SPECTRAL THEORY OF DAMPED QUANTUM CHAOTIC SYSTEMS

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ABSTRACT. We investigate the spectral distribution of the damped wave equation on a compact Riemannian manifold, especially in the case of a metric of negative curvature, for which the geodesic flow is Anosov (very chaotic). The final objective is to obtain conditions (in terms of the geodesic flow on X, the structure of the damping function) for which the energy of the waves decays exponentially fast, at least for smooth enough initial data. The spectrum of the equation amounts to a nonselfadjoint spectral problem.

Using semiclassical methods, we derive estimates and upper bounds for the high frequency spectral distribution, in terms of dynamically defined quantities, like the value distribution of the time-averaged damping. We also consider the toy model of damped quantized chaotic maps, for which we derive similar estimates, as well as a new upper bound for the spectral radius depending on the set of minimally damped trajectories.

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1. Introduction

1.1. **Spectrum of the damped wave equation.** Given a given a Riemannian manifold (X, g) and a *damping function* $a \in C^\infty(X, \mathbb{R}_+)$, we are interested in the solutions of the (linear) *damped wave equation* (DWE)

$$(\partial_t^2 - \Delta + 2a(x)\partial_t)v(x,t) = 0, \quad v(x,0) = v_0,\ \partial_t v(x,0) = v_1\,. \tag{1.1}$$

Equivalently, we want to solve the system

$$(i\partial_t + \mathcal{A})\,\boldsymbol{v}(t) = 0, \quad \mathcal{A} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & I \\ -\Delta & -2ia \end{pmatrix}, \quad \text{with } \boldsymbol{v}(t) = (v(t), i\partial v(t))\,. \tag{1.2}$$

Then, if $a \not\equiv 0$, the energy

$$E(v(t)) = \frac{1}{2}\left(\|\nabla v(t)\|^2 + \|\partial_t v(t)\|^2\right) \tag{1.3}$$

of any initial (v_0, v_1) will decay to zero.

To analyze the decay, we notice that $\mathcal{A}$ generates a strongly continuous semigroup on the space $\mathcal{H} \stackrel{\text{def}}{=} H^1(X) \times L^2(X)$, so the solution to (1.1,1.2) reads

$$\boldsymbol{v}(t) = e^{-it\mathcal{A}}\boldsymbol{v}(0), \quad \boldsymbol{v}(0) \in \mathcal{H}. \tag{1.4}$$

To analyze the decay of this solution it it natural to try to expand it in terms of the *spectrum* of $\mathcal{A}$. This spectrum is discrete, consisting of countably many complex eigenvalues $\{\tau_n\}$ with $\operatorname{Re}\tau_n \to \pm\infty$. It can be obtained by solving the generalized eigenvalue equation

$$P(\tau)u = \left(-\Delta - \tau^2 - 2ia\tau\right)u = 0. \tag{1.5}$$

To each eigenvalue τ_n corresponds a quasi-stationary mode $u_n(x)$ satisfying $P(\tau_n)u_n = 0$. Such a mode leads to a solution

$$v_n(t,x) = e^{-it\tau_n}u_n(x)$$

of the damped wave equation. Hence $\operatorname{Im}\tau_n$ represents the *quantum decay rate* of this stationary mode u_n. The corresponding eigenstate of $\mathcal{A}$ reads $\boldsymbol{u}_n = (u_n, \tau_n u_n)$.

Lemma 1. *If* $\operatorname{Re}\tau_n = 0$*, then* $-\operatorname{Im}\tau_n \in 2[a_{\min}, a_{\max}]$.
If $\operatorname{Re}\tau_n \neq 0$ *then* $-\operatorname{Im}\tau_n \in [a_{\min}, a_{\max}]$.

Proof. Expanding the imaginary part of $\langle u, P(\tau)u\rangle = 0$, we get

$$-\operatorname{Im}\left(\tau^2\right)\|u\|^2 - 2\operatorname{Re}\tau\langle u, au\rangle = 0.$$

If $\operatorname{Re}\tau \neq 0$, this simplifies into

$$-\operatorname{Im}\tau\,\|u\|^2 = \langle u, au\rangle.$$

If $\tau = it$, $\langle u, P(it)u\rangle = 0$ becomes

$$\|\nabla u\|^2 + t^2\|u\|^2 + 2t\langle u, au\rangle = 0\,.$$

Since $\langle u, au\rangle \geq 0$, we have $t \leq 0$, and then either $t = 0$ (which corresponds a constant solution), or $t < 0$, which implies

$$-t\,\|u\|^2 \leq 2\langle u, au\rangle \Longrightarrow -t \in 2[a_{\min}, a_{\max}].$$

□

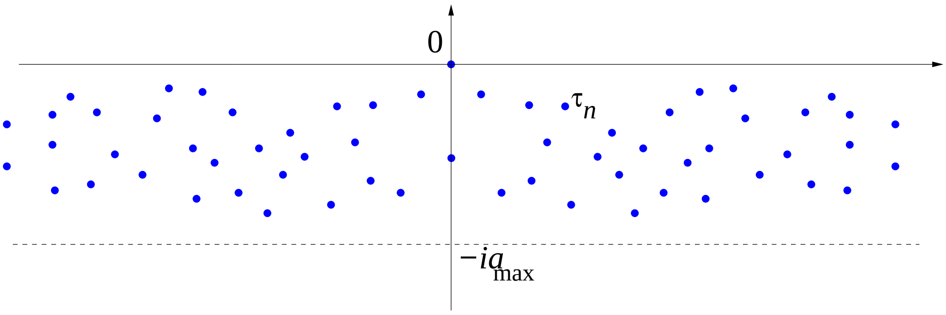

FIGURE 1.1. Spectrum of the damped wave equation.

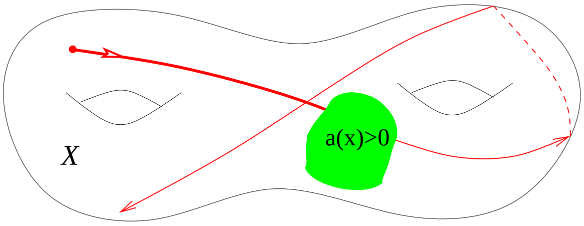

FIGURE 1.2. A damped geodesic.

The spectrum is symmetric w.r.to the imaginary axis:

$$(\tau, u) \text{ solution } \iff (-\bar{\tau}, \bar{u}) \text{ solution.}$$

1.1.1. *The condition of geometric control.* As we will see below, the distribution of the quantum decay rates $\operatorname{Im}\tau_n$ in the high frequency regime will be constrained by the ergodic averages of the damping through the geodesic flow on S^*X, that is the functions

$$\langle a\rangle_T = \frac{1}{T}\int_0^T a\circ\Phi^s\,ds, \quad \text{on } S^*X.$$

The damping $a(x)$ is said to satisfy the Geometric Control Condition (GCC) if there exists T_0 such that, for any $\rho\in S^*X$, the trajectory $\Phi^t(\rho)$ meets the damped region $\{a(x)>0\}$ for some time $t\in[0,T_0]$. As a consequence, there exists $c>0$ such that, for T large enough, the function $\langle a\rangle_T\geq c>0$ everywhere. We will show below (Thm 8) that, as a consequence, the quantum decay rates satisfy $\operatorname{Im}\tau_n\leq -c+o(1)$ when $\operatorname{Re}\tau_n\to\infty$.

Lebeau [Leb93] showed that GCC is equivalent with the uniform exponential decay of the energy for initial data in $\mathcal{H}$, namely, there exists $C>0$, $\gamma>0$ such that, for any data $\boldsymbol{v}(0)\in\mathcal{H}$,

$$E(\boldsymbol{v}(t))\leq C\,e^{-2\gamma t}\,E(\boldsymbol{v}(0))\leq C\,e^{-2\gamma t}\,\|\boldsymbol{v}(0)\|^2_{\mathcal{H}}\,. \tag{1.6}$$

In this GCC case, the optimal decay rate γ is given by $\min(G,a_-)$, where $G=\inf\{\operatorname{Im}\tau_n,\ \tau_n\neq 0\}$ is the spectral gap, while

$$a_-\stackrel{\text{def}}{=}\lim_{T\to\infty}\min_{S^*X}\langle a\rangle_T$$

is the minimal asymptotic damping. We will sketch a proof of this decay estimate in §1.3.3.

Koch and Tataru [KoTa94] studied the same question in a more general context (case of a manifold with boundaries, and of a damping taking place both in the "bulk" and on the boundary). They first showed that the minimum of $\langle a\rangle_T$ on S^*X really governs the decay of the semigroup on $\mathcal{H}$, up to compact subspaces.

Theorem 2. [KoTa94]*For each $\epsilon > 0$ and each $t > 0$ there exists a subspace $\mathcal{H}_{\epsilon,t} \subset \mathcal{H}$ of finite codimension such that*

$$\left\|e^{-it\mathcal{A}}\right\|_{\mathcal{H}_{\epsilon,t}\to\mathcal{H}} \leq \exp\left\{\max_{S^*X}\langle a\rangle_T\right\} + \epsilon.$$

Moreover, for any subspace $\mathcal{H}_1 \subset \mathcal{H}$ of finite codimension,

$$\exp\left\{\max_{S^*X}\langle a\rangle_T\right\} \leq \left\|e^{-it\mathcal{A}}\right\|_{\mathcal{H}_1\to\mathcal{H}}.$$

A consequence of this result is the characterization of the *Fredholm spectrum* of the semigroup. In the present situation of damped waves on a compact X withough boundary, their result states[1] that this Fredholm spectrum is given by the annulus $\{z \in \mathbb{C},\ e^{-ta_+} \leq |z| \leq e^{-ta_-}\}$.

In absence of GCC, one cannot have such a uniform exponential decay for arbitrary data in $\mathcal{H}$: one can cook up initial data in $\mathcal{H}$ with arbitrary slow decay, e.g. by using Gaussian beams of higher and higher frequencies localized along undamped geodesics. However, in case the spectral gap G is positive and one has algebraic resolvent estimates in the eigenvalue free strip, it is possible to prove the presence of such an exponential decay for *more regular data.* Typically, one can try to prove exponential decay estimates of the form

$$E(v(t)) \leq C_s\, e^{-\gamma_s t}\, \|\boldsymbol{v}\|_{\mathcal{H}^s}, \quad \forall \boldsymbol{v} \in \mathcal{H}^s. \tag{1.7}$$

Here $\mathcal{H}^s = H^{s+1} \times H^s$, $s > 0$, is a Sobolev space of "regular" data, and the decay rate $\gamma_s > 0$. The example of thin Gaussian beams then breaks down, because the $\mathcal{H}^s$ norms of such beams grows algebraically with the frequency. We provide the proof of such a decay with loss of derivatives in §1.3.1.

1.2. High frequency limit – semiclassical formulation and generalization.

The main question we want to address is the distribution of eigenvalues τ_n, and in particular of their imaginary parts, in the high-frequency limit $\operatorname{Re}\tau_n \gg 1$.

A semiclassical reformulation was used in ([Sjo00]): take $\hbar \ll 1$ and consider eigenvalues $\tau_n \approx \hbar^{-1}$, by writing

$$\tau = \frac{\sqrt{2z}}{\hbar} \quad \text{with} \quad z \in D(1/2, C\hbar). \tag{1.8}$$

The equation (1.5) becomes

$$(P(\hbar,z) - z)u = 0, \qquad P(\hbar,z) = -\frac{\hbar^2\Delta}{2} - i\hbar\sqrt{2z}\,a = -\frac{\hbar^2\Delta}{2} - i\hbar a + \mathcal{O}(\hbar^2). \tag{1.9}$$

More generally, we may consider operators of the type

$$P(\hbar,z) = -\frac{\hbar^2\Delta}{2} + i\hbar\operatorname{Op}_\hbar(q_z). \tag{1.10}$$

where $q = q_z \in S^1(X)$ depends holomorphically on $z \in D(1/2, C\hbar)$. This is the most general framework considered in [Sjo00]. The unboundedness of $q \in S^1(X)$ when $|\xi| \to \infty$ is not a real challenge, since one is interested in the region $|\xi| \approx 1$.

[1]Their result holds under the condition that, for any $T > 0$, the set of T-periodic geodesics has measure zero.

However, for simplicity of presentation we will assume that q is real valued, that it is independent of z, and that it is in the symbol class[2] $S^0(X)$. This will imply the convenient property that the "damping operator"

$$Q = \mathrm{Op}_\hbar(q)$$

will be uniformly bounded in $L^2(X)$, for $\hbar$ small enough. By abuse, we will refer to the function $q(x,\xi)$ as "the damping". We are then interested in the semiclassical distribution of the eigenvalues $z_n(\hbar) \in D(1/2, C\hbar)$ of the operator

$$P(\hbar) = -\frac{\hbar^2\Delta}{2} + i\hbar\,\mathrm{Op}_\hbar(q) \tag{1.11}$$

The principal symbol of this operator is $p_0 = \frac{|\xi|^2}{2}$, which generates the geodesic flow $\Phi^t = \exp(tH_{p_0})$ on T^*X, with unit speed on the energy shell $p_0^{-1}(1/2) = S^*X$. The function $iq(x,\xi)$ is the subprincipal symbol of $P(\hbar)$.

1.3. **From resolvent estimates to energy decay.** (see [Leb93] [Hit03][BuHi07][Chris09]).

1.3.1. *From the resolvent to the semigroup.* We want to expand the semigroup $e^{it\mathcal{A}}$ using the resolvent $(\tau - \mathcal{A})^{-1}$. This resolvent always satisfies the bound

$$\left\|(\tau - \mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \le \frac{1}{\mathrm{Im}\,\tau}, \quad \mathrm{Im}\,\tau > 0.$$

For some integer $k \ge 2$, we may write for any $\boldsymbol{v}(0) \in \mathcal{H}$ and any $t > 0$:

$$e^{-it\mathcal{A}}(I + i\mathcal{A})^{-k}\boldsymbol{v}(0) = \frac{1}{2i\pi}\int_{\mathbb{R}+i/2} e^{-it\tau}\,(1+i\tau)^{-k}\,(\tau - \mathcal{A})^{-1}\,\boldsymbol{v}(0)\,d\tau. \tag{1.12}$$

By inserting $(I + i\mathcal{A})^{-k}$ we ensure that the above integral converges absolutely in $\mathcal{H}$.

Now, assume that we have proved a resolvent estimate[3]

$$\left\|(\tau - \mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \lesssim \tau^N \quad \text{in the strip} \quad \{|\mathrm{Re}\,\tau| \ge C, \quad \mathrm{Im}\,\tau \ge -\gamma\} \tag{1.13}$$

for some height $\gamma > 0$ and power $N \ge 0$. In particular, this implies that $\mathcal{A}$ has finitely many eigenvalues such that $\mathrm{Im}\,\tau_n > -\gamma$. Up to taking a smaller γ, we may assume that the only eigenvalue in $\{\mathrm{Im}\,\tau > \gamma\}$ is the zero eigenvalue, corresponding to the constant function. This function has no effect on the energy, so we may remove it from the initial data $\boldsymbol{v}(0)$ and consider the evolution of

(1.14)

$$\boldsymbol{u}(0) = (I - \Pi_0)\boldsymbol{v}(0), \quad \Pi_0 = \frac{1}{2i\pi}\oint_0 (\tau - \mathcal{A})^{-1}\,d\tau \quad \text{the spectral projector on the constant function.}$$

The operator $(\tau - \mathcal{A})^{-1}(1 - \Pi_0)$ is holomorphic in $\{\mathrm{Im}\,\tau \ge -\gamma\}$. By taking $k \ge N+2$, we may deform the contour of integration of (1.15) into the line $\mathbb{R} - i\gamma$, keeping an

[2]The class $S^k(X)$ denotes the functions $a(x,\xi;\hbar)$ satisfying the following bounds: for any multi-indices $\alpha, \beta \in \mathbb{N}^d$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;\hbar)\right| \le C_{\alpha,\beta}\,\langle\xi\rangle^{k-|\beta|} \quad \text{uniformly for } \hbar \in (0,1),$$

where we used the standard notation $\langle\xi\rangle = \left(1 + |\xi|^2\right)^{1/2}$.

