abel's lemma gives the first point : $\forall r \in [0, R[$, $\exists r' \in]r, R]$ such that $(a_n r'^n)$ is bounded, which implies the normal convergence on K_r. For the second point, it's the contraposition of $a_n z^n \to 0 \Rightarrow (a_n z^n)$ bounded $\Rightarrow |z| \leq R$. □

Corollary 1.8. *With the same hypothesis,* $R = \sup\{r \geq 0 \mid \sum a_n r^n$ *converges*$\} = \inf\{r \geq 0 \mid \sum a_n r^n$ *diverges*$\} \in \overline{\mathbb{R}}^+$.

Proof. Let's note $R' = \sup\{r \geq 0 \mid \sum a_n r^n$ converges$\}$ and $R'' = \inf\{r \geq 0 \mid \sum a_n r^n$ diverges$\}$. First, $R' \leq R''$: if not, $R'' < R'$ and $\exists r \in]R'', R']$ such that $\sum a_n r^n$ converges, so we would have $(a_n r^n)$ bounded and convergence on Δ_r (cf. 1.4), and thus $R'' \geq r$, absurd. By the first point of the theorem, $R' \geq R$. By the second point, $R'' \leq R$. So we have $R \leq R' \leq R'' \leq R$, which gives the result. □

Remark 1.9.

- With the same kind of proof, one can show that we also have $R = \sup\{r \geq 0 \mid a_n r^n \to 0\}$.
- To sum up, if we note $\mathfrak{C}$ the domain of convergence of a power series which has a radius of convergence R, we have

$$\boxed{\Delta_R \subset \mathfrak{C} \subset K_R}$$

  and we have absolute convergence on Δ_R.

Definition 1.10. We call $\Delta_R = \{z \in \mathbb{C} \mid |z| < R\}$ the (open) disk of convergence.

Remark 1.11. We can't say anything *a priori* about the convergence of a power series on the circle C_R, as we will see in the examples.

Examples 1.12.

- $RCV(\sum z^n) = 1$ since the constant sequence (1) is bounded ($\Rightarrow RCV \geq 1$) and $\sum 1$ diverges ($\Rightarrow RCV \leq 1$). In fact there's no point in C_1 where there is oconvergence ($|z| = 1 \Rightarrow z^n \nrightarrow 0$).
- $RCV(\sum z^n/n) = 1$ since $(1/n)$ bounded ($\Rightarrow RCV \geq 1$) and $\sum 1/n$ diverges ($\Rightarrow RCV \leq 1$). Here, the only point of C_1 where the power series diverges is 1 : if $z = e^{i\theta} \neq 1$, $\sum z^n/n$ converges iff $\Re(\sum z^n/n)$ and $\Im(\sum z^n/n)$ converge, ie iff $\sum \cos(n\theta)/n$ and $\sum \sin(n\theta)/n$ converge. But we've already seen that the first one converges iff $e^{i\theta} \neq 1$, and the same proof shows that it's the same for the second one.

Exercise 1.13. [*Hadamard theorem*] Prove that this definition of the radius of convergence is equivalent to the first one :

$$\boxed{R = (\limsup |a_n|^{1/n})^{-1}}$$

2 Few methods to find the RCV

Proposition 2.1. *Let* $\sum a_n z^n$ *be a power series and* $z_0 \in \mathbb{C}$*. Then :*

- *If* $\sum a_n z_0^n$ *converges but* $\sum |a_n z_0^n|$ *diverges, then* $RCV = |z_0|$.
- *Same conclusion if* $\sum a_n z_0^n$ *diverges but* $a_n z_0^n \to 0$.

Proof. For the first point, we have $RCV \geq |z_0|$ (cf. 1.8), but we can't have $RCV > |z_0|$ (cf. 1.6). The second point is a consequence of 1.8 and 1.9. □

Proposition 2.2. *Let* $\sum a_n z^n$ *and* $\sum b_n z^n$ *be two power series, and* R_a, R_b *their RCV. We have* $a_n = O(b_n) \Rightarrow R_a \geq R_b$.

Proof. Let $z \in \Delta_{R_b}$, we have $a_n z^n = O(b_n z^n)$ and $\sum |b_n z^n|$ converges (cf. 1.6), so $\sum a_n z^n$ converges. By 1.8 we conclude $R_a \geq R_b$. □

Remark 2.3. We can't say that $(a_n z^n = O(b_n z^n)$ and $\sum b_n z^n$ converges$) \Rightarrow \sum a_n z^n$ converges because we're not in the case $b_n z^n \in \mathbb{R}^+$ for n big enough $(b_n z^n \in \mathbb{C})$.

Corollary 2.4. *With the same notations, we have* $a_n \sim b_n \Rightarrow R_a = R_b$.

