Minimal Surfaces in $\mathbb{R}^3$

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Abstract

The aim of this paper is to present some geometric results about minimal surfaces in Euclidean space $\mathbb{R}^3$. In the first part we recall some general facts about surfaces in $\mathbb{R}^3$. In the second part we give some basic examples of minimal surfaces and we construct a more complicated one using general properties of minimal surfaces.

1 Generalities about surfaces

We will first recall the definition and the basic properties of surfaces in $\mathbb{R}^3$. In this paper, the word "differentiable" means "of class C^∞".

Definition 1. *A subset $S \subset \mathbb{R}^3$ is called a regular surface if, for each point $p \in S$, there exists a neighbourhood V of p in $\mathbb{R}^3$, an open subset $U \subset \mathbb{R}^2$ and a map $X : U \to V \cap S$ such that*

- *X is differentiable,*
- *X is a homeomorphism (i.e. X has an inverse $X^{-1} : V \cap S \to U$ which is the restriction of a continuous map $F : W \to \mathbb{R}^2$ where W is an open subset of $\mathbb{R}^3$ contaning $V \cap S$),*
- *for each $q \in U$, the differential $\mathrm{d}_q X : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.*

The map X is called a parametrization or a system of (local) coordinates in (a neighbourhood of) p. The neighbourhood $V \cap S$ of p in S is called a coordinate neighbourhood.

Intuitively, a surface is a two-dimensional object which is smooth, which is locally a smooth deformation of an open subset of the plane. Self-intersection, angles and edges are not allowed by this definition of a regular surface.

The third condition in the definition allows us to define a tangent plane to S at each point $p \in S$: this is the set of vectors $v \in \mathbb{R}^3$ such that there exists a curve $\gamma : I \to S$ on S such that $\gamma(0) = p$ and $\gamma'(0) = v$. It will be denoted $\mathrm{T}_p S$. If X is a parametrization at p, then $\mathrm{T}_p S = \mathrm{Im}\, \mathrm{d}_q X$ where $p = X(q)$.

We can also define the notion of a differentiable map between two regular surfaces S_1 and S_2. A map $\varphi : S_1 \to S_2$ is said to be differentiable if for every parametrization $X : U \to S_2$ the map $\varphi \circ X : U \to S_2$ is differentiable (in the usual sense of differentiable functions defined on an open subset of $\mathbb{R}^2$). This is independent of the parametrization, since a change of parameters is a diffeomorphism. In this case we can define its differential $\mathrm{d}_p \varphi$ at a point $p \in S_1$: it is a linear map from $\mathrm{T}_p S_1$ to $\mathrm{T}_{\varphi(p)} S_2$.

On the tangent plane at p, we have a natural inner product which we will denote by $< , >_p$: the restriction of the inner product $< , >$ of $\mathbb{R}^3$. The map $p \mapsto < , >_p$ is called the metric of S. We also obtain a Euclidean norm on each tangent plane by setting

$$||v||_p = \sqrt{< v, v >_p}.$$

In all that follows we will omit the index p.

We can define a notion of length and a notion of area on a regular surface. If $\gamma : [t_0, t_1] \to S$ is a curve on S, its length is

$$\int_{t_0}^{t_1} ||\gamma'(t)|| \mathrm{d}t.$$

If Ω is a part of S, its area is

$$\iint_{X^{-1}(\Omega)} ||X_u \times X_v|| \, \mathrm{d}u \mathrm{d}v$$

if Ω is in a coordinate neighbourhood $X(U)$.

The notion of length allows us to define geodesics: they are curves that locally minimize length. More precisely, a curve γ on S is a geodesic if, for any couple of points (p, q) on γ that are sufficiently close to each other, the arc of γ going from p to q is the curve having the smallest length among all curves in S going from p to q. Being a geodesic is a local property: if we take two points p and q on a geodesic, then the arc of this geodesic from p to q is not necessarily minimizing (but this is always the case if p and q are sufficiently close).

The geodesics play the same role as the straight lines in the plane.

For example, on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$, the geodesics are the big circles, *i.e.* the circles whose center is the center of the sphere.

In a neighbourhood U in S of a point $p \in S$, we can define a differentiable map $N : U \to \mathbb{R}^3$ such that for all $q \in U$ the vector $N(q)$ is a unit vector and is normal to the tangent plane $\mathrm{T}_q S$. The map N is defined uniquely up to a sign. It is called the Gauss map.

Generally it is not possible to define the Gauss map on the whole surface S. If it is possible, the surface is said to be orientable. The sphere is an orientable surface (in fact we have $N(p) = p$). The Möbius strip is a non-orientable surface.

