Université Paris-Saclay • M2 Analyse Modélisation Simulation

Introduction to Semiclassical Analysis (2018-2019, 1er semestre)

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L^∞ estimates for quasimodes

In this problem we will consider several semiclassical families $(u_\hbar)_{\hbar\to 0}$; *these families will all be uniformly bounded in* $L^2(\mathbb{R}^d)$: $\|u_\hbar\|_{L^2} = \mathcal{O}(1)$.

For some order function $m \geq 1$ on $\mathbb{R}^{2d}$, we consider a real symbol $p \in S(m)$, which we assume to be independent of $\hbar$. We call $P_\hbar = \mathrm{Op}_\hbar^W(p)$ the quantum Hamiltonian.

Let a family of states $(u_\hbar)_{\hbar\to 0}$ satisfy the following estimate when $\hbar \to 0$:

$$(1) \qquad \|(P_\hbar - E_\hbar)u_\hbar\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\hbar)\,, \quad \text{with} \;\; E_\hbar \to E_0,$$

for some energy $E_0 \in \mathbb{R}$. We will call such a family $(u_\hbar)$ an $\mathcal{O}(h)$-quasimode of $P_\hbar$. Our goal is to obtain L^∞ bounds for such quasimodes.

1. Given a cutoff function $\chi \in C_c^\infty(\mathbb{R}^{2d})$, show that if a state $(u_\hbar)$ is an $\mathcal{O}(\hbar)$ quasimode of $P_\hbar$, then the state $v_\hbar \stackrel{\text{def}}{=} \mathrm{Op}_\hbar^W(\chi)u_\hbar$ is also an $\mathcal{O}(\hbar)$-quasimode of $P_\hbar$.
   Hint : show that $[P_\hbar, \mathrm{Op}_\hbar^W(\chi)] = \mathcal{O}(\hbar)_{L^2\to L^2}$.

An L^∞ bound for momentum-localized functions

2. Show that a family $(w_\hbar)_{\hbar\to 0}$ uniformly bounded in L^2 is not necessarily bounded in L^∞.
3. We say that a family $(w_\hbar)_{\hbar\to 0}$ of L^2 states is *localized in a bounded momentum region* if there exists $\psi \in C_c^\infty(\mathbb{R}^d)$, such that

   $$(2) \qquad \forall k \geq 0, \qquad \|(I - \psi(\hbar D))w_\hbar\|_{H^k(\mathbb{R}^d)} = \mathcal{O}(\hbar^\infty)\,.$$

   (here H^k are the usual Sobolev spaces). Show that for some $C > 0$, the states $w_\hbar$ satisfy the L^∞ bound :

   $$(3) \qquad \|w_\hbar\|_{L^\infty(\mathbb{R}^d)} \leq C\,\hbar^{-d/2}\,, \quad \forall h \in (0,1].$$

   Hint : use the semiclassical Fourier transform.

4. We say that a family $(v_\hbar)_{\hbar\to 0}$ is microlocalized in a compact region $K \Subset \mathbb{R}^{2d}$ if there exists $\chi_K \in C_c^\infty(\mathbb{R}^{2d})$ with[1] $\mathrm{supp}\chi_K \Subset K$, such that

   $$(4) \qquad (I - \mathrm{Op}_\hbar^W(\chi_K))v_\hbar = \mathcal{O}(\hbar^\infty) \quad \text{in } \mathscr{S}(\mathbb{R}^d)\,.$$

   Show that if $\chi \in C_c^\infty(\mathbb{R}^{2d})$ with $\mathrm{supp}\chi \Subset K$, and if $(v_\hbar)$ is any bounded family in L^2, then $\tilde{v}_\hbar \stackrel{\text{def}}{=} \mathrm{Op}_\hbar^W(\chi)v_\hbar$ is microlocalized in K.
   Hint : construct χ_K such that $\mathrm{supp}\chi_K \Subset K$ while $\chi_K = 1$ near $\mathrm{supp}\chi$.

1. $A \Subset B$ means that A is contained in a compact set, itself contained in the interior of B.

