Université Paris-Saclay • M2 Analyse Modélisation Simulation

Introduction to spectral theory (2016-2017, 1er semestre)

Stéphane Nonnenmacher

Examen intermédiaire (26 octobre 2016)

Exponential mixing of the angle-doubling map

Our goal is to analyse the long time dynamics of a simple map (or *transformation*) T defined on the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\begin{aligned} T :& \mathbb{T} \to \mathbb{T} \\ & x \mapsto T(x) = 2x \bmod \mathbb{Z}. \end{aligned}$$

This transformation T is often called the angle doubling map. Because $|T'(x)| > 1$ at each point $x \in \mathbb{T}$, this map is said to be *uniformly dilating*. For this reason, it is *strongly chaotic*. One aspect of this chaos is the *mixing property*, which will be examined below. The analysis of the long time iterates of T, namely the maps $T^n = T \circ \cdots \circ T$, $n \to \infty$, will lead us to study an associated *transfer operator*, on various functional spaces.

1. Show that the map T is C^∞ on $\mathbb{T}$. Here and below, the torus $\mathbb{T}$ can be identified with the unit interval $I = [0, 1)$, with periodic boundary conditions. Draw the graph of T using this identification
2. The long time behaviour of the map T is studied through the *correlation function* between two *test functions* $f, g : \mathbb{T} \to \mathbb{R}$: the correlation at *time* $n > 0$ between the two functions is defined as

$$C_{f,g}(n) \stackrel{\text{def}}{=} \int f(x)\, g \circ T^n(x)\, dx - \int f(x)dx \int g(x)dx\,, \tag{1}$$

all integrals being taken on $\mathbb{T} \simeq [0, 1)$. The *mixing property* of the dynamics T is equivalent with the fact that $C_{f,g}(n) \to 0$ when $n \to \infty$. Our goal is to investigate more precisely this mixing.

(a) For $f, g \in L^2(\mathbb{T})$, express the correlation function $C_{f,g}(t)$ in terms of scalar products involving f, g, the n-th iterate of the pull-back operator $A_T(f) \stackrel{\text{def}}{=} f \circ T$, and the constant function $e_0(x) = 1$.

(b) Recall the formula for the spectral radius of a bounded operator. Since we are interested in scalar products involving iterates A_T^n for n large, justify why it makes sense to study the spectrum of A_T on $L^2(\mathbb{T})$.

(c) Show that $A_T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is bounded.

(d) Show that this operator is an isometry. Compute its norm $\|A_T\|_{\mathcal{L}(L^2)}$. What can be deduced about the spectrum of A_T on L^2 ?

(e) We will see later that it will be more convenient to express the correlations $C_{f,g}(n)$ in terms of the adjoint operator $\mathcal{L}_T = A_T^*$. $\mathcal{L}_T$ is often called the **transfer operator** associated with the map T. Write down $C_{f,g}(n)$ in terms of the n-th power of $\mathcal{L}_T$.

(f) Write down explicitly the action of $\mathcal{L}_T$ on a function $f \in C^0(\mathbb{T})$.

3. To study the spectrum of A_T and $\mathcal{L}_T$ more closely, we will use the Fourier decomposition of $L^2(\mathbb{T})$. We define the **Fourier modes** on $\mathbb{T}$ by

$$e_k(x) = \exp(2i\pi kx), \quad k \in \mathbb{Z}, \quad x \in \mathbb{T}.$$

(a) Check that each e_k belongs to $C^\infty(\mathbb{T})$.

(b) Show that the modes $(e_k)_{k\in\mathbb{Z}}$ form an orthonormal family in $L^2(\mathbb{T})$. Deduce that the Fourier series of a function $f \in L^2(\mathbb{T})$,

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k\, e_k(x)\,, \quad \hat{f}_k = \langle e_k, f\rangle_{L^2}\,,$$

induces a unitary map between $f \in L^2(\mathbb{T})$ and $(\hat{f}_k) \in \ell^2(\mathbb{Z})$.

(c) Compute the action of A_T on the Fourier modes e_k.

(d) Deduce the action of $\mathcal{L}_T$ on e_k. *Hint :* distinguish two cases, according to the parity of k. Is $\mathcal{L}_T$ an isometry on $L^2(\mathbb{T})$?