Actually the map N has its values in the unit sphere $\mathbb{S}^2$. Consequently its differential $\mathrm{d}_p N$ is a map from $\mathrm{T}_p S$ to $\mathrm{T}_{N(p)}\mathbb{S}^2$. On the other hand the tangent space to $\mathbb{S}^2$ at a point $x \in \mathbb{S}^2$ is the set $x^\perp$ of all vectors of $\mathbb{R}^3$ that are orthogonal to x. Consequently we have

$$\mathrm{T}_{N(p)}\mathbb{S}^2 = N(p)^\perp = \mathrm{T}_p S$$

by definition of $N(p)$. Thus $\mathrm{d}_p N$ is an endomorphism of $\mathrm{T}_p S$.

The number $K(p) = \det(\mathrm{d}_p N)$ is called the Gaussian curvature of S at p. It does not depend on the choice of N, *i.e.* replacing N by $-N$ does not change K. The curvature of the plane or the cylinder is zero. The curvature of the sphere of radius r is equal to $1/r^2$ (since $N(x) = x/||x|| = x/r$). If S is the graph of a differentiable function f and if p is a local extremum of f, then S has positive (or zero) curvature at p; if p is a saddle point of f, then S has a negative (or zero) curvature.

If T is a triangle on S bounded by geodesics between three points in the neighbourhood of $p \in S$, then we have the Gauss-Bonnet formula:

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \int_T K$$

where the α_i are the angles of the vertices of T. In particular, if the Gaussian curvature is positive (respectively zero or negative) on T, then the sum of the angles is greater than (respectively equal to or less than) π. If we make the three points tend to p, then we have

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + K(p)\mathcal{A} + \mathrm{o}(\mathcal{A})$$

where $\mathcal{A}$ is the area of the triangle T.

The number $H(p) = \frac{1}{2}\operatorname{tr}(\mathrm{d}_p N)$ is called the mean curvature of S at p. It does depend on the choice of N, but the mean curvature vector $\vec{H}(p) = H(p)N(p)$ does not depend on the choice of N. Intuitively, this vector shows the direction in which we have to deform the surface in order to decrease the most its area. It shows the direction where the surface is the most "curved".

2 Minimal surfaces in $\mathbb{R}^3$

A surface S in $\mathbb{R}^3$ is called a minimal surface if its mean curvature is zero at every point.

This terminology is motivated by the following property. Minimal surfaces are critical points for area: a compact surface S with boundary is minimal if and only if, for any differentiable family (S_t) of compact surfaces with the same boundary as S such that $S_0 = S$, the area of S_t has a critical point at $t = 0$. This comes from the fact that the mean curvature appears in the area variation formula.

Moreover, if S is a minimal surface and if γ is a sufficiently small closed curve on S, then the part of S bounded by γ is the surface that has the smallest area among all the surfaces whose boundary is γ. Generally this is not the case if γ is not sufficiently small.

The curvature of a minimal surface is negative or zero, since the trace of $\mathrm{d}N$ is zero.

Minimal surfaces play an important role in physics. If we put a closed iron wire in water mixed with soap, then after taking it out of the water we obtain a soap film bounded by the wire, and this soap film is a minimal surface.

They can be conformally parametrized (*i.e.* in such a way that the angles are preserved) by a function $X : \Sigma \to \mathbb{R}^3$ (where Σ is a Riemann surface, *i.e.* a holomorphic surface) such that $\Delta X = 0$.

The simplest examples of minimal surfaces are the plane, the catenoid and the helicoid.

The catenoid is parametrized by

$$\left\{ \begin{array}{rcl} x &=& -\lambda \operatorname{ch} u \cos v \\ y &=& -\lambda \operatorname{ch} u \sin v \\ z &=& \lambda u \ , \end{array} \right.$$

where λ is a non-zero real number.

The helicoid is parametrized by

$$\left\{ \begin{array}{rcl} x &=& \lambda \operatorname{sh} u \sin v \\ y &=& -\lambda \operatorname{sh} u \cos v \\ z &=& -\lambda v \ , \end{array} \right.$$

where λ is a non-zero real number.

The catenoid $\mathcal{C}$ and the helicoid $\mathcal{H}$ are locally isometric, *i.e.* for all $p \in \mathcal{C}$ and for all $q \in \mathcal{H}$ there exist a neighbourhood U of p in $\mathcal{C}$, a neighbourhood V of q in $\mathcal{H}$ and a diffeomorphism $\varphi : U \to V$ such that $\varphi(p) = q$ and

$$< \mathrm{d}\varphi(\xi), \mathrm{d}\varphi(\eta) > = < \varphi, \eta >$$

for all tangent vectors ξ and η.

Moreover, if $x_{\mathcal{H}}$, $y_{\mathcal{H}}$ and $z_{\mathcal{H}}$ (respectively $x_{\mathcal{C}}$, $y_{\mathcal{C}}$ and $z_{\mathcal{C}}$) denote the coordinates of $\mathcal{H}$ (respectively $\mathcal{C}$), then the functions $x_{\mathcal{H}} + i x_{\mathcal{C}}$, $y_{\mathcal{H}} + i y_{\mathcal{C}}$ and $z_{\mathcal{H}} + i z_{\mathcal{C}}$ are holomorphic functions of $\zeta = u + iv$. The helicoid and the catenoid are said to be conjugate surfaces.