5. Show that, if a family $(v_\hbar)_{\hbar\to 0}$ is microlocalized in a compact set K, then it is localized in a bounded momentum region, in the sense of (2). Therefore $(v_\hbar)$ satisfies the bound (3).

A sharper L^∞ bound for quasimodes

We consider a compact region $K \Subset \mathbb{R}^{2d}$, and we now assume that the symbol $p(x,\xi)$ satisfies

$$\nabla_\xi p(\rho) \neq 0, \qquad \forall \rho \in \mathcal{E}_{E_0} \stackrel{\text{def}}{=} p^{-1}(E_0) \cap K,$$

where E_0 is the energy appearing in (1)).

We assume that $(u_\hbar)$ is microlocalized in K, and is an $\mathcal{O}(\hbar)$-quasimode of $P_\hbar$ as in (1). Our goal is to show that, due to the second condition, $u_\hbar$ satisfied the improved L^∞ bound :

$$\|u_\hbar\|_{L^\infty(\mathbb{R}^d)} = \mathcal{O}(\hbar^{-(d-1)/2}). \tag{5}$$

Analyzing the neighbourhood of $\mathcal{E}_{E_0}$.

6. Which geometric property of the Hamiltonian vector field X_p does the assumption $\nabla_\xi p \neq 0$ imply ?
7. Show that $|\nabla_\xi p| \geq c_0$ on $\mathcal{E}_{E_0}$, for some $c_0 > 0$.
8. Show that for $\epsilon > 0$ small enough, the energy slab $\mathcal{E}_{E_0\pm\epsilon} \stackrel{\text{def}}{=} p^{-1}([E_0-\epsilon, E_0+\epsilon]) \cap K$ is a neighbourhood of $\mathcal{E}_{E_0}$ of radius $\mathcal{O}(\epsilon)$ (that is, any point in this neighbourhood is at distance $\mathcal{O}(\epsilon)$ from $\mathcal{E}_{E_0}$).
Hint : Use the Taylor expansion of p based on points $\rho_0 \in \mathcal{E}_{E_0}$ to solve $p(\rho) = E_0+\epsilon$.
9. Deduce that for $\epsilon > 0$ small enough, $|\nabla_\xi p| \geq c_0/2$ in $\mathcal{E}_{E_0\pm\epsilon}$.
10. Show that the cutoff χ_K may be split into $\chi_K = \chi_{in} + \chi_{out}$, where $\chi_{in},\ \chi_{out} \in C_c^\infty(\mathbb{R}^{2d})$ and

$$\text{supp}\chi_{in} \Subset \mathcal{E}_{E_0\pm\epsilon}, \qquad \text{supp}\chi_{out} \Subset K \setminus \mathcal{E}_{E_0\pm\epsilon/2}\,.$$

Explain why, to prove the bound (5), it suffices to prove it for the states $u_{in} = \text{Op}_\hbar^W(\chi_{in})u_\hbar$ and $u_{out} = \text{Op}_\hbar^W(\chi_{out})u_\hbar$ (we will omit to indicate the $\hbar$-dependence of those states).

Analyzing the component microlocalized away from $\mathcal{E}_{E_0}$.

11. We start to deal with u_{out}. Consider a cutoff $\tilde{\chi}_{out} \in C_c^\infty(K \setminus \mathcal{E}_{E_0\pm\epsilon/2})$, such that $\tilde{\chi}_{out} = 1$ near $\text{supp}\chi_{out}$. Justify that for $\hbar$ small enough, the symbol $\tilde{q}(\hbar) \stackrel{\text{def}}{=} \frac{\tilde{\chi}_{out}}{p-E_\hbar}$ is well-defined and smooth (here $E_\hbar$ are the energies appearing in (1)). Using the quantization of $\tilde{q}(\hbar)$ and the quasimode property (1), show that $\|u_{out}\|_{L^2} = \mathcal{O}(\hbar)$.
12. Check that u_{out} satisfies the conditions (2), and deduce that $\|u_{out}\|_{L^\infty} = \mathcal{O}(\hbar^{-(d-2)/2})$.

