# ALGEBRAIC AND RATIONAL POINTS ON CUBIC SURFACES (AFTER D F CORAY)

## INTRODUCTION

We pose the following simple question.

**Let $\mathtt{q}(\mathtt{x_0},\dots,\mathtt{x_n})$ be a homogeneous form of degree $r$ with coefficients in a field $\mathtt{k}$. Let $\mathtt{K}$ be a finite extension of $\mathtt{k}$ of degree $d$ prime to $r$. Then, if $\mathtt{q}(\mathtt{x_0},\dots,\mathtt{x_n})$ has a non-trivial zero over $\mathtt{K}$, does it have one over $\mathtt{k}$ too?**

If we put $r = 3$, we get the following conjecture, which was apparently first propounded by *Cassels* and *Swinnerton-Dyer* (possibly with $n = 3$, but we omit that assumption here) and which is closely related to some problems mentioned by *B. Segre* in [Seg51]

**Cassels Swinnerton-Dyer Conjecture (CS):**

**Let $\mathtt{q}(\mathtt{x_0},\dots,\mathtt{x_n})$ be a homogeneous form of degree $r = 3$ with coefficients in a field $\mathtt{k}$. Then, if $\mathtt{q}(\mathtt{x_0},\dots,\mathtt{x_n})$ has a non-trivial zero over a finite extension $\mathtt{K}$ of degree $d$ prime to $r$, then so does it have over $\mathtt{k}$.**

*D. F. Coray* in his paper [Cor76] entitled *Algebraic Points on Cubic Hypersurfaces* has dealt with this conjecture in some special cases and under some additional hypotheses. In this brief article, we have chosen a part of that for discussion, which involves some geometrical arguments. We build up all the machineries gradually in 3 sections and in the $4th$ section we prove the main result. *Coray* also discusses **CS** when the field $\mathtt{k}$ is *local* and extends the result of **Theorem 4.1** over to *singular* surfaces, but we skip those topics here.

We want geometry to prevail in our discussions and so we reformulate the conjecture in the language of algebraic geometry as:

**Let $\mathtt{K}/\mathtt{k}$ be a finite extension of degree $d$ prime to $r$ and let $\mathtt{V} \subset \mathbb{P}^n_{\mathtt{k}}$ be a hypersurface of degree $r = 3$ defined over $\mathtt{k}$. Then $\mathtt{V}(\mathtt{K}) \neq \emptyset \Longrightarrow \mathtt{V}(\mathtt{k}) \neq \emptyset$.**

**§1. Some Background Material.** In this section we deal with some special cases and analogues of **CS** and also prove a few results, which will be used later on. We call an extension *simple* if it is generated by only one element.

When $r = 2$, a lower level analogue of **(CS)** is true.

**Theorem 1.1 (Springer):**

**Let $\mathtt{q}$ be a quadratic form with coefficients in $\mathtt{k}$, and let $\mathtt{K}/\mathtt{k}$ be a finite extension of odd degree (that is prime to 2). Then if $\mathtt{q}$ admits a non-trivial zero in $\mathtt{K}$, then so does it in $\mathtt{k}$.**

**Sketch of Proof:**

We present an elementary proof here. We know that every finite extension of odd degree can be split into a finite tower of simple extensions of odd degree. So, by induction, it is enough to prove the theorem for simple extensions of odd degree. Let us also assume that $\mathtt{q}(x_1,\dots,x_n)$ is in the diagonal form (assuming $char(\mathtt{k}) \neq 2$, of course). Now the proof is by induction on $[\mathtt{K} : \mathtt{k}] = d$, say. Let $\mathtt{K} = \mathtt{k}(\alpha)$ and let

$p(t)$ be the minimal polynomial of $\alpha$ over $\mathtt{k}$. Since $\mathtt{q}$ has a non-trivial zero over $\mathtt{K}$, we write

$$\mathtt{q}\big(g_0(t), \dots, g_n(t)\big) = p(t)h(t) \in \mathtt{k}[t]$$

where at least one $g_j \neq 0$ and let $m = max\ \{deg(g_j) : j = 0, \dots, n\} \leqslant d-1$. If there is a non-constant polynomial $f(t) \in \mathtt{k}[t]$ that divides all the $g_j{}'$s then we can pull that out and write it as

$$f^2(t)\mathtt{q}\big(g_0'(t), \dots, g_n'(t)\big) = p(t)h(t)$$

Now, $\mathtt{k}[t]$ is a PID and hence a UFD. $f^2(t)$ divides the LHS, so it should divide the RHS. Further, $p(t)$ irreducible over $\mathtt{k}$ and $deg(f) < deg(p)$ implies that $f^2(t) \mid h(t)$ over $\mathtt{k}[t]$. Removing the $f^2(t)$ factor from both sides we write the above equation as

$$\mathtt{q}\big(g_0'(t), \dots, g_n'(t)\big) = p(t)h'(t)$$

In this way we can remove all common non-constant polynomial factors. By abuse of notation, we write it once again as

$$\mathtt{q}\big(g_0(t), \dots, g_n(t)\big) = p(t)h(t) \quad \cdots\cdots\cdots\cdots \textit{(i)}$$

with the assumption that the $g_j{}'$s have no non-constant common factor in $\mathtt{k}[t]$. So the ideal generated by them in $\mathtt{k}[t]$ is the unit ideal and hence, they cannot have a common zero in any algebraic closure of $\mathtt{K}$. It is clear that $h(t)$ should be non-zero, using the fact that $\mathtt{q}$ is in the diagonal form, which we have already assumed. Now, the degree of $h(t) \leqslant 2m - d \leqslant 2(d-1) - d = d-2$, which is odd (since $d$ is). Now, if $h(t)$ were of even degree, $\mathtt{q}$ should be of odd degree. This would force some cancellation of higher degree terms in $\mathtt{q}(g_0(t), \dots, g_n(t))$ and so one could produce a non-trivial zero of $\mathtt{q}$ over $\mathtt{k}$ using the coefficients of the leading terms of the $g_i{}'$s, which get cancelled. So, assume that the degree of $h(t)$ is odd. Hence, $h(t)$ should have an irreducible factor of odd degree $\leqslant d-2$ as well, say $z(t)$. Since its degree is odd, it is non-constant. So reading equation $(i)$ modulo $z(t)$ we get a non-trivial zero of $\mathtt{q}$ over an extension of odd degree $\backsimeq \frac{\mathtt{k}[t]}{\big(z(t)\big)}$, its degree being $\leqslant d-2$. Hence, by the induction hypothesis we are through. ■

♦ We shall only deal with *projective varieties* $\mathtt{V} \subset \mathbb{P}^n_{\bar{\mathtt{k}}}$. It should be reminded that a *variety* is a *reduced scheme of finite type over* $\mathtt{k}$. Also, unless otherwise stated, a *form* is always assumed to be *homogeneous* and by a *point* we mean a *closed point.*

A 0-cycle $\mathtt{Z}$ is a linear combination of closed points with coefficients in $\mathbb{Z}$ on a $\mathtt{k}$-variety $\mathtt{V}$. Henceforth, by a cycle we mean a 0-cycle.