This fact is not specific to the helicoid and the catenoid. For any minimal surface S of coordinates x, y and z, there exists a minimal surface S^* of coordinates x^*, y^* and z^* which is locally isometric to S and such that the functions $x + ix^*$, $y + iy^*$ and $z + iz^*$ are holomorphic. The surface S^* is called the conjugate surface of S.

A way to obtain minimal surfaces is to solve the Plateau problem (Plateau was a Belgian physicist): if $\Gamma \in \mathbb{R}^3$ is a Jordan curve, we look for a minimal surface that is topologically a disk and whose boundary is Γ. There is always a solution.

We have the following property: if there exists a plane such that the orthogonal projection of Γ on this plane bounds a convex domain, then the Plateau problem has a unique solution, and this solution is a graph over this domain (this is a difficult theorem of analysis).

We will use this property to construct another example of a minimal surface: Scherk's surface. We consider the polygonal contour $ABCDEFGH$ with $A = (-1, -1, n)$, $B = (-1, 1, n)$, $C = (-1, 1, -n)$, $D = (1, 1, -n)$, $E = (1, 1, n)$, $F = (1, -1, n)$, $G = (1, -1, -n)$ and $H = (-1, -1, -n)$, where $n \in \mathbb{N}^*$ (see figure 1).

The projection of this contour on the plane $z = 0$ bounds a convex domain (a square). Thus there exists a function $(x, y) \mapsto z(x, y)$ whose graph is a solution of the Plateau problem. This solution contains the two arcs shown on figure 1 and intersecting at the origin O.

Since minimal graphs are solutions of the elliptic equation

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

(where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$), we have some convergence theorems. In our case, if n tends to infinity, then the graphs converge to a minimal graph whose boundary is the union of the four vertical straight lines (AH), (BC), (ED) and (FG). (The convergence is a uniform convergence over each compact of the convex domain.)

If a minimal surface is bounded by a straight line, then we can complete it by adding its image by the symmetry about this line. We can apply this property to the surface we have obtained with each of the straight lines (AH), (BC), (ED) and (FG). We can repeat this process with this new surface and with the new vertical lines that bound it. If we repeat this infinitely many times, we obtain a minimal surface that is periodic in the directions x and y, called Scherk's surface. Moreover, this surface, except the vertical lines it contains, is a graph over the part of the xy-plane shown on figure 2.

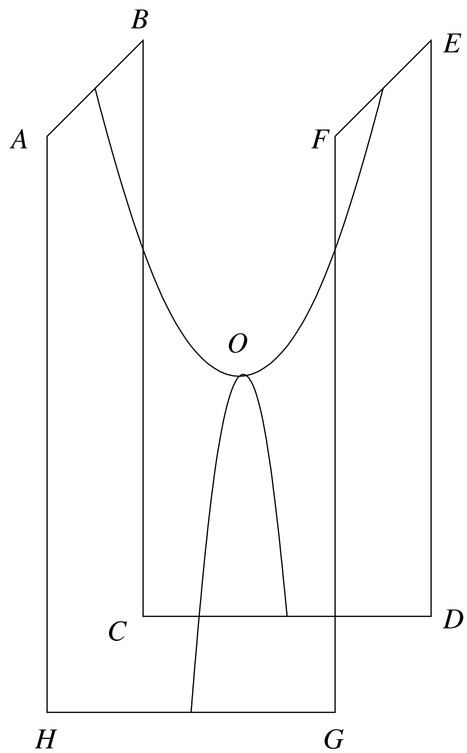

Figure 1: resolution of the Plateau problem for a polygonal contour.

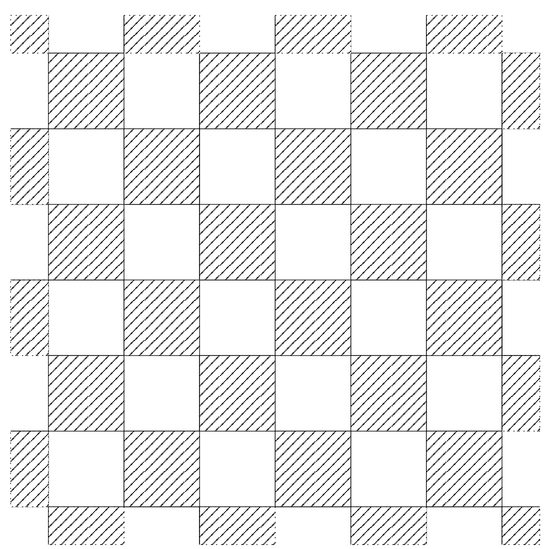

Figure 2: the projection of Scherk's surface on the xy-plane.

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