[3]Below $a(\tau) \lesssim b(\tau)$ will always mean that $a(\tau) \le Cb(\tau)$ for some constant $C > 0$.

absolutely converging integral:

$$(1.15)\quad (I+i\mathcal{A})^{-k}\, e^{-it\mathcal{A}}\boldsymbol{u}(0) = \frac{1}{2i\pi}\int_{\mathbb{R}-i\gamma} e^{-it\tau}\,(1+i\tau)^{-k}\,(\tau-\mathcal{A})^{-1}\,\boldsymbol{u}(0)\,d\tau$$

$$(1.16)\quad = \frac{e^{-t\gamma}}{2i\pi}\int_{\mathbb{R}} e^{-it\tau}\,(1+\gamma+i\tau)^{-k}\,(\tau-i\gamma-\mathcal{A})^{-1}\,\boldsymbol{u}(0)\,d\tau.$$

Using the bound (1.13), we obtain a uniform exponential decay:

$$\left\|(I+i\mathcal{A})^{-k}\, e^{-it\mathcal{A}}\boldsymbol{u}(0)\right\|_{\mathcal{H}} \lesssim e^{-t\gamma}\,\|\boldsymbol{u}(0)\|_{\mathcal{H}}\,.$$

This expression implies a uniform exponential decay for data $\boldsymbol{u}(0) = (I-\Pi_0)\boldsymbol{v}(0)$ in the Sobolev space $\big\{\boldsymbol{v}\in\mathcal{H},\ (1+i\mathcal{A})^k\boldsymbol{v}\in\mathcal{H}\big\} = \mathcal{H}^k$:

$$\|\boldsymbol{u}(t)\|_{\mathcal{H}} \lesssim e^{-t\gamma}\,\|\boldsymbol{u}(0)\|_{\mathcal{H}^k}\,.$$

On the space $\mathcal{H}_0 = (I-\Pi_0)\mathcal{H}$, the norm $\|\boldsymbol{u}\|^2_{\mathcal{H}}$ is equivalent to the energy $E(\boldsymbol{u})$, so the above estimate implies an exponential decay of the energy for smooth enough data:

$$(1.17)\qquad E(\boldsymbol{v}(t)) \lesssim e^{-2t\gamma}\,\|(I-\Pi_0)\boldsymbol{v}(0)\|^2_{\mathcal{H}^k}\,,\quad \forall \boldsymbol{v}(0)\in\mathcal{H}^k.$$

1.3.2. *Case of geometric control.* The resolvent $(\tau-\mathcal{A})^{-1}$ can be easily written in terms of the resolvent $R(\tau) \stackrel{\text{def}}{=} P(\tau)^{-1}$, $P(\tau) \stackrel{\text{def}}{=} (-\Delta-\tau^2-2ia\tau)$:

$$(1.18)\qquad (\tau-\mathcal{A})^{-1} = \begin{pmatrix} -R(\tau)(2ia+\tau) & -R(\tau) \\ -R(\tau)(2ia\tau+\tau^2)-1 & -\tau R(\tau) \end{pmatrix}.$$

In case the geometric control condition holds, Thm 8 implies that, for any $\epsilon>0$, there exists $C_\epsilon>0$ such that the following holds:

$$(1.19)\qquad \|R(\tau)\|_{L^2\to L^2} \lesssim \tau^{-1} \qquad \{\operatorname{Im}\tau \geq -a_- + \epsilon,\ |\operatorname{Re}\tau| \geq C_\epsilon\}\,.$$

Choose $0<\gamma<\min(G,a_-)$, such the only eigenvalue in $\{\operatorname{Im}\tau\geq\gamma\}$ is the constant function. We still have

$$(1.20)\qquad \|R(\tau)\|_{L^2\to L^2} \lesssim \tau^{-1} \qquad \{\operatorname{Im}\tau \geq -\gamma,\ |\operatorname{Re}\tau| \geq C\}\,.$$

The same estimate holds in the operator norm $H^s\to H^s$ for any $s\in\mathbb{R}$. By expanding $(-\Delta-\tau^2-2i\tau a)\,R(\tau)=I$ one gets, in the same region

$$(1.21)\qquad \|R(\tau)\|_{L^2\to H^2} \lesssim \tau \quad \text{so by interpolation} \quad \|R(\tau)\|_{L^2\to H^1} \lesssim 1.$$

By duality, we deduce

$$\|R(\tau)\|_{H^{-1}\to L^2} \lesssim 1.$$

From the identity $R(\tau)\,(-\Delta-\tau^2-2i\tau a) = I$, we get the following norm for the lower left entry in (1.18):

$$\left\|R(\tau)(2ia\tau+\tau^2)+1\right\|_{H^1\to L^2} = \|R(\tau)\Delta\|_{H^1\to L^2} \leq \|R(\tau)\|_{H^{-1}\to L^2}\,\|\Delta\|_{H^1\to H^{-1}} \lesssim 1.$$

From there we obtain the resolvent estimate

$$\left\|(\tau-\mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \lesssim 1, \qquad \{\operatorname{Im}\tau\geq-\gamma,\ |\operatorname{Re}\tau|\geq C\}\,.$$

Applying the method of the previous subsection, this implies an energy decay for data $\boldsymbol{v}(0)\in\mathcal{H}^2$:

$$E(\boldsymbol{v}(t)) \lesssim e^{-2\gamma t}\,\|(I-\Pi_0)\boldsymbol{v}(0)\|^2_{\mathcal{H}^2}\,.$$

This estimate is not optimal. As we will show in the next subsection, in case of geometric control one has a uniform exponential decay for initial data $\boldsymbol{v}(0)\in\mathcal{H}$.

1.3.3. *Geometric control - energy decay without loss of derivatives.* To show the exponential decay of the energy without a loss of derivative, one has to proceed a bit differently, namely by transforming the DWE into an inhomogeneous equation, and by invoking a Parseval identity w.r.to the time variable. We only sketch the argument, which first appeared in the context of obstacle scattering[Mora75].

Call $u(t,x)$ the solution of (1.1) with the data $\boldsymbol{u}(0) = (I - \Pi_0)\boldsymbol{v}(0)$. Apply a smooth cutoff in time $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 0$ for $t \leq 1$, $\chi(t) = 1$ for $t \geq 2$. The function $w(t,x) \stackrel{\text{def}}{=} \chi(t)u(t,x)$ satisfies the inhomogeneous equation

$$(\partial_t^2 - \Delta + 2a(x)\partial_t)\, w(t,x) = f(t,x),$$

where the RHS $f = \chi'' u + 2\chi' \partial_t u + 2a\chi' u$ is supported in $X \times [1,2]_t$. Apply a Fourier transform in the time variable:

$$w(t,x) = \int e^{-it\tau} \hat{w}(\tau, x) d\tau. \tag{1.22}$$

The Fourier transforms of w and v then satisfy the equation$P(\tau)\, \hat{w}(\tau,x) = \hat{f}(\tau,x)$. From there we get, using the holomorphy of $R(\tau)\hat{f}(\tau)$ in the strip $\{\operatorname{Im}\tau \geq -\gamma\}$:

$$e^{t\gamma}\, w(t) = \int_{\mathbb{R}} e^{-it\tau} R(\tau - i\gamma) \hat{f}(\tau - i\gamma)\, d\tau \tag{1.23}$$

We now recall the resolvent estimates (1.19,1.21). The RHS in (1.23) is the Fourier tranform of $\hat{w}(\tau - i\gamma) = R(\tau - i\gamma)\hat{f}(\tau - i\gamma) \in H^1_x$. Parseval's formula in the time-frequency variables reads

$$\begin{aligned} \|e^{\cdot\gamma} w(\cdot)\|_{L^2_t(\mathbb{R}, H^1_x)} &= \|\hat{w}(\cdot - i\gamma)\|_{L^2_\tau(\mathbb{R}, H^1_x)} \\ &\lesssim \left\| \hat{f}(\cdot - i\gamma) \right\|_{L^2_\tau(\mathbb{R}, L^2_x)} \\ &\lesssim \|f\|_{L^2_t(\mathbb{R}, L^2_x)}. \end{aligned}$$

On the second line we used (1.21), and in the third one the fact that $f(t)$ is compactly supported.

An easy computation (using Gronwall's inequality and the fact that f is supported in $t \in [1,2]$) shows that $\|f\|_{L^2_t(\mathbb{R}, L^2_x)} \lesssim \|\boldsymbol{u}(0)\|_{\mathcal{H}}$. Conversely, one can transform the L^2_t estimate for $e^{t\gamma} w(t)$ into a L^∞_t estimate, and obtain:

$$\|w(t)\|_{H^1_x} \lesssim e^{-\gamma t} \|\boldsymbol{u}(0)\|_{\mathcal{H}}, \quad \forall t \in \mathbb{R}. \tag{1.24}$$

Using (1.19), one can similarly obtain

$$\|\partial_t w(t)\|_{L^2_x} \lesssim e^{-\gamma t} \|\boldsymbol{u}(0)\|_{\mathcal{H}}. \tag{1.25}$$

Since $w(t) = u(t)$ for $t \geq 2$, we see that (1.24,1.25) provide a decay estimate of the energy for the initial data $\boldsymbol{v}(0)$:

$$E(\boldsymbol{v}(t)) \lesssim e^{-2\gamma t}\, \|(I - \Pi_0)\boldsymbol{v}(0)\|^2_{\mathcal{H}} \lesssim e^{-2\gamma t}\, E(\boldsymbol{v}(0)).$$

2. Semiclassical spectral distribution of $P(\hbar)$

2.1. **Weyl law for the real parts.** The rough distribution of the real parts $\operatorname{Re} z_n$ is not affected by the damping, it is asymptotically given by the same Weyl law as in the undamped case:

Theorem 3. *[Markus-Matsaev, Sjöstrand] For any $\epsilon > 0$ small enough (possibly depening on $\hbar$). Then the number of eigenvalues of $P(\hbar)$ in the strip $\mathcal{S}_\epsilon = \left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right] + i\mathbb{R}$, satisfies*

$$\#\left\{\operatorname{Spec} P(\hbar) \cap \mathcal{S}_\epsilon\right\} = (2\pi\hbar)^{-d} \operatorname{Vol}\left(\{|p_0(x,\xi) - 1/2| \leq \epsilon\}\right) + \mathcal{O}\left(\hbar^{-d+1}\right),$$

where Vol corresponds to the symplectic volume in T^*X. This estimate implies that, for $C > 0$ large enough,

$$\#\left\{\operatorname{Spec} P(\hbar) \cap \mathcal{S}_{C\hbar}\right\} \asymp \hbar^{d-1}. \tag{2.1}$$

Proof. (Sketch) One starts from the selfadjoint operator $P_0(\hbar) = -\frac{\hbar^2 \Delta}{2}$, for which we now that the result is OK. We then deform it into

$$\tilde{P}_0(\hbar) = P_0(\hbar) + \delta P,$$

where the self-adjoint perturbation δP is chosen such that $\tilde{P}_0$ has no eigenvalue in $C_0\hbar$-neighbourhoods of $E_\pm = \frac{1}{2} \pm \epsilon$, and δP has a trace norm $\asymp \hbar^{-d+1}$. This trace control shows that the eigenvalues of $\tilde{P}_0$ in $\mathcal{S}_\epsilon$ satisfies the above Weyl estimate. We then deform $\tilde{P}_0$ into the nonselfadjoint family

$$\tilde{P}_t(\hbar) = P(\hbar) + \delta P + it\hbar \operatorname{Op}(q), \quad t \in [0,1].$$

Provided the "forbidden width" $C_0\hbar$ is large enough, the resolvent of $\tilde{P}_t$ remains under control along the lines $\operatorname{Re} z = E_\pm$ for all $t \in [0,1]$, showing that $\tilde{P}_1$ has the same number of eigenvalues in $\mathcal{S}_\epsilon$ as $\tilde{P}_0$.

The last part (the most technical one) consists in showing that the difference between the number of eigenvalues of $P(\hbar)$ and $\tilde{P}_1 = P + \delta P$ in $\mathcal{S}_\epsilon$ is an $\mathcal{O}(\hbar^{-d+1})$. Here one uses the resolvent identity

$$(z - P)^{-1} = \left(z - \tilde{P}_1\right)^{-1} \left(1 + \hbar \left(z - \tilde{P}_1\right)^{-1} \delta P\right),$$

and a good control on the trace norm of $\left(z - \tilde{P}_1\right)^{-1} \delta P$ for z in the "forbidden strips". This allows to define and study the determinant

$$\det\left(1 + \hbar \left(z - \tilde{P}_1\right)^{-1} \delta P\right),$$

the vanishing of which corresponds to the eigenvalues of $P(\hbar)$. □

2.2. **Distribution of the imaginary parts.** From any eigenvalue/vector (z_n, u_n) of $P(\hbar)$, one can construct a solution $v(x,t) = e^{-izt/\hbar} u(x)$ for the time dependent Schrödinger equation $i\hbar\partial_t v - Pv = 0$. The solution v decays exponentially in time, which explains why the value

$$\operatorname{Im} z_n / \hbar$$

can be called the *quantum decay rate*[4] associated with u_n.

The first simple constraint for the quantum decay rates is the generalization of Lemma 1 to the damping function $q(x,\xi)$.

[4] *It is more standard to call $2 \operatorname{Im} z/\hbar$ the quantum decay rate: it is associated with the decay of the probability density $|v(x,t)|^2$. However, we will keep with the present definition.*

Lemma 4. *Take* $P(\hbar) = -\hbar^2\Delta/2 + i\hbar\,\mathrm{Op}_\hbar(q)$, $q \in S^0(X)$. *Then, for any* $\gamma > \max_{S^*X} q$ *and for any* $\hbar > 0$ *small enough, the following resolvent estimate holds:*

$$\forall z \in D(1/2, C\hbar) \cap \{\mathrm{Im}\, z \geq \hbar\gamma\}\,, \qquad \left\|(P(\hbar) - z)^{-1}\right\| \leq \frac{C_\gamma}{\hbar}.$$

A similar estimate holds if $\mathrm{Im}\, z/\hbar \leq \gamma < \min_{S^*X} q$.

As a consequence, all eigenvalues in $D(1/2, C\hbar)$ *satisfy*

$$\min_{S^*X} q - o_{\hbar\to 0}(1) \leq \frac{\mathrm{Im}\, z}{\hbar} \leq \max_{S^*X} q + o_{\hbar\to 0}(1).$$

Proof. Assume that $z \in D(1/2, C\hbar)$ and $\mathrm{Im}\, z/\hbar \geq \max_{S^*X} q + \epsilon$. We need to prove that there exists $C_1 > 0$ such that

$$\forall u \in L^2(X) \text{ normalized}, \qquad \|(P - z)\, u\| \geq C_1\hbar\, \|u\|\,.$$

We will use the fact that either the "real part" or the "imaginary part" of $(P - z)u$ is large.

Indeed, let us decompose

$$(P - z)\, u = (P_0 - 1/2)\, u - (\mathrm{Re}\, z - 1/2)\, u + i\hbar\, (Q - \mathrm{Im}\, z/\hbar)\, u,$$

so that (through the triangle inequality)

$$\begin{aligned} \|(P - z)\, u\| &\geq \|(P_0 - 1/2)\, u\| - |\mathrm{Re}\, z - 1/2|\, \|u\| - \hbar\, \|(Q - \mathrm{Im}\, z/\hbar)\, u\| \\ &\geq \|(P_0 - 1/2)\, u\| - (2C + \|Q\|)\, \hbar. \end{aligned}$$

Assume that, for some $C_0 > 2\,(2C + \|Q\|)$, we have $\|(P_0 - 1/2)\, u\| \geq C_0\hbar\, \|u\|\,$. Then we are done: $\|(P - z)\, u\| \geq \frac{C_0}{2}\hbar\, \|u\|\,$.

On the opposite, assuming $\|(P_0 - 1/2)\, u\| \leq C_0\hbar\, \|u\|$, we write

$$\|(P - z)\, u\| \geq |\langle u, (P - z)\, u\rangle| = \hbar\, |\langle u, (Q - \mathrm{Im}\, z/\hbar)\, u\rangle|\,. \tag{2.2}$$

We then construct a symbol $\tilde{q} \in S^0(X)$, such that $\tilde{q} = q$ on S^*X, while

$$\max_{T^*X} \tilde{q} = \max_{S^*X} \tilde{q} = \max_{S^*X} q, \quad \min_{T^*X} \tilde{q} = \min_{S^*X} \tilde{q} = \min_{S^*X} q.$$

We can write $q = \tilde{q} + k(p_0 - 1/2)$ for some symbol $k \in S^{-1}(X)$, and at the quantum level

$$Q = \tilde{Q} + K(P_0 - 1/2) + \mathcal{O}(\hbar), \quad K = \mathrm{Op}_\hbar(k). \tag{2.3}$$

We then insert this decomposition in (2.2), and use the triangle inequality:

$$\begin{aligned} \|(P - z)\, u\| &\geq \hbar \left|\langle u, \Big(\tilde{Q} - \mathrm{Im}\, z/\hbar + K(P_0 - 1/2) + \mathcal{O}(\hbar)\Big)\, u\rangle\right| \\ &\geq \hbar \left|\langle u, \Big(\tilde{Q} - \mathrm{Im}\, z/\hbar\Big)\, u\rangle\right| - \hbar\, |\langle u, (K(P_0 - 1/2) + \mathcal{O}(\hbar))\, u\rangle|\,. \end{aligned}$$

From the assumption on $\tilde{q}$ and the sharp Gårding inequality, the first term on the RHS is bounded below by $\hbar\epsilon - \mathcal{O}(\hbar^2)$. On the other hand from our assumption, and the boundedness of K, the second term is bounded below by $-\mathcal{O}(C_0\hbar^2)$, so for $\hbar$ small enough we have $\|(P - z)\, u\| \geq \frac{\epsilon}{2}\hbar\, \|u\|\,$. □

The bound of the above lemma only uses the microlocalization of u in an $\hbar$-neighbourhood of S^*X. One can be a bit more precise concerning this microlocalization, and show that the semiclassical wavefront set of $u = u(\hbar)$ is contained in S^*X.