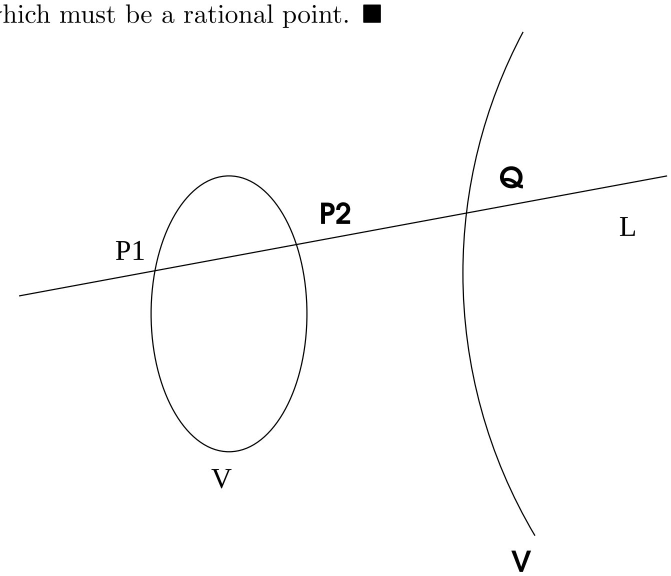
The ideas of the proofs of the results that follow in this section are borrowed from *Poincaré.* [Poi01]

**CS** is true when $r = 3, d = 2$.

**Lemma 1.2:** *If a cubic form* $\mathtt{q}$ *with coefficients in* $\mathtt{k}$ *has a non-trivial zero in a quadratic extension* $\mathtt{K}$*, then so does it have in* $\mathtt{k}$.

**Sketch of Proof:**

Let $\mathtt{P} \in \mathtt{V}$ be a non-trivial zero of $\mathtt{q}$ over $\mathtt{K}$. Assume that $\mathtt{K} = \mathtt{k}(\eta)$, where $\eta$ has degree 2 over $\mathtt{k}$. We assume that $\mathtt{P} = (1, a_1+b_1\eta, \ldots, a_n+b_n\eta)$, where at least one $b_i$ is non-zero (otherwise $\mathtt{P}$ is already a rational point). $L(t) = (1, a_1+b_1t, \ldots, a_n+b_nt)$ defines a parametrised $\mathtt{k}$-line which passes through $\mathtt{P}$ when $t = \eta$. So, if $L \subset \mathtt{V}$, we have lots of rational points anyway; otherwise, $\mathtt{q}(L(t))$ defines a non-zero polynomial in $t$ of degree 3. It has a solution $\mathtt{P}$ of degree 2. So there is a solution of degree 1, which must be a rational point. ■




**Lemma 1.3:** *Let $\mathtt{P}$ be a $\mathtt{K}$-point in $\mathbb{P}^n_{\mathtt{k}}$, where $[\mathtt{K} : \mathtt{k}] = d$; then the family of $\mathtt{k}$-divisors $\mathtt{F} \subset \mathbb{P}^n_{\mathtt{k}}$, of any given degree $l$, passing through the point $\mathtt{P}$, is determined by $d$ linear conditions (not necessarily independent) on the coefficients of $\mathtt{F}$.*

**Sketch of Proof:**

We assume that $\mathtt{P} = (1, a_1, \ldots, a_n)$ over $\mathtt{K}$. Then $\mathtt{P} \in \mathtt{F} \iff \mathtt{F}(1, \mathtt{a}_1, \ldots, \mathtt{a}_\mathtt{n}) = 0$. Let $\mathtt{W}$ be the vector space of all forms of degree $l$ over $\mathtt{k}$. Then, consider the $\mathtt{k}$-linear map,

$$\begin{aligned} \varphi : \mathtt{W} &\longrightarrow \mathtt{k}(\mathtt{P}) \\ \mathtt{F} &\longmapsto \mathtt{F}(1, \mathtt{a}_1, \ldots, \mathtt{a}_\mathtt{n}) \end{aligned}$$

The kernel of this map is precisely the linear space of forms passing through $\mathtt{P}$. Since $[\mathtt{k}(\mathtt{P}) : \mathtt{k}] = d$ the dimension of the kernel must be at least $dim\mathtt{W} - d$. ■

**Proposition 1.4:** *CS holds good for cubic curves* (i.e. when $n = 2$).

**Sketch of Proof:**

If $\mathtt{V}$ is $\mathtt{k}$-reducible then it has a linear factor and hence a rational point. So we may safely assume that $\mathtt{V}$ is irreducible. Let $\mathtt{P}$ be a point on the cubic curve $\mathtt{V}$ with coordinates in $\mathtt{K}$. Without loss of generality assume that $\mathtt{k}(\mathtt{P}) = \mathtt{K}$. Let the degree of $\mathtt{P}$ be $d = 3r + s$ ($s = 1$ or 2). Now, the dimension of the vector space of forms $\Gamma_l$ of degree $l$ in 3 variables is

$$\binom{l+(3-1)}{(3-1)} = \binom{l+2}{2}$$

We want them to pass through $\mathtt{P}$ and so by **Lemma 1.3** we need to put $d$ conditions. We are forced to further subtract $\binom{(l-3)+2}{2} = \binom{l-1}{2}$ dimensions as we do not want the $\Gamma_l{}'$s to contain $\mathtt{V}$ as a component. So we arrive at

