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Notes on the Minicourse *Entropy of Chaotic Eigenstates*

Stéphane Nonnenmacher

1. Overview and statement of results

1.1. Introduction. These lectures present a recent approach, mainly developed by Nalini Anantharaman and the author, aimed at studying the high-frequency eigenmodes of the Laplace−Beltrami operator $\Delta = \Delta_X$ on compact Riemannian manifolds (X, g) for which the sectional curvature is everywhere negative. It is well-known that the geodesic flow on such a manifold (which takes place on the unit cotangent bundle S^*X) is strongly chaotic, in the sense that it is uniformly hyperbolic (Anosov). This flow leaves invariant the natural smooth measure on S^*X, namely the Liouville measure μ_l (which is also the lift of the Lebesgue measure on X). Studying the eigenstates of Δ is thus a part of "quantum chaos." In these notes we extend the study to more general Schrödinger-like operators, such that the corresponding Hamiltonian flow on some compact energy shell $\mathcal{E}$ has the Anosov property. We also consider the case of quantized Anosov diffeomorphisms on the torus, which are popular toy models in the quantum chaos literature. To set up the problem we first stick to the Laplacian.

For a general Riemannian manifold (X, g) of dimension $d \geq 2$, there exist no explicit, not even approximate expression for the eigenmodes of the Laplacian. One way to "describe" these modes consists in comparing them (as "quantum" invariants) with "classical" invariants, namely probability measures on S^*X, invariant with respect to the geodesic flow. To this aim, starting from the full sequence of eigenmodes $(\psi_n)_{n\geq 0}$ one can construct a family of invariant probability measures on S^*X, called *semiclassical measures*. Each such measure μ_{sc} can be associated with a subsequence of eigenmodes $(\psi_{n_j})_{j\geq 1}$ which share, in the limit $j \to \infty$, the same macroscopic localization properties, both on the manifold X and in the velocity (or momentum) space: these macroscopic localization properties are "represented" by μ_{sc}. One (far-reaching) aim would be a complete classification of the semiclassical measures associated with a given manifold (X, g).

This goal is much too ambitious, starting from the fact that the set of invariant measures is itself not always well-understood. We will thus restrict ourselves to the

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This is the final form of the paper.

class of manifolds described above, namely manifolds (X, g) of negative sectional curvature. One advantage is that the classical dynamics is at the same time "irregular" (in the sense of "chaotic"), and "homogeneous." The geodesic flow on such a manifold is (semi)conjugated with a suspended flow over a simple symbolic dynamics (a subshift of finite type over a finite alphabet), which allows one to explicitly construct many different invariant measures. For instance, such a flow admits infinitely many isolated (unstable) periodic orbits γ, each of which carries a natural probability invariant measure μ_γ. The set of periodic orbits is so large that the measures $\{\mu_\gamma\}$ form a dense subset (in the weak-* topology) of the set of invariant probability measures. Hence, it would be interesting to know whether some high-frequency eigenmodes can be asymptotically localized near certain periodic orbits, leading to semiclassical measures of the form

$$\mu = \sum_\gamma p_\gamma \mu_\gamma, \qquad \text{with } \sum_\gamma p_\gamma = 1,\ p_\gamma \in [0, 1]. \tag{1.1}$$

This possibility was named "strong scarring" by Rudnick−Sarnak [27], in analogy with a weaker form of "scarring" observed by Heller on some numerically computed eigenmodes [17], namely a "nonrandom enhancement" of the wavefunction in the vicinity of a certain periodic orbit. In the same paper, Rudnick and Sarnak conjectured that such semiclassical measures do not exist for manifolds of negative curvature. More precisely, they formulated the Quantum Unique Ergodicity (QUE) conjecture

Conjecture 1.1 (Quantum Unique Ergodicity [27]**).** *Let (X, g) be a compact Riemannian manifold of negative curvature. Then there exist only one semiclassical measure, namely the Liouville measure μ_l.*

This conjecture rules out any semiclassical measure of the type (1.1). It also rules out linear combination of the form

$$\mu = \alpha \mu_l + (1 - \alpha) \sum_\gamma p_\gamma \mu_\gamma, \qquad \alpha \in [0, 1). \tag{1.2}$$

The name "quantum unique ergodicity" reminds of a classical notion: a dynamical system (map or flow) is *uniquely ergodic* if and only if it admits a unique invariant measure. In the present case, the classical system is not uniquely ergodic, but the conjecture is that its quantum analogue conspires to be so.

This conjecture was formulated several years after the proof of a general result describing "almost all" the eigenstates (ψ_n).

Theorem 1.2 (Quantum Ergodicity [9, 28, 30]**).** *Let (X, g) be a compact Riemannian manifold such that the geodesic flow is ergodic with respect to the Liouville measure μ_l. Then, there exists a subsequence $S \subset \mathbb{N}$ of density 1, such that the subsequence $(\psi_n)_{n \in S}$ is associated with μ_l.*[1]

The manifolds encompassed by this theorem include the case of negative curvature, but also more general ones (like manifolds where the curvature is negative outside a flat cylindrical part). The proof of this theorem is quite "robust." It has been generalized to many different ergodic systems: Hamiltonian flows ergodic on some compact energy shell [16], broken geodesic flows on some Euclidean domains [14, 32], symplectic diffeomorphism (possibly with discontinuities) on a compact

[1] A subsequence $S \subset \mathbb{N}$ is said to be of density 1 iff $\lim_{N \to \infty} \sharp\{n \in S, n \leq N\}/N = 1$.

phase space [6]. This result leaves open the possibility of *exceptional subsequences* $(\psi_{n_j})_{j\geq 1}$ (necessarily of density zero) of eigenmodes with different localization properties.

The QUE conjecture was motivated by partial results concerning a much more restricted class of manifolds, namely compact quotients of the hyperbolic disk $\Gamma\backslash\mathbb{H}$ for which Γ is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$.[2] Such manifolds admit a commutative algebra of self-adjoint Hecke operators $(T_k)_{k\geq 2}$ which all commute with the Laplacian. It thus makes sense to preferably consider an eigenbasis $(\psi_n)_{n\geq 0}$ made of joint eigenmodes of the Laplacian and the Hecke operators (called Hecke eigenmodes), and the associated semiclassical measures (called Hecke semiclassical measures).[3] Rudnick – Sarnak proved that the only Hecke semiclassical measure of the form (1.2) is the Liouville measure ($\alpha = 1$). Finally, Lindenstrauss [21] showed that for such manifolds, the only Hecke semiclassical measure is the Liouville measure, thus proving an arithmetic form of QUE. He used as an intermediate step a lower bound for the *Kolmogorov – Sinai* (*KS*) *entropy* of Hecke semiclassical measures, which he proved in a joint work with Bourgain.

Proposition 1.3 ([5])**.** *Let X be an arithmetic quotient $\Gamma\backslash\mathbb{H}$. Consider a Hecke semiclassical measure μ_{sc}. Then for any $\rho \in S^*X$ any small $\tau, \epsilon > 0$, the measure of the tube $B(\rho,\epsilon,\tau)$ of diameter ϵ around the stretch of trajectory $[\rho, g^\tau\rho]$ is bounded by*

$$\mu_{\mathrm{sc}}\big(B(\rho,\epsilon,\tau)\big) \leq C_\tau\, \epsilon^{2/9}.$$

As a consequence, for almost any ergodic component μ_{erg} of μ_{sc}, one has

$$H_{\mathrm{KS}}(\mu_{\mathrm{erg}}) \geq 2/9. \tag{1.3}$$

As we will see below, the KS entropy is an affine quantity, therefore $H_{\mathrm{KS}}(\mu_{\mathrm{sc}})$ also satisfies the same lower bound.

1.2. Entropy as a measure of localization. In the previous section we already noticed a relationship between phase space localization and entropy: a uniform lower bound on the measure of thin tubes implies a positive lower bound on the entropy of the measure. For this reason, it is meaningful to consider the KS entropy of a given invariant measure as a "quantitative indicator of localization" of that measure. In Section 3.1 we will give a precise definition of the entropy. For now, let us only provide a few properties valid in the case of Anosov flows [18, Chapter 4].

(1) $H_{\mathrm{KS}}(\bullet)$ is a real function defined on the set of invariant probability measures. It takes values in a finite interval $[0, H_{\max}]$ and is upper semicontinuous. In information-theoretic language, it measures the average complexity of the flow with respect to that measure. The maximum entropy $H_{\max} = H_{\mathrm{top}}(S^*X)$ is also the *topological entropy* of the flow on S^*X, which is a standard measure of the *complexity* of the flow.

(2) a measure μ_γ supported on a single periodic orbit has zero entropy.

(3) Since the flow is Anosov, at each point $\rho \in S^*X$ the tangent space splits into $T_\rho S^*X = E^u_\rho \oplus E^s_\rho \oplus E^0_\rho$, respectively the unstable, stable subspaces and the flow direction. Each of these subspaces is flow-invariant. Let us call $J^u(\rho) = J^u_1(\rho) =$

[2] More precisely, Γ is derived from an Eichler order in a quaternion algebra.

[3] It is widely believed that the spectrum of Δ on such a manifold is simple, in which case the restriction to Hecke eigenmodes is not necessary.

$\left|\det g^1_{\restriction E^u(\rho)}\right|$ the unstable Jacobian at the point ρ. Then, any invariant measure satisfies

$$H_{\mathrm{KS}}(\mu) \leq \int \log J^u(\rho)\,\mathrm{d}\mu(\rho). \qquad \text{(Ruelle inequality)} \tag{1.4}$$

The equality is reached iff $\mu = \mu_l$. In constant curvature, one has $H_{\mathrm{KS}}(\mu_l) = H_{\max}$.

(4) the entropy is affine: $H_{\mathrm{KS}}(\alpha\mu_1 + (1-\alpha)\mu_2) = \alpha H_{\mathrm{KS}}(\mu_1) + (1-\alpha)H_{\mathrm{KS}}(\mu_2)$.

Apart from the result of Bourgain−Lindenstrauss (relative to arithmetic surfaces), the first result on the entropy of semiclassical measures was obtained by N. Anantharaman:

Theorem 1.4 ([1])**.** *Let (X,g) be a manifold of negative sectional curvature. Then, there exists $c > 0$ such that any semiclassical measure μ_{sc} satisfies*

$$H_{\mathrm{KS}}(\mu_{\mathrm{sc}}) \geq c.$$

Furthermore, the flow restricted to the support of μ_{sc} has a nontrivial complexity: its topological entropy satisfies

$$H_{\mathrm{top}}(\operatorname{supp}\mu_{\mathrm{sc}}) \geq \frac{\Lambda^u_{\min}}{2},$$

where $\Lambda^u_{\min} = \lim_{t\to\infty}\inf_\rho \frac{1}{t}\log J^u_t(\rho)$ is the minimal volume expanding rate of the unstable manifold.

The lower bound $c > 0$ is not very explicit and is rather “small.” This is to be opposed to the lower bound controlling the complexity of the flow on $\operatorname{supp}\mu_{\mathrm{sc}}$, given in terms of the hyperbolicity of the flow. The lower bound on the KS entropy was improved by Anantharaman, Koch and the author:

Theorem 1.5 ([3,4])**.** *Let (X,g) be a d-dimensional manifold of negative sectional curvature. Then, any semiclassical measure μ_{sc} satisfies*

$$H_{\mathrm{KS}}(\mu_{\mathrm{sc}}) \geq \int \log J^u(\rho)\,\mathrm{d}\mu_{\mathrm{sc}}(\rho) - \frac{(d-1)\lambda_{\max}}{2}, \tag{1.5}$$

where $\lambda_{\max} = \lim_{t\to\infty}\sup_\rho (1/t)\log|dg^t(\rho)|$ is the maximal expansion rate of the flow.

In the particular case where X has constant curvature -1, this bound reads

$$H_{\mathrm{KS}}(\mu_{\mathrm{sc}}) \geq \frac{d-1}{2} = \frac{H_{\mathrm{top}}(S^*X)}{2}. \tag{1.6}$$

In the constant curvature case, the above bound roughly means that high-frequency eigenmodes of the Laplacian are *at least half-delocalized*. Still, the bound (1.5) is not very satisfactory when the curvature varies much across X; since $\int \log J^u\,\mathrm{d}\mu$ may be as small as $\Lambda_{\min}$, the right-hand side in (1.5) can become negative (therefore trivial) in case $\Lambda_{\min} < (d-1)\lambda_{\max}/2$. The following lower bound seems more natural:

Conjecture 1.6. *Let (X,g) be a manifold of negative sectional curvature. Then, any semiclassical measure μ_{sc} satisfies*

$$H_{\mathrm{KS}}(\mu_{\mathrm{sc}}) \geq \frac{1}{2}\int \log J^u(\rho)\,\mathrm{d}\mu_{\mathrm{sc}}(\rho). \tag{1.7}$$

This bound is identical with (1.6) in the case of curvature -1. Using a nontrivial extension of the methods developed in [4], it has been recently proved by G. Rivière in the case of surfaces ($d = 2$) of *nonpositive curvature* [24,25], and also by B. Gutkin for a certain class of quantized interval maps [15].

1.3. Generalization to Anosov Hamiltonian flows and symplectic maps. The conjecture 1.6 is weaker than the QUE conjecture 1.1. We expect the bound (1.7) to apply as well to more general classes of quantized chaotic dynamical systems, like Anosov Hamiltonian flows or symplectic diffeomorphisms on a compact phase space. In these notes we will extend the bound (1.5) to these more general Anosov systems (the first instance of this entropic bound actually appeared when studying the Walsh-quantized baker's map [2]). The central result of these notes is the following theorem.

Theorem 1.7. (i) *Let $p(x, \xi; \hbar)$ be a Hamilton function on T^*X (where the smooth manifold X may be either compact, or euclidean near infinity), with principal symbol p_0, such that the energy shell $\mathcal{E} = p_0^{-1}(0)$ is compact, and the Hamiltonian flow $g^t = \mathrm{e}^{tH_{p_0}}$ on $\mathcal{E}$ is Anosov (see Section 2.2). Let $P(\hbar) = \mathrm{Op}_\hbar(p)$ be the $\hbar$-quantization of p. Then, any semiclassical measure μ_{sc} associated with a sequence of null eigenmodes $(\psi_\hbar)_{\hbar\to 0}$ of $P(\hbar)$ satisfies the following entropic bound:*

$$H_{\mathrm{KS}}(\mu_{\mathrm{sc}}) \geq \int_{\mathcal{E}} \log J^u(\rho)\, \mathrm{d}\mu_{\mathrm{sc}}(\rho) - \frac{(d-1)\lambda_{\max}}{2}. \tag{1.8}$$

(ii) *Let $\mathcal{E} = \mathbb{T}^{2d}$ be the $2d$-dimensional torus, equipped with its standard symplectic structure. Let $\kappa \colon \mathcal{E} \to \mathcal{E}$ be an Anosov diffeomorphism, which can be quantized into a family of unitary propagators $\big(U_\hbar(\kappa)\big)_{\hbar\to 0}$ defined on (finite-dimensional) quantum Hilbert spaces $(\mathcal{H}_\hbar)$. Then, any semiclassical measure μ_{sc} associated with a sequence of eigenmodes $(\psi_\hbar \in \mathcal{H}_\hbar)_{\hbar\to 0}$ of $U_\hbar(\kappa)$ satisfies the entropic bound*

$$H_{\mathrm{KS}}(\mu_{\mathrm{sc}}) \geq \int \log J^u(\rho)\, \mathrm{d}\mu_{\mathrm{sc}}(\rho) - \frac{d\lambda_{\max}}{2}. \tag{1.9}$$

In Section 5 we will state more precisely what we meant by a "quantized torus diffeomorphism." Let us mention that the same proof could apply as well to Anosov symplectic maps on more general symplectic manifolds admitting some form of quantization. We restricted the statement to the $2d$-torus because simple Anosov diffeomorphisms on $\mathbb{T}^{2d}$ can be constructed, and their quantization is by now rather standard. As we explain below, their study has revealed interesting features regarding the QUE conjecture.

1.4. Counterexamples to QUE for Anosov diffeomorphisms. For the simplest Anosov diffeomorphisms on $\mathbb{T}^2$, namely the hyperbolic symplectomorphisms of the torus (generalizations of Arnold's "cat map"), the QUE conjecture is known to fail. Indeed, in [12, 13] counterexamples to QUE for hyperbolic symplectomorphisms on the 2-dimensional torus were exhibited, in the form of explicit semiclassical sequences of eigenstates of the quantized map, associated with semiclassical measures of type (1.2) with $\alpha = \frac{1}{2}$. We also showed that, for this particular map, semiclassical measures of the form (1.2) necessarily satisfy $\alpha \geq \frac{1}{2}$.