Proof. $\sim \Rightarrow O$. □

Proposition 2.5. *Suppose* $a_n \neq 0$ *for* n *big enough. Then (with* $1/0 = +\infty$ *and* $1/+\infty = 0$*) :*

$$\exists \lim \left|\frac{a_{n+1}}{a_n}\right| = l \in \overline{\mathbb{R}}^+ \;\Rightarrow\; RCV = \frac{1}{l}.$$

Proof. We have $\left|\dfrac{a_{n+1}z^{n+1}}{a_n z^n}\right| \to l|z|$. By De D'Alembert rule, $|z| < 1/l \Rightarrow \sum a_n z^n$ converges, and $RCV \geq 1/l$ (cf. 1.8). Similarly, if $|z| > 1/l$, $\sum a_n z^n$ diverges, and $RCV \leq 1/l$. □

Proposition 2.6. *Let* $\sum a_n z^n$ *a power series and* R *its RCV. Then for all* $\alpha \in \mathbb{R}$ *the RCV* R_α *of the power series* $\sum n^\alpha a_n z^n$ *is also* R.

Proof. Let $r < R$ and $\rho \in]r, R[$. We have

$$n^\alpha a_n r^n = \underbrace{n^\alpha \left(\frac{r}{\rho}\right)^n}_{\to 0} \underbrace{a_n \rho^n}_{\to 0} \;\Rightarrow\; (n^\alpha a_n r^n) \text{ bounded } \;\Rightarrow\; R_\alpha \geq R.$$

This is true for all $\sum a_n z^n$, and for all α, so we also have, with $\beta = -\alpha$,

$$R = RCV(\textstyle\sum n^\beta (n^\alpha a_n z^n)) \geq RCV(\sum n^\alpha a_n z^n) = R_\alpha.$$

□

Examples 2.7.
- By 2.5, $RCV(\sum z^n/n!) = +\infty$.
- By 2.5, $RCV(\sum n! z^n) = 0$.
- By 2.6, $RCV(\sum z^n/n^2) = 1$ and we have normal convergence on K_1.
- We can abusively note $\sum z^{2n}/5^n$ the power series defined by $a_{2n+1} = 0$ and $a_{2n} = 5^{-n}$ for all n. But we can't apply directly 2.5. However, it's clear that we have convergence on $\Delta_{\sqrt{5}}$ and divergence on its complementary, so $RCV = \sqrt{5}$.

Proposition 2.8. *Let* R_a *and* R_b *be the RCV of* $\sum a_n z^n$ *and* $\sum b_n z^n$. *Then* $R_{a+b} = RCV(\sum (a_n + b_n) z^n) \geq \mathfrak{m} = \min\{R_a, R_b\}$, *with equality if* $R_a \neq R_b$. *Moreover, on* $\Delta_\mathfrak{m}$, *we have*

$$\textstyle\sum (a_n + b_n) z^n = \sum a_n z^n + \sum b_n z^n.$$

Proof. For all $z \in \Delta_\mathfrak{m}$, $\sum a_n z^n$ and $\sum b_n z^n$ absolutely converges. Hence $\sum (a_n + b_n) z^n$ also does : $R_{a+b} \geq \mathfrak{m}$ and the additivity of limits of sequences gives the additivity formula. If $R_a < R_b$, for all $z \in \Delta_{R_b} \backslash K_{R_a}$ we have $a_n z^n \nrightarrow 0$ and $b_n z^n \to 0$, thus $(a_n + b_n) z^n \nrightarrow 0$, and $R_{a+b} \leq R_a = \mathfrak{m}$. □

Example 2.9. Let $\sum a_n z^n = \sum z^n$ and $\sum b_n z^n = \sum ((1/2)^n - 1)z^n$, we have $R_a = 1 = R_b$ (use 2.5 for the second one). As $a_n + b_n = (1/2)^n$, the domain of convergence of the $\sum (a_n + b_n)z^n$ is clearly Δ_2, so $R_{a+b} = 2 > \mathfrak{m}$.

The nest result is obvious :

Proposition 2.10. *For all $\lambda \in \mathbb{C}^*$, $\sum a_n z^n$ and $\sum \lambda a_n z^n$ have the same RCV R. Moreover, on Δ_R, we have*

$$\sum \lambda a_n z^n = \lambda \sum a_n z^n.$$

Proposition 2.11. *Let R_a and R_b be the RCV of $\sum a_n z^n$ and $\sum b_n z^n$. Then $R_{a\star b} = RCV((\sum a_n z^n) \star (\sum b_n z^n)) \geq \mathfrak{m} = \min\{R_a, R_b\}$. Moreover, on $\Delta_{\mathfrak{m}}$, we have*

$$(\sum a_n z^n) \star (\sum b_n z^n) = (\sum a_n z^n)(\sum b_n z^n).$$

Proof. For all $z \in \Delta_{\mathfrak{m}}$, $\sum |a_n z^n|$ and $\sum |b_n z^n|$ absolutely converges. Hence the Cauchy product $(\sum |a_n z^n|) \star (\sum |b_n z^n|)$ converges (cf. ch1). But

$$\forall n, \quad \left| \sum_{k=0}^{n} a_k z^k b_{n-k} z^{n-k} \right| \leq \sum_{k=0}^{n} |a_k z^k||b_{n-k} z^{n-k}|,$$

so we get $R_{a\star b} \geq \mathfrak{m}$ and the result given about the Cauchy product in chapter 1 gives the formula. □

Examples 2.12.