$$\binom{l+2}{2} - d - \binom{l-1}{2}$$

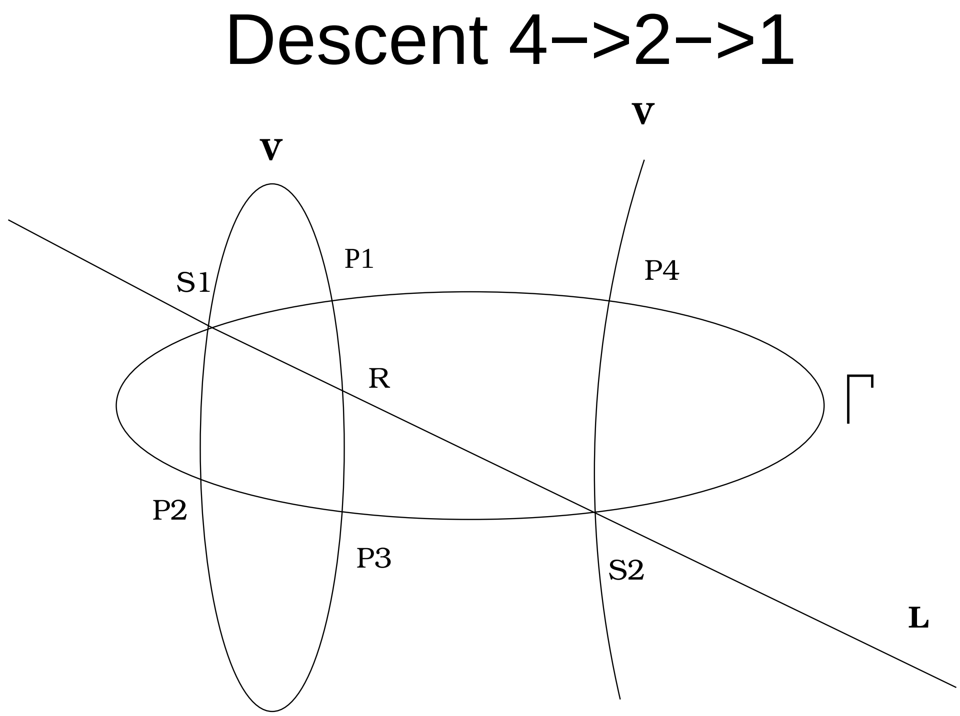
and when $l = r+1$, we get

$$\binom{(r+1)+2}{2} - (3r+s) - \binom{(r+1)-1}{2} = 3 - s > 0$$

So there exists a form and hence a $\mathtt{k}$-curve $\Gamma$ of degree $l$ which does not contain $\mathtt{V}$ as a component. Then

$$\Gamma \cdot \mathtt{V} = \sum_{\substack{\mathtt{P}\in\Gamma\cap\mathtt{V} \\ \mathtt{P}\text{ closed}}} len\left(\mathcal{O}_{\mathbb{P}^2,\mathtt{P}}\big/_{(f_\Gamma, f_{\mathtt{V}})}\right)\mathtt{P}$$

where $f_\Gamma$ and $f_{\mathtt{V}}$ are the local equations of $\Gamma$ and $\mathtt{V}$ respectively and $len\left(\mathcal{O}_{\mathbb{P}^2,\mathtt{P}}\big/_{(f_\Gamma, f_{\mathtt{V}})}\right)$ is its length as an $\mathcal{O}_{\mathbb{P}^2,\mathtt{P}}$-module, is a cycle defined over $\mathtt{k}$. The degree of $\Gamma \cdot \mathtt{V}$ is $deg_{\mathtt{k}}(\Gamma \cdot \mathtt{V}) = \sum len\left(\mathcal{O}_{\mathbb{P}^2,\mathtt{P}}\big/_{(f_\Gamma, f_{\mathtt{V}})}\right)[\mathtt{k}(\mathtt{P}) : \mathtt{k}] = deg(\Gamma).deg(\mathtt{V})$ (Bézout). Since $\Gamma$ has degree $l$ and $\mathtt{V}$ has degree 3, degree of $\Gamma \cdot \mathtt{V} = 3l = 3r+3$ and it contains $\mathtt{P}$ of degree $3r+s$. Then the residual $\mathtt{k}$-cycle has degree $3-s$, $s = 2$ or 1. In the former case we are through and in the latter we need to invoke **Lemma 1.2**. ■




We recall that a field k is called $C_1$ (or *quasi-algebraically closed*) if every form of degree $d$ in at least $d+1$ variables has a non-trivial zero in k. A well known theorem due to Chevalley-Warning says that every finite field is $C_1$. Hence we derive the following corollaries:

**Corollary 1.5:** *CS holds good for all $C_1$ fields, in particular for finite fields.*

**Sketch of Proof:**

**Proposition 1.4** tells us that CS is true if n $\leqslant 2$. When $n > 2$, there is nothing to prove. ■

Hence, the matter is settled over finite fields.

**Corollary 1.6:** *Let* $\mathtt{V} \subset \mathbb{P}^n_{\mathtt{k}}$ *be a cubic hypersurface defined over* k*. If* V *contains a* k*-divisor* D *of degree $d$ prime to 3, then* $\mathtt{V}(\mathtt{k}) \neq \emptyset$*.*

**Sketch of Proof:**

We can cut V by a generic k-plane $\Pi$ passing through D. Then $\mathtt{V} \cap \Pi$ is a cubic curve with a k-cycle D of degree $d$. Since $d$ is prime to 3 we can use **Proposition 1.4** to prove the result. ■

**§2. A Descent Argument.** In this section we shall confine our attention to the case of cubic surfaces (that is $n = 3$). With the following proposition we make inroads into the problem for the first time (with $n = 3$, of course).

**Proposition 2: (Cassels)** *Assume* **CS** *is true over* $\mathtt{k}$ *and its finite extensions for all degrees* $d < 3r+1$ $(3 \nmid d)$*. Then it is true for* $d_1 = 3r+1$ *if and only if it is so for* $d_2 = 3r+2$.

**Sketch of Proof:**

We assume that $\mathtt{P} = (1, \alpha, \beta, \delta) \in \mathtt{V}(\mathtt{K})$, with $[\mathtt{K} : \mathtt{k}] = d, (d = d_1$ or $d_2)$. At least one of $\alpha$, $\beta$ or $\delta$ is not in $\mathtt{k}$ and let us assume that $\alpha \notin \mathtt{k}$. Without loss of generality, we may assume that $\mathtt{K} = \mathtt{k}(\alpha)$. Otherwise, $\mathtt{k}(\alpha)$ would be an intermediate field extension and we could use induction hypothesis. Hence $\beta = a_0 + a_1\alpha + \cdots + a_{d-1}\alpha^{d-1}$ and $\delta = b_0 + b_1\alpha + \cdots + b_{d-1}\alpha^{d-1}$. This already shows that there is a $\mathtt{k}$-curve $\gamma(\theta) = (1, \theta, p(\theta), q(\theta))$ such that $p(\theta) = a_0 + a_1\theta + \cdots + a_{d-1}\theta^{d-1}$ and $q(\theta) = b_0 + b_1\theta + \cdots + b_{d-1}\theta^{d-1}$, of degree $\leqslant d-1$ that contains $\mathtt{P}$; but, with some foresight, we look for a curve of degree $m \leqslant 2r+1$.