In the case of toral symplectomorphisms on higher-dimensional tori $\mathbb{T}^{2d}$, Kelmer [19, 20] has exhibited semiclassical measures in the form of the Lebesgue measure on certain co-isotropic affine subspaces $\Lambda \subset \mathbb{T}^{2d}$ of the torus invariant through the map:

$$\mu_{\rm sc} = \mu_{L\restriction\Lambda}.$$

For another example of a chaotic map (the baker's map quantized *à la Walsh*), we were able to construct semiclassical measures of purely fractal (self-similar) nature.

Fact 1.8. *The above counterexamples to QUE all satisfy the entropy bound* (1.7), *and some of them* (*like the measure* (1.2) *with* $\alpha = \frac{1}{2}$) *saturate that bound.*

It is worth mentioning an interesting result obtained by S. Brooks [8] in the case of the "quantum Arnold's cat map". Brooks takes into accout the possibility to split any invariant measure μ into ergodic components:

$$\mu = \int_{\mathbb{T}^2} \mu_x \, d\mu(x),$$

where the probability measure μ_x, defined for μ-almost every point x, is ergodic. The affineness of the KS entropy ensures that

$$H_{\rm KS}(\mu) = \int_{\mathbb{T}^2} H_{\rm KS}(\mu_x)\, d\mu(x), \tag{1.10}$$

so to get a lower bound on $H_{\rm KS}(\mu)$ it is sufficient to show that "high-entropy" components have a positive weight in μ. Brooks's result reads as follows:

Theorem 1.9 ([8])**.** *Let* $\kappa\colon \mathbb{T}^2 \circlearrowleft$ *be a linear hyperbolic symplectomorphism, with positive Lyapunov exponent* λ (λ *is also equal to the topological entropy of* κ *on* $\mathbb{T}^2$). *Fix any* $0 < H_0 < \lambda/2$, *and consider any associated semiclassical measure* $\mu_{\rm sc}$. *Then the following inequality holds*:

$$\mu_{\rm sc}\{x \mid H_{\rm KS}(\mu_x) < H_0\} \leq \mu_{\rm sc}\{x \mid H_{\rm KS}(\mu_x) > \lambda - H_0\}.$$

This result directly implies (through (1.10)) the bound $H_{\rm KS}(\mu_{\rm sc}) \geq \lambda/2$, but it also implies (by sending $H_0 \to 0$) the above-mentioned fact that the weight of atomic components of $\mu_{\rm sc}$ is smaller or equal to the weight of its Lebesgue component.

1.5. Plan of the paper. These lectures reproduce most of the proofs of [3, 4] dealing with eigenstates of the Laplacian on manifolds of negative curvature. Yet, we extend the proofs in order to deal with more general Hamiltonian flows of Anosov type (for instance, adding some small potential to the free motion on X). This can be done at the price of using more general, "microlocal" partitions of unity, as opposed to the "local" partition of unity used in [3, 4] (which was given in terms of functions $\pi_k(x)$) only depending on the position variable. This microlocal setting is somehow more natural, since it does not depend on the way unstable manifolds project down to the manifold X. It is also necessary in the case of Anosov maps.

In Section 2 we recall the semiclassical tools we will need, starting with the $\hbar$-pseudodifferential calculus on a compact manifold, and including some exotic classes of symbols. We also define the main object of study, namely the semiclassical measures associated with sequences of null eigenstates $(\psi_\hbar)_{\hbar\to 0}$ of a family of Hamiltonians $\big(P(\hbar)\big)_{\hbar\to 0}$. In the central section 3 we provide the proof of Theorem 1.7(i) that is in the case of an Anosov Hamiltonian flow on a compact manifold X. We first recall the definition of entropies and pressures associated with invariant measures. We then introduce microlocal quantum partitions in Section 3.2, and their refinements used to define quantum entropies and pressures associated with the eigenstates $\psi_\hbar$. We try to provide some geometric intuition on the operators $\Pi_{\boldsymbol{\alpha}}$ defining these partitions. We then state the central hyperbolic dispersive estimate on the norms of these operators, deferring the proof to Section 4. We then introduce several versions of "entropic uncertainty principles," from the simplest to the most complex, microlocal form (Proposition 3.12). We then apply this microlocal EUP in order to bound from below quantum and classical pressures associated with our eigenmodes. Section 4 is devoted to the proof of the hyperbolic dispersive estimate. Here we adapt the proof of [23], which is valid in more general situations than the case of geodesic flows. Finally, in Section 5 we briefly recall the framework of quantized maps on the torus, and provide the details necessary to obtain Theorem 1.7(ii).

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2. Preliminaries and problematics

2.1. Semiclassical calculus on X. The original application of the methods presented below concern the Laplace−Beltrami operator on a smooth compact manifold X of negative sectional curvature. To deal with this problem, one needs to define a certain number of auxiliary operators on $L^2(X)$, which are $\hbar$-pseudodifferential operators on X (ΨDOs), or $\hbar$-Fourier integral operators on X. We will only recall the definition and construction of the former class.

The Hamiltonians mentioned in Theorem 1.7 also belong to some class of $\hbar$-pseudodifferential operators, but the manifold X on which they are defined is not necessarily compact any more. In this setting, the smooth manifold X can be taken as the Euclidean space $X = \mathbb{R}^d$, or be Euclidean near infinity, that is $X = X_0 \sqcup \big(\mathbb{R}^d \setminus B(0, R_0)\big)$, where $B(0, R_0)$ is the ball of radius R_0 in $\mathbb{R}^d$, and X_0 is a compact manifold, the boundary of which is smoothly glued to $\partial B(R_0)$.

2.1.1. *Symbol classes on T^*X and $\hbar$-pseudodifferential calculus.* Let us construct an $\hbar$-quantization procedure on a Riemannian manifold X. To a certain class of well-behaved functions $\big(f(\hbar)\big)_{\hbar\to 0}$ on T^*X (the physical *observables*, referred to as *symbols* in mathematics) one can associate, through a well-defined *quantization procedure* $\mathrm{Op}_\hbar$, a corresponding set of operators $\mathrm{Op}_\hbar\big(f(\hbar)\big)$ acting on $C_c^\infty(X)$. By "well-behaved" one generally refers to certain conditions on the regularity and growth of the function. There are many different types of classes of "well-behaved symbols"; we will be using the class

$$S^{m,k}(T^*X) = \{f(\hbar) \in C^\infty(T^*X), |\partial_x^\alpha \partial_\xi^\beta f(\hbar)| \leq C_{\alpha,\beta} \hbar^{-k} \langle\xi\rangle^{m-|\beta|}\}.$$

Here we use the "Japanese brackets" notation $\langle\xi\rangle \stackrel{\text{def}}{=} \sqrt{1+|\xi|^2}$. The estimates are supposed to hold uniformly for $\hbar \in (0,1]$ and $(x,\xi) \in T^*X$. The seminorms can be defined locally on coordinate charts of X; due to the factor $\langle\xi\rangle^{-|\beta|}$, this class is invariant with respect to changes of coordinate charts on X, and thus makes sense intrinsically on T^*X.

Some (important) symbols in this class are of the form

$$f(x,\xi;\hbar)=\hbar^{-k}f_0(x,\xi)+f_1(x,\xi,\hbar),\qquad f_1\in S^{m,k'},\qquad k'<k.$$

In that case, $\hbar^{-k}f_0(x,\xi)$ is called the *principal symbol* of f.

For $X=\mathbb{R}^d$, a symbol $f\in S^{m,k}$ can be quantized using the Weyl quantization: it acts on $\varphi\in\mathcal{S}(\mathbb{R}^d)$ through as the integral operator

$$\operatorname{Op}_\hbar^W(f)\varphi(x)=\int f\Big(\frac{x+y}{2},\xi\Big)\mathrm{e}^{\mathrm{i}\langle x-y,\xi\rangle/\hbar}\,\varphi(y)\frac{\mathrm{d}y\,\mathrm{d}\xi}{(2\pi\hbar)^d}. \tag{2.1}$$

If f is a real function, this operator is essentially self-adjoint on $L^2(\mathbb{R}^d)$.

If X is a more complicated manifold, one can quantize f by first splitting it into pieces localized on various coordinate charts $V_l\subset X$, through a finite partition of unity $1=\sum_l\phi_l$, $\operatorname{supp}\phi_l\subset V_l$:

$$f=\sum_l f_l,\qquad f_l=f\times\phi_l.$$

Each component f_l can be considered as a function on $T^*\mathbb{R}^d$, and be quantized through (2.1), producing an operator $\operatorname{Op}_\hbar^{\mathbb{R}}(f_l)$ acting on $C_c^\infty(\mathbb{R}^d)$. A wavefunction $\varphi\in C^\infty(X)$ will be cut into pieces $\tilde\phi_l\times\varphi$, where the cutoffs $\operatorname{supp}\tilde\phi_l\subset V_l$ satisfy $\tilde\phi_l\phi_l=\phi_l$.[4] Our final quantization is then defined as:

$$\operatorname{Op}_\hbar(f)\varphi=\sum_l\tilde\phi_l\times\operatorname{Op}_\hbar^{\mathbb{R}}(f_l)(\tilde\phi_l\times\varphi). \tag{2.2}$$

The image of the class $S^{m,k}(T^*X)$ through quantization is an operator algebra acting on $C_c^\infty(X)$, denoted by $\Psi^{m,k}(T^*X)$. This algebra has nice properties in the semiclassical limit $\hbar\ll1$. The product of two such operators behaves as a decoration of the usual multiplication:

$$\operatorname{Op}_\hbar(f)\operatorname{Op}_\hbar(g)=\operatorname{Op}_\hbar(f\sharp g), \tag{2.3}$$

where $f\sharp g\in S^{m+m',k+k'}$ admits an asymptotic expansion of the form

$$f\sharp g\sim\sum_{j\geq0}\hbar^j(f\sharp g)_j. \tag{2.4}$$

Here the first component $(f\sharp g)_0=f\times g$, while each $(f\sharp g)_j\in S^{m+m',k+k'}$ is a linear combination of derivatives $\partial^\gamma f\partial^{\gamma'}g$, with $|\gamma|,|\gamma'|\leq j$. In the case $X=\mathbb{R}^d$ and $\operatorname{Op}_\hbar$ is the Weyl quantization (2.1), $\sharp$ is called the Moyal product. In the case $f\in S(T^*X)=S^{0,0}(T^*X)$, $\operatorname{Op}_\hbar(f)$ can be extended into a continuous operator on $L^2(X)$, and the sharp Gårding inequality ensures that

$$\|\operatorname{Op}_\hbar(f)\|_{L^2}=\|f\|_\infty+\mathcal{O}_f(\hbar).$$

The quantization procedure $f\mapsto\operatorname{Op}_\hbar(f)$ is obviously not unique: it depends on the choice of coordinates on each chart, on the choice of quantization $\operatorname{Op}_\hbar^{\mathbb{R}}$, on the choice of cutoffs $\phi_j,\tilde\phi_j$. Fortunately, this non-uniqueness becomes irrelevant in the semiclassical limit.

[4]Throughout the text we will often encounter such embedded cutoffs. The property $\tilde\phi_l\phi_l=\phi_l$ will be denoted by $\tilde\phi_l\succ\phi_l$.

Proposition 2.1. *In the semiclassical limit $\hbar \to 0$, two $\hbar$-quantizations differ at most at subprincipal order*:

$$\forall f \in S^{m,k}(T^*X), \qquad \mathrm{Op}^1_\hbar(f) - \mathrm{Op}^2_\hbar(f) \in \Psi^{m,k-1}(T^*X).$$

2.2. From the Laplacian to more general quantum Hamiltonians.

2.2.1. *Rescaling the Laplacian.* One of our objectives is to study an eigenbasis $(\psi_n)_{n\geq 0}$ of the Laplace−Beltrami operator on some compact Riemannian manifold (X, g). To deal with the high-frequency limit $n \gg 1$, it turns out convenient to use a quantum mechanical point of view, namely rewrite the eigenmode equation

$$(\Delta + \lambda_n^2)\psi_n = 0, \qquad \lambda_n > 0$$

in the form

$$\Big(-\frac{\hbar_n^2\,\Delta}{2}\psi_n - \frac{1}{2}\Big)\psi_n = 0, \qquad \hbar_n = \lambda_n^{-1}. \tag{2.5}$$

This way, $\hbar_n$ appears as an effective Planck's constant (which is of the order of the *wavelength* of the state ψ_n). The rescaled Laplacian operator

$$-\frac{\hbar^2\Delta}{2} - \frac{1}{2} = P(\hbar)$$

is the $\hbar$-quantization $P(\hbar) = \mathrm{Op}_\hbar(p)$ of a certain classical Hamiltonian

$$p(x, \xi; \hbar) = p_0(x, \xi) + \hbar p_1(x, \xi) + \cdots \in S^{2,0}(T^*X).$$

The principal symbol $p_0(x, \xi) = |\xi|_x^2/2 - 1/2$ generates (through Hamilton's equations) the motion of a free particle on X. In particular, the Hamilton flow $g^t = \exp tX_{p_0}$ restricted to the energy shell $p_0^{-1}(0) = S^*X$ is the geodesic flow (in the following, we will often denote by $\mathcal{E}$ this energy shell).

Notation 2.2. The Laplacian eigenmodes will often be denoted by $\psi_\hbar$ instead of ψ_n, with the convention that the state $\psi_\hbar$ satisfies the eigenvalue equation

$$P(\hbar)\,\psi_\hbar = \Big(-\frac{\hbar^2\,\Delta}{2} - \frac{1}{2}\Big)\psi_\hbar = 0. \tag{2.6}$$

Definition 2.3. We will call $S \subset (0, 1]$ a countable set of scales $\hbar$, with only accumulation point at the origin. A sequence of states indexed by $\hbar \in S$ will be denoted by $(\varphi_\hbar)_{\hbar\in S}$, or sometimes, omitting the reference to a specific S, by $(\varphi_\hbar)_{\hbar\to 0}$.

2.2.2. *Anosov Hamiltonian flows.* In Theorem 1.7 we deal with more general Hamiltonians $p(x, \xi; \hbar)$ on T^*X, where X is compact or could also be the Euclidean space $\mathbb{R}^d$. The (real) symbol p is assumed to belong to a class $S^{m,0}(T^*X)$, and admit the expansion

$$p(x, \xi, \hbar) = p_0(x, \xi) + \hbar^\nu p_1(x, \xi, \hbar), \qquad p_0, p_1 \in S^{m,0}(T^*X),\ \nu > 0,$$

that is p_0 is the principal symbol of p. We could as well consider more general symbol classes, see for instance [11, Section 4.3] in the case $X = \mathbb{R}^d$. We assume that

(1) the energy shell $\mathcal{E} = p_0^{-1}(0)$ is compact, so that $\mathcal{E}_\epsilon \stackrel{\text{def}}{=} p_0^{-1}([-\epsilon, \epsilon])$ is compact as well for $\epsilon > 0$ small enough.
(2) the Hamiltonian flow $g^t = e^{tH_{p_0}}$ restricted to the energy shell $\mathcal{E}$ does not admit fixed points, and is of Anosov type.

The Hamiltonian p is quantized into an operator $P(\hbar) = \mathrm{Op}_\hbar(p) \in \Psi^{m,0}$. The first assumption above implies that, for $\hbar > 0$ small enough, the spectrum of $P(\hbar)$ near zero is purely discrete. We will focus on sequences of normalized null eigenstates $(\psi_\hbar)_{\hbar\to 0}$:

$$P(\hbar)\psi_\hbar = 0. \tag{2.7}$$

Remark 2.4. If $\psi_\hbar$ is a "quasi-null" eigenstate of $P(\hbar)$, that is if $P(\hbar)\psi_\hbar = E(\hbar)\psi_\hbar$ with $E(\hbar) = \mathcal{O}(\hbar^\nu)$, then it is a null eigenstate of $\tilde{P}(\hbar) \stackrel{\mathrm{def}}{=} P(\hbar) - E(\hbar)$, which admits the same principal symbol p_0 as $P(\hbar)$. As a result, Theorem 1.7 is also valid for such sequences of states.

2.3. $\hbar$-dependent singular observables. In the following we will have to use some classes of "singular" $\hbar$-dependent symbols.

2.3.1. "*Isotropically singular*" *observables.* For $\nu \in [0, \frac{1}{2})$, we will consider the class

$$S^{m,k}_\nu(T^*X) = \big\{ f(\hbar) \in C^\infty(T^*X), \big|\partial_x^\alpha \partial_\xi^\beta f(\hbar)\big| \leq C_{\alpha,\beta} \hbar^{-k-\nu|\alpha+\beta|} \langle\xi\rangle^{m-|\beta|} \big\}. \tag{2.8}$$

Such functions can strongly oscillate on scales $\geq \hbar^\nu$. The corresponding operators belong to an algebra $\Psi^{m,k}_\nu(T^*X)$ which can still be analyzed using an $\hbar$-expansion of the type (2.4). The main difference is that the higher-order terms $(f \sharp g)_j \in S^{m+m',k+k'+2j\nu}_\nu$. Similarly, the Gårding inequality reads, for $f \in S^{0,0}_\nu$:

$$\|\mathrm{Op}_\hbar(f)\|_{L^2} = \|f\|_\infty + \mathcal{O}_f(\hbar^{1-2\nu}),$$

where the implicit constant depends on a certain seminorm of f.