- We don't have the same result as for the addition if $R_a \neq R_b$: Let $\sum a_n z^n$ and $\sum b_n z^n$ be defined by $a_0 = 1/2, b_0 = -2$ and $a_n = -1/2^{n+1}, b_n = -3$ for $n \geq 1$. We have $\sum a_n z^n = 1 - \sum_{n\geq 0} z^n/2^{n+1}$, $\sum b_n z^n = 1 - 3\sum_{n\geq 0} z^n$, so $R_a = 2 \neq R_b = 1$. We also have
$$\sum a_n z^n = 1 - \frac{1/2}{1-(z/2)} = \frac{z-1}{z-2} \; \forall z \in \Delta_2,$$
$$\text{and } \sum b_n z^n = 1 - 3\frac{1}{1-z} = \frac{z-2}{z-1} \; \forall z \in \Delta_1.$$
Hence by 2.11 $(\sum a_n z^n) \star (\sum b_n z^n) = 1$ on Δ_1, so if we note $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, we have $c_0 = 1$ and $c_n = 0$ for $n \geq 1$. Thus $R_{a\star b} = RCV(\sum c_n z^n) = +\infty > \mathfrak{m}$.
- Let R, R' be the RCV of $\sum a_n z^n$ and $\sum s_n(a) z^n$. We have $\sum s_n(a) z^n = (\sum a_n z^n) \star (\sum z^n)$, hence $R' \geq \min\{1, R\}$. We also have
$\sum a_n z^n = \sum s_n(a) z^n - \sum s_{n-1}(a) z^n = \sum s_n(a) z^n - z \sum_{n\geq 1} s_n(a) z^n$,
which gives $R \geq R'$. Thus we have
$$\boxed{\min\{1, R\} \leq R' \leq R}$$
which gives $R = R'$ if $1 \geq R$.

3 Properties of the sum

We've already seen :

Theorem 3.1. *Let $\sum a_n z^n$ be a power series and R its RCV. $\sum a_n z^n$ normally converges on every K_r, $r < R$, which leads to the continuity of the sum function on Δ_R.*

Remark 3.2. If $\exists z_0 \in C_R$ such that $\sum a_n z_0^n$ *absolutely* converges, then we have normal convergence (and continuity) on K_R.

Theorem 3.3. *[Radial continuity] Let's suppose that $\sum a_n z_0^n$ converges for $z_0 \in C_R$. Then $\sum a_n z^n$ uniformally converges on $[0, z_0]$, ie $t \mapsto \sum a_n z_0^n t^n$ uniformally converges on $[0, 1]$.*

Proof. We note $s_n(t) = \sum_{k=0}^n a_k z_0^k t^k$ for $t \in [0, 1]$ and $r_n = \sum_{k=n+1}^\infty a_k z_0^k$. By Abel's formula we obtain

$$s_n(t) = \textstyle\sum_{k=0}^n (r_{k-1} - r_k)t^k = \underbrace{\displaystyle\sum_{k=0}^n (t^{k+1} - t^k) r_k}_{f_n(t)} - t^{n+1} r_n + r_{-1}.$$

For $\epsilon > 0$, $\exists N$ such that $n \geq N \Rightarrow |r_n| \leq \epsilon$, hence, forall $n \geq N,\ p \geq 1,\ t \in [0, 1]$

$$|f_{n+p}(t) - f_n(t)| \leq \sum_{k=n+1}^{n+p} |r_k|(t^k - t^{k+1}) \leq \epsilon(t^{n+1} - t^{n+p+1}) \leq \epsilon,$$
$$\text{and } |t^{n+1} r_n| \leq \epsilon,$$

so (s_n) uniformally converges. ☐

Remark 3.4. The Leibniz criterion can also be used in the case of a decreasing real sequence (a_n) which converges to zero. Suppose $R = 1$, then for $x \in [-1, 0]$, $\sum a_n x^n$ satisfies the hypothesis of the Leibliz criterion ; so we get $|\sum_{k=n+1}^\infty a_n x^n| \leq |a_n x^n| \leq a_n$ which proves the uniform convergence on $[-1, 0]$, and thus the continuity in -1.

We can deduce from the radial continuity a new result about the Cauchy product - compare with the one obtained in ch.1 :

Corollary 3.5. *Let $\sum c_n$ be the Cauchy product of $\sum a_n$ and $\sum b_n$. We suppose that $\sum a_n$, $\sum b_n$ and $\sum c_n$ converge to A, B and C. Then $C = AB$.*

Proof. The three power series $f(x) = \sum a_n x^n$, $g(x) = \sum b_n x^n$ and $h(x) = \sum c_n x^n$ have a RCV≥ 1, hence absolutely converge for $|x| < 1$ so we can apply the theorem of chapter 1 and get $f(x)g(x) = h(x)$ for these x. But by the radial continuity theorem we can apply the double limit theorem for $x \to 1$ to obtain the result. ☐

Definition 3.6. We call *derivative series* (resp. *primitive series*) of a power series $\sum a_n z^n$ the power series defined by $\sum (n+1) a_{n+1} z^n$ (resp. $\sum_{n\geq 1} (a_{n-1}/n) z^n$).