In order to construct it, we multiply all the coordinate functions of the curve by a polynomial $A_0(\theta) = c_0 + c_1\theta + \cdots + c_{2r}\theta^{2r}$, whose coefficients will be determined later and call the new curve $\Gamma(\theta)$.

$$\Gamma(\theta) = \big(A_0(\theta), A_1(\theta), A_2(\theta), A_3(\theta)\big)$$

where,

$$A_1(\theta) = \theta A_0(\theta) = c_0 + c_1\theta + \cdots + c_{2r+1}\theta^{2r+1}$$

$$A_2(\theta) = p(\theta)A_0(\theta) = c'_0 + c'_1\theta + \cdots + c'_{d-1}\theta^{d-1}$$

$$A_3(\theta) = q(\theta)A_0(\theta) = c''_0 + c''_1\theta + \cdots + c''_{d-1}\theta^{d-1}.$$

This defines two polynomials $A_2(\theta)$ and $A_3(\theta)$, whose coefficients depend linearly and homogeneously on those of $A_0(\theta)$. We, therefore, need to determine the coefficients $c_i \ \forall \ i = 0 \ldots 2r$, in such a way that the degrees of $A_2(\theta)$ and $A_3(\theta)$ do not exceed $2r+1$. This amounts to solving (non-trivially) a linear system of $2\big((d-1) - (2r+1)\big) = 2d - 4r - 4 \leqslant 2r$ homogeneous equations in $2r+1$ variables and this we know can be done.

Thus we have found a $\mathtt{k}$-curve $\Gamma$ which is the locus of $\big(A_0(\theta), A_1(\theta), A_2(\theta), A_3(\theta)\big)$ of degree $m \leqslant 2r+1$, passing through $\mathtt{P}$. $\Gamma$ is a parametrised curve with coefficients in $\mathtt{k}$. So we can safely assume that $\Gamma \not\subset \mathtt{V}$ (otherwise we have lots of rational points). We have a map of degree $m$,

$$\begin{aligned} \mathbb{A}^1 &\longrightarrow \mathbb{P}^3 \\ \theta &\longmapsto \big(A_0(\theta), A_1, (\theta), A_2(\theta), A_3(\theta)\big) \end{aligned}$$

Let $F$ be the form of $\mathtt{V}$ of degree 3. Then $F(\theta) = F\big(A_0(\theta), A_1(\theta), A_2(\theta), A_3(\theta)\big)$ is a non-zero polynomial in $\theta$ of degree $3m = deg(\mathtt{V}).deg(\Gamma)$. $\mathtt{P}$ is a solution of $F(\theta)$ of degree $d$. Hence there is at least one solution of degree $\leqslant 3m - d$ over $\mathtt{k}$ and prime to 3 (as $3m - d$ is). That defines a closed point of degree $\leqslant 3m - d$ on $\mathtt{V}$ over $\mathtt{k}$. Suppose that the degree of the point is equal to $3m - d$ (otherwise we can use the induction hypothesis, which will be evident from the calculations to follow). If $d = d_2$ then $\delta = 3m - d_2 \leqslant 3(2r+1) - (3r+2) = d_1$; and if $d = d_1$ then

$$\delta = 3m - d_1 = \begin{cases} d_2 & \text{if } m = 2r+1, \\ 3m - (3r+1) < d_1 & \text{if } m \leqslant 2r. \end{cases}$$

Since $3 \nmid \delta$, we can also find a $\mathtt{k}$-cycle of degree $\delta' \leqslant \delta$ not divisible by 3, and the assertion once again follows from the induction hypothesis. ■

**§3. Curves and Divisors.** In this section we shall prove a crucial theorem and also estimate the maximum possible genus of a curve lying on a non-singular surface embedded in a projective space. We denote the *normalisation* of a curve $\Gamma$ by $\tilde{\Gamma}$ [Har77]. For the convenience of the reader we recall the definition of a *perfect* field. A field is said to be *perfect* if its algebraic extensions are all separable. By *absolutely irreducible* (respectively *integral*) we mean *geometrically irreducible* (respectively *integral*).

The theorem presented below will be used under various circumstances in the arguments later on. That is why it is crucial !!

**Theorem 3.1:**

**Let $\mathtt{k}$ be a perfect field. Let $\Gamma \subset \mathbb{P}^n_{\mathtt{k}}$ be an absolutely integral $\mathtt{k}$-curve of degree $m$ and let the genus of $\tilde{\Gamma}$ be $g = dim_{\mathtt{k}} H^1(\tilde{\Gamma}, \mathcal{O}_{\tilde{\Gamma}})$. Suppose that $\tilde{\Gamma}$ contains a divisor $\mathtt{E}$ of degree $e$ and let $\delta = gcd\big(m, e, 2(g-1)\big)$. Then $\Gamma$ also contains a effective $\mathtt{k}$-cycle of degree $\theta = j\delta, \forall j$ such that $j\delta = \theta \geqslant g$.**

**Sketch of Proof:**

Since $\mathtt{k}$ is perfect, the normalised curve $\tilde{\Gamma}$ is smooth. The normalisation map $p : \tilde{\Gamma} \longrightarrow \Gamma$ induces two maps at the level of zero-cycles as follows:

$$\begin{aligned} p^\star : Z_0(\Gamma) &\longrightarrow Z_0(\tilde{\Gamma}) \qquad \text{(pull back)} \\ Q &\longmapsto \sum_{P \in p^{-1}(Q)} v_P(t) \cdot P \end{aligned}$$

where $t \in \mathcal{O}_Q$ is the local parameter at $Q$ *i.e.* $t \in K(\Gamma)$ & $v_Q(t) = 1$.

$$\begin{aligned} p_\star : Z_0(\tilde{\Gamma}) &\longrightarrow Z_0(\Gamma) \qquad \text{(push forward)} \\ Q &\longmapsto [\mathtt{k}(Q) : \mathtt{k}(p(Q))] p(Q) \end{aligned}$$