Lemma 5. *[Microlocalization on S^*X] Take $\chi \in C_c^\infty([1/2 \pm \epsilon])$ with $\chi(s) = 1$ near $s = 1/2$ and energy cutoff. Let u be an eigenvalue of $P(\hbar)$ with eigenvalue $z \in D(1/2, C\hbar)$. Then,*

$$(I - \chi(P_0))\, u = \mathcal{O}_{L^2}(\hbar^\infty).$$

Proof. Since the symbol $p - z$ is elliptic away from S^*X, there exists an symbol $k = k_z \in S^2(X)$ such that

$$k_z\,(p - z) = (1 - \chi(p_0))\,.$$

Working order by order, one can construct an operator $K = K_z \in \Psi^{-2}(X)$ of principal symbol $k(z)$, such that

$$K_z\,(P - z) = (I - \chi(P_0)) + r(z), \qquad r(z) \in \hbar^\infty \Psi^{-\infty}.$$

Applying K_z to the equation $(P - z)\, u = 0$ proves the lemma. □

2.3. **Shrinking the interval of quantum decay rates by using time evolution.**

2.3.1. *A factorization of the Schrödinger propagator.* The equation $(P - z)\, u = 0$ implies more than the microlocalization of Lemma 5. As we will see below, it implies a *propagation of singularities* (Egorov's theorem) which allows to get informations on both u and $\operatorname{Im} z$.

Let us give the rough idea. Consider a coherent state ψ_0 localized on a point $\rho_0 = (x_0, \xi_0) \in S^*X$, locally of the form

$$\psi_0(x) = \hbar^{-d/4} a(x)\, \exp\left\{ \frac{i\xi_0 \cdot x}{\hbar} - \frac{\langle (x - x_0), M_0(x - x_0)\rangle}{2\hbar} \right\},$$

where M_0 is a $d \times d$ symmetric matrix with $\operatorname{Re} M > 0$, and $a_0(x)$ is a smooth amplitude. Let us propagate it through the group,

$$\psi_t = e^{-itP/\hbar}\, \psi_0.$$

In the undamped case $q \equiv 0$, $\psi_t = \psi_t^0$ approximately remains a coherent state, localized at the point $\rho_t = \Phi^t(\rho_0)$ and modified matrix M_t. If we switch on the skew-adjoint part $i\hbar Q$, the state ψ_t remains an approximate coherent state localized on ρ_t, but with a modified *amplitude*: to lowest order, one has

$$\psi_t = C(\rho_0, t)\, \psi_t^0,$$

with the factor $C(\rho_0, t)$ given by the damping accumulated between the times 0 and t:

$$b(t, \rho_0) = \exp \int_0^t q\left(\Phi^s(\rho_0)\right)\, ds.$$

The imaginary subprincipal term $i\hbar Q$ thus has the effect to *damp* the state ψ_t. This property can be recovered from the following

Lemma 6. *Assume $q \in C_c^\infty(T^*X)$, and take $P = P_0 + i\hbar \operatorname{Op}_\hbar(q)$.*

For any fixed $t \in \mathbb{R}$, decompose the Schrödinger propagator $V^t = e^{-itP/\hbar}$ into

$$V^t = U^t\, B(t)\,,$$

where $U^t = e^{-itP_0/\hbar}$ is the undamped propagator. Then, the operator $B(t)$ is a PDO in $\Psi^0(X)$ of principal symbol

$$b(t) = e^{\int_0^t q \circ \Phi^s\, ds}.$$

Proof. From our assumption, $\|V^t\|_{L^2\to L^2} \leq e^{Ct}$ for any $t \in \mathbb{R}$. To show that $B(t) \stackrel{\text{def}}{=} U^{-t}V^t$ is a PDO, we differentiate w.r.to time:

$$\partial_t B(t) = \frac{-i}{\hbar} U^{-t} \left(P(\hbar) - P_0(\hbar)\right) V^t = U^{-t} Q V^t = U^{-t} Q U^t \, B(t).$$

Let us call $Q(t) \stackrel{\text{def}}{=} U^{-t} Q U^t$, so the above equation reads

$$\partial_t B(t) = Q(t)\, B(t), \quad B(0) = I\,.$$

Egorov's theorem shows that, for any fixed time $t \in \mathbb{R}$, $Q(t) \in \Psi^0(X)$, with principal symbol $q \circ \Phi^t$.

One can then use Beals's lemma [EvZw, Thm 8.19] to show that $B(t) \in \Psi^0(X)$.

It suffices to show that, for any indices $i_1, \ldots, i_N$, $j_1, \ldots, j_M$, we have (in local coordinates)

$$\left\| \mathrm{ad}_{x_{i_1}} \cdots \mathrm{ad}_{x_{i_N}} \, \mathrm{ad}_{D_{x_{j_1}}} \cdots \mathrm{ad}_{D_{x_{j_M}}} \; B(t) \right\|_{L^2(X)\to H^N_\hbar(X)} = \mathcal{O}_{L^2\to L^2}(\hbar^N).$$

Let's assume this estimate holds up to order $N-2$ for all M, and to order $N-1$ up to $M-1$. Call (schematically) $V_{N-1,M}(t) \stackrel{\text{def}}{=} \left(\mathrm{ad}_x\right)^{N-1} \left(\mathrm{ad}_D\right)^M \; B(t)$. Differentiate it w.r.to time:

$$\partial_t V_{N-1,M}(t) = \left(\mathrm{ad}_x\right)^{N-1} \left(\mathrm{ad}_D\right)^M \; Q(t)\, B(t),$$

Applying Leibnitz's rule $\mathrm{ad}_*(AB) = (\mathrm{ad}_* A)\, B + A\, \mathrm{ad}_* B$ recursively, we may separate the unique contribution where all ad_* act on $B(t)$, from the other ones:

$$\partial_t V_{N-1,M}(t) = Q(t) V_{N-1,M}(t) + E_{N-1,M}(t), \quad V_{N-1,M}(0) = 0\,. \tag{2.4}$$

The operator $E_{N-1,M}(t)$ is a linear combination of products $\left(\mathrm{ad}_x\right)^{j_1} \left(\mathrm{ad}_\xi\right)^{j_2} Q(t) \; \left(\mathrm{ad}_x\right)^{N-1-j_1} \left(\mathrm{ad}_\xi\right)^{M-j_2}$ where $j_1 + j_2 \geq 1$ derivatives act on $Q(t)$. From the fact that $Q(t) \in \Psi^0$ and the recurrence hypothesis, we see that $\|E_{N-1,M}(t)\|_{L^2\to H^{N-1}_\hbar} = \mathcal{O}(\hbar^{N-1})$. The equation (2.4) is solved by

$$V_N(t) = \int_0^t V(t;s)\, E_{N-1,M}(s)\, ds,$$

where $V(t;s)$ is the solution of $\partial_{t'} V(t';s) = Q(t')V(t';s)$ with condition $V(s;s) = I$. This implies the bound $\|V(t;s)\|_{H^{N-1}\to H^{N-1}} = \mathcal{O}(1)$, so we get $\|V_{N-1,M}(t)\|_{L^2\to H^{N-1}} = \mathcal{O}(\hbar^{N-1})$. This shows the hypothesis to order $N-1, M$.

One proceeds in a similar way to show the hypothesis at order N, M, knowing it at order $N-1, M$.

To investigate the symbol of $B(t)$, let us compare $B(t)$ with the (invertible) PDO $\tilde{B}(t) = \mathrm{Op}(e^{\int_0^t q\circ\Phi^s ds})$. Differentiating in time, we get

$$\partial_t \tilde{B}(t)^{-1} B(t) = \tilde{B}(t)^{-1} \left(Q(t) - \mathrm{Op}_\hbar(q\circ\Phi^t) + \mathcal{O}(\hbar) \right) B(t) = \tilde{B}(t)^{-1} \mathcal{O}_{L^2\to L^2}(\hbar)\, B(t)\,.$$

From Duhamel's formula and the initial condition $\tilde{B}(0)^{-1}B(0) = I$, we infer

$$\tilde{B}(t)^{-1} B(t) = I + \mathcal{O}_t(\hbar) \Longrightarrow B(t) = \tilde{B}(t) + \mathcal{O}_t(\hbar)\,.$$

Hence $B(t) \in \Psi^0$ has the principal symbol $b(t)$. $\square$

2.3.2. *From propagator estimate to refined constraints on the decay rates.* From the factorization of Lemma 6 one may obtain sharper constraints on quantum decay rates than in Lemma 4.

Take $P = P_0 + i\hbar \operatorname{Op}_\hbar(q)$, $q \in S^1(X)$, and u a normalized eigenstate of P,

$$Pu = zu\,, \quad z \in D(1/2, C\hbar).$$

We have shown in Lemma 5 that u is microlocalized on S^*X: for $\chi \in C_c^\infty((1/2\pm\epsilon))$, $\chi(s) = 1$ near $s = 1/2$, one has

$$u = \chi(P_0)u + \mathcal{O}(\hbar^\infty)\,.$$

Hence, if we replace Q by $\tilde{Q} = \chi_1(P_0)Q$, where $\chi_1 \in C_c^\infty(\mathbb{R})$, $\chi_1 = 1$ on $\operatorname{supp}\chi$, we also have

$$\tilde{P}u = zu + \mathcal{O}(\hbar^\infty), \quad .$$

Then, for any $t \in \mathbb{R}$, we apply the Schrödinger propagator $\tilde{V}^t = e^{-it\tilde{P}/\hbar}$ to u. From the above identity, we have

$$\tilde{V}^t u = e^{-itz/\hbar}u + \mathcal{O}_t(\hbar^\infty)\,.$$

On the other hand, using Lemma 6 for $\tilde{V}^t$, we get:

$$U^t\tilde{B}(t)u = e^{-itz/\hbar}u + \mathcal{O}_t(\hbar^\infty),$$

so that

$$\left\|\tilde{B}(t)u\right\|_{L^2\to L^2} = e^{t\operatorname{Im} z_n/\hbar}\,\|u\| + \mathcal{O}_t(\hbar^\infty).$$

This equation shows that the norm of $B(t)$ constrains the quantum decay rate:

$$e^{t\operatorname{Im} z/\hbar} - \mathcal{O}_t(\hbar^\infty) \le \left\|\tilde{B}(t)\right\| \le \max_{T^*X}\tilde{b}(t) + \mathcal{O}_t(\hbar), \tag{2.5}$$

where on the RHS we used the sharp Gårding inequality. Since the interval supporting χ_1 may be arbitrary small, we get for any t

$$e^{t\operatorname{Im} z/\hbar} \le \max_{S^*X} e^{\int_0^t q\circ\Phi^s ds} + o_{\hbar\to 0}(1). \tag{2.6}$$

This result shows that the imaginary parts $\operatorname{Im} z_n$ of the eigenvalues are constrained by the *ergodic averages* of the damping function q on the energy shell S^*X. Let us set up some notations. The *microcanonical* average of the damping on S^*X is denoted by

$$\bar{q} = \int_{S^*X} q(\rho)\,d\mu_L,$$

where μ_L is the (normalized) Liouville measure on S^*X.

The time (or ergodic) averages of q are denoted by

$$\langle q\rangle_T(\rho) = \frac{1}{T}\int_0^T q\circ\Phi^t(\rho)\,dt.$$

We will mostly restrict these averages to S^*X.

Fact 7. *[Birkhoff's theorem] These averages converge a.e. (w.r.to μ_L) to an L^∞ function*

$$\langle q\rangle_\infty(\rho) = \lim_{T\to\infty}\langle q\rangle_T(\rho).$$

If the flow is ergodic on S^*X, then $\langle q\rangle_\infty = \bar{q}$ a.e.

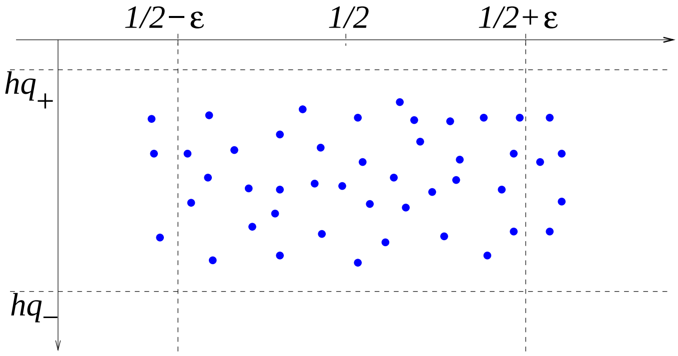

FIGURE 2.1. Eigenvalues of $P(\hbar)$ near $1/2$.

We also define the asymptotic extremal values of the damping on S^*X:

$$q_+ \stackrel{\rm def}{=} \lim_{T\to\infty} \sup_{\rho\in S^*X} \langle q\rangle_T = \sup_{\mu\in\mathcal{M}} \mu(q), \qquad q_- \stackrel{\rm def}{=} \lim_{T\to\infty} \inf_{\rho\in S^*X} \langle q\rangle_T = \inf_{\mu\in\mathcal{M}} \mu(q).$$

These are also the extremal values of the function $\langle q\rangle_\infty$.

From the bound (2.6) one obtains the following general bound on the distribution of the quantum decay rates.

Theorem 8. [Leb93, Sjo00] *For any $\epsilon > 0$ and for $\hbar < \hbar_\epsilon$, all eigenvalues $z_n \in D(1/2, C\hbar)$ lie in the strip $\left\{\frac{\operatorname{Im} z}{\hbar} \in [q_- - \epsilon, q_+ + \epsilon]\right\}$.*

Proof. From 2.6 we get for any $T > 0$

$$\operatorname{Im} z/\hbar \leq \max_{S^*X} \langle q\rangle_T + o_{\hbar\to 0}(1).$$

On the other hand, for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that for any $T \geq T_\epsilon$ one has

$$\max_{S^*X} \langle q\rangle_T \leq q_+ + \epsilon/2.$$

Taking $T = T_\epsilon$ and sending $\hbar \to 0$, we get

$$\operatorname{Im} z/\hbar \leq q_+ + \epsilon.$$

The lower bound is obtained by applying (2.6) for negative times. □

2.3.3. *Conjugating the generator P to time average the damping.* The proof of Thm 8 given by Sjöstrand in [Sjo00] does not use the propagator, but directly works on the operator $P(\hbar)$, by conjugating it by a well-chosen microlocal weight function $G_T = \operatorname{Op}_\hbar(g_T)$ indexed by a time parameter $T > 0$. The weight $g_T \in S^0(X)$ is constructed such as to take the following values near S^*X:

$$g_T = \frac{1}{2}\int_0^{T/2} \left(\frac{2s}{T} - 1\right) q\circ\Phi^s\, ds + \frac{1}{2}\int_{-T/2}^{0} \left(\frac{2s}{T} + 1\right) q\circ\Phi^s\, ds. \tag{2.7}$$

This weight has the following effect. The symbol of the conjugated operator

$$P_T \stackrel{\rm def}{=} e^{-G_T} P E^{G_T} = P + [P, G_T] + \mathcal{O}(\hbar^2) \tag{2.8}$$

reads

$$p_T = p - i\hbar\,\{p, g_T\} + \mathcal{O}(\hbar^2) = p_0 + i\hbar q - i\hbar H_{p_0}(g_T) + \mathcal{O}(\hbar^2), \tag{2.9}$$

$$\stackrel{\rm def}{=} p_0 + i\hbar q_T + \mathcal{O}(\hbar^2)\,. \tag{2.10}$$

The form of g_T near S^*X implies that the subprincipal symbol q_T is equal there to the damping function averaged over a time window $[-T/2, T/2]$:

$$q_T(\rho) = \langle q \rangle^T(\rho) \stackrel{\text{def}}{=} \frac{1}{T}\int_{-T/2}^{T/2} q \circ \Phi^s(\rho)\, ds \quad \text{near } S^*X.$$

One notices that the principal and subprincipal symbol are "a bit more commutative" than the original ones:

$$\{p_0, q_T\} = \mathcal{O}(1/T) \quad \text{near } S^*X,$$

which explains why this averaging procedure leads to better Weyl estimates.

One can then directly apply Lemma 4 to the operator P_T to obtain Thm 8, as well as the following

Corollary 9. *For any $\gamma > q_+$ and $\hbar$ small enough, we have the resolvent bound*

$$\left\|(P(\hbar) - z)^{-1}\right\| \leq \frac{C_\gamma}{\hbar}, \qquad \forall z \in D(1/2, C\hbar) \cap \{\operatorname{Im} z/\hbar \geq \gamma\}.$$

Let us apply this resolvent bound to the DWE operator $P(\tau)$ (1.5) and its semiclassical version (1.9), $P(\hbar, z) - z = \frac{\hbar^2}{2} P(\tau) = -\hbar^2\Delta/2 - i\hbar a + \mathcal{O}(\hbar^2)$. Let us assume the geometric control condition holds, so that $a_- > 0$. In this case, the above corollary shows that, for any $\epsilon > 0$, one has

$$\left\|(P(\hbar) - z)^{-1}\right\| \leq \frac{C_\gamma}{\hbar}, \qquad z \in D(1/2, C\hbar) \cap \{\operatorname{Im} z/\hbar \geq -a_- + \epsilon\}.$$

Undoing the semiclassical scaling, this proves the resolvent estimate announced in (1.19):

$$\left\|P(\tau)^{-1}\right\|_{L^2 \to L^2} \leq \frac{C'_\gamma}{\tau}, \quad \tau \in \{|\operatorname{Re} \tau| \geq C,\ \operatorname{Im} \tau \geq -a_- + \epsilon\}.$$

2.4. **Questions on the spectral distribution.** The following questions were raised in [Sjo00, AschLeb, Anan10] concerning the semiclassical distribution of the quantum decay rates $\operatorname{Im} z_n(\hbar)/\hbar$.