4. For each k a positive *odd integer*, denote by $\mathcal{V}_k$ the subspace of L^2 generated by the Fourier modes $\{e_{2^j k},\ j \in \mathbb{N}\}$, and similarly for $\mathcal{V}_{-k}$. We will call $\overline{\mathcal{V}_{\pm k}}$ the L^2-closures of these spaces.

(a) Show that A_T and $\mathcal{L}_T$ leave each subspace $\overline{\mathcal{V}_k}$ invariant. We will denote $\mathcal{L}_{T,k} \stackrel{\text{def}}{=} \mathcal{L}_T \restriction_{\overline{\mathcal{V}_k}}$.

(b) For k a given odd integer, show that $\mathcal{L}_{T,k}$ is unitarily equivalent with the shift operator S on $\ell^2(\mathbb{N})$: $S(\psi_j) = (\psi_{j+1})$, for any $(\psi_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

(c) deduce that $\mathcal{L}_{T,k}$ admits as point spectrum the open unit disk $D_1 = \{z \in \mathbb{C},\ |z| < 1\}$, and as spectrum the closed disk $\bar{D}_1 = \{z \in \mathbb{C},\ |z| \le 1\}$. What is the spectrum of $A_{T,k} \stackrel{\text{def}}{=} A_T \restriction_{\overline{\mathcal{V}_k}}$?

(d) For each $z \in D$, show that $\ker(\mathcal{L}_{T,k} - z)$ is 1-dimensional, and describe its eigenfunction $\phi_{k,z}(x)$ through its Fourier series. What is the dimension of $\ker(A_{T,k} - z)$?

5. Let us call $\mathcal{V}_0 = \mathbb{C}e_0$ the 1-dimensional space generated by the constant function e_0. Check that $\mathcal{V}_0$ is an eigenspace of both A_T and $\mathcal{L}_T$.

6. Show that $L^2(\mathbb{T})$ can be written as the closure of the following orthogonal direct sum :

$$L^2(\mathbb{T}) = \overline{\Big(\mathcal{V}_0 \oplus \bigoplus_{k>0 \text{ odd}} (\overline{\mathcal{V}_k} \oplus \overline{\mathcal{V}_{-k}})\Big)} \tag{2}$$

(a) What are the spectrum and the point spectrum of $\mathcal{L}_T$ on L^2 ? What are the discrete and essential spectra of $\mathcal{L}_T$?

(b) Same questions for $A_T : L^2 \to L^2$.

(c) show that for any $\rho \in (0,1)$, there exists $f, g \in L^2$ such that $|C_{f,g}(n)| \geq \rho^n \|f\|_{L^2} \|g\|_{L^2}$. *Hint* : use the eigenfunctions $\phi_{z,k}$.

As a result, for L^2 test functions there is no *uniform* exponential decay of the correlations.

7. In order to improve the decay of the correlations we will next assume that the test functions f, g are more regular than L^2, namely, we will assume f, g are in some Sobolev space $H^p(\mathbb{T})$ for some $p \in \mathbb{N}^*$. We will use the following norm on $H^p(\mathbb{T})$:

$$\|f\|_{H^p}^2 \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} (1+k^2)^p \, |\hat{f}_k|^2 \tag{3}$$

(a) In the case $p = 1$, check that the norm $\| \bullet \|_{H^1}$ on H^1 is equivalent with the norm $\|f\|_1 = \|f\|_{L^2} + \|f'\|_{L^2}$.

(b) Using the explicit expressions of A_T and $\mathcal{L}_T$, show that for any $f \in H^1(\mathbb{T})$, $\|(A_T f)'\|_{L^2} = 2\|f'\|_{L^2}$ and $\|(\mathcal{L}_T f)'\|_{L^2} \leq \frac{1}{2}\|f'\|_{L^2}$.
Hint : for the second inequality, you may use the identity $(a+b)^2 \leq 2(a^2+b^2)$.

(c) Deduce that A_T and $\mathcal{L}_T$ are bounded operators $H^1(\mathbb{T}) \to H^1(\mathbb{T})$. Using the norm $\| \bullet \|_1$ on H^1, show that $\|A_T\|_{\mathcal{L}(H^1)} \leq 2$ and $\|\mathcal{L}_T\|_{\mathcal{L}(H^1)} \leq 1$. Deduce bounds on the spectra of A_T and $\mathcal{L}_T$ on $H^1(\mathbb{T})$. Are A_T and $\mathcal{L}_T$, viewed as operators $H^1 \to H^1$, adjoint to each other ?