2.3.2. "*Anisotropically singular*" *observables.* We will also need to quantize observables which are "very singular" along certain directions, away from some specific submanifold (see for instance [29] for a presentation). Consider $\Sigma \subset T^*X$ a compact co-isotropic manifold of dimension $2d - D$ (with $D \leq d$). Near each point $\rho \in \Sigma$, there exist local canonical coordinates (y_i, η_i) such that $\Sigma = \{\eta_1 = \eta_2 = \cdots = \eta_D = 0\}$. For some index $\nu \in [0, 1)$, we define as follows a class of smooth symbols $f \in S^{m,k}_{\Sigma,\nu}(T^*X) \subset C^\infty(T^*X \times (0, 1])$:

- for any family of smooth vector fields $V_1, \ldots, V_{l_1}$ tangent to Σ and of smooth vector fields $W_1, \ldots, W_{l_2}$, we have in any neighbourhood Σ_ϵ of Σ:

$$\sup_{\rho\in\Sigma_\epsilon} |V_1 \cdots V_{l_1} W_1 \cdots W_{l_2} f(\rho)| \leq C\, \hbar^{-k-\nu l_2}.$$

- away from Σ we require $|\partial_x^\alpha \partial_\xi^\beta| = \mathcal{O}(\hbar^{-k}\langle\xi\rangle^{m-|\beta|})$.

Such a symbol f can be split into components f_j localized in neighbourhoods $\mathcal{V}_j \subset \Sigma_\epsilon$, plus an "external" piece $f_\infty \in S^{m,k}(T^*X)$ vanishing near Σ. Each piece f_j is Weyl-quantized in local adapted canonical coordinates (y, η) on $\mathcal{V}_j$ (as in (2.1)), and then brought back to the original coordinates (x, ξ) using Fourier integral operators. On the other hand, f_∞ is quantized as in (2.2). Finally, $\mathrm{Op}_{\Sigma,\hbar}(f)$ is obtained by summing the various contributions. The resulting class of operators is denoted by $\Psi^{m,k}_{\Sigma,\nu}(T^*X)$.

2.3.3. *Sharp energy cutoffs.* We will mostly use this quantization relative to the energy layer $\Sigma \stackrel{\mathrm{def}}{=} \mathcal{E} = p_0^{-1}(0)$, in order to define a family of sharp energy cutoffs. Namely, for some small $\delta > 0$ we will start from a cutoff $\chi_\delta \in C^\infty(\mathbb{R})$ such that $\chi_\delta(s) = 1$ for $|s| \le e^{-\delta/2}$, $\chi_\delta(s) = 0$ for $|s| \ge 1$. From there, we define, for each $\hbar \in (0,1]$ and each $n \ge 0$, the rescaled function $\chi^{(n)} \in C_c^\infty(\mathbb{R} \times (0,1])$ by

$$\chi^{(n)}(s,\hbar) \stackrel{\text{def}}{=} \chi_\delta(e^{-n\delta}\,\hbar^{-1+\delta}\,s). \tag{2.9}$$

The functions $\chi^{(n)} \circ p_0$ are "sharp" energy cutoffs, they belong to the class $S^{-\infty,0}_{\mathcal{E},1-\delta}$. We will always consider $n \le n_{\max} = C_\delta |\log \hbar|$, where the constant $C_\delta < \delta^{-1} - 1$, such that $\operatorname{supp}\chi^{(n)}$ is microscopic.

These cutoffs can be quantized in two ways:

(1) we may directly quantize the function $\chi^{(n)} \circ p_0$, into $\mathrm{Op}_{\mathcal{E},\hbar}(\chi^{(n)} \circ p_0) \in \Psi^{-\infty,0}_{\mathcal{E},1-\delta}$.
(2) or we can consider, using functional calculus, the operators $\chi^{(n)}\big(P(\hbar)\big)$. These operators (which generally differ from the previous ones) also belongs to $\Psi^{-\infty,0}_{\mathcal{E},1-\delta}$.

The sequence $(\chi^{(n)})_{0 \le n \le n_{\max}}$ is an increasing sequence of embedded cutoffs: for each n, we have $\chi^{(n+1)}\chi^{(n)} = \chi^{(n)}$ (equivalently, $\chi^{(n+1)} \succ \chi^{(n)}$). More precisely, we have here

$$\operatorname{dist}\big(\operatorname{supp}\chi^{(n)}, \operatorname{supp}(1-\chi^{(n+1)})\big) \ge \hbar^{1-\delta} e^{\delta n}(e^{\delta/2} - 1). \tag{2.10}$$

This distance between the supports implies the following

Lemma 2.5. *For any symbol $f \in S^{m,0}_{\mathcal{E},1-\delta}$ and any $0 \le n \le n_{\max}$, one has*

$$\big(\mathrm{Id} - \mathrm{Op}_{\mathcal{E},\hbar}(\chi^{(n+1)} \circ p_0)\big)\,\mathrm{Op}_{\mathcal{E},\hbar}(f)\;\mathrm{Op}_{\mathcal{E},\hbar}(\chi^{(n)} \circ p_0) = \mathcal{O}(\hbar^\infty). \tag{2.11}$$

The same property holds if we replace $\mathrm{Op}_{\mathcal{E},\hbar}(\chi^{(n)} \circ p_0)$ by $\chi^{(n)}\big(P(\hbar)\big)$.

Using the calculus of the class $S^{m,0}_{\mathcal{E},1-\delta}$, one can use the ellipticity of $P(\hbar)$ away from $\mathcal{E}$ to show that, if $(\psi_\hbar)$ is a sequence of null eigenstates of $P(\hbar)$, then

$$\Big(\mathrm{Id} - \chi^{(0)}\big(P(\hbar)\big)\Big)\psi_\hbar = \mathcal{O}(\hbar^\infty), \qquad \hbar \to 0. \tag{2.12}$$

That is, in the semiclassical limit the eigenstate $\psi_\hbar$ is microlocalized inside the energy layer of width $\hbar^{1-\delta}$ around $\mathcal{E}$.

2.4. Semiclassical measures. The $\hbar$-semiclassical calculus allows us to define what we mean by "phase space distribution of the eigenstate $\psi_\hbar$," through the notion of *semiclassical measure*. A Borel measure μ on the phase space T^*X can be fully characterized by the set of its values

$$\mu(f) = \int_{T^*X} f \, d\mu,$$

over *smooth test functions* $f \in C_c(T^*X)$. For each semiclassical scale $\hbar$, one can quantize a test function into a *test operator* $\mathrm{Op}_\hbar(f)$ (which is, as mentioned above, a continuous operator on $L^2(X)$). To any normalized state $\varphi \in L^2(X)$ we can then associate the linear functional

$$f \in C_c^\infty(T^*X) \mapsto \mu_{\hbar,\varphi}(f) \stackrel{\text{def}}{=} \langle \varphi, \mathrm{Op}_\hbar(f)\varphi\rangle.$$

$\mu_{\hbar,\varphi}$ is a distribution on T^*X, which encodes the localization properties of the state φ in the phase space, at the scale $\hbar$. Let us give an example. Using some local coordinate chart near $x_0 \in X$ and a function $\hbar^2 \ll c(\hbar) \ll 1$, we can define a Gaussian wavepacket by

$$\varphi_\hbar(x) \stackrel{\text{def}}{=} C_\hbar\, \chi(x) \exp\Big\{ -\frac{|x-x_0|^2}{c(\hbar)} + \mathrm{i}\frac{x\cdot\xi_0}{\hbar} \Big\}.$$

Here χ is a smooth cutoff equal to unity near x_0, which vanishes outside the coordinate chart, $C_\hbar$ is a normalization factor. When $\hbar \ll 1$, the distribution $\mu_{\hbar,\varphi_\hbar}$ associated with this wavefunction gets very peaked around the point $(x_0,\xi_0) \in T^*X$. If we had used the quantization at the scale $2\hbar$, the measure $\mu_{2\hbar,\varphi_\hbar}$ would have been peaked around $(x_0,\xi_0/2)$ instead.

Since the distribution $\mu_{\hbar,\varphi}$ is defined by duality with respect to the quantization $f \mapsto \mathrm{Op}_\hbar(f)$, it depends on the precise quantization scheme $\mathrm{Op}_\hbar$. In the case $X = \mathbb{R}^d$ and $\mathrm{Op}_\hbar$ is the Weyl quantization, the distribution $\mu_{\varphi,\hbar}$ is called the *Wigner distribution* associated with the state φ and the scale $\hbar$. Fortunately, as shown by Proposition 2.1, this scheme-dependence is irrelevant in the semiclassical limit.

Corollary 2.6. *For any $\varphi \in L^2$, consider the distributions $\mu^1_{\hbar,\varphi}, \mu^2_{\hbar,\varphi}$ defined by duality with two $\hbar$-quantizations $\mathrm{Op}^1_\hbar, \mathrm{Op}^2_\hbar$. Then, the following estimate holds in the semiclassical limit, uniformly with respect to $\varphi \in L^2$:*

$$\forall f \in C^\infty_{\mathrm{c}}(T^*X), \qquad \mu^1_{\hbar,\varphi}(f) - \mu^2_{\hbar,\varphi}(f) = \mathcal{O}_f(\hbar\|\varphi\|).$$

Let $S \subset (0,1]$ be a set of scales. For a given family of L^2-normalized states $(\varphi_\hbar)_{\hbar\in S}$, we consider the sequence of distributions $(\mu_{\hbar,\varphi_\hbar})_{\hbar\in S}$ on T^*X. It is always possible to extract a subset of scales $S' \subset S$, such that

$$\forall f \in C^\infty_{\mathrm{c}}(T^*X), \qquad \mu_{\hbar,\varphi_\hbar}(f) \xrightarrow{S'\ni\hbar\to 0} \mu_{\mathrm{sc}}(f),$$

with μ_{sc} a certain distribution on T^*X. One can show that μ_{sc} is a Radon measure on T^*X [11, Theorem 5.2]. From the above remarks, μ_{sc} does not depend on the precise scheme of quantization.

Definition 2.7. The measure μ_{sc} is called *the* semiclassical measure associated with the subsequence $(\varphi_\hbar)_{\hbar\in S'}$. It is also *a* semiclassical measure associated with the sequence $(\varphi_\hbar)_{\hbar\in S}$.

From now on, we will assume that $\varphi_\hbar = \psi_\hbar$ is a null eigenstate of the quantum Hamiltonian $P(\hbar)$ in Section 2.2: we will then call μ_{sc} a semiclassical measure of the Hamiltonian $P(\hbar)$.

Proposition 2.8. *Any semiclassical measure associated with a sequence $(\psi_\hbar)_{\hbar\to 0}$ of eigenstates of the Hamiltonian $P(\hbar)$ is a probability measure supported on the energy layer $\mathcal{E}$, which is invariant with respect to the geodesic flow g^t on $\mathcal{E}$.*

Proof. Possibly after extracting a subsequence, we assume that μ_{sc} is the semiclassical measure associated with a sequence of eigenstates $(\psi_\hbar)_{\hbar\in S}$. The support property of μ_{sc} comes from the fact that the operator $P(\hbar)$ is elliptic outside $\mathcal{E}$. As a result, for any $f \in C^\infty_{\mathrm{c}}(T^*X)$ vanishing near $\mathcal{E}$, one can construct a symbol $g \in S^{-\infty,0}(T^*X)$ such that

$$\mathrm{Op}_\hbar(f) = \mathrm{Op}_\hbar(g)P(\hbar) + \mathcal{O}_{L^2\to L^2}(\hbar^\infty).$$

Applying this equality to the eigenstates $\psi_\hbar$, we get $\|\mathrm{Op}_\hbar(f)\psi_\hbar\| = \mathcal{O}(\hbar^\infty)$, proving the support property of μ_{sc}.

To prove the flow invariance, we need to compare the quantum time evolution with the classical one. Denote by $U_\hbar^t = \exp\{-\mathrm{i}tP(\hbar)/\hbar\}$ the propagator generated by the Hamiltonian $P(\hbar)$: it solves the time-dependent Schrödinger equation, and thus provides the quantum evolution. Let us state **Egorov's theorem**, which is a rigorous form of quantum-classical correspondence in terms of observables:

$$\text{(2.13)}\quad \forall f \in C_c^\infty(T^*X), \forall t \in \mathbb{R},\qquad U_\hbar^{-t}\operatorname{Op}_\hbar(f)U_\hbar^t = \operatorname{Op}_\hbar(f\circ g^t) + \mathcal{O}_{f,t}(\hbar),\qquad \hbar\to 0.$$

Since $\psi_\hbar$ is an eigenstate of $U_\hbar^t$, we directly get

$$\mu_{\hbar,\varphi_\hbar}(f) = \mu_{\hbar,\varphi_\hbar}(f\circ g^t) + \mathcal{O}_{f,t}(\hbar) \implies \mu_{\mathrm{sc}}(f) = \mu_{\mathrm{sc}}(f\circ g^t). \qquad \square$$

These properties of semiclassical measures naturally lead to the following question:

Among all flow-invariant probability measures supported on $\mathcal{E}$, which ones appear as semiclassical measures associated with eigenstates of $P(\hbar)$?

To start answering this question, we will investigate the Kolmogorov–Sinai entropy of semiclassical measures. We will show that, in the case of an Anosov flow, the requirement of being a semiclassical measure implies a nontrivial lower bound on the entropy.

3. From classical to quantum entropies

3.1. Entropies and pressures [18].

3.1.1. *Kolmogorov–Sinai entropy of an invariant measure.* In this paper we will deal with several types of entropies. All of them are defined in terms of certain discrete probability distributions, that is *finite* sets of real numbers $\{p_i, i\in I\}$ satisfying

$$p_i \in [0,1],\qquad \sum_{i\in I} p_i = 1.$$

The entropy associated with such a set is the real number

$$\text{(3.1)}\qquad H(\{p_i\}) = \sum_{i\in I}\eta(p_i),\qquad \text{where } \eta(s) \stackrel{\text{def}}{=} -s\log s,\ s\in[0,1].$$

Our first example is the entropy $H(\mu,\mathcal{P})$ associated with a g^t-invariant probability measure μ on the energy shell $\mathcal{E}$ and a finite measurable partition $\mathcal{P} = (E_1,\ldots,E_K)$ of $\mathcal{E}$. That entropy is given by

$$\text{(3.2)}\qquad H(\mu,\mathcal{P}) = H(\{\mu(E_k)\}) = \sum_{k=1}^K \eta\big(\mu(E_k)\big).$$

One can then use the flow g^t in order to *refine* the partition $\mathcal{P}$. For each integer $n\geq 1$ we define the n-th refinement $\mathcal{P}^{\vee n} = [\mathcal{P}]_0^{n-1}$ as the partition composed of the sets

$$E_{\boldsymbol{\alpha}} \stackrel{\text{def}}{=} g^{-n+1}E_{\alpha_{n-1}}\cap\cdots\cap g^{-1}E_{\alpha_1}\cap E_{\alpha_0},$$

where $\boldsymbol{\alpha} = \alpha_0\cdots\alpha_{n-1}$ can be any sequence of length n with symbols $\alpha_i\in\{1,\ldots,K\}$. In general many of the sets $E_{\boldsymbol{\alpha}}$ may be empty, but we will nonetheless sum over all sequences of a given length n. More generally, for any $m \in \mathbb{Z}, n \geq 1$, we consider the partition $[\mathcal{P}]_m^{m+n-1}$ made of the sets

$$g^{-m} E_{\boldsymbol{\alpha}} = g^{-m-n+1} E_{\alpha_{n-1}} \cap \cdots \cap g^{-m} E_{\alpha_0}, \qquad |\boldsymbol{\alpha}| = n.$$

From this refined partition we obtain the entropy $H(\mu, [\mathcal{P}]_m^{m+n-1}) = H_m^{m+n-1}(\mu)$.