Analyzing the component microlocalized near $\mathcal{E}_{E_0}$.

13. We now consider the state u_{in}, microlocalized in $\mathcal{E}_{E_0\pm\epsilon}$. We remind that $|\nabla_\xi p| \geq c_0/2$ in this region. Hence at each point $\rho \in \mathcal{E}_{E_0\pm\epsilon}$, at least one component of the vector $\nabla_\xi p(\rho)$ does not vanish.
Show that $\mathcal{E}_{E_0\pm\epsilon}$ can be covered by a finite union of open sets [2] $(V_j)_{j=1,\dots,J}$, such that in each V_j there is some index $k = k(j)$ for which $|\partial_{\xi_k} p(\rho)| \geq c_1$ across V_j, for some uniform constant $c_1 > 0$.
14. Show that one can construct cutoffs $\chi_j \in C_c^\infty(V_j)$, such that $\chi_{in} = \sum_{j=1}^J \chi_j$. Explain why, in order to prove (5), it suffices to prove it for each state $v_j = \mathrm{Op}_\hbar^W(\chi_j) u_\hbar$, $j = 1, \dots, J$.

    From now on we fix $j \in \{1, \dots, J\}$, and study v_j. To alleviate the notations, we assume that $k(j) = 1$.
15. Explain why $\partial_{\xi_1} p$ keeps the same sign throughout V_j. Up to a change of direction of the coordinates, we may assume that $\partial_{\xi_1} p > c_1$ in V_j.

Setting up a $(d-1)$-dimensional Schrödinger equation

16. We decompose the coordinates into $x = (x_1, x')$, $\xi = (\xi_1, \xi')$. Show that if one fixes the coordinates x and ξ' so that (x, ξ') belongs to the projection of V_j along the ξ_1-axis, then equation $p(x, \xi_1, \xi') = E_0$ admits a single solution ξ_1, which depends in a smooth way of (x, ξ') ; we will write $\xi_1 = a_j(x, \xi')$. Show that the quotient $q_j(x,\xi) \stackrel{\mathrm{def}}{=} \frac{p(x,\xi)-E_0}{\xi_1 - a_j(x,\xi')}$ is smooth and positive.
Hint : I recommend to make a picture of the situation.
17. One may extend $a_j(x, \xi')$ from V_j into a function in $C_b^\infty(\mathbb{R}^{2d-1}, \mathbb{R})$, and $q_j(x, \xi)$ into an elliptic symbol in $S(1)$. Show that
$$P_\hbar \,\mathrm{Op}_\hbar^W(\xi_j) = \mathrm{Op}_\hbar^W(q_j)\,\mathrm{Op}_\hbar^W(\xi_1 - a_j)\mathrm{Op}_\hbar^W(\chi_j) + \hbar R_j,$$
for some $R_j \in \Psi_\hbar(1)$.
18. Using the ellipticity of q_j and the quasimode assumption, deduce that
$$\mathrm{Op}_\hbar^W(\xi_1 - a_j) u_j(x) = f_j(x), \quad \text{with} \quad \|f_j\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\hbar). \tag{6}$$
19. Using the explicit expression of $\mathrm{Op}_\hbar^W(\xi_1)$, show that the above equation can be interpreted as a time-dependent, inhomogeneous Schrödinger equation on $\mathbb{R}^{d-1} \ni x'$, where the classical Hamiltonian $a_j(x_1, x', \xi')$ and the inhomogeneous term $f_j(x_1, x')$ both depend on the « time » parameter x_1. $u_j(x_1, x')$ is hence viewed as a « time »-dependent function in $L^2(\mathbb{R}^{d-1})$, solution of this Schrödinger equation.
Hint : check that the operator $\mathrm{Op}_\hbar^W(a_j)$ can be interpreted as an x_1-dependent operator $\mathrm{Op}_\hbar^W(a_j(x_1, \bullet))$ acting on $L^2(\mathbb{R}^{d-1})$.
20. Show that the « time »-dependent norm $\|u_j(x_1)\|_{L^2(\mathbb{R}^{d-1})}$ satisfies, for any pair $x_1 \geq y_1$:
$$\|u_j(x_1)\|_{L^2(\mathbb{R}^{d-1})} \leq \|u_j(y_1)\|_{L^2(\mathbb{R}^{d-1})} + \frac{1}{\hbar}\int_{y_1}^{x_1} \|f_j(y_1)\|_{L^2(\mathbb{R}^{d-1})}\, dy_1\,.$$
Obtain a similar bound if $x_1 \leq y_1$.
Hint : compute the derivative of $\|u_j(x_1)\|^2_{L^2(\mathbb{R}^{d-1})}$ with respect to x_1.