(1) is there asymptotic distribution of the quantum decay rates when $\hbar \to 0$? More precisely, for a given interval $I \subset [q_-, q_+]$ and $1 \gg \epsilon \gg \hbar$, does the ratio

$$\frac{\#\left\{n,\ |\operatorname{Re} z_n - 1/2| \leq \epsilon,\ \frac{\operatorname{Im} z_n}{\hbar} \in I\right\}}{\#\{n,\ |\operatorname{Re} z_n - 1/2|\}}$$

have a limit when $\hbar \to 0$? Is this distribution related with the value distribution of $\langle q \rangle_\infty$ (which Asch-Lebeau call the *geometric distribution*)? If there is not a unique limit distribution, how do the various limits look like?

(2) What are the possible accumulation points of the quantum decay rates? In particular, are there sequences of decay rates $(\operatorname{Im} z(\hbar)/\hbar)_{\hbar \to 0}$ converging to the classical extremal values $q_\pm$? (that is, is the result of Thm 8 sharp?)

(3) [Inverse problem] [Anan10] Can one recover the damping function q from the knowledge of $z_n(\hbar)$? Note that we can hope to recover q only up to cohomology equivalence $q \equiv q + \{p_0, b\}$, with b is any smooth function. A positive answer is conjectured for manifolds X of negative curvature and simple length spectrum: a trace formula should allow to recover all the averages $\int_\gamma q$ from the spectrum, which is equivalent of recovering the cohomology class of q.

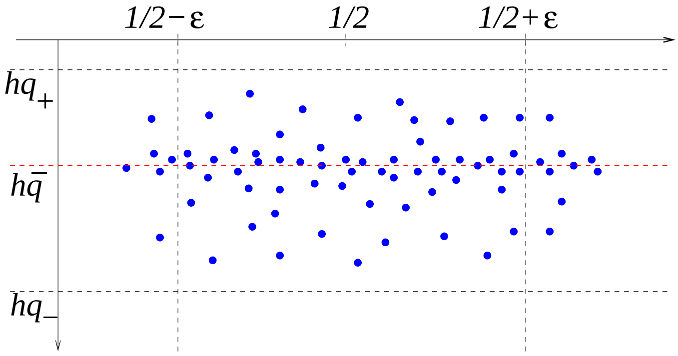

FIGURE 2.2. Case of an ergodic flow: concentration of the eigenvalues on the line $\{\operatorname{Im} z/h = \bar{q}\}$.

(4) [Inverse problem 2] [Anan10]Is the knowledge of the *imaginary parts* $\operatorname{Im} z_n(\hbar)$ sufficient to recover q? In particular, if *all* imaginary parts $\frac{\operatorname{Im} z_n(\hbar)}{\hbar} \to \bar{q}$, does this imply that q is cohomologous to $\bar{q}$? A positive answer is conjectured in [Anan10], in the case of the twisted Laplacian (3.17).

Sjöstrand [Sjo00] addressed question 1. To express the result, we need to consider the essential supremum and infimum of the function $\langle q \rangle_\infty$,

$$q_{ess+} = \text{ess-sup}\langle q \rangle_\infty, \quad q_{ess-} = \text{ess-inf}\langle q \rangle_\infty,$$

and notice that

$$q_- \leq q_{ess-} \leq \bar{q} \leq q_{ess+} \leq q_+,$$

where all inequalities may be strict.

Theorem 10. [Sjo00] *For any* $C > 0$, $\epsilon > 0$,

$$\#\left\{n,\ |\operatorname{Re} z_n - 1/2| \leq C\hbar,\ \frac{\operatorname{Im} z_n}{\hbar} \notin [q_{ess-} - \epsilon, q_{ess+} + \epsilon]\right\} = o(\hbar^{-d+1}).$$

Comparing this bound with the Weyl law (2.1) shows that almost all *the quantum decay rates are contained in the interval* $[q_{ess-} - \epsilon, q_{ess+} + \epsilon]$.

Corollary 11. [Sjo00]*If the geodesic flow on* (X, g) *is* ergodic*, then almost all quantum decay rates have values close to* $\bar{q}$.

In this case, the answer to the question 1 is positive, and the limit distribution $\delta_{\bar{q}}$ *is equal to the geometric distribution.*

2.5. **Eigenvalue counting and phase space volumes.** How is the distribution of the quantum decay rates related with the so-called *geometric distribution*, namely the value distribution of $\langle q \rangle_T$ on S^*X? This can be understood by a simple argument. We recall the Weyl law of Thm 3, which counts eigenvalues with $\operatorname{Re} z$ close to 1/2. If we want to restict ourselves to the eigenvalues in the domain

$$\Omega(C, \alpha) \stackrel{\text{def}}{=} \{\operatorname{Re} z \in [1/2 \pm C\hbar]\,,\ \operatorname{Im} z/\hbar \geq \alpha\}\,,$$

it is tempting to believe that they are associated to the phase space region

$$\mathcal{R}_T(C, \alpha) = p_0^{-1}\left([1/2 \pm C\hbar]\right) \cap q_T^{-1}\left([\alpha, q_+]\right),$$

and then estimate the number of these eigenvalues through the phase space volume of the region $\mathcal{R}_T(C, \alpha)$. If P were a normal operator, meaning that P_0 and Q_T would commute, the correspondence would be effective. However, due to the nonnormality of P, we will see that such phase space arguments can only provide *upper bounds* for the number of eigenvalues. One good reason to average over time the original operator is indeed to make the operator "less nonnormal", by reducing the commutator between real and imaginary parts:

$$[P_0, Q]\,\chi(P_0 - 1/2) = \mathcal{O}(\hbar), \qquad [P_0, Q_T]\,\chi(P_0 - 1/2) = \mathcal{O}(\hbar/T). \tag{2.11}$$

In general there are several ways to get upper bounds on the distribution of the $\operatorname{Im} z_n/\hbar$.

One possibility is to connects the eigenvalues of P_T with the singular values of a related operator, and then use Weyl's inequalities. This will be the approach we will pursue in the case of quantum maps. In the next section we will use a different approach, apparently more flexible in the case of a Schrödinger flow.

2.6. Eigenvalue counting using perturbations with controlled trace norm.

2.6.1. *General strategy.* A standard method to estimate the number of eigenvalues of P in the region $\Omega(C, \alpha)$ is to perturb P by an operator δP, such that the perturbed operator

$$\tilde{P} = P + \delta P$$

does not have eigenvalues in the region $\Omega(C, \alpha)$, the operator norm of δP is under control, and the *trace norm* of δP is also controlled in terms of a phase space volume of a region $\mathcal{R}_T(C', \alpha')$. The method is similar with the one used to get the Weyl law of Thm 3.

More precisely, for $z \in \Omega(C, q_{\max})$ we write the identity

$$(z - P) = \left(z - \tilde{P}\right)(1 + K(z)), \qquad K(z) = \left(z - \tilde{P}\right)^{-1} \delta P.$$

We want to control both $\left\|\left(z - \tilde{P}\right)^{-1}\right\|$ for $z \in \Omega(C, \alpha)$ and the trace norm $\|\delta P\|_{tr}$, so that both the operator and trace norms of $K(z)$ are under control for $z \in \Omega(C, \alpha)$. The eigenvalues are then given by solving

$$d(z) = 0, \quad \text{where} \quad d(z) \stackrel{\text{def}}{=} \det(1 + K(z)), \quad z \in \Omega(C, \alpha).$$

Consider a disk $D(z_0, \gamma r) \subset \Omega(C, \alpha)$, with $z_0 \in \Omega(C, q_{\max})$ and $\gamma = 1 + \epsilon$. One can use Jensen's formula to bound the number of zeros of $d(z)$ in the slightly smaller disk $D(z_0, r)$. Indeed, if $d(z_0) \neq 0$, this formula yields we have

$$\sup_{z \in D(z_0, \gamma r)} \log|d(z)| \geq \log|d(z_0)| + n(z_0, r)\log\gamma,$$

where $n(z_0, r)$ is the number of zeros of $d(z)$ in the disk $D(z_0, r)$. The upper bound on $\log d(z)$ is provided by our control of the traces:

$$\log|d(z)| = \log|\det(1 + K(z))| \leq \|K(z)\|_{tr},$$

which will be effective for $z \in D(z_0, \gamma r)$, due to a good control on the resolvent:

$$\left\|(\tilde{P}_T - z)^{-1}\right\| \lesssim \hbar^{-1}, \quad z \in D(z_0, \gamma r).$$

We also need a good lower bound on $|d(z_0)|$. For this we use the formula

$$(1 + K(z_0))^{-1} = 1 + (z_0 - P)^{-1}\,\delta P$$

which leads to

$$\begin{aligned} -\log|d(z_0)| &= \log\left|\det\left(1 + (z_0 - P)^{-1}\,\delta P\right)\right| \\ &\leq \left\|(z_0 - P)^{-1}\,\delta P\right\|_{tr} \\ &\leq \left\|(z_0 - P)^{-1}\right\|\,\|\delta P\|_{tr}\,. \end{aligned}$$

We already control $\|\delta P\|_{tr}$. Since we assumed $\operatorname{Im} z_0/\hbar > q_{\max}$, we have the estimate of Lemma 4:

$$\left\|(z_0 - P)^{-1}\right\| \lesssim \hbar^{-1}. \tag{2.12}$$

We thus get the upper bound

$$n(z_0, r) \leq \frac{\|\delta P\|_{tr}}{\log\gamma}\left(\max_{z\in D(z_0,\gamma r)}\left\|\left(z - \tilde{P}\right)^{-1}\right\| + \left\|(z_0 - P)^{-1}\right\|\right) \tag{2.13}$$

$$\lesssim \frac{\|\delta P\|_{tr}}{\hbar\,\log\gamma}. \tag{2.14}$$

This upper bound for eigenvalues in a disk $D(z_0, r)$ can be easily adapted to count eigenvalues in rectangles of the form $\Omega(C,\alpha)$.

2.6.2. *Explicit construction of the perturbation* δP [Sjo00]. Let us now describe more precisely the perturbation δP considered in [Sjo00]. Instead of working directly with the operator P, it makes more sense (and brings reward) to work with the conjugated operator P_T of (2.8).

Recall that the subprincipal symbol q_T in (2.9), on S^*X, may take values in $[q_- - \epsilon, q_+ + \epsilon]$. In particular, for $\alpha < q_+$ the region $\{\rho \in S^*X,\ q_T \geq \alpha\}$ is nonempty. To set up the perturbation $\delta P = \delta P_T$, we construct a function $\tilde{q}_T \leq q_T$, such that $\tilde{q}_T = q_T$ in regions where $q_T(\rho) \leq \alpha - 2\epsilon$, while $\tilde{q}_T \leq \alpha - \epsilon$ in regions where $q_T(\rho) \geq \alpha - \epsilon$ (at least in some nbhd of S^*X). Replacing q by $\tilde{q}$ thus amounts to "cut" the large values of q_T. A first guess $\delta P = i\hbar\left(\tilde{Q}_T - Q_T\right)$ would ensure the invertibility of $\tilde{P} - z$ for $z \in \Omega(C,\alpha)$ and would control $\|\delta P\|$, but we would have no control on the trace. Since we work near the energy $1/2$, we could take instead

$$\delta P = \delta P_T = i\hbar\,\chi(P_0 - 1/2)\left(\tilde{Q}_T - Q_T\right)\chi(P_0 - 1/2),$$

with $\chi \geq 0$ a smooth cutoff localized in $[-\epsilon, \epsilon]$ and equal to unity in $[-\epsilon/2, \epsilon/2]$. This perturbation is trace class. However, since we are counting eigenvalues on a rectangle of width $\sim \hbar$, it is crucial to minimize this trace by cutting off in energy intervals of width $\hbar$. For this we take $\chi \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ (with $\hat{f}$ of compact support), and define the perturbation

$$\delta P_T = i\hbar\chi\left(\frac{P_0 - 1/2}{\hbar}\right)\left(\tilde{Q}_T - Q_T\right)\chi\left(\frac{P_0 - 1/2}{\hbar}\right).$$

This operator is not a PDO, but one can nevertheless estimate its norms:

$$\|\delta P_T\| \leq \hbar\,\max\chi^2\max_{S^*X}|\tilde{q}_T - q_T| + o(\hbar), \tag{2.15}$$

$$\|\delta P_T\|_{tr} \leq C_d\,\hbar^{2-d}\,\|\chi\|^2_{L^2}\int_{S^*X}(q_T - \tilde{q}_T)\,dL + o(\hbar^{2-d}). \tag{2.16}$$

Remark 12. To get the second bound, one needs to assume that either $\operatorname{supp}\hat{\chi} \subset (-T_{\min}/2, T_{\min}/2)$, where $T_{\min}$ is the length of the shortest closed geodesics, or that the union of all closed geodesics in S^*X has measure zero.

If we could analyze $\tilde{P}_T$ as a PDO, its symbol would be

$$\tilde{p}_T = p_0 + i\hbar\,\hat{q}_T + \mathcal{O}(\hbar^2), \qquad \hat{q}_T \stackrel{\text{def}}{=} q_T + \chi\left(\frac{p_0 - 1/2}{\hbar}\right)^2 (\tilde{q}_T - q_T)\,.$$

Assuming $\chi(E) \geq 1$ for $|E| \leq C$, the construction of $\tilde{q}$ implies that

$$\hat{q}_T(\rho) \leq \alpha - \epsilon \quad \text{on} \quad p_0^{-1}\left([1/2 \pm C\hbar]\right).$$

Hence, for $z \in \Omega(C/2, \alpha)$, we have $|z - \tilde{p}(\rho)| \geq \theta\hbar$, with $\theta = \min(\epsilon, C/2)$. Eventhough the symbol $\hat{p}_T$ is not in a decent symbol class, one can prove (following the proof of Lemma 4) the requested resolvent bound

$$\left\| \left(z - \tilde{P}_T\right)^{-1} \right\| = \mathcal{O}\left(\frac{1}{\theta\hbar}\right), \qquad z \in \Omega(C/2, \alpha). \tag{2.17}$$

2.6.3. *Ending the proof of Thm 10.* We can now combine the arguments of the previous section, inserted into the bound (2.16). We recall the estimate (2.12) for $\operatorname{Im} z_0/\hbar \geq q_+ + c$, $\operatorname{Re} z \in [1/2 \pm C\hbar]$. If we take a disk $D(z_0, \gamma r) \subset \Omega)C/2, \hbar)$, the bound (2.17) holds inside this disk. Importantly, the implied constants are uniform w.r.to T (the dependence on the derivatives of q_T appears in lower-order terms in $\hbar$). Once c, θ have been chosen, the trace norm of δP (see (2.16)) is proportional to the integral

$$\int (q_T - \tilde{q}_T)\, dL \leq \max\,(q_T - \tilde{q}_T)\, \operatorname{Vol}\{\rho \in S^*X,\ q_T(\rho) \geq \alpha - 2\epsilon\}\,, \tag{2.18}$$

since we assumed that $\tilde{q}_T = q_T$ in the region $\{q_T \leq \alpha - 2\epsilon\}$. We now recall that q_T is equal to the time average on S^*X. The definition of the essential supremum shows that if $\alpha > q_{ess+}$, then

$$\operatorname{Vol}\{\rho \in S^*X,\ \langle q\rangle_T(\rho) \geq \alpha\} \xrightarrow{T\to\infty} 0.$$

On the other hand, the maximal difference $\max(q_T - \tilde{q}_T)$ can be assumed uniformly bounded w.r.to T. Hence, if $\alpha - 2\epsilon > q_{ess+}$, for any $\delta > 0$ we may choose $T = T(\delta)$ such that $\int (q_T - \tilde{q}_T)\, dL \leq \delta$. As a result, we obtain

$$n(z_0, r) \leq C\,\delta\,\hbar^{1-d}, \tag{2.19}$$

with $C > 0$ independent of $T(\delta)$. This proves the part of the theorem for counting ingenvalues in $\{\operatorname{Im} z/\hbar > q_{ess+} + \epsilon\}$. The case of the eigenvalues $\{\operatorname{Im} z/\hbar < \alpha < q_{ess-} - \epsilon\}$ is performed similarly, now constructing $\tilde{q}_T \geq q_T$ so that $\hat{q}_T \geq \alpha + \epsilon$ in $p_0^{-1}\left([1/2 \pm C\hbar]\right)$.□

3. Damped waves on Anosov manifolds

In order to get more precise informations on the distribution of the eigenvalues $z_n(\hbar)$, one needs to make specific assumptions on the classical dynamics, namely the geodesic flow on X. For instance, the case of a completely integrable dynamics has been considered by Hitrik-Sjöstrand. The less rigid case of a KAM system some invariant tori has been studied by Hitrik-Sjöstrand-Vu Ngoc. In these cases, one can transform the Hamiltonian flow into a normal form near the torus, which leads to a precise description of the spectrum "generated" by the torus, which lives in some region of $D(1/2, C\hbar)$.