We know that, $deg(p^\star(D)) = deg(p).deg(D)$. Degree of $p$ is $[K(\tilde{\Gamma}) : K(\Gamma)]$ and since $K(\tilde{\Gamma}) = K(\Gamma)$, $p$ is a map of degree 1. So $p^\star$ preserves the degree of cycles. It is clear from the definition of $p_\star$ that it also preserves the degree of cycles. We take an absolutely integral complete intersection of $\Gamma$ with a $\mathtt{k}$-hyperplane and pull it back to $\tilde{\Gamma}$ via the map $p^\star$ to obtain a $\mathtt{k}$-divisor $\mathtt{M}$ of degree $m$. It is known that we can choose a $\mathtt{k}$-divisor $\kappa$ in the canonical class and its degree is $2g-2$. Now, we write $\delta = pm + qe + r(2g-2)$ and set $\mathtt{D} = p\mathtt{M} + q\mathtt{E} + r\kappa$. Then $\mathtt{D}$ is a $\mathtt{k}$-divisor of degree $\delta$. Now by the *Riemann-Roch* on curves [Har77] [$\ell(\mathtt{D}) - \ell(\mathtt{K} - \mathtt{D}) = \mathtt{deg}(\mathtt{D}) + 1 - \mathtt{p_a}$] we get,

$$\ell(j\mathtt{D}) \geqslant j\delta + 1 - g$$

So whenever $j\delta = \theta \geqslant g$, $\ell(j\mathtt{D}) > 0$. Hence $j\mathtt{D}$ is linearly equivalent to an effective $\mathtt{k}$-divisor $\Theta$ of degree $\theta$ on $\tilde{\Gamma}$. This projects down onto an effective $\mathtt{k}$-cycle of same degree $\theta$ on $\Gamma$ via $p_\star$. ■

As a by-product we obtain the following beautiful result:

**Corollary 3.2:** *Every smooth projective curve of genus 0 and odd (respectively even) degree has a rational point (respectively a rational pair of points).*

**Estimate of genus.** In this part we find an estimate of the maximum possible value of the genera of curves on a cubic surface. This is needed because we shall be using **Theorem 3.1** quite often and there we need $\theta$ to be greater than the genus of the curve to claim the existence of positive cycles.

**Lemma 3.3:** *The arithmetic genus of an absolutely integral curve $\Gamma$ of degree $3l$, lying on a non-singular cubic surface $\mathtt{V}$ does not exceed*

$$\tfrac{3l(l-1)}{2} + 1.$$

**Sketch of Proof:**

The proof becomes neat if we make use of the following formula,

$$p_a(\Gamma) - dim_{\mathtt{k}} H^0(\Gamma, \mathcal{O}_\Gamma) = \tfrac{\Gamma\cdot(\Gamma+\mathtt{K})}{2}$$

where $\mathtt{K} = -\mathtt{H}$ is the canonical class of $\mathtt{V}$, $\mathtt{H}$ being a plane section. Since $\Gamma$ is an absolutely integral projective curve $dim_{\mathtt{k}} H^0(\Gamma, \mathcal{O}_\Gamma) = 1$. Now, let $\Gamma_0$ be an absolutely integral complete intersection with degree $3l$. Then ${\Gamma_0}^2 = (l\mathtt{H})^2 = 3l^2$. So we get,

$$p_a(\Gamma_0) - 1 = \tfrac{{\Gamma_0}^2 + \Gamma_0\cdot(-\mathtt{H})}{2} = \tfrac{3l(l-1)}{2}$$

Therefore, we obtain $p_a(\Gamma_0) = \frac{3l(l-1)}{2} + 1$. So it is enough to show that for a fixed degree $3l$, genus of an absolutely integral curve is maximal on complete intersections.

We recall the *Hodge Index Theorem* [Mum66] which says that if $\mathtt{H}$ is an ample divisor on a surface and $\mathtt{D}$ another divisor such that $\mathtt{D}\cdot\mathtt{H} = 0$, then $\mathtt{D}^2 \leqslant 0$. Now, $\mathtt{H}$ is ample on $\mathtt{V}$ and $(\Gamma - \Gamma_0)\cdot\mathtt{H} = 0$. So we have, $(\Gamma - \Gamma_0)^2 \leqslant 0$

$$\begin{aligned} \Rightarrow \Gamma^2 &\leqslant 2(\Gamma\cdot\Gamma_0) - {\Gamma_0}^2 \\ &= 2.l.3l - 3l^2 \quad [\Gamma_0 = l\mathtt{H}] \\ &= 3l^2 \\ &= {\Gamma_0}^2 \end{aligned}$$

This proves our assertion. ■

This result can also be extended to singular surfaces of degree $\mu$ ($\mu = 3$ here) with finitely many singular points if we use a result of *Max Noether* [Noe82] which says that for any given degree $m = \mu l$, the genus of curves of degree $\mu l$ is maximal on complete intersections. So, the first part of the proof, that is the calculation of the genus in case of complete intersections, goes through unchanged. For the second part we need to use the result of *Max Noether*.

Hence we obtain the following lemma (we shall not use it anywhere).

**Lemma 3.4:** *The geometric genus of an absolutely integral curve $\Gamma$ of degree $3l$, lying on a cubic surface $\mathtt{V}$, with only finitely many singular, points does not exceed* $\frac{3l(l-1)}{2}+1$.

**§4. The Descent on Non-singular Cubic Surfaces.** We are now ready to prove the main result. (**CS** with $n=3, \mathtt{k}$ perfect and $\mathtt{V}$ non-singular).

**Theorem 4.1:**
**Let $\mathtt{V} \subset \mathbb{P}^3_{\mathtt{k}}$ be a non-singular cubic surface defined over a perfect field $\mathtt{k}$ and containing a point $\mathtt{P}$ with co-ordinates in an algebraic extension $\mathtt{K}/\mathtt{k}$ of degree $d$ prime to 3. Then there is an extension $\mathtt{L}/\mathtt{k}$ with degree 1 (that is, in $\mathtt{k}$ itself), 4 or 10, such that $\mathtt{V}(\mathtt{L}) \neq \emptyset$.**

**Proof:**
In view of **Corollary 1.5** we may assume that $\mathtt{k}$ is infinite. The argument is by induction on the degree $d$ of the extension. If $d=1$, there is nothing to prove. So, we may assume that $d>1$ and also that $\mathtt{k}(\mathtt{P})=\mathtt{K}$ [otherwise $\mathtt{k} \subsetneqq \mathtt{k}(\mathtt{P}) \subsetneqq \mathtt{K}$ and each intermediate extension is of degree strictly less than $d$ and prime to 3. So we can use induction hypothesis]. Since $\mathtt{k}$ is perfect $\mathtt{P}$ has $d$ distinct conjugates. We shall try to pass a surface $\mathtt{F}_l$ of degree $l$ through $\mathtt{P}$. Let $\Gamma = \mathtt{F}_l \cap \mathtt{V}$. There is a unique integer $l$ such that

$$p_a(\Gamma) \leqslant \frac{3l(l-1)}{2}+1 \leqslant d < \frac{3l(l+1)}{2}+1 = \binom{l+3}{3} - \binom{(l-3)+3}{3}$$

The last term above indicates, in some sense, the "degree of freedom" that we have on the coefficients of $\mathtt{F}_l$ of degree $l$ not containing $\mathtt{V}$ as a component. We want $\mathtt{F}_l$'s to pass through $\mathtt{P}$ as well and so, this term better be strictly greater than $d$. Let $f=\frac{3l(l+1)}{2}-d$. It is the extra degree of freedom, that is, the number of extra conditions that we can further impose on the surfaces. By the above set of inequalities we get $f \geqslant 0$. But $3 \nmid d \Rightarrow 3 \nmid f$ and so we have $f \geqslant 1$. The idea of the proof is to apply **Theorem 3.1** to the curve $\Gamma$ cut out by $\mathtt{F}_l$.