From the concavity of the logarithm, one easily gets

$$\forall m, n \geq 1, \qquad H(\mu, [\mathcal{P}]_0^{n+m-1}) \leq H(\mu, [\mathcal{P}]_0^{n-1}) + H(\mu, [\mathcal{P}]_n^{n+m-1}). \tag{3.3}$$

If the measure μ is g^t-invariant, this has for consequence the *subadditivity property*:

$$H(\mu, \mathcal{P}^{\vee(n+m)}) \leq H(\mu, \mathcal{P}^{\vee n}) + H(\mu, \mathcal{P}^{\vee m}) \tag{3.4}$$

It thus makes sense to consider the limit

$$H_{\mathrm{KS}}(\mu, \mathcal{P}) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \frac{1}{n} H(\mu, \mathcal{P}^{\vee n}) = \lim_{n\to\infty} \frac{1}{2n} H(\mu, [\mathcal{P}]_{-n+1}^{n}),$$

the *Kolmogorov–Sinai entropy* of the invariant measure μ, associated with the partition $\mathcal{P}$. The KS entropy *per se* is defined by maximizing over the initial (finite) partition $\mathcal{P}$:

$$H_{\mathrm{KS}}(\mu) \stackrel{\mathrm{def}}{=} \sup_{\mathcal{P}} H_{\mathrm{KS}}(\mu, \mathcal{P}).$$

For an Anosov flow, this supremum is actually reached as soon as the partition $\mathcal{P}$ has a sufficiently small diameter (that is, its elements E_k have uniformly small diameters).

3.1.2. *Pressures associated with invariant measures.* Let us come back to our probability distribution $\{p_i, i \in I\}$. We may associate to it a set of weights, that is of positive real numbers $\{w_i > 0, i \in I\}$, making up a weighted probability distribution. The pressure $p(\{p_i\}, \{w_i\})$ associated with this weighted distribution is the real number[5]

$$p(\{p_i\}, \{w_i\}) \stackrel{\mathrm{def}}{=} -\sum_i p_i \log(w_i^2 p_i) = H(\{p_i\}) - 2 \sum_{i\in I} p_i \log w_i.$$

For instance, in the case of a flow-invariant measure on $\mathcal{E}$ and a partition $\mathcal{P}$, we can select weights w_k on each component E_k, and define the pressure

$$p(\mu, \mathcal{P}, w) \stackrel{\mathrm{def}}{=} H(\mu, \mathcal{P}) - 2 \sum_k \mu(E_k) \log w_k.$$

We want to refine this pressure using the flow. The weights corresponding to the nth refinement can be simply defined as

$$w_{\boldsymbol{\alpha}} = \prod_{j=0}^{n-1} w_{\alpha_j}, \qquad |\boldsymbol{\alpha}| = n.$$

The refined pressure is denoted by $p_0^{n-1}(\mu, \mathcal{P}, w)$. From the subadditivity of the entropies (3.5) one easily draws the subadditivity of the pressures:

$$p_0^{n+m-1}(\mu, \mathcal{P}, w) \leq p_0^{n-1}(\mu, \mathcal{P}, w) + p_0^{m-1}(\mu, \mathcal{P}, w). \tag{3.5}$$

[5]The factor -2 appearing in front of the second term is convenient for our future aims.

3.1.3. *Smoothed partitions near $\mathcal{E}$.* The definition of $H(\mu,\mathcal{P})$ can be expressed in terms of characteristic functions over the partition $\mathcal{P}$. Indeed, if $\mathbb{1}_k$ is the characteristic function on E_k, then the function

$$\mathbb{1}_{\boldsymbol{\alpha}} = (\mathbb{1}_{\alpha_{n-1}} \circ g^{n-1}) \times \cdots \times (\mathbb{1}_{\alpha_1} \circ g) \times \mathbb{1}_{\alpha_0} \tag{3.6}$$

is the characteristic function of $E_{\boldsymbol{\alpha}}$. In formula (3.2) we can then replace $\mu(E_k)$ by

$$\mu(\mathbb{1}_k) \stackrel{\text{def}}{=} \int_{\mathcal{E}} \mathbb{1}_k \, \mathrm{d}\mu.$$

Let us assume that the invariant measure μ does not charge the boundary of the E_k (this is always possible by slightly shifting the boundaries of the E_k). Then, for any $\epsilon > 0$, we can approximate the characteristic function $\mathbb{1}_k$ by a smooth function $\pi_k \in C_c^\infty(\mathcal{E}_\epsilon, [0,1])$ supported in a small neighbourhood $\widetilde{E}_k$ of E_k, and such that these K functions form a *smooth partition of unity near* $\mathcal{E}$:

$$\sum_{k=1}^{K} \pi_k(\rho) = \chi_{\epsilon/2}(\rho), \qquad \operatorname{supp} \chi_{\epsilon/2} \subset \mathcal{E}_\epsilon, \qquad \chi_{\epsilon/2} = 1 \text{ near } \mathcal{E}_{\epsilon/2}. \tag{3.7}$$

One can extend the definition of the entropy to the smooth partition $\mathcal{P}_{\mathrm{sm}} = \{\pi_k\}_{k=1,\ldots,K}$ and its refinements through the flow. From the assumption $\mu(\partial\mathcal{P}) = 0$, for any $\epsilon' > 0$ and any $n \geq 1$ we can choose $\mathcal{P}_{\mathrm{sm}}$ such that

$$|H(\mu, \mathcal{P}_{\mathrm{sm}}^{\vee n}) - H(\mu, \mathcal{P}^{\vee n})| \leq \epsilon' \,.$$

To prove a lower bound on the entropy $H_{\mathrm{KS}}(\mu)$ therefore amounts to proving a lower bound on $(1/n) H(\mu, \mathcal{P}_{\mathrm{sm}}^{\vee n})$, uniform with respect to $n \geq 1$ and the smoothing $\mathcal{P}_{\mathrm{sm}}$ of $\mathcal{P}$. The advantage of using a smoothed partition $\mathcal{P}_{\mathrm{sm}}$ is that it is fit for quantization.

3.2. Quantum partitions of unity.

3.2.1. *Definition.* From the smoothed partition $\mathcal{P}_{\mathrm{sm}} = \{\pi_k\}_{k=1,\ldots,K}$ we form a *quantum partition of unity* $\mathcal{P}_{\mathrm{sm,q}} = \{\Pi_k = \mathrm{Op}_\hbar(\widetilde{\pi}_k)\}_{k=1,\ldots,K}$, where $\mathrm{Op}_\hbar$ is the $\hbar$-quantization (2.2), and the symbols $\widetilde{\pi}_k \in S^{-\infty,0}(T^*X)$ satisfy the following properties:

(1) for each k, the symbol $\widetilde{\pi}_k$ is real, supported on $\widetilde{E}_k$, and admits $\sqrt{\pi_k}$ as principal symbol. The operator Π_k is thus self-adjoint.

(2) the family $\mathcal{P}_{\mathrm{sm,q}} = \{\Pi_k\}_{k=1,\ldots,K}$ is a quantum partition of unity microlocally near $\mathcal{E}$:

$$\sum_{k=1}^{K} \Pi_k^2 = \mathrm{Op}_\hbar(\widetilde{\chi}_{\epsilon/2}) + \mathcal{O}(\hbar^\infty), \tag{3.8}$$

where $\widetilde{\chi}_{\epsilon/2} \in S^{-\infty,0}(T^*X)$ satisfies

$$\widetilde{\chi}_{\epsilon/2}(\rho) \equiv 1 \text{ near } \mathcal{E}_{\epsilon/2}, \qquad \operatorname{supp} \widetilde{\chi}_{\epsilon/2} \subset \mathcal{E}_\epsilon, \qquad \|\mathrm{Op}_\hbar(\widetilde{\chi}_{\epsilon/2})\| = 1 + \mathcal{O}(\hbar^\infty)\,.$$

Notice that $\widetilde{\chi}_{\epsilon/2}$ has for principal symbol $\chi_{\epsilon/2}$ of (3.7).

Remark. Had we simply taken $\Pi_k = \mathrm{Op}_\hbar(\sqrt{\pi_k})$, the above properties would hold only up to remainders $\mathcal{O}(\hbar)$. By iteratively adjusting the higher-order symbols in $\widetilde{\pi}_k$ (and $\widetilde{\chi}_{\epsilon/2}$), we can enforce these properties to any order in $\hbar$.

3.2.2. *Refined quantum partitions.* In the classical framework, the n-refinement of the partition $\mathcal{P}_{\rm sm} = \{\pi_k\}_{k=1,\dots,K}$ was obtained by considering the products $\pi_{\alpha_{n-1}} \circ g^{n-1} \times \cdots \times \pi_{\alpha_0}$, for all sequences $\boldsymbol{\alpha}$ of length n. Egorov's theorem shows that the quantum observable $\mathrm{Op}_\hbar(\pi_{\alpha_j} \circ g^j)$ resembles the quantum evolution $U^{-j}\,\mathrm{Op}_\hbar(\pi_{\alpha_j})U^j$, where $U = U_\hbar = \mathrm{e}^{-\mathrm{i}P(\hbar)/\hbar}$ is the Schrödinger propagator (at time unity). For this reason, we define as follows the elements of the n-refined quantum projection:

$$\Pi_{\boldsymbol{\alpha}} \stackrel{\rm def}{=} U^{-n+1}\Pi_{\alpha_{n-1}}U\cdots U\Pi_{\alpha_2}U\Pi_{\alpha_1}U\Pi_{\alpha_0}, \qquad \boldsymbol{\alpha} = \alpha_0\cdots\alpha_{n-1}. \tag{3.9}$$

We first need to check that these operators still make up a quantum partition of unity near $\mathcal{E}$.

Proposition 3.1. *Take $n_{\max} = [C_\delta|\log\hbar|]$ as in Section* 2.3.3. *Then, for each $1 \leq n \leq n_{\max}$, the family of operators $\mathcal{P}^{\vee n}_{\rm sm,q} = \{\Pi_{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}| = n\}$ forms a quantum partition of unity microlocally near $\mathcal{E}$, in the following sense. For any symbol $\chi \in S^{-\infty,0}(T^*X)$ supported inside $\mathcal{E}_{\epsilon/2}$, we have*

$$\begin{aligned} \forall n \leq n_{\max}, \qquad \sum_{\alpha_0,\dots,\alpha_{n-1}} \Pi^*_{\boldsymbol{\alpha}}\Pi_{\boldsymbol{\alpha}} = S_n, \qquad \|S_n\| = 1 + \mathcal{O}(\hbar^\infty), \\ (\mathrm{Id} - S_n)\,\mathrm{Op}_\hbar(\chi) = \mathcal{O}(\hbar^\infty). \end{aligned} \tag{3.10}$$

Proof. The statement is obvious in the case $\{\Pi_k\}_{k=1,\dots,K}$ forms a full resolution of identity (that is, if the left-hand side in (3.8) is equal to the identity modulo $\mathcal{O}(\hbar^\infty)$), as was the case in [1, 3, 4] and will be the case in Section 5. One can then sum over $\sum_{\alpha_{n-1}} \Pi^2_{\alpha_{n-1}} = \mathrm{Id} + \mathcal{O}(\hbar^\infty)$, then over α_{n-2}, etc., to finally obtain $S_n = \mathrm{Id} + \mathcal{O}(\hbar^\infty)$.

In the case of a microlocal partition near $\mathcal{E}$, the sum over the index α_{n-1} leads to a product $\Pi_{\alpha_{n-2}}\,\mathrm{Op}_\hbar(\widetilde{\chi}_{\epsilon/2})\Pi_{\alpha_{n-2}}$, where $\widetilde{\chi}_{\epsilon/2}$ is the symbol appearing in (3.8). To "absorb" the factor $\mathrm{Op}_\hbar(\widetilde{\chi}_{\epsilon/2})$, we will insert intermediate cutoffs at each time. We recall that $\chi \prec \mathbb{1}_{\mathcal{E}_{\epsilon/2}} \prec \widetilde{\chi}_{\epsilon/2}$. We consider a sequence of cutoffs $(\chi_j \circ p_0)_{1\leq j\leq n_{\max}}$ such that $\mathbb{1}_{\mathcal{E}_{\epsilon/2}} \prec \chi_1 \circ p_0 \prec \chi_2 \circ p_0 \cdots \prec \chi_{n_{max}} \circ p_0 \prec \widetilde{\chi}_{\epsilon/2}$. Since $n_{\max} \sim |\log\hbar|$, the χ_j will necessarily depend on $\hbar$, their derivatives growing like $\hbar^{-\nu}$ for some small $\nu > 0$, so that $\chi^j \circ p_0 \in S^{-\infty,0}_\nu$. The calculus in $S^{-\infty,0}_\nu$ and $\chi_j \prec \chi_{j+1}$ show that, for any $1 \leq k \leq K$:

$$\forall 1 \leq j \leq n_{\max} - 1, \qquad \big(\mathrm{Id} - \chi_{j+1}\big(P(\hbar)\big)\big)\,\Pi_k\chi_j\big(P(\hbar)\big) = \mathcal{O}(\hbar^\infty). \tag{3.11}$$

This implies that we can indeed insert intermediate cutoffs in $\Pi_{\boldsymbol{\alpha}}$ with no harm:

$$\begin{aligned} \Pi_{\boldsymbol{\alpha}}\,\mathrm{Op}_\hbar(\chi) = U^{-n+1}\Pi_{\alpha_{n-1}}U\Pi_{\alpha_{n-2}}U\chi_{n-2}\big(P(\hbar)\big)\cdots U\chi_2\big(P(\hbar)\big)\Pi_{\alpha_1} \\ \times U\chi_1\big(P(\hbar)\big)\Pi_{\alpha_0}\,\mathrm{Op}_\hbar(\chi) + \mathcal{O}(\hbar^\infty). \end{aligned}$$

We also have

$$\forall j, \qquad \big(\mathrm{Id} - \mathrm{Op}_\hbar(\widetilde{\chi}_{\epsilon/2})\big)\Pi_k\chi_j\big(P(\hbar)\big) = \mathcal{O}(\hbar^\infty). \tag{3.12}$$

This equation, and the fact that $\chi_j\big(P(\hbar)\big)$ commutes with the propagator U, results in

$$\begin{aligned}\sum_{\alpha_{n-2}} U^*\Pi_{\alpha_{n-2}} \operatorname{Op}_\hbar(\widetilde{\chi}_{\epsilon/2})\Pi_{\alpha_{n-2}} U\chi_{n-2}\big(P(\hbar)\big) & \\ &= \sum_{\alpha_{n-2}} U^*\Pi^2_{\alpha_{n-2}}\chi_{n-2}\big(P(\hbar)\big)U + \mathcal{O}(\hbar^\infty)\\ &= U^*\operatorname{Op}_\hbar(\widetilde{\chi}_{\epsilon/2})\chi_{n-2}\big(P(\hbar)\big)U + \mathcal{O}(\hbar^\infty)\\ &= U^*\chi_{n-2}\big(P(\hbar)\big)U + \mathcal{O}(\hbar^\infty)\\ &= \chi_{n-2}\big(P(\hbar)\big) + \mathcal{O}(\hbar^\infty).\end{aligned}$$

Using (3.11) at each step, the summation over $\alpha_{n-3}, \alpha_{n-4}, \ldots$ finally brings us to

$$\sum_{\boldsymbol{\alpha}} \Pi^*_{\boldsymbol{\alpha}}\Pi_{\boldsymbol{\alpha}} \operatorname{Op}_\hbar(\chi) = \sum_{\alpha_0} \Pi_{\alpha_0}\chi_1\big(P(\hbar)\big)\Pi_{\alpha_0}\operatorname{Op}_\hbar(\chi) + \mathcal{O}(\hbar^\infty).$$

The equation $\big(\mathrm{Id} - \chi_1\big(P(\hbar)\big)\big)\,\Pi_{\alpha_0}\operatorname{Op}_\hbar(\chi) = \mathcal{O}(\hbar^\infty)$ ends the proof. □

3.3. From the refined operators $\boldsymbol{\Pi_\alpha}$ to a quantum symbolic measure. Let us turn back to the sequence of eigenstates $(\psi_\hbar)_{\hbar\to 0}$ associated with the semiclassical measure $\mu_{\rm sc}$. The proof of Proposition 2.8 shows that for any cutoff $\chi \in S^{-\infty,0}$, $\chi \equiv 1$ near $\mathcal{E}$, we have

$$\operatorname{Op}_\hbar(\chi)\psi_\hbar = \psi_\hbar + \mathcal{O}(\hbar^\infty).$$

As a result, for any $1 \le n \le n_{\max}$ we have $\sum_{|\boldsymbol{\alpha}|=n}\|\Pi_{\boldsymbol{\alpha}}\psi_\hbar\|^2 = 1 + \mathcal{O}(\hbar^\infty)$. Therefore, modulo an $\mathcal{O}(\hbar^\infty)$ error, the set $\{\|\Pi_{\boldsymbol{\alpha}}\psi_\hbar\|^2, |\boldsymbol{\alpha}| = n\}$ forms a discrete probability distribution. The proof shows that these weights also satisfy a compatibility condition (up to a negligible error):

$$\text{(3.13)}\quad \forall n \le n_{\max}, \forall\boldsymbol{\alpha} = \alpha_0\cdots\alpha_{n-1},\qquad \|\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar\|^2 = \sum_{\alpha_{n-1}}\|\Pi_{\alpha_0\cdots\alpha_{n-1}}\psi_\hbar\|^2 + \mathcal{O}(\hbar^\infty).$$

If we forget the errors $\mathcal{O}(\hbar^\infty)$, we can interpret the weights $\|\Pi_{\boldsymbol{\alpha}}\psi_\hbar\|^2$ in terms of a certain probability measure $\mu_\hbar$ on the symbolic space $\Sigma = \{1, \ldots, K\}^{\mathbb{Z}}$. Namely, each $\|\Pi_{\boldsymbol{\alpha}}\psi_\hbar\|^2$ corresponds to the weight of that measure on the *cylinder* $[\cdot\alpha_0\alpha_1\cdots\alpha_{n-1}]$:

$$\mu_\hbar([\cdot\boldsymbol{\alpha}]) \stackrel{\rm def}{=} \|\Pi_{\boldsymbol{\alpha}}\psi_\hbar\|^2. \tag{3.14}$$

Formally, $\mu_\hbar$ can be defined as an equivalence class of $\hbar$-dependent probability measures taking values on cylinders of lengths $n \le n_{\max} = C_\delta|\log\hbar|$, the equivalence relation consisting in equality up to errors $\mathcal{O}(\hbar^\infty)$. We will call such a measure a **symbolic measure**.