Remark 3.7. We know that they have the same RCV than $\sum a_n z^n$, thanks to 2.6 and 2.4 : $\sum (n+1) a_{n+1} z^n$ converges iff $z \sum (n+1) a_{n+1} z^n = \sum_{n\geq 1} n a_n z^n$ converges ; and $\sum_{n\geq 1} (a_{n-1}/n) z^n = z \sum (a_n/(n+1)) z^n$ with $a_n/(n+1) \sim a_n/n$.

Theorem 3.8. *Let $\sum a_n x^n$ (real variable) be a power series, f its sum, g (resp. F) the sum of its derivative (resp. primitive) series and R its RCV. Then, on $]-R, R[$, f is $\mathcal{C}^1$ with $f' = g$, and F is the only primitive of f such that $F(0) = 0$.*

Remark 3.9. This implies

$$\forall x \in]-R, R[, \ \sum_{n=0}^\infty \frac{a_n}{n+1} x^{n+1} = \int_0^x \left(\sum_{n=0}^\infty a_n t^n \right) dt.$$

Proof. Replacing f by F, the first assertion immediately gives the second one. But if we note $f_n(x) = a_n x^n$, we have f_n $\mathcal{C}^1$ with $f_n'(x) = n a_n x^{n-1}$ for $n \geq 1$ ($f_0' = 0$). Hence $\sum f_n'$ is the derivatives series of $\sum a_n x^n$ which normally converges on each $[-r, r] \subset]-R, R[$ (cf. 3.7), and we know that it implies : $\sum f_n$ $\mathcal{C}^1$ on $[-r, r]$ and $f' = (\sum f_n)' = \sum f_n' = g$. We conclude with the fact that $]-R, R[= \cup_{0<r<R}[-r, r]$. □

Corollary 3.10. *The sum function f of a power series $\sum a_n x^n$ with RCV$= R$ is $\mathcal{C}^\infty$ on $]-R, R[$, and $f^{(p)}$ is the sum function of*

$$\sum \frac{(n+p)!}{n!} a_{n+p} x^n.$$

The RCV of these power series is also R.

Remark 3.11. This implies

$$\boxed{\forall p, \quad \frac{f^{(p)}(0)}{p!} = a_p}$$

Corollary 3.12. *If we have $\sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} b_n x^n$ on $]-R, R[$ (both power series converging on this interval), then $a_n = b_n$ for all n.*

Proof. The difference of the sum functions is 0. Hence, all its derivatives at 0 are 0. □

4 RPS functions

Definition 4.1. Given a complex number z_0 and a function $f : \mathcal{U} \to \mathbb{C}$ defined on a neighborhood $\mathcal{U} \subset \mathbb{C}$ of z_0, we say that f is *representable by a power series* (=*RPS*) or *analytic* at z_0 if $\exists r > 0$ and a power series $\sum a_n z'^n$ with RCV$\geq r$ such that $\Delta(z_0, r) = \{z \in \mathbb{C} \mid |z_0 - z| < r\} \subset \mathcal{U}$ and

$$\forall z \in \Delta(z_0, r),\ f(z) = \sum a_n (z - z_0)^n.$$

Remark 4.2.

- For $f = \mathbb{R} \to \mathbb{C}$ and $z_0 = x_0$, replace $\mathcal{U}$ by an interval $I \ni x_0$ and $\Delta(z_0, r)$ by $\Delta(x_0, r) \cap \mathbb{R} =]-r + x_0, x_0 + r[= I(x_0, r)$.
- Most results will be given relatively to $z_0 = 0$, but only for convenience. The generalization is just the consequence of

  f RPS at $z_0 \Leftrightarrow f(z_0 + \bullet)$ is RPS at 0.

Definition 4.3. $f : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ is said to be *analytic* if f is RPS at any point of $\mathcal{U}$.

Proposition 4.4. *Let f be representable by $\sum a_n z^n$ at 0 on $\Delta(0, r)$. Then f is analytic on $\Delta(0, r)$.*

Proof. Let $z_0 \in \Delta(0, r)$ and $\rho = r - |z_0|$. For $z \in \Delta(z_0, \rho)$ we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n ((z - z_0) + z_0)^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(a_n \binom{n}{m} z_0^{n-m} (z - z_0)^m \right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} a_n \binom{n}{m} z_0^{n-m} \right) (z - z_0)^m \end{aligned}.$$

The last equality is a consequence of the Fubini theorem given in ch1 with $a_{m,n} = a_n\binom{n}{m}z_0^{n-m}(z-z_0)^m$ (with the convention $\binom{n}{m} = 0$ if $m > n$). We just have for example to check that $\sum_n \sum_m |a_{m,n}|$ is finite :

$$\sum_n \sum_m |a_{m,n}| = \sum_{n=0}^{\infty} |a_n|(|z-z_0| + |z_0|)^n = \sum_{n=0}^{\infty} |a_n| r'^n,$$

with $0 \le r' < \rho + |z_0| = r$ hence $\sum |a_n| r'^n$ converges and we have the result. $\square$

Remark 4.5. For $z_0 \in \Delta(0,r)$, it's important to notice that f is RPS at z_0 on the *bigger* disk centered at z_0 and contained in $\Delta(0,r)$, which is $\Delta(z_0, r - |z_0|)$.