I. We first consider the case, where $\Gamma$ is absolutely integral. The reducible case will be dealt with later on. The proof is broken up into the following cases:
   A. $d > \frac{3l(l-1)}{2}+1$

      There is no harm in assuming $\mathtt{P}$ to be non-singular on $\Gamma$. If $\mathtt{P}$ were a point with multiplicity greater than 1 then so would be its conjugates and hence the geometric genus of $\Gamma$ would drop at least by $d$, but $p_a(\Gamma) \leqslant \frac{3l(l-1)}{2}+1 < d$ in this case. This says that all its conjugates are also non-singular on $\Gamma$. So we have a Weil $\mathtt{k}$-divisor of degree $d$ defined on $\tilde{\Gamma}$ by $\mathtt{P}$.

      (i) $f \geqslant 3$ $(3 \nmid f \Rightarrow f \geqslant 4)$

With some foresight, we try to produce a $\mathtt{k}$-divisor of degree 3 on $\tilde{\Gamma}$. Let $\Sigma$ be the intersection of $\mathtt{V}$ with a $\mathtt{k}$-line, such that $Supp(\Sigma)$ is $\{Q_1, Q_2, Q_3\}$. We achieve our goal using the following claim.

**Claim:** *There exists an* $\mathtt{F}_l$*, whose intersection with* $\mathtt{V}$*, is a non-singular and absolutely integral* $\mathtt{k}$*-curve on* $\mathtt{V}$ *that passes through* $\mathtt{P}$ *and* $\Sigma$*.*

**A few words about the Claim:** We request readers to refer to a paper [KA79] by *Kleiman* and *Altman* which contains the following theorem:

*Let* $\mathtt{k}$ *be an infinite field. Let* $Z$ *be a subscheme of* $\mathbb{P}^n_{\mathtt{k}}$ *and* $X$ *a subscheme of* $Z$*. Let* $\mathcal{I}$ *denote the ideal in* $\mathcal{O}_{\mathbb{P}}$ *of the closure* $\bar{X}$ *of* $X$ *in* $\mathbb{P}$ *and fix a positive integer* $n$ *such that* $\mathcal{I}(n)$ *is generated by its global sections. Assume that* $Y = Z - \bar{X}$ *is smooth (respectively that* $Z$ *is absolutely integral). Then an intersection of any number of (respectively of* $codim((X, Z) - 1)$*) sections of* $Z$ *by independent general hypersurfaces of* $\mathbb{P}$ *with degree* $n+1$ *containing* $X$ *is smooth off* $\bar{X}$ *(respectively absolutely integral).*
Granting this result, we are almost through. Of course, we need to worry about small values of $l$, which we hope, can be resolved "manually" !!

Using the claim, we conclude that we have a $\mathtt{k}$-divisor defined by $\mathtt{P}$ of degree $d$ prime to 3 and another $\mathtt{k}$-divisor $\Sigma$ of degree 3 on $\tilde{\Gamma}$. So we have a $\mathtt{k}$-divisor of degree 1 on $\tilde{\Gamma}$ and since the genus of $\tilde{\Gamma}$ is at most $\frac{3l(l-1)}{2} + 1$, by **Theorem 3.1** we have an effective $\mathtt{k}$-cycle of degree $\sum\limits_{\mathtt{Q}} n_{\mathtt{Q}}[\mathtt{k}(\mathtt{Q}) : \mathtt{k}] = \frac{3l(l-1)}{2} + 1 < d$ on $\Gamma$ as well. Since $3 \nmid \frac{3l(l-1)}{2} + 1$ there is at least one $\mathtt{Q}$ such that $[\mathtt{k}(\mathtt{Q}) : \mathtt{k}]$ is prime to 3 and so corresponding to that there is an extension $\mathtt{K}'/\mathtt{k}$ of degree $d' < d$ which contains the closed point $\mathtt{Q}$ of degree $d'$ over $\mathtt{k}$.

(ii) $f < 3$

The argument is similar to the previous case, except that we do not need to introduce the additional set of 3 points. Set

$$\delta = gcd(3l, d, 2g - 2) \mid l$$

We choose $\theta = j\delta$ as in **Theorem 3.1** such that $\theta \in [g, g - 1 + 2\delta] \cap \mathbb{Z}$. There are $2\delta$ integers in this interval and two of them are divisible by $\delta$. Since $\delta$ and 3 are coprime, at least one of the two integers divisible by $\delta$ is not divisible by 3. We choose that multiple of $\delta$. We land up with a non-trivial zero in the corresponding field extension of $\mathtt{k}$, the degree of which is also strictly less than $d$ because,

$$(g-1) + 2\delta \leqslant (g-1) + 2l \leqslant \frac{3l(l-1)}{2} + 2l = \frac{3l(l+1)}{2} - l < d$$

provided $f < l$, which is true if $l \geqslant 3$ ($f < 3$ in this case).
If $l = 2$, $\delta$ can only be 1 or 2 and $d \geqslant 4$. If $d = 4$, we cannot do anything about it. So assume that $d \geqslant 5$. Now, $g \leqslant \frac{3.2.(2-1)}{2} + 1 = 4$. So with $\delta = 1$ or 2 we can always choose $\theta = j\delta$ as in **Theorem 3.1** to be 4 and hence an effective cycle of degree 4 on $\Gamma$.