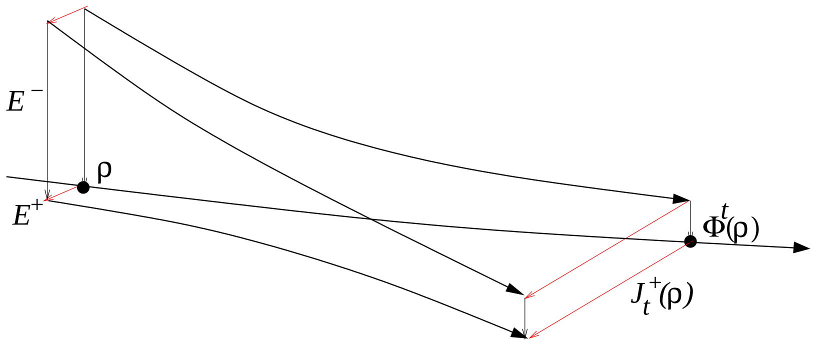

FIGURE 3.1. Structure of the Anosov flow near an orbit $\Phi^t(\rho)$

Asch-Lebeau [AschLeb] addressed the question 2 for the case of the 2D standard sphere, where they show that, if the damping function q is real analytic, then (under some generic condition) there exists $\epsilon_1 > 0$ such that, for $\hbar$ small enough, all eigenvalues $z_n \in D(1/2, C\hbar)$ satisfy

$$\operatorname{Im} z_n/\hbar \in [q_- + \epsilon_1, q_+ - \epsilon_1] .$$

In view of Thm 8, the above property can be called a *spectral gap*. It shows that, even in the semiclassical limit, the range of the quantum decay rates is strictly smaller than the range of classical decay rates. We will see that such a phenomenon may occur also in the case of Anosov geodesic flows.

3.0.4. *Anosov manifolds.* The opposite case of a *fully chaotic flow* was considered by Anantharaman [**?**] and Schenck [Schenck-pressure]. Such a flow is obtained if the manifold (X, g) has a negative sectional curvature everywhere: this negative curvature is responsible for the uniform hyperbolicity of the flow, and conjugated with the compactness of X one obtains an Anosov flow: with respect to the invariant Liouville measure on S^*X, the flow is ergodic and exponentially mixing flow; it mimicks a stochastic system. In a word, it is a strongly chaotic flow. A bit paradoxically, the long time properties of such a strongly chaotic flow are rather well understood, compared with less chaotic ones. Uniform hyperbolicity means that on each point $\rho \in S^*X$, there exists a splitting of the tangent space

$$T_\rho S^*X = \mathbb{R}H_p(\rho) \oplus E^+(\rho) \oplus E^-(\rho),$$

where $H_p(\rho)$ is the Hamiltonian vector field, $E^\pm(\rho)$ are the unstable and stable subspaces at the point ρ. They both have dimension $d-1$, are uniformly transverse. The families $\{E^\pm(\rho),\, \rho \in S^*X\}$ form the unstable/stable distributions; they are invariant w.r.to the flow ($d\Phi^t_\rho E^\pm(\rho) = E^\pm(\Phi^t(\rho))$), Hölder continuous, and are characterized by the following contraction property: there exists $C, \lambda > 0$ such that

$$\forall \rho \in S^*X,\ \forall v \in E^\mp(\rho),\ \forall t > 0, \quad \left\| d\Phi^{\pm t}_\rho v \right\| \leq C\, e^{-\lambda t}\, \|v\| ,$$

That is, we have exponential contraction along the direction E^- in the future, exponential contraction along E^+ in the past.

Two important quantities will play a role:

$$\text{(3.1)} \qquad \text{the maximal expansion rate} \quad \lambda_{\max} \stackrel{\text{def}}{=} \lim_{t\to\infty} \sup_{\rho \in S^*X} \frac{1}{t} \log \left\| d\Phi^t_\rho \right\|$$

$$\text{the unstable Jacobian} \quad J^+(\rho, t) = \det \left(d\Phi^t \restriction_{E^+(\rho)} \right), \quad t > 0.$$

The Jacobian depends on the choice of the norm chosen on the spaces $T_\rho S^*X$, so it is not intrinsic. However, its long time asymptotics, or its value along periodic geodesics, will be independent of this choice. For $t > 0$ large enough, this Jacobian grows exponentially. In practice, the contraction inside E^- is not isotropic, but proceeds slower or faster along certain directions of E^-. The various contraction rates are given by the positive Lyapunov exponents $\lambda_j(\rho) > 0$. The parameter λ is a lower bound for the positive exponents.

The unstable Jacobian is often presented in its infinitesimal version,

$$\varphi^+(\rho) = \lim_{t\to 0} \frac{\log J^+(\rho, t)}{t}.$$

3.0.5. *Manifolds of constant negative curvature.* A particular class of Anosov manifolds consists in quotients of the d-dimensional hyperbolic space $\mathbb{H}^d$ by co-compact subgroups of its isometry group. These manifolds have a constant curvature $-\Lambda^2$[5]; one can then show that, using the natural norms on $T_\rho S^*X$, that for such manifolds

$$\lambda_{\max} = \Lambda, \quad J^+(\rho, t) = e^{(d-1)\Lambda t}, \quad \varphi^+(\rho) = \Lambda(d-1).$$

Also, in each fiber $E^+(\rho)$ the positive Lypapunov exponents are all equal to Λ: the contraction/expansion is both isotropic and homogeneous. The homogeneity of the hyperbolicity simplifies the results, and makes them "optimal" compared with the case of variable curvature.

3.1. **Fractal Weyl upper bounds for the distribution of quantum decay rates.**

3.1.1. *Averaging up to the Ehrenfest time.* Assuming X has constant negative curvature $-\Lambda^2$, Anantharaman [Anan10] was able to improve Sjöstrand's method for the proof of Thm 10 (which for such flows takes the form of Corol. 11), by letting the averaging time T explicitly depend on $\hbar$:

$$T = T_{Ehr} = (1-\epsilon)\,\frac{\log 1/\hbar}{\lambda_{\max}}. \tag{3.2}$$

This time is called the *Ehrenfest time.* For an observable $f \in S^0(X)$ supported in a thin neighbourhood of S^*X, the classically evolved observable $f \circ \Phi^t$ remains in the "decent" symbol class $S^0_{1/2-\epsilon}(X)$ for $|t| \leq T_{Ehr}$. We recall that for $\delta \in [0, 1/2)$, the symbols $g \in S^0_\delta$ generally depend in $\hbar$, they may become more singular when $\hbar \to 0$, but in a controlled way:

$$\forall \alpha \in \mathbb{N}^d, \forall \rho \in T^*X, \quad |\partial^\alpha g(\rho)| \leq C_\alpha\, \hbar^{-\delta|\alpha|}.$$

In this class one can still use pseudodifferential calculus, and the expansions in powers of $\hbar$ still make sense.

One can easily construct a microlocal weight g_T taking values (2.7) near S^*X, and which remains in the class $S^0_{1/2-\epsilon}$ if $T \leq T_{Ehr}$. The resulting symbol q_T in (2.9) will also belong to this class.

[5]Usually one normalizes the metrics on $\mathbb{H}^d$ so that the curvature is -1. However, we prefer to keep track of this curvature, that is of the hyperbolicity of the flow, in our notations.

3.1.2. *Large deviation estimates for Anosov flows – constant negative curvature.* In the case $T = T_{Ehr}$, the counting estimate (2.13), the resolvent estimates (2.12,2.17) and the trace norm estimate (2.16) are still valid, as well as the connection (2.18) with the volume of the region $\{q_T \geq \alpha - 2\epsilon\}$.

Now, compared with a generic ergodic system, the Anosov property of the flow results in more precise convergence of the volumes when $T \to \infty$. Such bounds are called large deviation estimates. For any smooth observable q on S^*X, these estimates take the following form ine the case of a manifold of constant curvature $-\Lambda^2$.

For any closed interval I,

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathrm{Vol}\{\rho \in S^*X,\ \langle q\rangle_T(\rho) \in I\} \leq \sup_{s\in I} H(s) - (d-1)\Lambda. \tag{3.3}$$

The function $s \mapsto H(s)$ is called the *rate function* for these large deviations. It can be defined as follows:

$$\forall \alpha \in \mathbb{R}, \qquad H(\alpha) \stackrel{\text{def}}{=} \sup\{h_{KS}(\mu),\ \mu \in \mathcal{M},\ \mu(q) = \alpha\}, \tag{3.4}$$

where $\mathcal{M}$ is the set of probability measures on S^*X invariant w.r.to the flow Φ^t. This rate function is continuous on $[q_-, q_+]$, strictly concave and real-analytic on (q_-, q_+), equal to $-\infty$ outside $[q_-, q_+]$. Its maximum is reached at $H(\bar{q}) = \Lambda(d-1)$.

There is an alternative definition for it. For any observable f, the *topological pressure* $\mathcal{P}(f)$ associated with f and the geodesic flow on S^*X can be defined as

$$\mathcal{P}(f) = \sup\{h_{KS}(\mu) + \mu(f),\ \mu \in \mathcal{M}\}.$$

In the case of an Anosov flow, the pressure also reflects the statistics of long periodic orbits:

$$\mathcal{P}(f) = \lim_{T\to\infty} \frac{1}{T} \log \sum_{\gamma: l(\gamma)\in[T,T+1]} e^{\int_0^{l(\gamma)} f(\Phi^t\rho_\gamma)\,dt},$$

where we sum over periodic orbits $\gamma \subset S^*X$ of period $T \leq l(\gamma) \leq T+1$, and ρ_γ is any point on γ.

Obviously, for any $\beta \in \mathbb{R}$, we have

$$\mathcal{P}(\beta q) = \sup_{\alpha\in\mathbb{R}} (\alpha\beta + H(\alpha)).$$

The above expression is a Legendre transform. Conversely, the rate function can be obtained through the inverse Legendre transform

$$H(\alpha) = \inf_{\beta\in\mathbb{R}} (\mathcal{P}(\beta q) - \beta\alpha).$$

In particular, the extremal values are given by

$$H(q_+) = \lim_{\beta\to+\infty} \mathcal{P}(\beta q) - \beta q_+, \qquad H(q_-) = \lim_{\beta\to-\infty} \mathcal{P}(\beta q) - \beta q_-.$$

3.1.3. *Fractal Weyl upper bounds on Anosov manifolds – constant curvature.* The norm bound (2.16) can be sharpened to include higher orders in $\hbar$, leading to

$$\|\delta P_T\|_{tr} \leq C_d\, \hbar^{2-d} \left[\|f\|_{L^2}^2 \int_{S^*X} (q_T - \tilde{q}_T)\, dL + \sum_{k=1}^{N-1} \hbar^k D^{2k} \hat{f}^2(0) \int_{S^*X} \left|D^{2k}(q_T - \tilde{q}_T)\right| dL + \mathcal{O}(\hbar^{N(1-2\epsilon)}\right.$$

This estimate is valid for symbols $q_T, \tilde{q}_T \in S^0_{1/2-\epsilon}$. The bound (2.18) can be easily generalized to the derivatives $D^{2k}(q_T - \tilde{q}_T)$, leading to

$$\|\delta P_T\|_{tr} \leq C_{d,q,f}\, \hbar^{2-d}\, \mathrm{Vol}\{\rho \in S^*X,\ q_T(\rho) \geq \alpha - 2\epsilon\}, \quad T = T_{Ehr},$$

valid for $\hbar$ small enough. Finally, the volume estimate (3.3) applied to the time $T = T_{Ehr}$, for $q_+ > \alpha > \alpha - 2\epsilon > \bar{q}$, leads to the trace bound:

$$\begin{aligned} \|\delta P_T\|_{tr} &\leq C_{d,q,f}\, \hbar^{2-d}\, e^{(\epsilon + H(\alpha-2\epsilon) - \Lambda(d-1))T_{Ehr}} \\ &\leq C_{d,q,f}\, \hbar^{2-d}\, \hbar^{-(\epsilon+H(\alpha-2\epsilon)-\Lambda(d-1))\frac{1-\epsilon}{\Lambda}} \\ &\leq C_{d,q,f}\, \hbar^{2-d}\, \hbar^{d-1-\frac{H(\alpha)}{\Lambda} - C\epsilon}. \end{aligned}$$

Since $H(\alpha) < \Lambda(d-1)$ for $\alpha > \bar{q}$, the exponent in the second factor is positive, so we have gained a power of $\hbar$ compared with the case of finite T. On the other hand, the bounds for the resolvents involved in (2.13) are unchanged, so one gets, using the same notations, the following bound for the number of eigenvalues in $D(z_0, r) \subset \{\mathrm{Im}\, z/\hbar \geq \alpha + \epsilon\}$:

$$n(z_0, r) \leq C_\epsilon\, \hbar^{-\frac{H(\alpha)}{\Lambda} - C\epsilon}.$$

One again, this estimate in a disk can be modified to give an estimate in $\Omega(C, \alpha)$. We have then obtained *fractal Weyl upper bounds* for the number of eigenvalues with $\mathrm{Im}\, z/\hbar > \alpha > \bar{q}$.

Theorem 13. [Anan10]*Assume X has constant negative curvature $-\Lambda^2$. For any $\alpha \geq \bar{q}$, $\epsilon > 0$, one has for $\hbar$ small enough*

$$\#\left\{n,\ |\mathrm{Re}\, z_n(\hbar) - 1/2| \leq C\hbar,\ \frac{\mathrm{Im}\, z_n(\hbar)}{\hbar} \geq \alpha\right\} \leq \hbar^{-\frac{H(\alpha)}{\Lambda} + \epsilon}, \tag{3.5}$$

where $H(\alpha)$ is the rate function defined in (3.4).

Here $(d-1)\Lambda = \lambda^u$ can be interpreted as the (uniform) infinitesimal unstable Jacobian, it is also the topological entropy of the flow.

3.1.4. *Case of nonconstant negative curvature.* In case the infinitesimal unstable Jacobian $\varphi^+(\rho)$ is nonconstant, the large deviation estimates (3.6) should be replaced by the following expressions. For I a closed interval,

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathrm{Vol}\{\rho \in S^*X,\ \langle q\rangle_T(\rho) \in I\} \leq \sup\left\{\tilde{H}(s),\ s \in I\right\}, \tag{3.6}$$

with the new rate function

$$\tilde{H}(s) = \sup\left\{h_{KS}(\mu) - \mu(\varphi^+),\ \mu \in \mathcal{M},\ \mu(q) = s\right\}, \tag{3.7}$$

which generalizes $H(s) - \Lambda(d-1)$ in the constant curvature case. This function satisfies

$$\tilde{H}(s) \leq \mathcal{P}(-\varphi^+) = 0,$$

with equality reached only for the measure $\mu = dL$, and therefore $s = \bar{q}$.

A global Ehrenfest time for the flow Φ^t on S^*X is

$$T_{Ehr} = \frac{(1-\epsilon)\log 1/\hbar}{\lambda_{\max}},$$

where $\lambda_{\max}$ is the largest expansion rate (3.1).

By using the same method as above, one obtains a fractal upper bound for $\alpha > \bar{q}$:
(3.8)

$$\#\left\{n,\ |\operatorname{Re} z_n(\hbar) - 1/2| \leq C\hbar,\ \frac{\operatorname{Im} z_n(\hbar)}{\hbar} \geq \alpha\right\} \leq \hbar^{-\frac{\tilde{H}(\alpha)}{\lambda_{\max}}+(d-1)-\epsilon}, \qquad \hbar < \hbar_{\alpha,\epsilon}.$$

This estimate can be improved in dimension 2 [Anan10] by letting the time T depend on the phase space points (that is, using local Ehrenfest times).

3.2. **Finer spectral gaps for Anosov manifolds.** We have seen above that upper bounds for the number of eigenvalues in $\hbar$-boxes $\Omega(C,\alpha)$ could be obtained by working directly on the generator $P(\hbar)$ of the quantum dynamics, namely by conjugating this generator with an appropriate microlocal weight. One then obtained a conugated generator P_T which is still a nonselfadjoint PDO, upon which one can apply standard counting methods (as in §2.6). Grossly speaking, this conjugation had the aim to replace the subprincipal symbol q of P, which is generally not invariant w.r.to Φ^t, by a subprincipal symbol q_∞ which would be invariant under the flow. If this were possible, we would ideally get at the quantum level a subprincipal symbol $\mathrm{Op}_\hbar(q_\infty)$ commuting with P_0, so that the distribution of the imaginary parts would be given by the Weyl law for $\mathrm{Op}(q_\infty)$.