From 3.10, we get a necessary condition for f to be RPS :

Proposition 4.6. *If $f : I \to \mathbb{C}$ is representable by $\sum a_n x^n$ at 0, then $\exists r > 0$ such that $I(0,r) \subset I$, with f $\mathcal{C}^\infty$ on $I(0,r)$. Moreover we necessarily have $a_n = f^{(n)}(0)/n!$.*

Example 4.7. of a function which is *not* RPS :

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 0 & \text{if } x \le 0 \\ \exp(-1/x^2) & \text{if } x > 0 \end{cases}$$

By induction, one can prove that f is $\mathcal{C}^\infty$ on $\mathbb{R}$ with all derivatives $= 0$ for all $x \le 0$ and $f^{(p)}(x) = P_p(1/x)\exp(1/x^2)$ for $x > 0$, P_p being a polynomial. Hence if f representable by $\sum a_n x^n$, $a_n = f^{(n)}(0)/n! = 0 \Rightarrow f = 0$ on $I(0,r)$ for $r > 0$, which is false.

Definition 4.8. For $f : I \subset \mathbb{R} \to \mathbb{C}$ $\mathcal{C}^\infty$ we note for all $a, x \in I$

$$T_n(f, a, \bullet) : x \mapsto \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

the *Taylor polynomial* of f at a,

$$R_n(f, a, \bullet) : x \mapsto f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

the *Taylor remainder* of f at a, and

$$T(f, a, \bullet) : x \mapsto \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

the *Taylor series* of f at a.

Corollary 4.9. *A function $f : I \to \mathbb{C}$ is RPS at 0 iff $\exists r > 0$ such that $I(0,r) \subset I$ such that f is $\mathcal{C}^\infty$ on $I(0,r)$ and*

$$\forall x \in I(0,r),\ R_n(f, 0, x) \xrightarrow{n\infty} 0.$$

In such a case, f is representable by its Taylor series at 0.

Remark 4.10.

- Of course we have the same result replacing 0 by a - just use $f_a = f(\bullet + a)$.

- About the Taylor remainder : one can prove by induction, using integrations by parts, that we have, for f $\mathcal{C}^{n+1}$:

$$\boxed{R_n(f,a,x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt}$$

This implies, for example, that

$$\begin{aligned} |R_n(f,a,x)| &\leq \int_a^x \left|\frac{(x-t)^n}{n!}\right| |f^{(n+1)}(t)|dt \\ &\leq \max_{[(a,x)]} |f^{(n+1)}| \int_a^x \left|\frac{(x-t)^n}{n!}\right| dt \\ &= \max_{[(a,x)]} |f^{(n+1)}| \left|\int_a^x \frac{(x-t)^n}{n!} dt\right| \end{aligned}$$

because the sign of $x-t$ is constant on $[(a,x)]$ ($=[a,x]$ if $a \leq x$, $=[x,a]$ if not). Hence we have

$$\boxed{|R_n(f,a,x)| \leq \max_{[(a,x)]} |f^{(n+1)}| \frac{|a-t|^{n+1}}{(n+1)!}}$$

This gives a sufficient condition for f $\mathcal{C}^\infty$ to be RPS at a :

$$\exists r>0,\ \exists M \geq 0,\ \forall x \in [a-r,a+r],\ \forall n,\ |f^{(n)}(t)| \leq M.$$

($|a-t|^{n+1}/(n+1)! \to 0$ since $RCV(\sum z^n/n!) = +\infty$).

Proposition 4.11. *Let $\sum a_n z^n$ a power series with RCV$= R > 0$, sum function f. We suppose $a_0 \neq 0$. Then $1/f$ is RPS at 0.*

Proof. We can suppose $a_0 = 1$ (consider $f \leftarrow f/a_0$). Let's first prove

Lemma 4.12. $RCV(\sum u_n z^n) > 0 \ \Leftrightarrow\ \exists q > 0,\ |u_n| \leq q^n.$

Proof. For $\Rightarrow$, we note $r = RCV(\sum u_n z^n) > 0$. Fix $r' \in]0,r[$: we have $(u_n r'^n)$ bounded by some constant $M \geq 1$, and we get $\forall n,\ |u_n| \leq M(1/r')^n \leq q^n$ with $q = M/r'$. For the other implication we have $u_n = O(q^n)$, hence $RCV(\sum u_n z^n) \geq RCV(\sum q^n z^n) = 1/q > 0$. □

If $1/f$ is RPS $\sum b_n z^n$ on $\Delta(0,R') \subset \Delta(0,R)$, we get (cf. 2.11) on $\Delta(0,R')$

$$(\sum a_n z^n) \star (\sum b_n z^n) = (\sum a_n z^n)(\sum b_n z^n) = 1 \tag{1}$$

which implies (cf. 3.12)

$$b_0 = 1 \text{ and } \forall n \geq 1,\ b_n = -a_1 b_{n-1} - \cdots - a_n b_0.$$

Let $q > 0$ such that $|a_n| \leq q^n$ and let's prove by induction that $|b_n| \leq q'^n$ with $q' = 2q$. This is true for $n = 0$ and if $|b_{n-1}| \leq q'^{n-1}$, we have

$$|b_n| \leq \sum_{k=1}^n |a_k||b_{n-k}| \leq \sum_{k=1}^n q^k q'^{n-k} = \sum_{k=1}^n \frac{1}{2^k} q'^n \leq q'^n.$$