The defining property $\Pi_k = \operatorname{Op}_\hbar(\widetilde{\pi}_k)$ shows that each element $\mu_\hbar([\cdot\alpha_0]) = \|\Pi_{\alpha_0}\psi_\hbar\|^2$ approximately represents the microlocal weight of the state $\psi_\hbar$ inside the element E_k of the partition. Further on, for any fixed $n \ge 1$, Egorov's theorem (2.13) and the composition rule (2.3) show that the refined operators $\Pi_{\boldsymbol{\alpha}}$ are still "good" pseudodifferential operators:

$$\Pi_{\boldsymbol{\alpha}} = \operatorname{Op}_\hbar(\pi_{\boldsymbol{\alpha}}) + \mathcal{O}_n(\hbar),$$

where $\mathcal{P}_{\rm sm}^{\vee n} = \{\pi_{\boldsymbol{\alpha}}\}$ is the n-refinement of the smooth partition $\mathcal{P}_{\rm sm}$, as in (3.6). As a result, $\mu_\hbar([\cdot\boldsymbol{\alpha}])$ approximately represents the weight of $\psi_\hbar$ inside the refined partition element $E_{\boldsymbol{\alpha}}$.

From the assumption on the sequence $(\psi_\hbar)$, the symbolic measure $\mu_\hbar$ is obviously related with the semiclassical measure $\mu_{\rm sc}$: for any fixed $n \geq 1$ we have

$$\forall \boldsymbol{\alpha}, |\boldsymbol{\alpha}| = n, \qquad \mu_\hbar([\cdot\boldsymbol{\alpha}]) \xrightarrow{\hbar\to 0} \mu_{\rm sc}(\pi_{\boldsymbol{\alpha}}). \tag{3.15}$$

This limit assumes that the "time" n is fixed when taking $\hbar \to 0$. For our purposes, it will be crucial to extend the analysis to times n of logarithmic order in $\hbar$. Before doing so, let us give a crude description of the sets $E_{\boldsymbol{\alpha}}$ when $n = |\boldsymbol{\alpha}|$ grows. Let us assume that the elements E_k are approximately "isotropic" with respect to the stable and unstable directions. The inverse flow g^{-t} has the effect to compress along the unstable directions, and expand along the stable ones. As a result, the set $E_{\alpha_0\alpha_1} = g^{-1}E_{\alpha_1} \cap E_{\alpha_0}$ will be narrower than E_{α_0} along the unstable direction, but keep approximately the same size in the stable one. Iterating this procedure, the set $E_{\alpha_0\cdots\alpha_{n-1}}$ will become very anisotropic for large n: its size along the stable directions will remain comparable with that of E_{α_0}, while its "unstable volume" will be diminished by a factor $J_n^u(\alpha_0\cdots\alpha_{n-1})^{-1}$, where we use the **coarse-grained unstable Jacobian**

$$J_n^u(\alpha_0\cdots\alpha_{n-1}) \stackrel{\rm def}{=} \prod_{i=0}^{n-1} J^u(\alpha_i), \qquad J^u(k) \stackrel{\rm def}{=} \min_{\rho\in E_k} J^u(\rho). \tag{3.16}$$

The factor $J_n^u(\boldsymbol{\alpha})^{-1}$ decreases exponentially with n according to the minimal $(d-1)$-dimensional unstable expansion rate $\Lambda^u_{\rm min}$:

$$\forall n, \forall \boldsymbol{\alpha}, |\boldsymbol{\alpha}| = n, \qquad J_n^u(\boldsymbol{\alpha})^{-1} \leq C \mathrm{e}^{-n(\Lambda^u_{\rm min}-\epsilon)}, \tag{3.17}$$

so the sets $E_{\boldsymbol{\alpha}}$ become very thin along the unstable direction. This anisotropy is as well visible on the refined smooth functions $\pi_{\boldsymbol{\alpha}}$ or the refined symbols $\widetilde{\pi}_{\boldsymbol{\alpha}}$.

3.4. Egorov theorem up to the Ehrenfest time. The Egorov theorem (2.13) can be extended up to times $t \sim C|\log\hbar|$, provided the constant C is not too large. The breakdown occurs when the classically evolved function $f \circ g^t$ shows fluctuations of size unity across a distance $\sim h^{1/2}$: such a function is no more a "nice quantizable observable" (see Section 2.3).

Let us start from a symbol $f \in S^{-\infty,0}(T^*X)$ supported on the energy layer $\mathcal{E}_\epsilon$. We have called $\lambda_{\max}$ the maximal expansion rate of the flow on $\mathcal{E}$. Assume that $\lambda_\epsilon = \lambda_{\max} + \mathcal{O}(\epsilon)$ is larger than the maximal expansion rate on $\mathcal{E}_\epsilon$. It implies that the derivatives of the flow are controlled as follows:

$$\forall t \in \mathbb{R}, \forall \rho \in \mathcal{E}_\epsilon, \qquad \|\partial^\alpha g^t(\rho)\| \leq C_\alpha \mathrm{e}^{\lambda_\epsilon|\alpha t|}.$$

As a result, for any symbol $f \in S^{-\infty,0}$ supported inside $\mathcal{E}_\epsilon$, its classical evolution $f_t = f \circ g^t$ satisfy

$$\forall t \in \mathbb{R}, \forall \rho \in \mathcal{E}_\epsilon, \qquad \|\partial^\alpha f_t(\rho)\| \leq C_{f,\alpha} \mathrm{e}^{\lambda_\epsilon|\alpha t|}.$$

For $t \sim C|\log\hbar|$ the right hand sides become of order $\hbar^{-C\lambda_\epsilon|\alpha|}$. Therefore, if we want f_t to belong to a reasonable symbol class (see Section 2.3), we must restrict the values of C. Let us define the time

$$T_{\epsilon,\hbar} \stackrel{\rm def}{=} \frac{(1-\epsilon)|\log\hbar|}{2\lambda_\epsilon},$$

which is about half of what is generally called the *Ehrenfest time* $T_{\rm E} = |\log\hbar|/\lambda_{\max}$. Take any $\nu \in [(1-\epsilon)/2, \frac{1}{2})$. The above estimates show that for any choice of sequence $\big(t(\hbar)\big)_{\hbar\to 0}$ satisfying $|t(\hbar)| \leq T_{\epsilon,\hbar}$, the family of functions $\big(f_{t(\hbar)}\big)_{\hbar\to 0}$ belongs to the class $S_\nu^{-\infty,0}(T^*X)$ defined in (2.8). In other words, any seminorm of that class is uniformly bounded over the set $\{f_t,\, |t|\leq T_{\epsilon,\hbar},\, \hbar\in(0,1]\}$. It is then not surprising that Egorov's theorem holds true up to the time $T_{\epsilon,\hbar}$.

Proposition 3.2 ([3, Proposition 5.1])**.** *Fix $\epsilon>0$ and $\nu\in\big((1-\epsilon)/2,\frac12\big)$. Take $f\in S^{-\infty,0}$ supported inside $\mathcal{E}_\epsilon$. Then, for any $\hbar\in(0,1]$ and any time $t=t(\hbar)$ in the range $|t|\leq T_{\epsilon,\hbar}$, we have*

$$U^{-t}\operatorname{Op}_\hbar(f)U^t=\operatorname{Op}_\hbar(\tilde f_t)+\mathcal{O}(h^\infty),\qquad \text{with } \tilde f_t-f_t\in S_\nu^{-\infty,-(1+\epsilon)/2},\ f_t=f\circ g^t\in S_\nu^{-\infty,0}. \tag{3.18}$$

Proof. This proposition was essentially proved in [7] in the case of symbols on $T^*\mathbb{R}^d$ driven by some (appropriate) Hamiltonian flow. In that paper, the $\hbar$-expansion of the symbol $\tilde f_t$ was explicitly computed up to any fixed order $\hbar^L$, and the L^2 norm of the remainder was estimated. In [3, Section 5.2] we used the fact that

$$U^{-t}\operatorname{Op}_\hbar(f)U^t-\operatorname{Op}_\hbar(f_t)=\int_0^t \mathrm{d}s\, U^{-s}\operatorname{Diff} f_{t-s}\, U^s,$$

where

$$\operatorname{Diff} f_s=\frac{\mathrm{i}}{h}[P(\hbar),\operatorname{Op}_\hbar(f_s)]-\operatorname{Op}_\hbar(\{p,f_s\})\in\Psi_\nu^{-\infty,-(1+\epsilon)/2},$$

uniformly for $|t|\leq T_{\epsilon,\hbar}$. The Calderon−Vaillancourt theorem on $\Psi_\nu^{-\infty,-(1+\epsilon)/2}$ then implies that

$$\|U^{-t}\operatorname{Op}_\hbar(f)U^t-\operatorname{Op}_\hbar(f_t)\|\leq C\,|t|\,h^{(1+\epsilon)/2}. \tag{3.19}$$

In order to prove that $\tilde f_t\in S_\nu^{-\infty,0}$ one can proceed as in [7], that is compute the $\hbar$ expansion of $\tilde f_t$ order by order, taking into account that the quantization is performed on the manifold X, so that higher-order terms also depend on the various choices of local charts and cutoffs. We will not do so in any detail here, since we will mostly use the inequality (3.19). □

We will apply this proposition to the operators $U^{-j}\Pi_k U^j$: in the range $|j|\leq T_{\epsilon,\hbar}$ they are still pseudodifferential operators in some class $S_\nu^{-\infty,0}$. The products of these operators can also be analyzed:

Proposition 3.3. *Take any $1>\epsilon>0$ and $\nu\in[(1-\epsilon)/2,\frac12)$. Then the family of symbols $\{\widetilde\pi_{\boldsymbol\alpha},|\boldsymbol\alpha|\leq T_{\epsilon,\hbar}\}$ belongs to a bounded set in the class $S_\nu^{-\infty,0}$. Furthermore, the product operators $\Pi_{\boldsymbol\alpha}$ satisfy $\Pi_{\boldsymbol\alpha}-\operatorname{Op}_\hbar(\widetilde\pi_{\boldsymbol\alpha})\in\Psi_\nu^{-\infty,-(1+\epsilon)/2}$.*

Proof. A similar result was proved in [24, Theorem 7.1]. We already know that the symbols $\widetilde\pi_{\alpha_j}\circ g^j$ composing $\widetilde\pi_{\boldsymbol\alpha}$ belong to the class $\Psi_\nu^{-\infty,0}$. Any finite product of those symbols also remains in that class. We need to check that the symbol $\widetilde\pi_{\alpha_0\cdots\alpha_j}$ remains uniformly bounded in the class when increasing j until $T_{\epsilon,\hbar}$.

We start by applying Egorov's theorem to the operator $\Pi_{\alpha_{n-1}}$, then multiply by $\Pi_{\alpha_{n-2}}$:

$$U^{-1}\Pi_{\alpha_{n-1}}U\Pi_{\alpha_{n-2}}=\operatorname{Op}_\hbar(\widetilde\pi_{\alpha_{n-1}}\circ g\times\widetilde\pi_{\alpha_{n-2}})+R_2,\qquad R_2\in\Psi^{-\infty,-1}.$$

The function $\widetilde\pi_{\alpha_{n-2}\alpha_{n-1}}\overset{\text{def}}{=}\widetilde\pi_{\alpha_{n-1}}\circ g\times\widetilde\pi_{\alpha_{n-2}}$ is supported in a "rectangle" and satisfies $\|\partial^\beta\widetilde\pi_{\alpha_{n-2}\alpha_{n-1}}\|\leq C_\beta \mathrm{e}^{\lambda_\epsilon|\beta|}$. Applying the same procedure (evolution and multiplication), we construct a sequence of symbols

$$\widetilde\pi_{\alpha_{n-j}\cdots\alpha_{n-1}}\overset{\text{def}}{=}\widetilde\pi_{\alpha_{n-j+1}\cdots\alpha_{n-1}}\circ g\times\widetilde\pi_{\alpha_{n-j}}$$

and operators

$$U^{-j}\Pi_{\alpha_{n-1}}U\Pi_{\alpha_{n-2}}U\cdots\Pi_{\alpha_{n-j}}.$$

The symbols are supported in small rectangles, similar with the elements $E_{\boldsymbol\alpha}$ of the refined partition $\mathcal{P}^{\vee j}$. One iteratively shows that

$$\|\partial^\beta\widetilde\pi_{\alpha_{n-j}\cdots\alpha_{n-1}}\|\leq C_\beta e^{\lambda_\epsilon|\beta|j},$$

with constants C_β uniform with respect to j. Therefore, as long as $j\leq T_{\epsilon,\hbar}$, the symbol $\widetilde\pi_{\alpha_{n-j+1}\cdots\alpha_{n-1}}\in S^{-\infty,0}_{\nu_j}$ (with uniform constants), where $\nu_j=\lambda_\epsilon j/\log(1/\hbar)$. At the same time, $\widetilde\pi_{\alpha_{n-j}}\in S^{-\infty,0}_0$. As a result,

$$U^{-1}\operatorname{Op}_\hbar(\widetilde\pi_{\alpha_{n-j+1}\cdots\alpha_{n-1}})U\operatorname{Op}_\hbar(\widetilde\pi_{\alpha_{n-j}})=\operatorname{Op}_\hbar(\widetilde\pi_{\alpha_{n-j}\cdots\alpha_{n-1}})+R_j,$$

and the remainder $R_j\in\Psi^{-\infty,-1+\nu_j}_{\nu_j}$ satisfies

$$\|R_j\|_{L^2\circlearrowleft}\leq C\hbar^{1-\nu_j},\qquad j=2,\ldots,n,$$

with a uniform constant C. The sum of all remainders thus satisfies

$$\sum_{j=2}^n R_j\in\Psi^{-\infty,-1+\nu_n}_{\nu_n},\qquad\Big\|\sum_{j=2}^n R_j\Big\|\leq\sum_{j=2}^n C\hbar^{1-\nu_j}\leq\widetilde C\hbar^{1-\nu_n}.\qquad\square$$

Corollary 3.4. *Take any $1>\epsilon>0$ and $\nu\in[(1-\epsilon)/2,\frac12)$. Let $\boldsymbol\alpha,\boldsymbol\beta$ be two sequences of length $n\leq T_{\epsilon,\hbar}$. Then the symbols $\widetilde\pi_{\boldsymbol\alpha},\widetilde\pi_{\boldsymbol\beta}\circ g^{-n}$ belong to $S^{-\infty,0}_\nu$, and so does their product. The operator*

$$\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}\stackrel{\rm def}{=}\Pi_{\boldsymbol\alpha}U^n\Pi_{\boldsymbol\beta}U^{-n}=U^{-n+1}\Pi_{\alpha_{n-1}}U\Pi_{\alpha_{n-2}}U\cdots\Pi_{\alpha_0}U\Pi_{\beta_{n-1}}U\cdots U\Pi_{\beta_0}U^{-n}$$

belongs to $\Psi^{-\infty,0}_\nu$, and satisfies

$$\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}=\operatorname{Op}_\hbar(\widetilde\pi_{\boldsymbol\alpha}\times\widetilde\pi_{\boldsymbol\beta}\circ g^{-n})+\Psi^{-\infty,-1+2\nu}_\nu.$$

These results show that, for times $n\leq T_{\epsilon,\hbar}$, the operators $\Pi_{\boldsymbol\alpha}$ (resp. $\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}$) are "quasiprojectors" on refined rectangles $E_{\boldsymbol\alpha}\in\mathcal{P}^{\vee n}$ (resp. in the rectangles $E_{\boldsymbol\alpha}\cap g^n(E_{\boldsymbol\beta})$ of the "isotropic" refined partition $\mathcal{P}^{n-1}_{-n}$). Using the fact that $\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}=U^n\Pi_{\boldsymbol\beta\boldsymbol\alpha}U^{-n}$, we also draw the

Corollary 3.5. *Take ϵ,ν as above. Then, for any sequence $\boldsymbol\alpha$ of length $|\boldsymbol\alpha|\leq 2T_{\epsilon,\hbar}$, the operator norm $\|\Pi_{\boldsymbol\alpha}\|=\|\widetilde\pi_{\boldsymbol\alpha}\|_\infty+\mathcal{O}(\hbar^{1-2\nu})$, which can be close to unity.*

3.5. Hyperbolic dispersive estimates. We will now consider operators $\Pi_{\boldsymbol\alpha}$ for sequences $\boldsymbol\alpha$ *longer than* $2T_{\epsilon,\hbar}$. We recall that $J^u_n(\boldsymbol\alpha)$ is the coarse-grained unstable Jacobian along orbits following the path $\boldsymbol\alpha$ (see (3.16)). Given some small $\delta>0$, we have constructed in Section 2.3.3 cutoffs $\chi^{(m)}$ supported on intervals of lengths $2e^{m\delta}\hbar^{1-\delta}$, from which we built up sharp energy cutoffs. Our major dynamical result is the following dispersive estimate [3]. We provide its proof in Section 4.