The construction presented in §2.3.3 and improved in §3.1.1, had the objective to approach the function q_∞ as much as possible, by averaging q over the time evolution. We saw that the full average $\langle q\rangle^\infty$ is not a nice function on S^*X, since it is not well-defined everywhere. Furthermore, on an Anosov manifold this function is equal to $\bar{q}$ almost everywhere. In any case, in order to keep q_T in a decent symbol class one needs to bound the averaging time by the Ehrenfest time T_{Ehr}, resulting in a function q_T still far from commuting with the flow. As a result, we could only obtain upper bounds on the number of quantum decay rates $\operatorname{Im} z/\hbar$ away from the typical value $\bar{q}$.

These upper bounds leave the possibility for these quantum decay rates to take values in the full classical range $[q_-, q_+]$. We remind the question 2 in §2.4: can quantum decay rates accumulate, when $\hbar \to 0$, to any value in this interval (that is, to any classically allowed value)?

3.3. **A pressure estimate on the propagator.** In this section we will show that direct estimates on the propagator $V^t = e^{-itP/\hbar}$ can lead to nontrivial constraints on the quantum decay rates, namely the fact that all quantum decay rates belong to a strictly smaller interval $[q_- + \epsilon_-, q_+ - \epsilon_+]$ when $\hbar$ is small enough.

What are these estimates?

We recall Lemma 6 of §2.3. It implies that, when restricting oneself to a thin energy window, the norm of the propagator at time t is given by

$$\left\|V^t\right\|_{L^2\to L^2} = \|B(t)\|_{L^2\to L^2} = \|b(t)\|_{L^\infty} + \mathcal{O}_t(\hbar).$$

Here we recall that $b(t) = e^{\int_0^t q\circ\Phi^s\,ds}\chi(p_0)$ is the principal symbol of $B(t) \in \Psi^0$. The asymptotic bound $\sup_{S^*X} b(t) \leq e^{t(q_+ + \epsilon)}$ for $t > T_\epsilon$ large enough, lead us to the bound

$$\|B(t)\|_{L^2\to L^2} \leq e^{t(q_+ + 2\epsilon)}, \quad t > T_\epsilon \text{ finite},$$

and then directly to the bound of Thm 8.

In order to improve on this estimate, one would need to let the time t depend on $\hbar$. The above argument can be pursued for times t as large as $T_{Ehr}/2$, where T_{Ehr} is the Ehrenfest time (3.2). This way, the operator $B(t)$ still belongs to a decent PDO class $\hbar^{-\beta}\Psi^0_{1/2-\epsilon}$, and admits the function $b(t) = e^{t/2\langle q\rangle_{t/2}}\chi(p_0)$ as a leading symbol. This shows that, even for this logarithmic time, the bound on

$$\|B(t)\|_{L^2\to L^2} = \|b(t)\|_{L^\infty} + \mathcal{O}_t(\hbar^\epsilon), \quad t \leq T_{Ehr}/2,$$

is still valid.

In order to get new information on the extremal decay rates, one actually needs to study the propagator V^t (or the operator $B(t)$) *beyond (half) the Ehrenfest time.* The relevant times scales will still be logarithmic in $\hbar$, but they will be of the form

$$t \sim K \log 1/\hbar, \quad \text{with } K > 0 \text{ arbitrary large.} \tag{3.9}$$

For such large times, the operator $B(t)$ is no more a PDO, and its norm is a priori unrelated with the function $b(t)$. For these long times, we will actually use the representation $V^1\chi(P_0) = U^1 B(1)$, that is integrate the damping over time 1. The first estimate we will present on $\|V^t\chi(P_0)\|$ will be obtained in a very different manner from the previous symbol calculus. It will use a phase space partition of unity, and a crucial *hyperbolic dispersion estimate.*

3.3.1. *Using a quantum partition of unity.* In order to estimate the norm $\|V^t\chi(P_0)\|$ for $t \sim \mathcal{K}\log 1/\hbar$, we will propagate an arbitrary state $\psi \in L^2(X)$, normalized to unity. The cutoff $\chi(P_0)$ has the effect to microlocalize the state $\psi_0 = \chi(P_0)$ in a thin neighbourhood of S^*X. Then, the strategy consists in splitting ψ_0 into finitely many pieces, each one microlocalized in a small domain in $p_0^{-1}([1/2 \pm \epsilon])$. For this we use an open cover

$$p_0^{-1}([1/2 \pm \epsilon]) \subset \bigcup_{j=1}^{J} W_j,$$

each W_j being an open subset of T^*X of small diameter. One then constructs a smooth partition of unity $\{\pi_j \in C_c^\infty(W_j, [0,1]),\ j = 1, \ldots, J\}$ adapted to this open cover:

$$\sum_{j=1}^{J} \pi_j = 1 \quad \text{near } p_0^{-1}([1/2 \pm \epsilon]).$$

This partition can be quantized into $\Pi_j \stackrel{\text{def}}{=} \mathrm{Op}_\hbar(\pi_j)$, which is a partition of unity microlocally near S^*X:

$$\sum_{j=1}^{J} \Pi_j \equiv I \text{ microlocally near } p_0^{-1}([1/2 \pm \epsilon]). \tag{3.10}$$

Our aim is to estimate $\|V^t\psi_0\|$. A crucial property is the fact that $V^t\psi_j$ remains microlocalized inside $p_0^{-1}([1/2 \pm \epsilon])$ even for a "long logarithmic" time t.

Lemma 14. *Consider two cutoffs $\chi_1, \chi_2 \in C_c^\infty([1/2 \pm \epsilon])$, with $\chi_2 \equiv 1$ near $\operatorname{supp}\chi_1$. Then,*

$$\chi_2(P_0)\, V^t\, \chi_1(P_0) = V^t\, \chi_1(P_0) + \mathcal{O}_{L^2\to L^2}(\hbar^\infty),$$

uniformly for $|t| \leq \mathcal{K}\log 1/\hbar$.

Proof. EXERCISE. Hint: insert a family of nested cutoffs for times $1, 2, \ldots, t$. $\square$

Using this property, we can split $V^t\psi_0$ using the partition of unity (3.10). Assuming $t = n \in \mathbb{N}$, we may write

$$V^n\psi_0 = \sum_{\alpha_0,\alpha_1,\ldots,\alpha_n} V_{\alpha_0\alpha_1\cdots\alpha_n}\psi_0 + \mathcal{O}(\hbar^\infty), \quad V_{\alpha_0\cdots\alpha_n} \stackrel{\text{def}}{=} \Pi_{\alpha_n} V \Pi_{\alpha_{n-1}} \cdots V\Pi_{\alpha_1} V \Pi_{\alpha_0},$$

where all indices α_j run over $1,\ldots,J$. For a bounded time t , each product $V_{\alpha_0\alpha_1\cdots\alpha_n}$ can be written as $V_{\alpha_0\alpha_1\cdots\alpha_n} = V^n\Pi_{\alpha_0\alpha_1\cdots\alpha_n}$, where $\Pi_{\alpha_0\alpha_1\cdots\alpha_n}$ is a PDO which "projects" on points sharing the "history" $\alpha_0\cdots\alpha_n$. These are the points $\rho \in W_{\alpha_0}$ which will be in W_{α_j} at time j for any $0 \le j \le n$: they define the subset

$$W_{\alpha_0\cdots\alpha_n} = W_{\alpha_0} \cap \Phi^{-1}W_{\alpha_1} \cap \Phi^{-2}W_{\alpha_2} \cap \cdots \Phi^{-n}W_{\alpha_n}.$$

This description leads to the simple norm estimates for finite $n = |\boldsymbol{\alpha}|$:

$$\|V_{\boldsymbol{\alpha}}\| = \mathcal{O}(\hbar^\infty) \text{ if } W_{\boldsymbol{\alpha}} = \emptyset \tag{3.11}$$

$$\|V_{\boldsymbol{\alpha}}\| \le b(\boldsymbol{\alpha}) + \mathcal{O}(\hbar) \text{ otherwise}, \quad b(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{W_{\boldsymbol{\alpha}}} b(n).$$

Fixing some large but finite N, we select the sequences $\boldsymbol{\alpha} = \alpha_0\cdots\alpha_N$ effectively corresponding to physical trajectories, forming the set of sequences $\mathcal{A}_N$. Below these allowed sequences will be renamed β_j.

3.3.2. *Local adapted Fourier expansion and a hyperbolic dispersion estimate.* To improve on the bound (3.11) for logarithmic times, consists in splitting the state $\psi_{\alpha_0} = \Pi_{\alpha_0}\psi_0$ into an adapted Fourier expansion. Namely, we may choose local Darboux coordinates (y,η) in W_{α_0}, such that $\eta_1 = p_0 - 1/2$ is the energy variable, $W_{\alpha_0} \subset \{|y| \le \epsilon,\ |\eta| \le \epsilon\}$, and, most importantly, such that the local "momentum" Lagrangian leaves $\Lambda\{(y,\eta),\ \eta = \eta_0\}$ are close from tangent to the weak unstable foliation $\mathbb{R}H_{p_0} \oplus E^+$. To each leaf $\Lambda_{\eta_0}$is associated a "local momentum state" e_{η_0}, microlocally supported near W_{α_0}, which we can assume to be L^2-normalized. The adapted Fourier expansion reads

$$\psi_{\alpha_0} = \int_{|\eta|\le\epsilon} \psi(\eta)\, e_\eta \,\frac{d\eta}{(2\pi\hbar)^{d/2}} + \mathcal{O}(\hbar^\infty), \tag{3.12}$$

with the bound $\|\psi\|_{L^1} = \mathcal{O}(1)$. We will then specifically propagate the "momentum states" e_η individually through $V_{\boldsymbol{\alpha}}$. Because e_η is a WKB (or Lagrangian) state supported on a Lagrangian leaf close to unstable, its image through $\Pi_{\alpha_1}V$ will also be such a Lagrangian state, localized on the leaf $\Phi^1(\Lambda_\eta)\cap W_{\alpha_1}$. It turns out that one can precisely describe the Lagrangian state $V_{\boldsymbol{\alpha}}e_\eta$, even for times $n = |\boldsymbol{\alpha}| \sim \mathcal{K}\log 1/\hbar$: it will be a localized on some Lagrangian Λ_n exponentially close to the unstable manifold. In this process we are helped by the expanding dynamics along E^+.

The amplitude of the Lagrangian state $V_{\boldsymbol{\alpha}}e_\eta$ can be described to any order in $\hbar$. Its sup-norm depends on both the *damping* accumulated along the trajectory, and the *instability* of the dynamics:

$$\|V_{\boldsymbol{\alpha}}e_\eta\| \le C\, b(\boldsymbol{\alpha})\, J^+(\boldsymbol{\alpha})^{-1/2}, \quad |\boldsymbol{\alpha}| \le \mathcal{K}\log 1/\hbar, \tag{3.13}$$

where $J^+(\boldsymbol{\alpha}) = \min_{\rho\in W(\boldsymbol{\alpha})} J_n^+(\rho)$ measures the (local) accumulated unstable expansion. This new factor $J^+(\boldsymbol{\alpha})^{-1/2}$ is due to the following mechanism. At each step the dynamics V stretches the state (which is supported inside W_{α_j}) along the unstable manifold, increasing its "unstable volume" by a factor $\sim J_1^+(\alpha_j)$. Then, the cutoff $\Pi_{\alpha_{j+1}}$ truncates this volume into $W_{\alpha_{j+1}}$, effectively reducing the norm of the state by a factor $\sim J_1^+(\alpha_j)^{-1/2}$. From the estimate (3.13) and the expansion (3.12) we deduce the following *hyperbolic dispersion estimate*

$$\|V_{\boldsymbol{\alpha}}\| \leq C\,\hbar^{-d/2}\, b(\boldsymbol{\alpha})\, J^+(\boldsymbol{\alpha})^{-1/2}, \qquad |\boldsymbol{\alpha}| \leq \mathcal{K}\log 1/\hbar. \tag{3.14}$$

This type of estimate was first proved by Anantharaman in the case of undamped waves on an Anosov manifold X [Anan08, AN1], and used there to show that the Kolmogorov-Sinai entropy of semiclassical measures associated with eigenstates of Δ_X cannot be too small. The adaptation to the case of the damped wave equation was written by Schenck in [Schenck-pressure].

3.3.3. *Summing over the paths: link with the pressure.* One can then add up the bounds (3.14), taking into account also (3.11), to get the following bound for the norm of $V^{nN}\psi_0$, where N is the fixed integer used to define the admissible sequences in $\mathcal{A}_N$, while n can be logarithmic in $1/\hbar$:

$$\begin{aligned}\left\|V^{nN}\psi_0\right\| &\leq \sum_{\beta_1,\dots,\beta_n\in\mathcal{A}_N} \|V_{\beta_1\cdots\beta_n}\psi_0\| + \mathcal{O}(\hbar^\infty)\\ &\leq C\,\hbar^{-d/2} \sum_{\beta_1,\dots,\beta_n\in\mathcal{A}_N} b(\beta_1\cdots\beta_n)\, J^+(\beta_1\cdots\beta_n)^{-1/2} + \mathcal{O}(\hbar^\infty).\end{aligned}$$

Here we call β_j the allowed sequences $\alpha_1\cdots\alpha_N$. Now, provided the diameters of the open cover $\{W_j\}$ is small enough, and provided the time N used to select the admissible sequences is large enough, the above sum can be estimated in terms of a certain topological pressure :

$$\sum_{\beta_1,\dots,\beta_n\in\mathcal{A}_N} b(\beta_1\cdots\beta_n)\, J^+(\beta_1\cdots\beta_n)^{-1/2} \leq e^{nN\left(\mathcal{P}(q-\varphi^+/2)+\epsilon\right)}.$$

Now, if we choose $nN \sim \mathcal{K}\log 1/\hbar$ with $\mathcal{K}$ large enough, the prefactor $C\hbar^{-d/2}$ becomes smaller than $e^{nN\epsilon}$, and the remainder $\mathcal{O}(\hbar^\infty)$ can be absorbed as well, so one gets

$$\left\|V^{nN}\psi_0\right\| \leq e^{nN\left(\mathcal{P}(q-\varphi^+/2)+2\epsilon\right)}, \quad n \sim \mathcal{K}\log 1/\hbar.$$

Taking into account the fact that eigenstates u_n with $z_n \in D(1/2, C\hbar)$ are microlocalized on S^*X, one then obtains the following bound on the quantum decay rates.

Theorem 15. [Schenck-pressure] *Let X be an Anosov manifold. Then, for any $\epsilon > 0$ and $\hbar < \hbar_\epsilon$, all eigenvalues $z_n(\hbar) \in D(1/2, C\hbar)$ satisfy*

$$\frac{\operatorname{Im} z_n(\hbar)}{\hbar} \leq \mathcal{P}(q-\varphi^+/2) + \epsilon. \tag{3.15}$$

Furthermore, for any $\gamma > \mathcal{P}(q-\varphi^+/2)$ there exists $N > 0$ such that, for $\hbar$ small enough, the following resolvent estimate holds:

$$\forall z \in D(1/2, C\hbar) \cap \{\operatorname{Im} z/\hbar \geq \gamma\}, \qquad \left\|(P(\hbar)-z)^{-1}\right\| \lesssim \hbar^{-N}.$$

The proof of the resolvent estimate (inspired by [NZ3]) is left to the reader.

This pressure bound is nontrivial (that is, improves the bound $q_+ + \epsilon$) provided this topological pressure satisfies

$$\mathcal{P}(q-\varphi^+/2) < q_+. \tag{3.16}$$

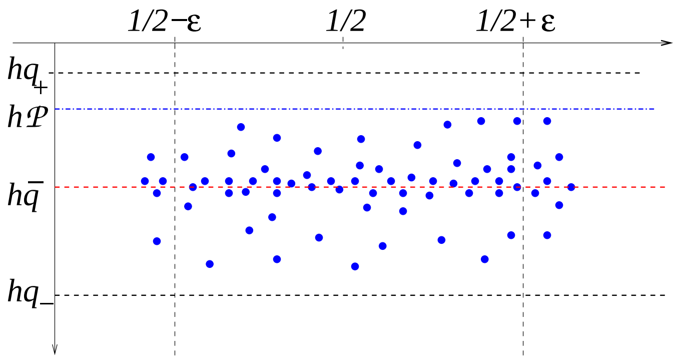

FIGURE 3.2. Improved spectral gap in case the pressure $\mathcal{P}(q - \log \varphi^+/2) < q_+$.

For a uniform damping $q \equiv \bar{q}$, $\mathcal{P}(\bar{q} - \varphi^+/2) = \bar{q} + \mathcal{P}(-\varphi^+/2)$ is always larger than $\bar{q}$. So, the condition (3.16) can be satisfied only for *sufficiently inhomogeneous* damping functions.