Hence by the lemma we have $RCV(\sum b_n z^n) = R_b > 0$ and the formula 1 proves that that the sum function of $\sum b_n z^n$ is equal to $1/f$ on $\Delta(0, \min\{R, R_b\})$. □

Remark 4.13. About the composition of two RPS functions : Suppose $f(z) = \sum a_n z^n$ on $\Delta(0, R)$ and $g(z) = \sum b_n z^n$ on $\Delta(0, R')$ with $b_0 = 0 = g(0)$: then $\exists \rho < R'$ such that $z \in \Delta(0, \rho) \Rightarrow g(z) \in \Delta(0, R)$ by continuity of g, and for $z \in \Delta(0, \rho)$, we have $f(g(z)) = \sum_n a_n g(z)^n$. But, by Cauchy product, g^n is RPS on $\Delta(0, \rho)$, and we can note $g(z)^n = \sum_p b_{n,p} z^p$ for some complex numbers $b_{n,p}$. Hence,

$$f(g(z)) = \textstyle\sum_n \sum_p a_n b_{n,p} z^p = \sum_p (\sum_n a_n b_{n,p}) z^p$$

if we can apply the Fubini theorem to the double series $(a_n b_{n,p})$.

5 Classical examples

Definition 5.1. We note $\exp(z) = e^z$, $\cos z$ and $\sin z$ the sum functions of the following power series :

$$\sum \frac{z^n}{n!}\ ,\ \sum \frac{(-1)^n}{(2n)!} z^{2n} \ \text{ and } \ \sum \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

Remark 5.2.

- The three power series have RCV$= \infty$: we already know that for the first one. But if we note these series respectively $\sum a_n z^n$, $\sum b_n z^n$ and $\sum c_n z^n$ ($a_n = 1/n!$) we remark that $|b_n| \leq a_n$ and $|c_n| \leq a_n$.
- Following this definition, we clearly have, for $z \in \mathbb{C}$,
$\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$.

Proposition 5.3. *We have the following facts :*

1. *The derivative series of* $\exp$, $\sin$ *and* $\cos$ *are respectively* $\exp$, $\cos$ *and* $-\sin$.
2. *For all* $z, z' \in \mathbb{C}$, $e^{z+z'} = e^z e^{z'}$.
3. *For all* $z \in \mathbb{C}$, $\cos z + i \sin z = e^{iz}$.
4. *For all* $z \in \mathbb{C}$, $e^z = \lim\limits_{n\infty} (1 + \frac{z}{n})^n$.

Proof. The first point is a consequence of 3.8. For 2, we use the Cauchy product (and $RCV(\sum z^n/n!) = \infty$, so we have absolute convergence everywhere) to get

$$e^z e^{z'} = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!} z^k \frac{(-1)^{n-k}}{(n-k)!} z'^{n-k} \right) = \sum_{n\geq 0} \frac{(z+z')^n}{n!} = e^{z+z'}.$$

With the notations of 5.2, $b_n + i c_n = i^n a_n$ so we get 3. Let's prove 4 : we note $E = \mathbb{N} \subset \mathbb{R}$ and for all $k \in \mathbb{N}$ (with the convention $\binom{n}{k} = 0$ if $k > n$),

$$\alpha_k \left| \begin{array}{ccc} E & \to & \mathbb{C} \\ n & \mapsto & \displaystyle\binom{n}{k} \frac{1}{n^k} z^k \end{array} \right. ,$$

so we get for all $n \in E$,

$$A(n) = \left(1 + \frac{z}{n}\right)^n = \sum_{k=0}^{+\infty} \alpha_k(n).$$

Let's try to apply the double-limit theorem for $n \to \infty$: we first have for all $k \geq 0$

$$\forall n \geq k, \ \ \alpha_k(n) = \frac{z^k}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \xrightarrow{n\infty} \frac{z^k}{k!}.$$

But we also have

$$\forall n \geq k, \ \ |\alpha_k(n)| = \left| \frac{z^k}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \right| \leq \frac{|z|^k}{k!},$$

and this inequality is also true for $n < k$: we have the normal convergence (cf. $\sum |z|^n/n!$ converges). The double-limit theorem gives the result. □

Remark 5.4. As a consequence of 5.2 and 3 we have

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Lemma 5.5. *We note* $\mathcal{E} = \{x \in \mathbb{R}^+ \mid \cos(x) = 0\}$. *Then* $\exists \alpha = \inf \mathcal{E} \in \mathbb{R}_+^*$.

Proof. We just have to prove that $\mathcal{E} \neq \varnothing$. If not, $\cos x > 0$ for all $x \geq 0$ (cf. $\cos 0 = 1$ and $\cos$ is continuous). This would imply the strict convexity of $-\cos$ on $\mathbb{R}_+$, which cannot happen since for all $x \in \mathbb{R}_+$, $-\cos x < 0$ (the only negative convex functions on $\mathbb{R}_+$ are the constant functions). □

Definition 5.6. The constant 2α will be noted π.