8. We will also consider the subspace of $H^p(\mathbb{T})$ made of functions orthogonal to the constant function, $\dot{H}^p(\mathbb{T}) \stackrel{\text{def}}{=} H^p(\mathbb{T}) \cap \{e_0\}^\perp$.

(a) explain why $\dot{H}^p(\mathbb{T})$ is a closed subspace of $H^p(\mathbb{T})$, and thus a Hilbert space, equipped with the norm (3).

(b) Check that on $\dot{H}^p$, the norm (3) is equivalent with the *homogeneous norm*

$$\|f\|_{\dot{H}^p}^2 \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}\setminus 0} k^{2p} \, |\hat{f}_k|^2 \, . \tag{4}$$

9. We will again use the decomposition (2) to analyze the spectrum of the operator $\mathcal{L}_T$ on H^p, which we will denote by $\mathcal{L}_T^{(p)}$.

(a) Show that for any $p \geq 1$, 1 belongs to the point spectrum of $\mathcal{L}_T^{(p)}$.

(b) For k an odd integer and $z \in D_1$, recall the eigenfunction $\phi_{z,k} \in L^2$ of $\mathcal{L}_{T,k}$ described above. Show that $\phi_{z,k}$ belongs to $H^p(\mathbb{T})$ if and only if $|z| < 2^{-p}$.

(c) Deduce that the spectrum of $\mathcal{L}_T^{(p)}$ contains the disk $\bar{D}_{2^{-p}}$.

(d) show that $\mathcal{L}_T^{(p)}$ can be decomposed as the sum of two operators acting on each subspace of the orthogonal decomposition $H^p = \mathcal{V}_0 \oplus \mathcal{V}_0^\perp$, with $\mathcal{V}_0^\perp = \dot{H}^p$:

$$\mathcal{L}_T^{(p)} = \Pi_{\mathcal{V}_0} \oplus \dot{\mathcal{L}}_T^{(p)} \Pi_{\mathcal{V}_0^\perp} \,. \tag{5}$$

Here $\Pi_{\mathcal{V}_0}$ (resp. $\Pi_{\mathcal{V}_0^\perp}$ is the orthogonal projector on $\mathcal{V}_0$ (resp. on $\mathcal{V}_0^\perp$), while $\dot{\mathcal{L}}_T^{(p)} \stackrel{\text{def}}{=} \mathcal{L}_T^{(p)} \restriction_{\dot{H}^p}$.

(e) Using the norm (4), show that

$$\forall f \in \dot{H}^p, \qquad \|\dot{\mathcal{L}}_T^{(p)} f\|_{\dot{H}^p} \leq 2^{-p} \, \|f\|_{\dot{H}^p} \,.$$

(f) Deduce that the spectrum of $\dot{\mathcal{L}}_T^{(p)}$ is equal to the disk $\bar{D}_{2^{-p}}$.

(g) What are the discrete spectrum, resp. the essential spectrum, of the operator $\mathcal{L}_T^{(p)}$? Such an operator is said to be *quasicompact*.

10. We can now go back to the analysis of the correlations $C_{f,g}(n)$, for $f, g \in H^p(\mathbb{T})$.

(a) For any $n \geq 1$, write down a decomposition of $(\mathcal{L}_T^{(p)})^n$ similar with (5), and insert it in the expression for $C_{f,g}(n)$.

(b) Using the above the spectral analysis of $\mathcal{L}_T^{(p)}$, prove that $C_{f,g}(n)$ decays exponentially as $n \to \infty$, uniformly for $f, g \in H^p(\mathbb{T})$: in other words, show that there is a uniform decay rate $\gamma_p > 0$ and $C > 0$ such that,

$$\forall f, g \in H^p, \ \forall n \geq 0, \quad |C_{f,g}(n)| \leq C \, e^{-n\gamma_p} \, \|f\|_{H^p} \, \|g\|_{H^p} \,.$$

This bound means that the dynamical system T is exponentially mixing. Such a mixing property is typical of very chaotic systems.

(c) Show that for two test functions $f, g \in C^\infty(\mathbb{T})$, the correlation $C_{f,g}(n)$ decays faster than any exponential. One says that the decay of the correlations is *superexponential*.