Proposition 3.6. *Choose $\delta>0$ small, leading to the constant C_δ of Section 2.3.3. Then, there exists $\hbar_\delta>0$ and $C>0$ such that, for any $0<\hbar\leq\hbar_\delta$, any integers $n,m\in[0,C_\delta\log(1/\hbar)]$ and any sequence $\boldsymbol\alpha$ of length n, the following estimate holds*:

$$\big\|\Pi_{\boldsymbol\alpha}\,\chi^{(m)}\big(P(\hbar)\big)\big\|\leq Ce^{m\delta/2}\,\hbar^{-(d-1+\delta)/2}J^u_n(\boldsymbol\alpha)^{-1/2}.\tag{3.20}$$

From the bound (3.17) on the coarse-grained Jacobians, we see that (3.20) becomes sharper than the obvious bound $\|\Pi_{\boldsymbol\alpha}\chi^{(n)}\big(P(\hbar)\big)\| \leq 1 + \mathcal{O}(\hbar^\infty)$ for times

$$n \geq T_1 \stackrel{\text{def}}{=} \frac{(d-1)\log(1/h)}{\Lambda^u_{\min}} > 2T_{\epsilon,\hbar}. \tag{3.21}$$

If we specialize Proposition 3.6 to the case $n \approx 4T_{\epsilon,\hbar}$ and insert $U^{-n/2}$ on the right, we obtain the following

Corollary 3.7. *Take $\delta > 0$ small. For $0 < \hbar < h_\delta$, take $\boldsymbol\alpha, \boldsymbol\beta$ two arbitrary sequences of length $n = \lfloor 2T_{\epsilon,\hbar}\rfloor$. Then,*

$$\big\|\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}\,\chi^{(n)}\big(P(\hbar)\big)\big\| \leq C\,\hbar^{-(d-1+c\delta)/2} J^u_n(\boldsymbol\alpha)^{-1/2} J^u_n(\boldsymbol\beta)^{-1/2}, \tag{3.22}$$

with uniform constants $C, c > 0$.

It is this estimate which we will use in Section 3.8.

3.5.1. *A remark on the sharpness of* (3.22). Let us give a handwaving argument to show that, in the case of a surface ($d = 2$) of constant curvature, the upper bound (3.22) is close to being sharp. This argument was made rigorous in the case of the toy model studied in [2]. Since our argument is sketchy, we set all "small constants" (ϵ, δ) to zero.

The operator $\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}$ is the product of two quasiprojectors, $\Pi_{\boldsymbol\alpha}$ associated with the "thin stable" rectangle $E_{\boldsymbol\alpha}$, which has length $\lesssim \hbar$ along the unstable direction, and $\Pi_{\boldsymbol\beta\cdot}$ associated with the "thin unstable" rectangle $E_{\boldsymbol\beta\cdot}$ which has length $\lesssim \hbar$ along the stable direction. The intersection $E_{\boldsymbol\beta\cdot\boldsymbol\alpha}$ has length $\lesssim \hbar$ along the two directions transverse to the flow, which are symplectically conjugate to each other. As a result, the refined smoothed characteristic function $\pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}$ does not belong to any "nice" symbol class, and the norm of the operator $\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}$ is not connected with the sup-norm of $\pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}$.

Since $E_{\boldsymbol\alpha}$ has symplectic volume $\lesssim \hbar$, the "essential rank" of $\Pi_{\boldsymbol\alpha}$ is of order $\mathcal{O}(1)$: $\Pi_{\boldsymbol\alpha}$ resembles a projector on a subspace spanned by *finitely many* normalized "stable states" $s^i_{\boldsymbol\alpha}$ localized in $E_{\boldsymbol\alpha}$,

$$\Pi_{\boldsymbol\alpha} \approx \sum_i s^i_{\boldsymbol\alpha} \otimes s^{i*}_{\boldsymbol\alpha}.$$

Similarly, $\Pi_{\boldsymbol\beta\cdot}$ effectively projects on $\mathcal{O}(1)$ normalized "unstable states" $u^j_{\boldsymbol\beta}$ localized in $E_{\boldsymbol\beta\cdot}$. The stable and unstable directions are symplectically conjugate to each other, so that stable and unstable states behave like position vs momentum states in the phase space $\mathbb{R}^2$. The product operator

$$\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha} = \Pi_{\boldsymbol\alpha}\Pi_{\boldsymbol\beta\cdot} \approx \sum_{i,j} \langle s^i_{\boldsymbol\alpha}, u^j_{\boldsymbol\beta}\rangle s^i_{\boldsymbol\alpha} \otimes u^j_{\boldsymbol\beta}$$

involves the overlaps between the two families of states, which are all of order $\hbar^{1/2}$. It is thus natural to expect $\|\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}\| \sim \hbar^{1/2}$, which is the order of the estimate (3.22).

3.6. Quantum entropy and pressure.

3.6.1. *Back to the symbolic measure $\mu_\hbar$.* We now turn back to the the symbolic measure $\mu_\hbar$ defined in Section 3.3. We recall that for a fixed sequence $\boldsymbol\alpha$, $\mu_\hbar([\cdot\boldsymbol\alpha])$ approximately measures the weight of the state $\psi_\hbar$ inside the rectangle $E_{\boldsymbol\alpha}$. This interpretation is actually possible as long as $\Pi_{\boldsymbol\alpha}$ can be interpreted as a quasiprojector on this rectangle, that is for $n \le 2T_{\epsilon,\hbar}$. Under this condition, we have seen that the only upper bounds at our disposal are trivial:

$$\mu_\hbar([\cdot\boldsymbol{\alpha}]) \le 1, \qquad n = |\boldsymbol{\alpha}| \le 2T_{\epsilon,\hbar}.$$

On the other hand, Proposition 3.6 implies that the weights of longer cylinders satisfy nontrivial bounds:

$$\mu_\hbar([\cdot\boldsymbol{\alpha}]) \le C\hbar^{-(d-1+c\delta)} J_n^u(\boldsymbol{\alpha})^{-1}, \qquad |\boldsymbol{\alpha}| = n \le C_\delta \log \hbar^{-1}. \tag{3.23}$$

Corollary 3.8. *For times $n > T_1$ (see (3.21)), the measure $\mu_\hbar$ is necessarily distributed over many cylinders of length n. This corresponds to a dispersion phenomenon: the state $\psi_\hbar$ cannot be concentrated in $\mathcal{O}(1)$ boxes $E_{\boldsymbol{\alpha}}$, since each such box has a volume $\ll \hbar^{d-1}$.*

Following Section 3.1, the distribution of the weights $\{\mu_\hbar([\cdot\boldsymbol{\alpha}]), |\boldsymbol{\alpha}| = n\}$ can be characterized by an entropy. Since $\mu_\hbar$ was built from the quantum state $\psi_\hbar$, it is natural to call this entropy a **quantum entropy**:

$$H_0^{n-1}(\psi_\hbar, \mathcal{P}_{\mathrm{sm,q}}) \stackrel{\mathrm{def}}{=} H_0^{n-1}(\mu_\hbar) = \sum_{|\boldsymbol{\alpha}|=n} \eta\big(\mu_\hbar([\cdot\boldsymbol{\alpha}])\big). \tag{3.24}$$

One can associate a **quantum pressure** to the state $\psi_\hbar$, the quantum partition $\mathcal{P}_{\mathrm{sm,q}}$ and a set of weights $\{w_k, k = 1, \ldots, K\}$. Below we will be dealing with weights of the form $w_{\boldsymbol{\alpha}} = J_n^u(\boldsymbol{\alpha})^{1/2}$. The quantum pressures will also be denoted by $p_0^{n-1}(\psi_\hbar, \mathcal{P}_{\mathrm{sm,q}}, w) = p_0^{n-1}(\mu_\hbar, w)$.

Upper bounds (3.23) on the weights of "long" cylinders have direct consequences on the values of the quantum entropies:

$$H_0^{n-1}(\psi_\hbar, \mathcal{P}_{\mathrm{sm,q}}) \ge n\Lambda^u_{\min} - (d-1+c\delta)\log\hbar^{-1} - \log C, \qquad n \le C_\delta \log \hbar^{-1}.$$

The right-hand side is positive (and thus makes up a nontrivial lower bound) only for $n > T_1$, that is for "long" times. A similar lower bound on the entropy of "long times" was used in [1] to deduce nontrivial information on the values of the entropies at "short" times, and finally a lower bound on the KS entropy.

3.7. Entropic uncertainty principles. In [3, 4] we used a different strategy, which we describe below. Instead of using the upper bounds (3.23) at "long" times $C_\delta \log \hbar^{-1}$, we rather use the bound (3.22) corresponding to "moderately long" times $4T_{\epsilon,\hbar}$. The strategy consists in interpreting the operator on the left-hand side as a "block matrix element" associated with two different quantum partitions. Through a certain *Entropic Uncertainty Principle* (EUP), the bound (3.22) then leads to a lower bound on the pressures $p_0^{n-1}(\mu_\hbar, w)$ at "moderately long times" $n \approx 2T_{\epsilon,\hbar}$ (that is, right below the Ehrenfest time). Using an approximate subadditivity of those pressures, we then get a lower bound for finite time pressures.

The central piece of this method resides in a certain entropic uncertainty principle. Before giving the precise version used for our aims, we first give the simplest example of such a "principle," first proven by Maassen and Uffink [22].

Proposition 3.9 (EUP, level 1 (finite-dimensional projectors)). *Consider two orthonormal bases in the Hilbert space $\mathbb{C}^N$, $e = \{e_i\}_{i=1,\ldots,N}$ and $f = \{f_j\}_{j=1,\ldots,N}$. For any $\psi \in \mathbb{C}^N$ of unit norm, consider the two probability distributions $\{|\langle e_i \mid \psi\rangle|^2, i = 1, \ldots, N\}, \{|\langle f_j \mid \psi\rangle|^2, j = 1, \ldots, N\}$. Then the entropies* *associated with these two distributions satisfy the inequality*

$$H(\psi, e) + H(\psi, f) \geq -2 \log\Big(\max_{i,j} |\langle e_i, f_j \rangle|\Big).$$

For instance, take $e = \{e_i\}$ the standard basis on $\mathbb{C}^N$, and for $f = \{f_j\}$ the "discrete momentum states," related with $\{e_i\}$ through the discrete Fourier transform. All matrix elements satisfy $|\langle e_i, f_j \rangle| = N^{-1/2}$, so the inequality reads

$$H(\psi, e) + H(\psi, f) \geq \log N.$$

The inequality shows that the distributions of "position amplitudes"$\langle e_i, \psi \rangle$ on the one hand, of "momentum amplitudes"$\langle f_j, \psi \rangle$, cannot be both arbitrarily localized. It is hence a form of "uncertainty principle."

If we call ρ_i (resp. τ_j) the orthogonal projector on the state e_i (resp. f_j), then each overlap $|\langle e_i, f_j \rangle|$ can be interpreted as the operator norm $|\langle e_i, f_j \rangle| = \|\tau_j \rho_i^*\|$. This interpretation allows to generalize this "uncertainty principle" as follows:

Proposition 3.10 (EUP, level 2 (quantum partition)). *On a Hilbert space $\mathcal{H}$, consider two quantum partitions of unity, that is two finite sets of bounded operators $\rho = \{\rho_i, i \in I\}, \pi = \{\pi_j, j \in J\}$ satisfying the identities*

$$\sum_{i \in I} \rho_i^* \rho_i = \mathrm{Id}, \qquad \sum_{j \in J} \tau_j^* \tau_j = \mathrm{Id}.$$

To any normalized state $\psi \in \mathcal{H}$ we associate the probability distributions $\{\|\rho_i \psi\|^2, i \in I\}, \{\|\tau_j \psi\|^2, j \in J\}$. Then, the entropies associated with these two distributions satisfy

$$H(\psi, \rho) + H(\psi, \tau) \geq -2 \log\Big(\max_{i \in I, j \in J} \|\tau_j \rho_i^*\|\Big).$$

Because the instability of the flow may not be uniform, the coarse-grained Jacobians $J_n^u(\boldsymbol{\alpha})$ may vary a lot among all n-sequences $\boldsymbol{\alpha}$. For this reason, the estimates (3.22) may also strongly depend on the sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}$. To counterbalance these variations, it is convenient to use *pressures* instead of entropies (see Section 3.1.2).

Proposition 3.11 (EUP, level 2 (quantum weighted partition)). *On a Hilbert space $\mathcal{H}$, consider two quantum partitions of unity $\rho = \{\rho_i, i \in I\}, \tau = \{\tau_j, j \in J\}$ as in Proposition 3.10 and two families of weights $v = \{v_i > 0, i \in I\}$, $w = \{w_j > 0, j \in J\}$. To any normalized state $\psi \in \mathcal{H}$ correspond the probability distributions $\{\|\rho_i \psi\|^2, i \in I\}, \{\|\tau_j \psi\|^2, j \in J\}$.*

Then, the pressures associated with these distributions and weights satisfy

$$p(\psi, \rho, v) + p(\psi, \tau, w) \geq -2 \log\Big(\max_{i \in I, j \in J} v_i w_j \|\tau_j \rho_i^*\|\Big).$$

This version is almost sufficient for our aims. Yet, the quantum partitions of unity we are using are localized near the energy shell $\mathcal{E}$ (see (3.8), (3.10)), and the estimate (3.20) starts with a sharp energy cutoff. For these reasons, the version we will need is of the following form.

Proposition 3.12 (EUP, level 3 (microlocal weighted partition)). *On a Hilbert space $\mathcal{H}$, consider two approximate quantum partitions of unity, that is two finite sets of bounded operators $\rho = \{\rho_i, i \in I\}, \tau = \{\tau_j, j \in J\}$ satisfying the identities*

$$\sum_{i \in I} \rho_i^* \rho_i = S_\rho, \qquad \sum_{j \in J} \tau_j^* \tau_j = S_\tau,$$

and two families of weights $v = \{v_i, i \in I\}, w = \{w_j, j \in J\}$ *satisfying* $V^{-1} \leq v_i, w_j \leq V$ *for some* $V \geq 1$.

We assume that for some $0 < \varepsilon \leq \min(|I|^{-2}V^{-2}, |J|^{-2}V^{-2})$ *the above sum operators satisfy* $0 \leq S_{\rho/\tau} \leq 1+\varepsilon$. *Besides, let* S_{c_1}, S_{c_2} *be two Hermitian operators on* $\mathcal{H}$ *satisfying* $0 \leq S_{c_*} \leq 1+\varepsilon$, *and related with the above partitions as follows*:

$$\|(S_{c_2} - 1)\rho_i S_{c_1}\| \leq \varepsilon, \qquad \forall i \in I, \tag{3.25}$$

$$\|(S_{\rho/\tau} - 1)S_{c_1}\| \leq \varepsilon. \tag{3.26}$$

Let us define the "cone norm"

$$c_{\text{cone}} \stackrel{\text{def}}{=} \max_{i\in I, j\in J} v_i w_j \|\tau_j \rho_i^* S_{c_2}\|. \tag{3.27}$$

Then, for any $\psi \in \mathcal{H}$ *satisfying*

$$\|\psi\| = 1, \qquad \|(\mathrm{Id} - S_{c_1})\psi\| \leq \varepsilon, \tag{3.28}$$

the pressures of ψ *with respect to the weighted partitions* (ρ, v) *and* (τ, w) *satisfy the bound*

$$p(\psi, \rho, v) + p(\psi, \tau, w) \geq -2\log(c_{\text{cone}} + 3|I|V^2\varepsilon) + \mathcal{O}(\epsilon^{1/5}).$$

The implied constant is independent of the weighted partitions or the cutoff operators S_{c_*}.