In [Schenck10] Schenck explains how to construct damping functions q for which the above condition is satisfied. In view of the original damping function $a(x)$, the idea is to consider a damping function $q \leq 0$, which vanishes on a rather large set $U \subset S^*X$, so that the ultimately undamped set

$$K = \bigcap_{t \in \mathbb{R}} \Phi^t(U),$$

consisting of the points $\rho \in S^*X$ which never enter the damping region, is nontrivial: this implies that $q_+ = 0$. This set K is flow-invariant and closed, so it makes sense to define the pressure $\mathcal{P}(-\varphi^+/2, \Phi^t \restriction_K)$ associated with the flow Φ^t restricted on K. Schenck proves that

$$\lim_{t \to \infty} \mathcal{P}(tq - \varphi^+/2) = \mathcal{P}(-\varphi^+/2, \Phi^t \restriction_K).$$

Hence, provided the set K is *thin enough* for the pressure $\mathcal{P}(-\varphi^+/2, \Phi^t \restriction_K)$ to be negative, one also gets a negative pressure $\mathcal{P}(tq - \varphi^+/2)$ if $t \gg 1$.

This possibility of an upper bound $\mathcal{P}(q - \varphi^+/2) < q_+$ shows that the fractal upper bound (3.8) cannot always be sharp.

3.4. **An arithmetic example in dimension** 2**[Anan10].** In order to study the existence of quantum decay rates $\operatorname{Im} z_n/\hbar$ away from $\bar{q}$, a specific example was investigated by Anantharaman, namely a twisted Laplacian expressed in terms of a harmonic 1-form ω on a surface X of constant curvature -1:

$$P(\hbar) = -\frac{\hbar^2 \Delta}{2} + i\hbar \langle \omega, \frac{\hbar d \bullet}{i} \rangle + \hbar^2 \frac{\|\omega\|_x^2}{2} = -\frac{\hbar^2 \Delta}{2} + i\hbar \operatorname{Op}_\hbar(\langle \omega_x, \xi \rangle) + \mathcal{O}(\hbar^2). \tag{3.17}$$

This operator is of the form (1.9), with the damping function

$$q(x, \xi) = \langle \omega_x, \xi \rangle \in S^1(X), \tag{3.18}$$

depending on both position and momentum. This damping function (restricted to S^*X) satisfies $q_- = -q_+ < \bar{q} = 0 < q_+$, and the relevant pressures

$$q_+ \leq \mathcal{P}(q) \leq q_+ + 1 \quad \text{and} \quad \mathcal{P}(q - \varphi^+/2) = \mathcal{P}(q) - 1/2.$$

Using the exact Selberg trace formula, together with the degeneracies of the length spectrum for arithmetic surfaces, Anantharaman obtains the following lower bounds for the density of eigenvalues away from the real axis.

Theorem 16. [Anan10]*Assume X is an arithmetic surface of curvature -1, and take the damping function (3.18) for some harmonic 1-form ω.*

Then, for any $\beta \in (0,1)$ and $\epsilon > 0$, if $\hbar < \hbar_{\beta,\epsilon}$ there exist eigenvalues of the twisted Laplacian (3.17) in the box $\left\{ |\operatorname{Re} z - 1/2| \leq \hbar^{1-\beta},\ \frac{\operatorname{Im} z}{\hbar} \geq \mathcal{P}(q) - \frac{1}{2} - \frac{1+\epsilon}{2\beta} \right\}$.

Notice that lower bound is positive only for q_+ large enough.

Remark 17. If we multiply the above damping q by $t > 0$, in the large t limit one has the estimate

$$\mathcal{P}(tq) = tq_+ + o_{t\to\infty}(1),$$

corresponding to a limiting rate function $H(q_+) = 0$. As a result, the pressure involved in Thm 15 satisfies

$$\mathcal{P}(tq - \varphi^+/2) = tq_+ - \frac{1}{2} + o_{t\to\infty}(1)\,.$$

In this regime the above theorem shows that there exist eigenvalues in the region

$$\left\{ |\operatorname{Re} z - 1/2| \leq \hbar^{1-\beta},\ tq_+ - \frac{1}{2} - \frac{1+\epsilon}{2\beta} \leq \frac{\operatorname{Im} z}{\hbar} \right\},$$

which lies strictly above the real line for $t \gg 1$. On the other hand, according to Thm 15, for $t \gg 1$ and $\hbar$ small enough there are no eigenvalue in the strip

$$\left\{ |\operatorname{Re} z - 1/2| \leq \hbar^{1-\beta},\ tq_+ - \frac{1}{2} + \epsilon \leq \frac{\operatorname{Im} z}{\hbar} \right\}.$$

4. Spectral study of damped quantum maps

As we have seen , the spectral analysis of $P(\hbar)$ in a disk $D(1/2, C\hbar)$ only involves the classical and quantum dynamics near S^*X. In order to obtain optimal results concerning the eigenvalue distribution, we were forced to modify the subprincipal symbol $\operatorname{Op}_\hbar(q)$ in order to localize it on S^*X (for instance in §2.3.2), or project in a $\hbar$-thin energy layer to minimize a trace (in §2.6.2). These procedures are necessary because the global operator $P(\hbar)$ does simultaneously act on all energy shells, but it somehow obscures the dynamical ingredients of our proofs, which only refer to the flow on S^*X. For this reason, we will in this section consider a simplified Anosov systems, which are quantized Anosov maps on the torus. This model has proved convenient in the past in the study of quantum chaos, both on numerical and analytical aspects [DEG03]. In this section, we will consider a damped version of these quantized Anosov maps, aimed at modelling the damped wave system studied above [NonSche08].

4.1. Damped quantum maps.

In classical dynamics, one may study the flow on S^*X by setting up a Poincaré section $\Sigma \subset S^*X$, that is a finite set of hypersurfaces in S^*X transverse to the flow, and consider the Poincaré return map $\kappa_\Sigma : \Sigma \to \Sigma$. If the flow Φ^t is symplectic, the section Σ is naturally equipped with a symplectic structure, which is preserved by the map κ_Σ.

For our toy model, we will construct by hand a symplectomorphism[6] κ defined on a compact phase space Σ (typically, the torus $\mathbb{T}^{2d-2}$), which will mimic the properties of κ_Σ. At the quantum level, we will replace the Schrödinger propagator $e^{-itP_0/\hbar}$ on $L^2(X)$ by a *quantum map*, that is a family of unitary operators $U_\hbar = U_\hbar(\kappa)$ defined on Hilbert spaces $\mathcal{H}_\hbar$ of dimensions $N_\hbar \sim C\,\hbar^{1-d}$, which satisfy some form of Egorov theorem (one can view $U_\hbar$ as a discrete type of Fourier Integral Operator)[7].

We will select κ to be of Anosov type. Such maps are easy to construct on $\mathbb{T}^{2d-2}$, by taking hyperbolic automorphisms or their Hamiltonian perturbations.

In Lemma 6 we have seen that the damping function q has the effect to modify the propagator by a PDO:

$$e^{-iP/\hbar} = e^{-itP_0/\hbar} B(1), \quad \sigma_{pr}\left(B(1)\right) = b(1) = e^{\int_0^1 q\circ\Phi^s\,ds}.$$

To introduce a damping in our system, we will simply choose some positive function b on Σ, and define our damped quantum map as

$$M_\hbar \stackrel{\text{def}}{=} U_\hbar(\kappa)\, B, \qquad B = \mathrm{Op}_\hbar(b)\,. \tag{4.1}$$

This function plays the same role as $b(1)\restriction_{S^*X}$. We are now interested in the distribution of the $N_\hbar$ eigenvalues $\lambda_j(\hbar)$ of $M_\hbar$ (counted with multiplicities). From the formal analogy

$$\{\lambda_j(\hbar)\} \longleftrightarrow \left\{e^{-iz_n(\hbar)/\hbar},\ \mathrm{Re}\, z_n \in [1/2 \pm \pi\hbar]\right\},$$

we see that the quantum decay rates are the numbers $\log|\lambda_j(\hbar)|$. Furthermore, studying the eigenvalues z_n in an $\hbar$-strip near 1/2 is equivalent with studying the full spectrum of $M_\hbar$.

4.2. **Spectral bounds for damped quantum (Anosov) maps.** In this framework, we can now state and prove the analogues of the spectral results mentioned in previous sections.

The first result, namely the "horizontal" Weyl law of Thm 3, simply results from the "kinematics" of this toy model. Namely, then number of eigenvalues of $M_\hbar$ is given by the dimension $N_\hbar \sim C_\Sigma\,\hbar^{1-d}$ of the quantum Hilbert space. This is of the same order as the number of eigenvalues z_n in the strip $\{|\mathrm{Re}\, z_n - 1/2| \leq \pi\hbar]\}$.

The analogue of Lemma 4 is also obvious. Namely, since the symbol $b(x,\xi) \in [b_{\min}, b_{\max}]$ with $b_{\min} > 0$, one has from the sharp Gårding inequality

$$\|B\| = b_{\max} + \mathcal{O}(\hbar), \quad \left\|B^{-1}\right\| = b_{\min}^{-1} + \mathcal{O}(\hbar).$$

(here the norms are the Hilbert norms on $\mathcal{H}_\hbar$). Since $U_\hbar$ is unitary, we directly get

$$\|M_\hbar\| = b_{\max} + \mathcal{O}(\hbar), \quad \left\|M_\hbar^{-1}\right\| = b_{\min}^{-1} + \mathcal{O}(\hbar),$$

so that all eigenvalues of $M_\hbar$ satisfy

$$|\lambda_j(\hbar)| \in [b_{\min} - \mathcal{O}(\hbar), b_{\max} + \mathcal{O}(\hbar)]\,.$$

[6] That is κ is an invertible map $\mathcal{M} \to \mathcal{M}$, which preserves the symplectic structure on $\mathcal{M}$.

[7] The quantization procedure $\kappa \mapsto U_\hbar(\kappa)$ is not unique, one usually uses some "recipe". The parameter $\hbar$ cannot take all possible values: for instance, on $\mathbb{T}^{2d}$ it must satisfy the condition $(2\pi\hbar)^{-1} \in \mathbb{N}$. The limits $\hbar \to 0$ will be taken along these discrete sequences.

4.2.1. *Time averaging.* Using Egorov's theorem, the iteration of the map $M_\hbar$ reads

$$\begin{aligned}(M_\hbar)^n &= UBUB\cdots UB\\ &= U^n(U^{-n+1}BU^{n-1})\cdots(U^{-2}BU^2)(U^{-1}BU)B\\ &= U^n\operatorname{Op}(b^{(n)}) + \mathcal{O}(\hbar^\infty)\end{aligned} \tag{4.2}$$

where $b^{(n)}\in S^0$ admits the principal symbol

$$b_0^{(n)} = b\circ\kappa^{n-1}b\circ\kappa^{n-2}\cdots b\circ\kappa b = e^{n\langle\log b\rangle_n},$$

where we use the notation $\langle\bullet\rangle_n$ for the (arithmetic) time average from the time 0 through $n-1$. From this expression we easily recover the analogue of Thm 8:

Theorem 18. *For $\hbar>0$ small enough all eigenvalues of $M_\hbar$ satisfy*

$$|\lambda_j(\hbar)|\in[b_- - \epsilon, b_+ + \epsilon],$$

where

$$\log b_+ = \lim_{n\to\infty}\max_{\mathcal{M}}\langle\log b\rangle_n,\qquad \log b_- = \lim_{n\to\infty}\min_{\mathcal{M}}\langle\log b\rangle_n.$$

4.2.2. *Fractal Weyl upper bounds.* Here we state the analogue of Thm 13 for Anosov damped maps. The method is the one used by Schenck [Schenck-map].

The expression (4.2) can be pushed forward until the time $n\approx T_{Ehr}/2$, keeping $b^{(n)}\in S_{1/2-\epsilon}$. A similar decomposition holds on the opposite side,

$$(M_\hbar)^n = \operatorname{Op}_\hbar(b^{(-n)})U^n + \mathcal{O}(\hbar^\infty),$$

where $b^{(-n)}\in\hbar^* S_{1/2-\epsilon}$ admits the principal symbol

$$b_0^{(-n)} = b\circ\kappa^{-n}b\circ\kappa^{-n+1}\cdots b\circ\kappa^{-1}.$$

As a result, we may write

$$(M_\hbar)^{2n} = U^n\operatorname{Op}_\hbar(b^{(-n,n)})U^n + \mathcal{O}(\hbar^\infty),\quad b^{(-n,n)} = b^{(n)}\sharp b^{(-n)}\in\hbar^* S_{1/2-\epsilon}. \tag{4.3}$$

The singular values of $M_\hbar^{2n}$ are given by the spectrum of

$$A^{(-n,n)} \overset{\text{def}}{=} \sqrt{\operatorname{Op}_\hbar(b^{(-n,n)})^*\operatorname{Op}_\hbar(b^{(-n,n)})}, \tag{4.4}$$

a positive PDO with leading symbol $e^{2n\langle\log b\rangle^{2n}}$. Then, for $\alpha>\overline{\log b}$, the number of singular values $s_j(2n)>e^{2n\alpha}$ is approximately given by

$$(2\pi\hbar)^{-d+1}\operatorname{Vol}\left\{\rho,\ \langle\log b\rangle^{2n}\geq\alpha\right\}\sim\hbar^{-d+1}e^{2n\tilde H(\alpha)}\sim\hbar^{-d+1-\tilde H(\alpha)/\lambda_{\max}},$$

where we used the rate function

$$\tilde H(s) = \sup\left\{h_{KS}(\mu)-\mu(\log J^+),\ \mu\,\text{invariant},\ \mu(\log b)=s\right\}, \tag{4.5}$$

and $J^+ = J_1^+$ is the unstable Jacobian of the map. Like for an Anosov flow, this function is strictly convex on $[\log b_-,\log b_+]$, negative on this interval except for the value $\tilde H(\log\bar b)=0$.

Let $J=J(\alpha)$ be the number of eigenvalues $|\lambda_j|\geq e^\alpha$. Weyl's inequalities read

$$\alpha J\leq\sum_{j=1}^J\log|\lambda_j|\leq\frac{1}{2n}\sum_{j=1}^J\log s_j(2n).$$

Call $\beta = \frac{1}{2n}\log s_J(2n)$, so that $J \approx (2\pi\hbar)^{-d+1+\tilde{H}(\beta)/\lambda_{\max}}$, while the higher concentration of smaller singular values implies that

$$\frac{1}{2n}\sum_{j=1}^{J}\log s_j(2n) \leq J(\beta+\epsilon).$$

As a result, $\alpha \leq \beta+\epsilon$, so that $\tilde{H}(\alpha) \leq \tilde{H}(\beta+\epsilon)$. We thus have obtained the following fractal Weyl upper bound on $J(\alpha)$, similar with the bound in (3.8):

Theorem 19. *Take $\alpha > \log \bar{b}$. Then, for any $\epsilon > 0$ and $\hbar > 0$ small enough,*

$$\#\left\{\lambda \in \mathrm{Spec}(M_\hbar),\ |\lambda| \geq e^{\alpha}\right\} \leq \hbar^{-d+1-\frac{\tilde{H}(\alpha)}{\lambda_{\max}}-\epsilon},$$

where $\tilde{H}(\alpha)$ is the rate function given in (4.5).

4.3. **A topological pressure condition for a gap.** The analogue of the pressure bound of Thm 15 can be proved for the damped quantum map $M_\hbar$:

Theorem 20. *For any $\epsilon > 0$ and $\hbar > 0$ small enough, all eigenvalues of $M_\hbar$ satisfy*

$$|\lambda_j(\hbar)| \leq e^{\mathcal{P}(\log b - \log J^+/2)+\epsilon}.$$

As in the case of the flows, this upper bound can be smaller than b_+ only if $\log b$ is sufficiently inhomogeneous. In the next section we will prove an alternative upper bound for the spectral radius of $M_\hbar$, which can be nontrivial even for weakly inhomogeneous dampings. This upper bound (Thm 23 below) will only depend on the structure of the set of weakest damping (that is the set of points for which $\langle \log b\rangle_\infty = \log b_+$).