Corollary 5.7. *We have the following facts :*

1. *For all* $x \in \mathbb{R}$, $\cos^2 x + \sin^2 x = 1$.
2. $e^{i\pi/2} = i$, *which implies* $\forall x \in \mathbb{R}, \cos(x+\frac{\pi}{2}) = -\sin x$ *and* $\sin(x+\frac{\pi}{2}) = \cos x$.
3. $e^{i\pi} = -1$, *which implies* $\forall x \in \mathbb{R}, \cos(\pi-x) = -\cos x$ *and* $\sin(\pi-x) = \sin x$.
4. $e^{i2\pi} = 1$, *which implies the* 2π*-periodicity of the functions of the real variable* $x \mapsto \sin x, \cos x$.

Proof. Using the continuity and the algebraic properties of $\tau : z \mapsto \bar{z}$, we have for all $z \in \mathbb{C}$,

$$\overline{\exp z} = \tau\left(\lim_{n\infty} \sum_{k=0}^{n} \frac{z^k}{k!}\right) = \lim_{n\infty}\left(\tau\left(\sum_{k=0}^{n} \frac{z^k}{k!}\right)\right) = \lim_{n\infty}\left(\sum_{k=0}^{n} \frac{\bar{z}^k}{k!}\right) = \exp \bar{z}.$$

Hence for $z = ix \in i\mathbb{R}$, by 5.3.2, we have $(e^{ix})^{-1} = e^{-ix} = \overline{e^{ix}}$, which gives $|e^{ix}| = 1$ and then 1. But $\cos(\pi/2) = 0$, so 5.3.3 implies $e^{i\pi/2} = i$. Then $e^{i\pi} = (e^{i\pi/2})^2 = -1$ and $e^{i2\pi} = (e^{i\pi/2})^4 = 1$. Just take the real and imaginary parts of $e^{ix}e^{i\lambda\pi} = e^{i(x+\lambda)\pi}$ for $\lambda \in \{1/2, 1, 2\}$ to obtain the complementary assertions in 2, 3 and 4. □

Remark 5.8.

- More generally for $a, b \in \mathbb{R}$, the classical trigonometric formulas
$$\begin{cases} \cos(a+b) = \cos a \cos b - \sin a \sin b \\ \sin(a+b) = \cos a \sin b + \sin a \cos b \end{cases}$$
are a consequence of $e^{ia}e^{ib} = e^{i(a+b)}$.

- The hyperbolic sine and cosine are defined as follow for $z \in \mathbb{C}$:

$$\begin{cases} \sinh z = -i\sin(iz) = \sum \dfrac{z^{2n+1}}{(2n+1)!} = \dfrac{e^z - e^{-z}}{2} \\ \cosh z = \cos(iz) = \sum \dfrac{z^{2n}}{(2n)!} = \dfrac{e^z + e^{-z}}{2} \end{cases}$$

generalizing the definition known for $x \in \mathbb{R}$.

Example 5.9. There's a classical way to calculate the sum of power series of the form $\sum P(n)z^n/n!$ for a given polynomial $P \in \mathbb{C}[X]$. First the RCV is $+\infty$ by De D'Alembert rule. Then the idea is to decompose P on the base $\{1, X, X(X-1), \ldots, X(X-1)\ldots(X-d+1)\}$ if $\deg P = d$. Practically, with $\prod_{i=0}^{-1}(X-i) = 1$,

$$\begin{aligned} \deg P = d \quad &\Rightarrow \quad \exists!(a_0, \ldots, a_d) \in \mathbb{C}^{d+1} \mid P = \sum_{k=0}^{d} a_k \prod_{i=0}^{k-1}(X-i) \\ &\Rightarrow \quad \sum_{n\geq 0} \frac{P(n)}{n!} z^n = \sum_{k=0}^{d} a_k \sum_{n\geq 0} \frac{n\ldots(n-k+1)}{n!} z^n \\ &\Rightarrow \quad \sum_{n\geq 0} \frac{P(n)}{n!} z^n = \sum_{k=0}^{d} a_k \sum_{n\geq k} \frac{n\ldots(n-k+1)}{n!} z^n \\ &\Rightarrow \quad \sum_{n\geq 0} \frac{P(n)}{n!} z^n = \sum_{k=0}^{d} a_k \sum_{n\geq k} \frac{z^n}{(n-k)!} = \sum_{k=0}^{d} a_k z^k e^z \end{aligned}$$

Theorem 5.10. *The function $x \in \mathbb{R} \mapsto -\ln(1-x)$ is representable by the power series $\sum_{n\geq 1} x^n/n$ on $]-1,1[$.*

Proof. More precisely, we have : the primitive series of $\sum z^n$ (which has RCV$=1$) is $\sum_{n\geq 1} z^n/n$. Hence we have the result since $\ln$ is defined on $\mathbb{R}_+^*$ as the primitive F of $x \mapsto 1/x$ such that $F(1) = 0$. □

Definition 5.11. We define the *complex logarithm* as the sum of the power series $-\sum_{n\geq 1}(1-z)^n/n$, defined on $\Delta(1,1)$, and we note it $\ln z$.