Proof. The proof is a slight adaptation of the one given in [3, Section 6] (in the case $\mathcal{U} = \mathrm{Id}$). One considers a bounded operator $T\colon \mathcal{H}^{|I|} \to \mathcal{H}^{|J|}$, and studies the norm of T as an operator $l_p^{(v)}(\mathcal{H}^{|I|}) \mapsto l_q^{(w)}(\mathcal{H}^{|J|})$, with the weighted norms $\|\Psi\|_p^{(v)} = (\sum_i v_i^{p-2}\|\Psi_i\|^p)^{1/p}$ and similarly for $\|T\Psi\|_q^{(w)}$. Notice that the weights are irrelevant when $p = q = 2$. An auxiliary bounded operator $O\colon \mathcal{H} \to \mathcal{H}$ is used to define a cone of states:

$$\mathcal{C}(O, \vartheta) = \{\Psi \in \mathcal{H}^{|I|}, \|O\Psi_i - \Psi_i\| \leq \vartheta\|\Psi\|_2, i \in I\}.$$

The proof of [3, Theorem 6.3], which uses a Riesz−Thorin interpolation argument, shows that for any Ψ in the cone $\mathcal{C}(O, \vartheta)$, one has:

$$\forall t \in [0,1], \qquad \|T\Psi\|_{2/(1-t)}^{(w)} \leq (c_O(T) + |I|V^2\vartheta\|T\|_{2,2})^t \|T\|_{2,2}^{1-t}\|\Psi\|_{2/(1+t)}^{(v)}, \quad \text{where } c_O(T) = \max_{i,j} v_i w_j \|T_{ji}O\|. \tag{3.29}$$

We now apply this result to the specific choice

$$\Psi_i \stackrel{\text{def}}{=} \rho_i\psi, \qquad T_{ji} \stackrel{\text{def}}{=} \tau_j\rho_i^*, \qquad O \stackrel{\text{def}}{=} S_{c_2},$$

where the state ψ satisfies (3.28), that is, the cone $\mathcal{C}(S_{c_1}, \varepsilon)$ is not empty (in the opposite case, the statement of the theorem is empty). The relations (3.26) then imply that

$$\|(S_{\rho/\tau} - 1)\psi\| \leq 3\varepsilon. \tag{3.30}$$

As a consequence, the state components $(T\Psi)_i = \tau_i S_\rho \psi = \tau_i\psi + \mathcal{O}(\varepsilon)$.

The same duality argument as in [3, Lemma 6.5] shows that the $l^2 \to l^2$ norm of the operator T takes the value $\|T\|_{2,2} = \big\|\sqrt{S_\rho\sqrt{S_\tau}}\big\|$. Using the spectral theorem, one easily deduces that $\|\sqrt{S_{\rho/\tau}}\psi - \psi\| \leq \sqrt{3\varepsilon}$, so that $\|\sqrt{S_\rho}\sqrt{S_\tau}\psi - \psi\| \leq 4\sqrt{\varepsilon}$, and hence $\|T\|_{2,2} \in [1 - 4\sqrt{\varepsilon}, 1 + 2\varepsilon]$.

The l^2 norm of Ψ is $\|\Psi\|_2^2 = \langle\psi, S_\rho\psi\rangle \in [1-3\varepsilon, 1+\varepsilon]$. From (3.25) and the fact that ψ is in the cone $\mathcal{C}(S_{c_1}, \epsilon)$, we easily get $\|(O-1)\rho_i\psi\| \leq 2\varepsilon$, so that $\Psi \in \mathcal{C}(O, \vartheta)$ for $O = S_{c_2}$, $\vartheta = 2\varepsilon/(1-3\varepsilon)$. We are now in a position to apply (3.29) to the above data. The constant $c_O(T)$ is equal to the c_{cone} in the statement of the proposition, so we get

$$\forall t \in [0,1], \qquad \|T\Psi\|^{(w)}_{2/(1-t)} \leq (c_{\text{cone}} + 3|I|V^2\varepsilon)^t \|T\|_{2,2}^{1-t}\, \|\Psi\|^{(v)}_{2/(1+t)}.$$

Let us now expand this expression when $t \to 0$. We first split the sum $\sum_i \|\Psi_i\|^{2/(1+t)}$ between the terms $\|\Psi_i\| \geq \epsilon$ and the remaining ones:

$$\begin{aligned}\sum_i v_i^{2t/(1+t)} \|\Psi_i\|^{2/(1+t)} &= \sum_{i,>} \|\Psi_i\|^2 - t\sum_{i,>} \|\Psi_i\|^2 (\log v_i^2 \|\Psi_i\|^2) + \mathcal{O}\big((t\log\varepsilon)^2\big) \\ &\qquad\qquad + \sum_{i,<} v_i^{-2t/(1+t)} \|\Psi_i\|^{2/(1+t)} \\ &= \|\Psi\|_2^2 + tp(\Psi, v) + \mathcal{O}\big((t\log\varepsilon)^2\big) + \mathcal{O}(|I|\varepsilon).\end{aligned}$$

We now take the logarithm of this expression, and use $\|\Psi\|_2^2 = 1 + \mathcal{O}(\varepsilon)$ to get

$$\log\|\Psi\|^{(v)}_{2/(1+t)} = \frac{t}{2} p(\Psi, v) + \mathcal{O}\big((t\log\varepsilon)^2 + |I|\varepsilon + t^2 p(\Psi, v)^2\big).$$

We can perform the same manipulations on the left-hand side of (3.29), noticing that $\|T\Psi\|_2^2 = \langle S_\tau S_\pi \psi, S_\pi\psi\rangle \in [1-10\varepsilon, 1+10\varepsilon]$:

$$\log\|T\Psi\|^{(w)}_{2/(1-t)} = -\frac{t}{2} p(T\Psi, w) + \mathcal{O}\big((t\log\varepsilon)^2 + |J|\varepsilon + t^2 p(T\Psi, w)^2\big).$$

We notice that both pressures satisfy simple bound $|p(\bullet,\bullet)| \leq \log(V^2|I|) \leq 3|\log\varepsilon|$, and (from (3.30)) the estimate $(T\Psi)_i = \tau_i\psi_\hbar + \mathcal{O}(\varepsilon)$. Inserting these estimates in the logarithm of (3.29) and using $\|T\|_{2,2} \in [1-4\sqrt{\varepsilon}, 1]$, we get

$$tp(\psi_\hbar, \tau, w) + tp(\psi_\hbar, \rho, v) \geq -2t\log(c_{\text{cone}} + 3|I|V^2\varepsilon) + \mathcal{O}(\sqrt{\varepsilon} + (t\log\varepsilon)^2 + (|J|+|I|)\varepsilon).$$

We now need to make some assumptions on the values of t to make the remainder small. If we take $t = \varepsilon^{1/4}$, a simple power counting shows that

$$p(\psi_\hbar, \tau, w) + p(\psi_\hbar, \rho, v) \geq -2\log(c_{\text{cone}} + 3|I|V^2\varepsilon) + \mathcal{O}\big(\varepsilon^{1/4}(\log\varepsilon)^2\big). \qquad \square$$

3.8. Our application of the entropy uncertainty principle. We now apply the EUP to our semiclassical framework. Our choice of quantum partitions is determined by the requirement that the operators $\tau_j\rho_i^* S_{c_2}$ are of the form $\Pi_{\boldsymbol\beta\cdot\boldsymbol\alpha}\chi^{(n)}\big(P(\hbar)\big)$, with $\boldsymbol\alpha, \boldsymbol\beta$ two sequences of length n close to the Ehrenfest time. We will thus take

$$\begin{aligned} &\text{the time} \qquad n = \lfloor 2T_{\epsilon,\hbar} \rfloor, \\ &\tau = \{\Pi_{\boldsymbol\alpha} = \Pi_{\alpha_{n-1}}(n-1)\cdots\Pi_{\alpha_0}, |\boldsymbol\alpha| = n\}, \\ &\rho = \{\Pi^*_{\boldsymbol\beta\cdot} = \Pi_{\beta_{-n}}(-n)\cdots\Pi_{\beta_{-1}}(-1), |\boldsymbol\beta| = n\}, \\ &v = w = \{J_n^u(\boldsymbol\beta)^{1/2}, |\boldsymbol\beta| = n\}, \\ &S_{c_1} = \chi^{(0)}\big(P(\hbar)\big), \qquad S_{c_2} = \chi^{(n)}\big(P(\hbar)\big). \end{aligned} \tag{3.31}$$

The weights v, w have been selected in order to balance the variations of the upper bounds in (3.22). Both the cardinals $|I| = |J| = K^n$ and the upper bound $v_i, w_j \leq e^{n\lambda_{\max}(d-1)/2}$ are $\mathcal{O}(\hbar^{-M})$ for some $M > 0$. Hence, we may take the small paremeter $\varepsilon = \hbar^L$ for some fixed exponent $L \gg M$.

With this choice, the assumption (3.26) is satisfied for $\hbar$ small enough according to Proposition 3.1. The assumption (3.25) can be checked by inserting the increasing sequence of cutoffs $\{\chi^{(j)}(P(\hbar)), 1 \leq j \leq n-1\}$ along the sequence $\Pi^*_{\boldsymbol{\beta}}$, similarly as in the proof of Proposition 3.1. The assumption (3.28) holds if one takes $\psi = \psi_\hbar$ a null eigenstate of $P(\hbar)$, see (2.12).

The coefficient c_{cone} is then estimated by the hyperbolic dispersive estimate of Corollary 3.7:

$$c_{\text{cone}} \leq C_{\text{cone}}(\hbar) \stackrel{\text{def}}{=} C\,\hbar^{-(d-1+c\delta)/2}, \qquad \hbar \leq \hbar_\delta. \tag{3.32}$$

The application of Proposition 3.12 to these data gives the following result.

Proposition 3.13 (Applied entropic uncertainty principle). *Take the weighted quantum partitions* $(\rho, v), (\tau, w)$ *defined in* (3.31), *and* L *a large positive number. Then, there exists* $\hbar_L > 0$, $C > 0$ *such that the pressures of the eigenstates* $(\psi_\hbar)$ *associated with these weighted partitions satisfy the inequality*:

$$p(\psi_\hbar, \rho, v) + p(\psi_\hbar, \tau, w) \geq -2\log C_{\text{cone}}(\hbar) + C\hbar^{L/3}, \qquad \hbar < \hbar_L. \tag{3.33}$$

Our next task will be to relate the pressures associated to the "long time" partitions ($n = \lfloor 2T_{\epsilon,\hbar}\rfloor$), to pressures associated with "finite time" partitions (n_0 independent of $\hbar$).

3.9. Approximate subadditivity of the quantum pressure. In (3.33) appear two pressures, associated with two types of refined quantum partitions. The partition $\tau = \{\Pi_{\alpha_{n-1}}(n-1)\cdots\Pi_{\alpha_0}, |\boldsymbol{\alpha}| = n\}$ corresponds to the definition of the symbolic measure $\mu_\hbar$ in (3.14), so that the pressure $p(\psi_\hbar, \tau, w)$ can be expressed as the refined pressure $p_0^{n-1}(\mu_\hbar, w)$. On the other hand, the probability weights involved in $p(\psi_\hbar, \rho, v)$ can be rewritten (after relabelling the sequences and using the fact that $\psi_\hbar$ is an eigenmode) $\rho_{\boldsymbol{\beta}} = \|\Pi_{\beta_{n-1}}(-n)\cdots\Pi_{\beta_0}(-1)\psi_\hbar\|^2$, so they correspond to a backwards evolution.[6] We express these weights in terms of a "backwards symbolic measure" $\tilde{\mu}_\hbar$ similar with $\mu_\hbar$:

$$\tilde{\mu}_\hbar([\cdot\beta_0\cdots\beta_{n-1}]) \stackrel{\text{def}}{=} \|\Pi_{\beta_{n-1}}(-n)\cdots\Pi_{\beta_0}(-1)\psi_\hbar\|^2, \tag{3.34}$$

so that $p(\psi_\hbar, \rho, v) = p_0^{n-1}(\tilde{\mu}_\hbar, v)$. For fixed $n > 0$, and any $|\boldsymbol{\beta}| = n$, we have $\tilde{\mu}_\hbar([\cdot\boldsymbol{\beta}]) \xrightarrow{\hbar\to 0} \mu_{\text{sc}}(\pi_{\beta_{n-1}\cdots\beta_0})$.

The bound (3.33) concerns the quantum pressures of $\mu_\hbar$ and $\tilde{\mu}_\hbar$ at the time $n = \lfloor 2T_{\epsilon,\hbar}\rfloor$ close to the Ehrenfest time. These pressures cannot be directly connected with the pressures of the semiclassical measure μ_{sc}. For this aim, we need to deduce from (3.33) a lower bound for the pressures of $\mu_\hbar$ and $\tilde{\mu}_\hbar$ at *finite* times n_0. This connection will be done by using an approximate version of the subadditivity property (3.5). We present the computations in the case of $\mu_\hbar$, the case of $\tilde{\mu}_\hbar$ being identical up to exchanging forward and backward evolution.

The symbolic measure $\mu_\hbar$ was defined on cylinders $[\cdot\boldsymbol{\alpha}]$, up to $\mathcal{O}(\hbar^\infty)$ errors which are uniformly controlled as long as $|\boldsymbol{\alpha}| \leq C|\log\hbar|$. This definition was possible thanks to the (approximate) compatibility condition (3.13) . The property (3.3) can be extended to the measure $\mu_\hbar$:

$$\forall n, n_0 \leq C|\log\hbar|, \qquad H_0^{n+n_0-1}(\mu_\hbar) \leq H_0^{n-1}(\mu_\hbar) + H_n^{n+n_0-1}(\mu_\hbar) + \mathcal{O}(\hbar^\infty). \tag{3.35}$$

[6] Notice that, using the fact that $\psi_\hbar$ is an eigenmode of U, these weights can also be written as $\|\Pi_{\beta_{n-1}}(0)\cdots\Pi_{\beta_0}(n-1)\psi_\hbar\|^2$, which is of the same form as $\|\tau_{\beta_{n-1}\cdots\beta_0}\psi_\hbar\|^2$, except for the ordering of the operators.

The second term on the right-hand side can be written as $H_n^{n+n_0-1}(\mu_\hbar) = H_0^{n_0-1}(\sigma_*^{-n}\mu_\hbar)$, where σ refers to a shift of indices. Shift-invariance would mean that $\sigma^{-n}\mu_\hbar = \mu_\hbar$, which would allow us to replace this entropy with $H_0^{n_0-1}(\mu_\hbar)$. How far are we from this invariance? To answer this question we need to compare the weights

$$\mu_\hbar(\sigma^{-n}[\cdot\beta_0\cdots\beta_{n_0-1}]) \stackrel{\text{def}}{=} \sum_{\alpha_0,\dots,\alpha_{n-1}} \mu_\hbar([\cdot\alpha_0\cdots\alpha_{n-1}\beta_0\cdots\beta_{n_0-1}]), \tag{3.36}$$

with the weights $\mu_\hbar([\cdot\beta_0\cdots\beta_{n_0-1}])$. This is achieved in the following

Lemma 3.14. *Let $(\mu_\hbar)_{\hbar\to 0}$, $(\tilde\mu_\hbar)_{\hbar\to 0}$ be the symbolic measures associated with the eigenstates $(\psi_\hbar)_{\hbar\to 0}$. Fix some $n_0 \geq 0$, and take n in the range $[0, 2T_{\epsilon,\hbar} - n_0]$. Then, for any $\boldsymbol\beta$ of length n_0, we have*

$$\mu_\hbar(\sigma^{-n}[\cdot\boldsymbol\beta]) = \mu_\hbar([\cdot\boldsymbol\beta]) + \mathcal{O}(\hbar^{\epsilon/2}), \qquad \tilde\mu_\hbar(\sigma^{-n}[\cdot\boldsymbol\beta]) = \tilde\mu_\hbar([\cdot\boldsymbol\beta]) + \mathcal{O}(\hbar^{\epsilon/2}).$$

This lemma means that the measures $\mu_\hbar$, $\tilde\mu_\hbar$ are approximately shift-invariant in the semiclassical limit.