4.4. **A topological entropy condition for a spectral gap.**

4.4.1. *Splitting the eigenstate into weakly and strongly damped components.* Let us come back to the decomposition (4.3) for the "optimal" time $n = T_{Ehr}/2$. The PDO $\mathrm{Op}_\hbar(b^{(-n,n)})$ is usually not positive, but it can be put in polar form as

$$\mathrm{Op}_\hbar(b^{(-n,n)}) = W^{(-n,n)}\,A^{(-n,n)},\quad A^{(-n,n)} \stackrel{\mathrm{def}}{=} \left(\mathrm{Op}(b^{(-n,n)})^*\,\mathrm{Op}(b^{(-n,n)})\right)^{1/2},$$

$$W^{(-n,n)} = \mathrm{Op}(b^{(-n,n)})\,(A^{(-n,n)})^{-1} \text{ unitary},$$

and both A and W are in $\hbar^*\Psi_{1/2-\epsilon}$. $A^{(-n,n)}$ still has leading symbol $b_0^{(-n,n)} = e^{2n\langle\log b\rangle^{2n}}$. In particular, we have

$$\left\|A^{(-n,n)}\right\| = e^{2n(\log b_+ + o(1))}. \tag{4.6}$$

We now fix some $\alpha \in (\overline{\log b}, \log b_+)$ and construct an orthogonal projector

$$\Pi_{+,\alpha} \stackrel{\mathrm{def}}{=} \mathbb{1}_{A^{(-n,n)} \geq e^{2n\alpha}}.$$

The rank of this projector has been estimated above using the large deviation estimates:

$$\mathrm{rank}\,\Pi_{+,\alpha} \sim (2\pi\hbar)^{-d+1}\,\mathrm{Vol}\left\{\rho \in \Sigma,\ \langle \log b\rangle^{2n} \geq \alpha\right\} \sim \hbar^{-d+1}e^{2n\tilde{H}(\alpha)} \sim \hbar^{-d+1-\tilde{H}(\alpha)/\lambda_{\max}}.$$

We may then write

$$M_\hbar^{2n} = U^n W^{(-n,n)}\,A^{(-n,n)} U^n.$$

Let us now decompose the square of this operator using the projector $\Pi_{+,\alpha}$ and its supplement $\Pi_{-,\alpha} = I - \Pi_{+,\alpha}$:

$$
\begin{aligned}
(4.7) \qquad M_\hbar^{4n} &= U^n W^{(-n,n)}\, A^{(-n,n)} U^{2n} W^{(-n,n)}\, A^{(-n,n)} U^n \\
&= U^n W^{(-n,n)}\, A^{(-n,n)} \left(\Pi_+ + \Pi_-\right) U^{2n} W^{(-n,n)} \left(\Pi_+ + \Pi_-\right) A^{(-n,n)} U^n .
\end{aligned}
$$

The RHS gives 4 terms. For each term containing at least one factor Π_-, we use the obvious bound

$$
(4.8) \qquad \left\| A^{(-n,n)} \Pi_- \right\| \leq e^{2n\alpha}.
$$

On the other hand, to estimate the term containing the factor

$$
U_{++}^{2n} \stackrel{\text{def}}{=} \Pi_+ U^{2n} W^{(-n,n)} \Pi_+,
$$

we will use a crucial *hyperbolic dispersion estimate*, which depends on the size of the set $\{\rho \in \Sigma,\ \langle \log b \rangle^{2n} \geq \alpha\}$. To state this estimate we need some notations. Due to uniform hyperbolicity, there exists $0 < \nu_{\min} \leq \lambda_{\max}$ such that, for t large enough,

$$
(4.9) \qquad J_t^+(\rho) \geq e^{t(d-1)\nu_{\min}} = \hbar^{-(d-1)\frac{\nu_{\min}}{2\lambda_{\max}}} \quad \text{everywhere on } \Sigma.
$$

Now, let us assume that the rate function $\tilde{H}$ satisfies:

$$
(4.10) \qquad \tilde{H}(\log b_+) < (d-1)\left(\frac{\nu_{\min}}{2} - \lambda_{\max}\right).
$$

This condition depends on the set of weakest damping, $\mathcal{K} = \overline{\bigcup \{\operatorname{supp}\mu,\ \mu(\log b) = \log b_+\}}$:

$$
\tilde{H}(\log b_+) = \sup\left\{ H_{KS}(\mu) - \mu(\log J^+),\ \operatorname{supp}\mu \subset \mathcal{K} \right\} = \mathcal{P}(-\log J^+, \Phi^t \restriction_{\mathcal{K}}).
$$

By continuity of $\tilde{H}$, assuming (4.10) it is possible to choose $\alpha \in (\log \bar{b}, \log b_+)$ such that

$$
(4.11) \qquad \beta(\alpha) \stackrel{\text{def}}{=} (d-1)\left(\frac{\nu_{\min}}{2} - \lambda_{\max}\right) - \tilde{H}(\alpha) > 0.
$$

Remark 21. In case of constant hyperbolicity (all positive Lyapounov exponents equal $\Lambda = \lambda_{\max}$), we have $\tilde{H}(s) = H(s) - \Lambda(d-1)$ with $H(s) \geq 0$ a restricted topological entropy. The conditions (4.10,4.11) are replaced by

$$
(4.12) \qquad H(\log b_+) < \Lambda \frac{d-1}{2}, \qquad \beta(\alpha) = \Lambda \frac{d-1}{2} - H(\alpha) > 0 .
$$

Under the above assumptions, we will prove in §4.4.2 the following hyperbolic dispersion estimate.

Proposition 22. *Assume the set of minimal damping satisfies the condition (4.10), and the paramter $\alpha \in (\overline{\log b}, \log b_+)$ is sufficiently close to $\log b_+$ so as to satisfy (4.11). Then, in the limit $\hbar \to 0$, one has the operator bound*

$$
\left\| U_{++}^{2n} \right\| \leq e^{-2n[\beta(\alpha) - o(1)]}, \quad n = T_{Ehr}/2.
$$

Inserting this estimate and (4.6,4.8) in the identity (4.7), we get

$$
\begin{aligned}
\left\| M_\hbar^{4n} \right\| &\leq e^{4n(\log b_+ + o(1))} \left\| U_{++}^{2n} \right\| + 2e^{2n(\log b_+ + \alpha + o(1))} + e^{4n\alpha} \\
&\lesssim e^{4n(\log b_+ + o(1))} \left(e^{-2n\beta(\alpha)} + e^{2n(\alpha - \log b_+)} + e^{4n(\alpha - \log b_+)} \right).
\end{aligned}
$$

The third term on the RHS is obviously subdominant w.r.to the second one. We may optimize this upper bound over α : the optimal value of the exponent is reached for the parameter α_c solving

$$\beta(\alpha_c) = (d-1)\left(\frac{\nu_{\min}}{2} - \lambda_{\max}\right) - \tilde{H}(\alpha_c) = \log b_+ - \alpha_c,$$

and the optimal value is

$$\gamma_c = \frac{\log b_+ + \alpha_c}{2} < \log b_+.$$

This leads to the following spectral radius estimate, which is the main result of this section.

Theorem 23. *Assume that the set of minimal damping satisfies the condition (4.10). Then, for any $\epsilon > 0$ and $\hbar$ small enough, all eigenvalues of $M_\hbar$ satisfy*

$$|\lambda_j(\hbar)| \leq e^{\gamma_c + \epsilon}.$$

We also have "for free" a resolvent estimate (inspired from [NZ3]):

$$(\lambda - M_\hbar)^{-1} = \lambda^{-1} \sum_{j \geq 0} M_\hbar^j \lambda^{-j}$$

converges for $|\lambda| \geq e^\gamma > e^{\gamma_c}$, and has a norm

$$\left\|(\lambda - M_\hbar)^{-1}\right\| \leq |\lambda|^{-1}\left(1 - (e^\gamma/|\lambda|)^{2T_E}\right)^{-1} \sum_{j=0}^{2T_E - 1} \left\|M_\hbar^j\right\| |\lambda|^{-j}.$$

Let us only use the trivial bound $\left\|M_\hbar^j\right\| \leq b_+^j$. If $|\lambda| < b_+$ (that is, λ is in the "spectral gap" zone), the sum on the RHS is of order $(b_+/|\lambda|)^{2T_E} = \exp\left\{\frac{2\log\hbar^{-1}}{\lambda_{\max}} \log(b_+/|\lambda|)\right\}$. For $|\lambda| = b_+$ we get a logarithmic bound. Altogether, we have an algebraic bound on the resolvent for λ in an annulus:

$$\left\|(\lambda - M_\hbar)^{-1}\right\| \lesssim \hbar^{-\frac{2\log(b_+/|\lambda|)}{\lambda_{\max}}} \log\hbar^{-1}, \quad e^{\gamma_c + \epsilon} \leq |\lambda| \leq e^{\log b_+}. \tag{4.13}$$

4.4.2. *A hyperbolic dispersion estimate for U_{++}^{2n}.* In this section we prove Proposition 22, that is obtain a nontrivial upper bound for the norm

$$\left\|U_{++}^{2n}\right\|_{L^2 \to L^2} = \left\|\Pi_{+,\alpha} U^{2n} W^{(-n,n)} \Pi_{+,\alpha}\right\|, \qquad n = T_{Ehr}/2.$$

To estimate this norm, it will be useful to replace the projector Π_+ by a smoothed microlocal projector obtained by the anti-Wick (positive) quantization of some symbol χ_+.

Lemma 24. *There exists a symbol $\chi_+ = \chi_{+,\alpha} \in S^0_{1/2-\epsilon}$ supported on $\{\rho \in \mathcal{M},\ \langle \log b \rangle^{2n} \geq \alpha - 2\delta\}$, such that*

$$\Pi_+ = \mathrm{Op}^+(\chi_+)\,\Pi_+ + \mathcal{O}(\hbar^\infty).$$

Using this lemma, we have

$$\begin{aligned}
\left\|U_{++}^{2n}\right\| &= \sup_{\|\psi_1\| = \|\psi_2\| = 1} \left|\langle \psi_2, \Pi_+ U^{2n} W^{(-n,n)} \Pi_+ \psi_1 \rangle\right| \\
&= \sup_{\|\psi_1\| = \|\psi_2\| = 1} \left|\langle \mathrm{Op}^+(\chi_+)\Pi_+\psi_2, U^{2n} W^{(-n,n)}\, \mathrm{Op}^+(\chi_+)\Pi_+\psi_1 \rangle\right| + \mathcal{O}(\hbar^\infty) \\
&\leq \sup_{\|\psi_1\| = \|\psi_2\| = 1} \left|\langle \mathrm{Op}^+(\chi_+)\psi_2, U^{2n} W^{(-n,n)}\, \mathrm{Op}^+(\chi_+)\psi_1 \rangle\right| + \mathcal{O}(\hbar^\infty)
\end{aligned}$$

We thus need to bound from above

$$\begin{aligned}
&\langle U^{-n}\operatorname{Op}^+(\chi_+)\psi_2, U^n W^{(-n,n)}\operatorname{Op}^+(\chi_+)\psi_1\rangle \\
&\qquad = \int \frac{d\rho_1\, d\rho_2}{(2\pi\hbar)^{2(d-1)}}\,\overline{\psi_2(\rho_2)}\,\psi_1(\rho_1)\,\chi_+(\rho_2)\,\chi_+(\rho_1)\langle U^{-n}e_{\rho_2}, U^n W^{(-n,n)}e_{\rho_1}\rangle,
\end{aligned}$$

with $e_\rho \in \mathcal{H}_\hbar$ an essentially normalized coherent state at ρ, and $\psi(\rho) = \langle e_\rho, \psi\rangle$.

We are able to propagate coherent states e_ρ up to times $|n| \leq T_{Ehr}/2$. Firstly, applying $W^{(-n,n)} \in \Psi_{1/2-\epsilon}$ will essentially only multiply e_{ρ_1} by a global phase. Then, applying U^t will transport e_{ρ_1} to states localized at $\kappa^t(\rho_1)$, and deform the state along the unstable direction. By the time $t = n \approx T_{Ehr}/2$ the state $U^n e_{\rho_1}$ is a Lagrangian state supported on an *unstable leaf* of volume $\sim \hbar^{(d-1)/2} J_n^+(\rho_1) \leq 1$.

Similarly, the state $U^{-n}e_{\rho_2}$ is a Lagrangian state supported on a *stable leaf* of volume $\sim \hbar^{(d-1)/2} J_n^+(\rho_2) \leq 1$. These two leaves intersect at at most one point, and are uniformly transverse to each other. As a result, the overlap of these two states is bounded by:

$$\left|\langle U^{-n}e_{\rho_2}, U^n W^{(-n,n)}e_{\rho_1}\rangle\right| \leq C\hbar^{(d-1)/2}\,\frac{1}{\hbar^{(d-1)/2}\sqrt{J_n^+(\rho_1)J_n^+(\rho_2)}}.$$

and this bound is essentially sharp. On the other hand, the volume estimate for $\operatorname{supp}\chi_+$ reads

$$\operatorname{Vol}\operatorname{supp}\chi_{+,\alpha} \leq e^{2n\left[\tilde{H}(\alpha-2\delta)+0\right]},$$

where the rate function $\tilde{H}(s) \leq 0$ is given in (4.5). We thus obtain a factorized bound:

$$\begin{aligned}
&\langle U^{-n}\operatorname{Op}^+(\chi_+)\psi_2, U^n W^{(-n,n)}\operatorname{Op}^+(\chi_+)\psi_1\rangle \\
&\qquad \leq \int \frac{d\rho_1}{(2\pi\hbar)^{(d-1)}}\,|\psi_1(\rho_1)|\,\frac{\chi_+(\rho_1)}{\sqrt{J_n^+(\rho_1)}}\int \frac{d\rho_2}{(2\pi\hbar)^{(d-1)}}\,|\psi_2(\rho_2)|\,\frac{\chi_+(\rho_2)}{\sqrt{J_n^+(\rho_2)}}.
\end{aligned}$$

To use the L^2 bound on ψ_j, let us apply Cauchy-Schwarz to each factor:

$$\begin{aligned}
\int \frac{d\rho_1}{(2\pi\hbar)^{(d-1)}}\,|\psi_1(\rho_1)|\,\frac{\chi_+(\rho_1)}{\sqrt{J_n^+(\rho_1)}} &\leq \sqrt{\int \frac{d\rho_1}{(2\pi\hbar)^{(d-1)}}\,|\psi_1(\rho_1)|^2}\sqrt{\int \frac{d\rho_1}{(2\pi\hbar)^{(d-1)}}\,\frac{\chi_+(\rho_1)^2}{J_n^+(\rho_1)}} \\
&\leq C\,\hbar^{-(d-1)/2}\,\|\psi_1\|\sqrt{\int_{\operatorname{supp}\chi_+} d\rho_1 (J_n^+(\rho_1))^{-1}}. \\
&\leq C\,\hbar^{-(d-1)/2}\sqrt{\operatorname{Vol}\operatorname{supp}\chi_+(\inf J_n^+)^{-1}}. \qquad (4.14)
\end{aligned}$$

From the lower bound (4.9) on J_n^+ and (4.14) we get the main result of this subsection,

$$\left\|U_{++}^{2n}\right\| \leq \hbar^{-(d-1)}\,\hbar^{(d-1)\frac{\nu_{\min}}{2\lambda_{\max}}}\,\hbar^{-\left[\tilde{H}(\alpha-2\delta)/\lambda_{\max}+0\right]} = e^{-2n[\beta(\alpha)-\mathcal{O}(\delta)]}, \qquad (4.15)$$

where we used the parameter $\beta(\alpha)$ of 4.11. □

4.5. **A spectral gap for the DWE.** Similar spectral gap estimates can be proved in the case of the damped wave equation on an Anosov manifold. Compared with the case of Anosov maps, the main supplementary difficulty in the proofs consists in appropriately localizing in the energy direction, as was the case in the proof of Thms 10 and 13.

Theorem 25. *[Work in progress] Consider the operator $P(\hbar) = -\hbar^2\Delta/2 - i\hbar a$ on an Anosov manifold, with damping function $a(x) \geq 0$. Assume that the uncontrolled set $\mathcal{K} = \overline{\bigcup\{\operatorname{supp}\mu,\ \mu(a) = 0\}}$ is sufficiently "thin" such that*

$$\mathcal{P}(-\varphi^+, \Phi^t\restriction_{\mathcal{K}}) < (d-1)\left(\frac{\nu_{\min}}{2} - \lambda_{\max}\right), \tag{4.16}$$

where φ^+ is the infinitesimal Jacogian, and $\nu_{\min}$ is a lower bound for the growth of the unstable Jacobian as in (4.9). Then, there exists $\gamma > 0$ such that, for $\hbar > 0$ small enough, all eigenvalues of $P(\hbar)$ in $D(1/2, C\hbar)$ satisfy

$$\operatorname{Im} z_n/\hbar \leq -\gamma.$$

Besides, there exists $N > 0$ such that, for $\hbar$ small enough, we have the following resolvent estimate:

$$\forall z \in D(1/2, C\hbar) \cap \{\operatorname{Im} z/\hbar \geq -\gamma\}, \qquad \left\|(P(\hbar) - z)^{-1}\right\| \lesssim \hbar^{-N}.$$

Corollary 26. *[Work in progress] Under the condition (4.16), there exists $\gamma > 0$, $C > 0$ such that the region $\{|\operatorname{Re}\tau| \geq C,\ \operatorname{Im}\tau \geq -\gamma\}$ does not contain any eigenvalue of the DWE, and one has the following resolvent estimate in that region:*

$$\|R(\tau)\|_{L^2\to L^2} \lesssim \tau^{N-2}.$$

This resolvent estimate, which directly provides a similar estimate for $(\tau - \mathcal{A})^{-1}$, can then be used as in §1.3 to prove an exponential decay of the energy for smooth enough data.

Remark 27. Burq and Hitrik [BuHi07] considered the DWE in the *stadium billiard*, in cases where the damping function is positive in the half-disks but may vanish in some part of the rectangle. The (broken) geodesic flow in the billiard is ergodic and mixing, but it includes marginally stable orbits, namely the 1D family of *bouncing ball orbits* inside the rectangle. In the situation they consider, the set $\mathcal{K}$ of uncontrolled trajectories consists of a subfamily of boucing ball orbits, and the topological pressure $\mathcal{P}(-\log\varphi^+, \Phi^t\restriction_{\mathcal{K}}) = 0$ (in particular the condition (4.16) is not satisfied). In this situation, they can prove that the energy decays *algebraically* for smooth enough data.

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