B. $d = \frac{3l(l-1)}{2} + 1, f \geqslant 9$ ($3 \nmid f \Rightarrow f \geqslant 10$)

Since the geometric genus can be as large as $d$ in this case, we need a further trick. Once again we choose a $\mathtt{k}$-set of 3 points $\{Q_1, Q_2, Q_3\}$ and impose on $\mathtt{F}_l$ the further condition that it should meet $\mathtt{V}$ at $Q_1, Q_2, Q_3$ with multiplicity at least 2. This can be done provided the extra degree of freedom $f \geqslant 9$. [If $\mathfrak{m}_Q$ is the maximal ideal corresponding to a closed point $Q$ and if $h$ is the form defining $\mathtt{F}_l$, then we need, $h \in \mathfrak{m}_Q^2$. This represents 3 conditions as we shall have to adjust the terms $1, x$ and $y$ in $h$. There are 3 points $Q_1, Q_2, Q_3$. So we need 9 conditions altogether]. The descent fails in the cases of $d = 4$ and $d = 10$ precisely because of this assumption on $f$. [$f < 9 \Rightarrow l \leqslant 3$. $l = 2$ gives $d = 4, f = 5$ and $l = 3$ gives $d = 10, f = 8$].
If we carry out the above mentioned construction, the curve $\Gamma$ acquires 3 double points and so the geometric genus drops at least by 3. Thus we have

$$g \leqslant \frac{3l(l-1)}{2} + 1 - 3 = \frac{3l(l-1)}{2} - 2.$$

Now as $d = \frac{3l(l-1)}{2} + 1$ we can explicitly compute $\delta$,

$$\delta = gcd\left(3l, \frac{3l(l-1)}{2} + 1, 2g-2\right) = \begin{cases} 1 & \text{if l} \not\equiv 2\ (mod\ 4)\ , \\ 2 & \text{if l} \equiv 2\ (mod\ 4)\ . \end{cases}$$

The term $2g - 2$ does not play any role as it is always even. So we obtain a descent from $d$ to $d'$, where
$d' = \frac{3l(l-1)}{2} - 2$ or $\frac{3l(l-1)}{2} - 1$. $d'$ is clearly prime to 3.

That takes care of the proof when $\Gamma$ is absolutely integral. Unfortunately, this does not complete the proof as $\Gamma$ need not necessarily be irreducible.

II. Now suppose that the cycle defined by $\mathtt{P}$ is contained in a $\mathtt{k}$-irreducible component $C$ such that $C \times_{\mathtt{k}} \bar{k} = C_1 \cup \cdots \cup C_r$, with $C_j$ integral over $\bar{k}$, $j = 1, \ldots, r$. We have the following two subcases:

A. $\mathbf{r = 1}$

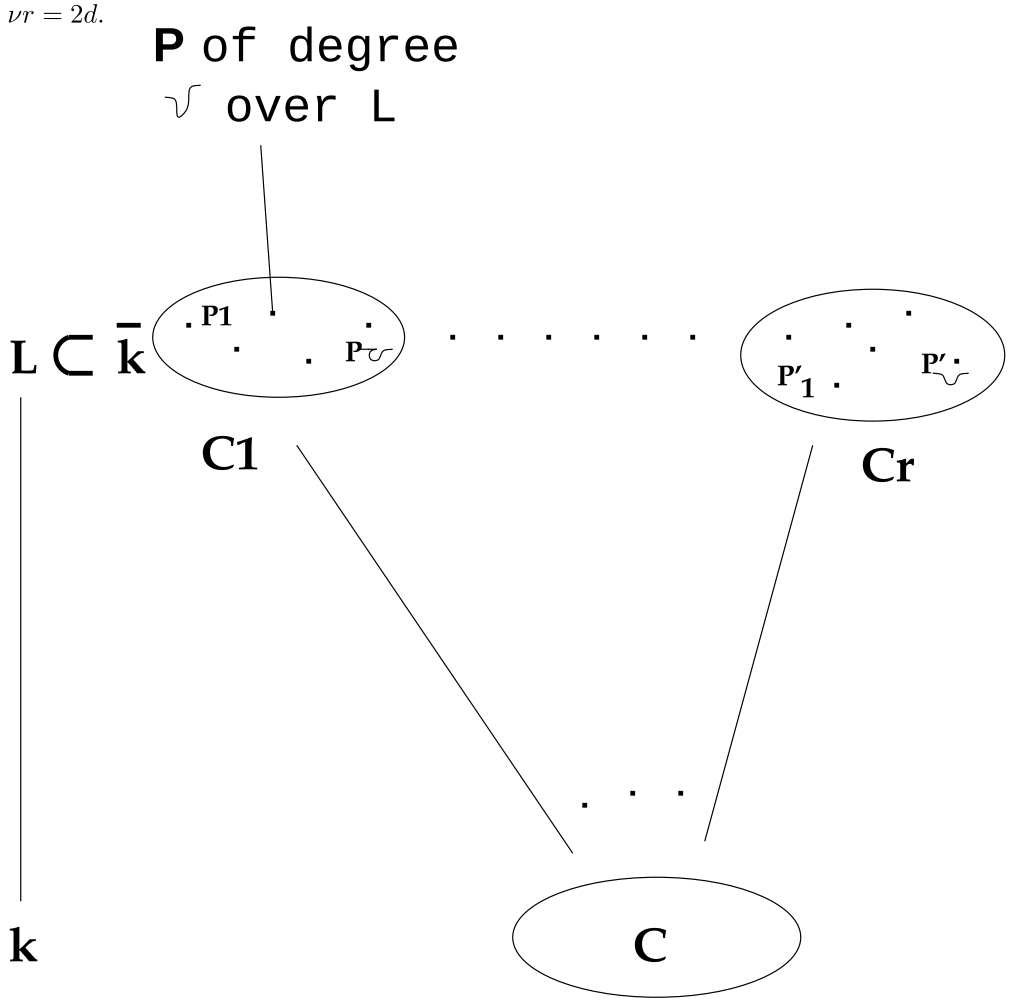
In view of **Corollary 1.6** we may assume that the degree $m$ of $C$ is a multiple of 3, $m = 3\lambda$ say, $\lambda < l$. Now the geometric genus $g$ of $C$ satisfies

$$g \leqslant \frac{3\lambda(\lambda-1)}{2} + 1 < \frac{3l(l-1)}{2} + 1 \leqslant d$$

and so we have reduced it to the case IA of the previous part of the proof ($C$ itself being absolutely integral now).

B. $r \geqslant 2$

It is clear that the action of the Galois group $\mathfrak{G}$ is transitive on the set $\{C_1, \ldots, C_r\}$. Hence each $C_j$ contains an equal number $\nu$ of conjugates of $\mathtt{P}$. Let us suppose that $\mathtt{P} \in C_1$; we conclude that $C_1$ is defined over a field $\mathtt{L}$, where $\mathtt{L}/\mathtt{k}$ is an extension of degree $r$ ($C \times_{\mathtt{k}} \bar{\mathtt{k}} = C_1 \cup \cdots \cup C_r$). If $C_1$ is the only component containing $\mathtt{P}$, then none of the conjugates of $\mathtt{P}$ belongs to more than one component of $C$ and so we have $\nu r = d$. Since $3 \nmid d$, both $\nu$ and $r$ are prime to 3; and since $\mathtt{P}$ has exactly $\nu$ conjugates on $C_1$, $\mathtt{P}$ defines a point of degree $\nu$ over $\mathtt{L}$. [The attached diagram can be more illustrative]. A similar argument goes through if $\mathtt{P}$ belongs to precisely two components of $C$, except that we now have $\nu r = 2d$.