Proposition 5.12. *We have*
- *for all $z \in \Delta(1,1)$, $\exp(\ln z) = z$;*
- *for all $z \in \Delta(0, \ln 2)$, $\ln(\exp z) = z$.*

Proof. Following 4.13, we write, for $z \in \Delta_1$,

$$\ln^n(1-z) = (-1)^n (\textstyle\sum_{k\geq 1} z^k/k)^n = (-1)^n \sum_{k\geq 0} a_{k,n} z^k,$$

and we set $b_{k,n} = (-1)^n a_{k,n} z^k/n!$. We have $|b_{k,n}| = a_{k,n}|z|^k/n!$ because $a_{k,n} \geq 0$ (cf. $\alpha_n \geq 0, \beta_n \geq 0 \Rightarrow \sum_{k=0}^{n} \alpha_k \beta_{n-k} \geq 0$), hence the series $\sum_{k\geq 0} |b_{k,n}|$ converges to $(-1)^n \ln^n(1-|z|)/n!$. Since the series $\sum(-\ln(1-|z|))^n/n!$ converges, we can apply the Fubini's theorem, which gives (cf. 4.13) :

$$\exp(\ln(1-z)) = \sum_{k\geq 0} \left(\sum_{n\geq 0} \frac{a_{k,n}}{n!} \right) z^k = \sum_{k\geq 0} c_k z^k.$$

The point is that we know that this quantity is $1-x$ if $z = x \in]-1,1[$. Thus, by 3.12, we have $c_0 = 1$, $c_1 = -1$ and $c_k = 0$ if $k > 1$. Finally we get the result

$$\exp(\ln(1-z)) = 1-z.$$

For the other assumption, we first remark that the left member is well defined :

$$z \in \Delta(0, \ln 2) \Rightarrow |e^z - 1| = |\textstyle\sum_{n\geq 1} z^n/n!| \leq \sum_{n\geq 1} |z|^n/n! = e^{|z|} - 1 \in [0, 1[.$$

Then we write

$$\ln(\exp z) = \ln(1 - (1 - e^z)) = \textstyle\sum_{n\geq 1}\sum_{k\geq 0} b_{k,n}.$$

with this time $b_{k,n} = (-1)^n a_{k,n} z^k/n$, where

$$(-1)^n \textstyle\sum_{k\geq 0} a_{k,n} z^k = (1 - e^z)^n = (-1)^n (\sum_{p\geq 1} z^p/p!)^n$$

Again, by induction (and using the definition of the coefficients of the Cauchy product), one can show that $a_{k,n} \geq 0$. This implies $|b_{k,n}| = a_{k,n}|z|^k/n$ and thus

$$\textstyle\sum_{k\geq 0} |b_{k,n}| = \sum_{k\geq 0} a_{k,n}|z|^k/n = (-1)^n(1 - e^{|z|})^n/n = (e^{|z|} - 1)^n/n$$

with $e^{|z|} - 1 \in [0, 1[\subset] - 1, 1[$. Hence $\sum_{n\geq 1}(e^{|z|} - 1)^n/n$ converges and we can, here again, apply the Fubini's theorem. The end of the proof is the same as in the first case, using the known results when $z = x \in] - \infty, \ln 2[$. ☐

Proposition 5.13. *For all* $x \in] - 1, 1[$ *:*

1. $\arctan(x) = \displaystyle\sum_{n\geq 0} \frac{(-1)^n}{2n+1} x^{2n+1}$
2. $\operatorname{arctanh}(x) = \displaystyle\sum_{n\geq 0} \frac{x^{2n+1}}{2n+1} = \frac{1}{2}\ln\frac{1+x}{1-x}$
3. $\forall \alpha \notin \mathbb{N}$, $(1+x)^\alpha = \displaystyle\sum_{n\geq 0} \binom{\alpha}{n} x^n$ *with* $\displaystyle\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$ *and* $\displaystyle\binom{\alpha}{0} = 1.$
4. $\displaystyle\frac{1}{\sqrt{1-x^2}} = \sum_{n\geq 0} \frac{(2n)!}{2^{2n}(n!)^2} x^{2n}$
5. $\arcsin(x) = \displaystyle\sum_{n\geq 0} \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{2n+1}$
6. $\displaystyle\frac{1}{\sqrt{1+x^2}} = \sum_{n\geq 0} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} x^{2n}$
7. $\operatorname{arcsinh}(x) = \displaystyle\sum_{n\geq 0} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{2n+1}$

Proof. 1 and 2 follow from 3.8 ; 5 and 7 follow from 3.8 and 4 and 6, which follow from 3. So, let's prove 3 : the only power series which can represent $x \mapsto (1+x)^\alpha$ is the one given, which is the Taylor series of ϕ. The power series $\sum \binom{\alpha}{n} x^n$ has RCV$= 1$ by the ratio test and if we note S its sum function we have

$$S'(x) = \sum_{n\geq 0} \binom{\alpha}{n+1} (n+1)x^n = \sum_{n\geq 0} \binom{\alpha}{n} (\alpha - n)x^n = \alpha S(x) - xS'(x).$$

Hence, since $S(0) = 1$, $S(x) = (1+x)^\alpha$ for all $x \in] - 1, 1[$. ☐