2. That is, $\mathcal{E}_{E_0\pm\epsilon} \subset \bigcup_{j=1}^J V_j$.

21. Show that the above inequality implies, for any x_1, y_1 :

$$\|u_j(x_1)\|_{L^2(\mathbb{R}^{d-1})} \leq \|u_j(y_1)\|_{L^2(\mathbb{R}^{d-1})} + \frac{\sqrt{|x_1 - y_1|}}{\hbar} \|f_j\|_{L^2(\mathbb{R}^d)} .$$

Using the estimate (6) for f_j and the uniform boundedness of $\|u_j\|_{L^2(\mathbb{R}^d)}$, show that for I any bounded interval, we have for some $C_I > 0$:

$$\|u_j(x_1)\|_{L^2(\mathbb{R}^{d-1})} \leq C_I, \qquad \forall x_1 \in I.$$

We will assume that I contains the projection of V_j along (x, ξ').

22. For $x_1 \in I$, we want to apply the bound (3) to the $d-1$-dimensional state $u_j(x_1)$. To do this, check that $u_j(x_1)$ is localized in a compact ξ'-momentum region, in the sense of (2), uniformly w.r.t. x_1.
23. Deduce that $\|u_j(x_1)\|_{L^\infty(\mathbb{R}^{d-1})} = \mathcal{O}(\hbar^{-(d-1)/2})$, uniformly w.r.t. $x_1 \in I$. Show that this bound actually holds for all $x_1 \in \mathbb{R}$. Deduce finally that $\|u_\hbar\|_{L^\infty} = \mathcal{O}(\hbar^{-(d-1)/2})$.

Examples of states saturating the bounds

24. We start by an example for the bound (3). Take some function $\varphi \in C^\infty(\mathbb{R}^d)$, and define the state

$$v_\hbar(x) \stackrel{\text{def}}{=} (\mathcal{F}_\hbar \varphi)(x), \quad \hbar \in (0, 1] \tag{7}$$

Show that this state is localized in a bounded momentum region in the sense of (2).

25. Compute $v_\hbar(x)$ explicitly, and check that the bound (3) is sharp for this state.
26. Give a (nontrivial) example of a symbol $p(x, \xi)$ such that $\partial_\xi p = 0$ everywhere. What is the quantization $P_\hbar = \mathrm{Op}_\hbar^W(p)$?
27. Show that the state $v_\hbar$ is an $\mathcal{O}(\hbar)$-quasimode for this Hamiltonian $P_\hbar$.
28. We now turn to the simplest symbol satisfying $\partial_\xi p \neq 0$, namely $p_0(x, \xi) = \xi_1$. Decomposing the coordinate $x = (x_1, x')$, consider two functions $\varphi \in C^\infty(\mathbb{R}^d)$ and $\psi \in C_c^\infty(\mathbb{R})$, and define

$$u_\hbar(x) \stackrel{\text{def}}{=} \psi(x_1)\, (\mathcal{F}_\hbar \varphi)(x'),$$

where $\mathcal{F}$ is now the semiclassical Fourier transform on $\mathbb{R}^{d-1}$.
Check that $(u_\hbar)$ is an $\mathcal{O}(\hbar)$-quasimode of $P_{0,\hbar} = \mathrm{Op}_\hbar^W(p_0)$.

29. Show that $(u_\hbar)$ is microlocalized in some compact region K in $\mathbb{R}^{2d}$.
30. Compute $\|u_\hbar(x)\|_{L^\infty}$ explicitly, and check that this state saturates the bound (5).