We may assume that the degree $\mu$ of $C_1$ is a multiple of 3, say $\mu = 3\lambda$, in view of **Corollary 1.6**. $\Gamma$ is a curve of degree $3l$ and $C$ is a $\mathtt{k}$-irreducible component of $\Gamma$ which further breaks down as $C_1 \cup \cdots \cup C_r$

with each $C_j$ integral over $\bar{k}$, and the degree of $C_1$ being $\mu$. Since Galois automorphisms preserve dimension and degree of algebraic sets, all $C_j$'s have the same degree $\mu$.

$$3l = deg(\Gamma) \geqslant deg(C) = \sum_j deg(C_j) = r\mu = 3r\lambda.$$

We combine all these data to obtain the following set of equalities and inequalities.

$$[\mathtt{L} : \mathtt{k}] = r \leqslant r\mu \leqslant 3l \leqslant \frac{3l(l-1)}{2} < d.$$

except when $l = 2$, which is a trivial case. We have thus reduced the problem to the previous case, that is IIA. On $C_1$ we can use **Theorem 3.1** to find an $\mathtt{L}$-cycle of degree $\theta_1 = j\delta_1 \geqslant g_1$, where $\delta_1 = gcd(3\lambda, \nu, 2g_1 - 2) \mid \lambda$, $g_1$ being the geometric genus of $C_1$. This induces a $\mathtt{k}$-cycle of degree $\theta = r\theta_1$ on $C$. We need to check that $\theta < d$ and $3 \nmid \theta$. The argument now is in the same vein as I(A)ii. The chosen $j$ should be such that $\theta_1 = j\delta_1 \leqslant g_1 - 1 + 2\delta_1$. Thus, we get

$$\theta = r\theta_1 < r(g_1 - 1 + 2\delta_1) \leqslant r\left(\frac{3\lambda(\lambda-1)}{2} + 2\lambda\right) = \frac{r\lambda(3\lambda+1)}{2} \leqslant \frac{3l(l-1)}{2} < d$$

[The last inequality but one is easy to see when $\lambda \geqslant 2$; $\lambda = 1 \Rightarrow \mu = 3$ and so by **Theorem 3.1** $C_1$ has a rational point over $\mathtt{L}$]. Observe that, $\theta_1 \in [g_1, g_1 - 1 + 2\delta_1] \cap \mathbb{Z}$ and $[g_1, g_1 - 1 + 2\delta_1]$ contains 2 multiples of $\delta_1$. As $3 \nmid \delta_1$ at least one of them is not divisible by 3. We choose that value of $\theta_1$ and as $3 \nmid r$, $\theta = r\theta_1$ is also prime to 3.

The following claim finishes the proof.

**Claim:** *The point* $\mathtt{P}$ *cannot belong to more than 2 distinct components.*

**Proof of Claim:**

Assume the contrary. Suppose $\mathtt{P}$ belongs to $s > 2$ distinct components. Then each point accounts for at least $s - 1$ intersections and hence the total number of intersections, $\mathtt{i} = \sum_{j<k} C_j \cdot C_k \geqslant (s-1)d$

On the other hand we know that

$$\mathtt{i} = p_a(C) - \sum_{j=1}^{r} p_a(C_j) + (r-1).$$

[just formally replacing $p_a(C)$ by $\frac{C(C+K)}{2} + 1$ and using the fact that $C$ breaks up into $C_j$'s, we can derive the above expression]. Therefore, even if all the $p_a(C_j)$'s are zero, the maximum value that $\mathtt{i}$ can attain is

$$p_a(C) + (r-1) \leqslant \frac{3l(l-1)}{2} + 1 + (r-1) = \frac{3l(l-1)}{2} + r$$

Remember that we have assumed $s > 2$. So we get the following set of inequalities.

$$2d \leqslant (s-1)d \leqslant \mathtt{i} \leqslant \frac{3l(l-1)}{2} + r < \frac{3l(l-1)}{2} + d$$

So we find that $d < \frac{3l(l-1)}{2}$, which contradicts our hypothesis,

$$d \geqslant \tfrac{3l(l-1)}{2} + 1.$$

This finishes the proof of the claim and exhausts all possibilities in the proof of **Theorem 4.1**. Hence, we have proved **Theorem 4.1.** **Q.E.D.**

## CONCLUSION

The arguments here are clearly very tricky at some places and the magic does not quite take place when $r \geqslant 4$, with $n = 3$. There do exist counter-examples and for that interested readers are requested to refer to [Kol03]. Counter-examples also exist in the case $r < n$ [Sha72], a condition, under which an affirmative answer would have been less surprising. At this point, I should put forward my apologies for the sloppy language, not always conforming to the *Grothendieck-style* of algebraic geometry. Finally, it should be mentioned that someone has claimed that we can actually obtain a descent in the case $d = 10$ as well (in the proof of **Theorem 4.1** we got stuck at $d = 4$ or 10 !!)

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## References


[Cor76] D. F. Coray. *Algebraic points on cubic hypersurfaces.* Acta Arithmetica 30, 1976.

[Har77] R. Hartshorne. *Algebraic Geometry.* Springer-Verlag, 1977.

[KA79] S. L. Kleiman and A. B. Altman. *Bertini theorems for hypersurface sections containing a subscheme.* Communications in Algebra 7(8), 1979.

[Kol03] J. Kollár. *Specialization of zero-cycles.* math.AG/0205148 (on the arXiv server), 2003.

[Mum66] D. Mumford. *Lectures on Curves on an Algebraic Surface.* Annals of Mathematics Studies, 1966.

[Noe82] M. Noether. *Zur Grundlegung de Theorie der algebraischen Raumeurven.* Abh. Akad. Wissenschaften zu Berlin, 1882.

[Poi01] H. Poincaré. *Sur les propriétés arithmétiques des courbes algébriques.* J. Math. Pures. Appl. 7, 1901.

[Seg51] B. Segre. *On the rational solutions of homogeneous cubic equations in four variables.* J. of London. Math. Soc. 19, 1951.

[Sha72] S. S. Shatz. *Profinite Groups, Arithmetic and Geometry.* Annals of Mathematics Studies, 1972.


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