Proof. We only give the proof for the measure $\mu_\hbar$. Each term on the right-hand side of (3.36) reads

$$\langle \Pi_{\boldsymbol\beta}(n)\Pi_{\boldsymbol\alpha}\psi_\hbar, \Pi_{\boldsymbol\beta}(n)\Pi_{\boldsymbol\alpha}\psi_\hbar\rangle = \langle \Pi_{\boldsymbol\alpha}^*|\Pi_{\boldsymbol\beta}(n)|^2\Pi_{\boldsymbol\alpha}\psi_\hbar, \psi_\hbar\rangle.$$

We first present a short (but false) proof. In order to use (3.10), we try to bring together the operator $\Pi_{\boldsymbol\alpha}$ and its Hermitian conjugate, such as to let appear the sum $\sum_{|\boldsymbol\alpha|=n}\Pi_{\boldsymbol\alpha}^*\Pi_{\boldsymbol\alpha}$. An error appears while commuting $\Pi_{\boldsymbol\alpha}$ with $|\Pi_{\boldsymbol\beta}(n)|^2$. Still, from Proposition 3.3 we know that for any $|\boldsymbol\alpha| = n$, $|\boldsymbol\beta| = n_0$ with $n+n_0 \leq 2T_{\epsilon,\hbar}$, the operators $\Pi_{\boldsymbol\alpha}(-n/2)$ and $|\Pi_{\boldsymbol\beta}(n/2)|^2$ belong to the class $\Psi_\nu^{-\infty,0}$, with $\nu \in [(1-\epsilon)/2, \frac{1}{2})$. As a consequence, their commutator satisfies $[|\Pi_{\boldsymbol\beta}(n/2)|^2, \Pi_{\boldsymbol\alpha}(-n/2)] \in \Psi_\nu^{-\infty,-1+2\nu}$, an operator of norm $\mathcal{O}(\hbar^{1-2\nu})$. By unitarity of the evolution, the commutator $[|\Pi_{\boldsymbol\beta}(n)|^2, \Pi_{\boldsymbol\alpha}]$ has the same norm. We thus get

$$\sum_{\boldsymbol\alpha} \mu_\hbar([\cdot\boldsymbol\alpha\boldsymbol\beta]) = \langle |\Pi_{\boldsymbol\beta}(n)|^2 S_n\psi_\hbar, \psi_\hbar\rangle + \sum_{\boldsymbol\alpha}\mathcal{O}(\hbar^{1-2\nu}) = \mu_\hbar([\cdot\boldsymbol\beta]) + \mathcal{O}(K^n\hbar^{1-2\nu}).$$

The remainder in the right-hand side is small if n is uniformly bounded, but it becomes very large if $n \approx 2T_{\epsilon,\hbar}$!

To remedy this problem one actually needs to *successively* group together the pairs of operators $\Pi_{\alpha_j}(j)$ and perform the sum over α_j. One starts by grouping together the $\Pi_{\alpha_{n-1}}$:

$$\begin{aligned}
&\langle \Pi_{\boldsymbol\beta}(n)\Pi_{\boldsymbol\alpha}\psi_\hbar, \Pi_{\boldsymbol\beta}(n)\Pi_{\boldsymbol\alpha}\psi_\hbar\rangle \\
&\qquad = \langle \Pi_{\alpha_{n-1}}^*(n-1)|\Pi_{\boldsymbol\beta}(n)|^2\Pi_{\alpha_{n-1}}(n-1)\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar, \Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar\rangle \\
&\qquad = \langle |\Pi_{\alpha_{n-1}}(n-1)|^2|\Pi_{\boldsymbol\beta}(n)|^2\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar, \Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar\rangle \\
&\qquad\qquad + \langle \Pi_{\alpha_{n-1}}^*(n-1)[|\Pi_{\boldsymbol\beta}(n)|^2, \Pi_{\alpha_{n-1}}(n-1)]\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar, \Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar\rangle.
\end{aligned}$$

The commutator in the right-hand side is an operator of norm $\mathcal{O}(\hbar^{1-2\nu})$ for the reasons indicated above. The second overlap is then bounded from above by $\|\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar\|^2\mathcal{O}(\hbar^{1-2\nu})$, where the implied constant does not depend on $\boldsymbol\alpha$. Using the quantum partition of order $n-1$, we can sum this second term over $\boldsymbol\alpha$, to obtain an error $\mathcal{O}(\hbar^{1-2\nu})$. The first term on the right-hand side can be summed over α_{n-1}, to produce

$$\begin{aligned}\langle \mathrm{Op}_\hbar(\widetilde{\chi}_{\epsilon/2})(n-1)|\Pi_{\boldsymbol{\beta}}(n)|^2\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar, \Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar\rangle \\ = \langle |\Pi_{\boldsymbol{\beta}}(n)|^2\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar, \Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar\rangle + \mathcal{O}(\hbar^\infty),\end{aligned}$$

where we used the fact that $\widetilde{\chi}_{\epsilon/2} \equiv 1$ on the microsupport of $\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar$ (see the proof of Lemma 3.1). Apart from the errors, we are now left with the sum

$$\sum_{\alpha_0,\ldots,\alpha_{n-2}} \langle |\Pi_{\boldsymbol{\beta}}(n)|^2\Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar, \Pi_{\alpha_0\cdots\alpha_{n-2}}\psi_\hbar\rangle.$$

This sum can be treated as above, by bringing $\Pi_{\alpha_{n-2}}(n-2)$ to the left, commuting it with $|\Pi_{\boldsymbol{\beta}}(n)|^2$, and summing over α_{n-2}. It produces another error $\mathcal{O}(\hbar^{1-2\nu})$. Iterating this procedure down to α_0, we get

$$\sum_{\boldsymbol{\alpha}} \mu_\hbar([\cdot\boldsymbol{\alpha\beta}]) = \mu_\hbar([\cdot\boldsymbol{\beta}]) + \mathcal{O}(n\hbar^{1-2\nu}).$$

By selecting ν appropriately, we get the statement of the lemma. □

Coming back to the subadditivity equation (3.35) in the case n_0 fixed, $n = \lfloor 2T_{\epsilon,\hbar}\rfloor - n_0$, we get

$$\begin{aligned}H_0^{n-1}(\mu_\hbar) &\leq H_0^{n-n_0-1}(\mu_\hbar) + H_0^{n_0-1}(\sigma_*^{-n}\mu_\hbar) + \mathcal{O}(\hbar^\infty) \\ &\leq H_0^{n-n_0-1}(\mu_\hbar) + H_0^{n_0-1}(\mu_\hbar) + \mathcal{O}_{n_0}(\hbar^{\epsilon/3}).\end{aligned}$$

Here we used the fact that the function $\eta(s)$ satisfies $|\eta(s+s') - \eta(s)| \leq \eta(|s'|)$. It is also easy to check that the "potential part" of the pressure $p(\mu_\hbar, w)$ satisfies this inequality:

$$\begin{aligned}&\sum_{|\boldsymbol{\alpha}|=n}\sum_{|\boldsymbol{\beta}|=n_0} \mu_\hbar([\cdot\boldsymbol{\alpha\beta}])\log w_{\boldsymbol{\alpha\beta}} \\ &\qquad = \sum_{|\boldsymbol{\alpha}|=n} \mu_\hbar([\cdot\boldsymbol{\alpha}])\log w_{\boldsymbol{\alpha}} + \sum_{|\boldsymbol{\beta}|=n_0} \sigma^{-n}\mu_\hbar([\cdot\boldsymbol{\beta}])\log w_{\boldsymbol{\beta}} + \mathcal{O}(\hbar^\infty) \\ &\qquad = \sum_{|\boldsymbol{\alpha}|=n} \mu_\hbar([\cdot\boldsymbol{\alpha}])\log w_{\boldsymbol{\alpha}} + \sum_{|\boldsymbol{\beta}|=n_0} \mu_\hbar([\cdot\boldsymbol{\beta}])\log w_{\boldsymbol{\beta}} + \mathcal{O}(\hbar^{\epsilon/2}),\end{aligned}$$

so we finally get the approximate pressure subadditivity

$$p_0^{n-1}(\mu_\hbar, w) \leq p_0^{n-n_0-1}(\mu_\hbar, w) + p_0^{n_0-1}(\mu_\hbar, w) + \mathcal{O}_{n_0}(\hbar^{\epsilon/3}). \tag{3.37}$$

Taking the Euclidian quotient $n = qn_0 + r$, $r < n_0$, we can iterate this process q times and obtain:

$$p_0^{n+n_0-1}(\mu_\hbar, w) \leq p_0^{r-1}(\mu_\hbar, w) + q\, p_0^{n_0-1}(\mu_\hbar, w) + \mathcal{O}_{n_0}(q\hbar^{\epsilon/3}).$$

A similar approximate subadditivity holds for the pressures $p(\tilde{\mu}_\hbar, v)$ associated with the "backwards" symbolic measure $\tilde{\mu}_\hbar$. We have thus obtained the following

Proposition 3.15. *Let $\mu_\hbar^u, \tilde{\mu}_\hbar^u$ be the associated forward and backwards symbolic measures associated with $(\psi_\hbar)$. Fix some $n_0 > 0$, and for $\hbar < \hbar_0$ split the*

Ehrenfest time $n = \lfloor 2T_{\epsilon,\hbar} \rfloor$ into $n = qn_0 + r$, $r \in [0, n_0)$. Then, the pressures associated with these measures and the weights $w_k = v_k = J^u(k)^{1/2}$ satisfy the following lower bound:

$$
\begin{aligned}
(3.38) \quad q\big(p_0^{n_0-1}(\mu_\hbar, w) + p_0^{n_0-1}(\tilde\mu_\hbar, v)\big) + p_0^{r-1}(\mu_\hbar, w) + p_0^{r-1}(\tilde\mu_\hbar, v)& \\
\geq -(d-1+c\delta)|\log\hbar| + \mathcal{O}_{n_0}(\hbar^{\epsilon/4}).&
\end{aligned}
$$

Hence, approximate subadditivity has enabled us to transfer the bound on pressures for $n = \lfloor 2T_{\epsilon,\hbar} \rfloor$ into a bound on pressures at finite times n_0, r.

3.10. Back to the classical pressure. Using the fact that the pressures $p_0^{r-1}(\bullet)$ are uniformly bounded, we divide (3.38) by $qn_0 = 2T_{\epsilon,\hbar} - \mathcal{O}(1) = |\log\hbar| \times \big(1 + \mathcal{O}(\epsilon)\big)/\lambda_{\max}$. For $\hbar < \hbar_\epsilon$, we get

$$
\begin{aligned}
\frac{p_0^{n_0-1}(\mu_\hbar, w)}{n_0} + \frac{p_0^{n_0-1}(\tilde\mu_\hbar, v)}{n_0} &\geq \frac{-(d-1+c\delta)|\log\hbar|}{qn_0} + \mathcal{O}_{n_0}(|\log\hbar|^{-1}) \\
&\geq -(d-1+c\delta)\lambda_{\max}\big(1+\mathcal{O}(\epsilon)\big).
\end{aligned}
$$

The right-hand side does not depend on $\hbar$, so it is still valid once we take the semiclassical limit of the left-hand side. From the properties of $\mu_\hbar$ and $\tilde\mu_\hbar$, the latter is equal to the ratio $2p_0^{n_0-1}(\mu_{\rm sc}, v, \mathcal{P}_{\rm sm})/n_0$, where we recall that $\mu_{\rm sc}$ is the semiclassical measure associated with $(\psi_\hbar)$, while $\mathcal{P}_{\rm sm}$ is the smoothed partition (3.7). We thus obtained the following lower bound on the classical pressure:

$$
\frac{p_0^{n_0-1}(\mu_{\rm sc}, v, \mathcal{P}_{\rm sm})}{n_0} \geq -\frac{(d-1)\lambda_{\max}}{2} + \mathcal{O}(\delta, \epsilon).
$$

Importantly, the implied constants do not depend on the degree of smoothness of the partition (that is, on the derivatives of the functions π_k), so we may send $\delta \to 0$ and get rid of the smoothing (the pressure is continuous in this limit, due to our assumption $\mu(\partial\mathcal{P}) = 0$). We thus obtain the following lower bound on the pressure associated with the *sharp* partition $\mathcal{P}$:

$$
\frac{p_0^{n_0-1}(\mu_{\rm sc}, v, \mathcal{P})}{n_0} \geq -\frac{(d-1)\lambda_{\max}}{2} + \mathcal{O}(\epsilon).
$$

We recall that ϵ majorizes the diameter of $\mathcal{P}$. The first part of the pressure is the entropy $H(\mu_{\rm sc}, \mathcal{P}^{\vee n_0})$, while the second part is given by the sum

$$
-\sum_{|\boldsymbol{\alpha}|=n_0} \mu_{\rm sc}(E_{\boldsymbol{\alpha}}) \log J^u_{n_0}(\boldsymbol{\alpha}) = -n_0 \sum_{k=1}^K \mu_{\rm sc}(E_k) \log J^u(k),
$$

where we remind that $J^u(k) = \min_{\rho\in E_k} J^u(\rho)$. We can now let $n_0 \to \infty$, and get

$$
H_{\rm KS}(\mu_{\rm sc}, \mathcal{P}) \geq -\frac{(d-1)\lambda_{\max}}{2} + \sum_{k=1}^K \mu_{\rm sc}(E_k) \log J^u(k) + \mathcal{O}(\epsilon).
$$

By taking finer and finer partitions (that is, $\epsilon \to 0$), we finally get the bound (1.8) for the semiclassical measure $\mu_{\rm sc}$. $\square$

4. Proof of the hyperbolic dispersive estimate

4.1. Decomposition into adapted Lagrangian states. The proof of Proposition 3.6 starts from an arbitrary $\Psi \in L^2$ with $\|\Psi\|_{L^2} = 1$. The localized state $\psi_{\alpha_0} \stackrel{\text{def}}{=} \Pi_{\alpha_0}\chi^{(n)}\big(P(\hbar)\big)\Psi$ will then be decomposed into a linear combination of "nice" Lagrangian states. To construct these "nice states," we need to consider, on each neighbourhood $\widetilde{E}_k \supset E_k$, a coordinate chart $\{(y_i, \eta_i), i = 0, \dots, d-1\}$ adapted to the dynamics of the geodesic flow. These coordinates are required to satisfy the following properties:

(1) $\widetilde{E}_k$ is contained in the polydisk $D(\epsilon, \epsilon) = \{(y, \eta), |y| \le \epsilon, |\eta| \le \epsilon\}$.
(2) the coordinate $\eta_0 = p_0(x, \xi)$, so that the energy shells are given by $\{\eta_0 = \text{const}\}$, and the conjugate variable y_0 represents the time along the trajectory.
(3) the planes $\{\eta = \text{const}\}$ are close to the local weak unstable manifolds W^{u0}_ϵ in $\widetilde{E}_k$. For this aim, we let the plane $\{\eta = 0\}$ coincide with the local unstable manifold $W^{u0}_\epsilon(\rho_k)$ for some arbitrary point $\rho_k \in E_k \cap \mathcal{E}$.

Definition 4.1. We say that a Lagrangian leaf $\Lambda \subset \widetilde{E}_k$ belongs to the γ_1-*cone* if it is represented, in the chart $\{(y, \eta)\}$, as

$$\Lambda = \big\{\big(y, \mathrm{d}S(y)\big), |y| \le \epsilon\big\}, \qquad \text{with } \sup_{|y|\le\epsilon} \|\mathrm{d}^2 S(y)\| \le \gamma_1. \tag{4.1}$$

Fixing some $\gamma_1 > 0$, there exists $\epsilon_{\gamma_1} > 0$ such that, provided the diameters of the $\widetilde{E}_k$ are all smaller than ϵ_{γ_1}, then the γ_1-cone contains all the local unstable manifolds $W^{u0}_\epsilon(\rho)$, $\rho \in \widetilde{E}_k$, while all local stable manifolds $W^s_\epsilon(\rho)$ are uniformly transverse to this cone. We call such a cone an *unstable γ_1-cone*.

Let $\mathcal{U}_k$ be a semiclassical Fourier Integral Operator (FIO) associated with the change of coordinates

$$(x, \xi) \in T^*X \to (y, \eta) \in \mathbb{R}^{2d}, \tag{4.2}$$

unitary microlocally near $\widetilde{E}_k \times D(\epsilon, \epsilon)$. This means that for any $\psi_k \in L^2(X)$, $\|\psi\| = \mathcal{O}(1)$, microlocalized inside $\widetilde{E}_k$, we have

$$\|\psi\|_{L^2(X)} = \|\mathcal{U}_k\psi\|_{L^2(\mathbb{R}^d)} + \mathcal{O}(\hbar^\infty),$$

and the function $\mathcal{U}_k\psi_k(y)$ is microlocalized inside $D(\epsilon, \epsilon)$. Hence, using the cutoff

$$\chi \in C_0^\infty(\{|y| \le 2\epsilon\}), \qquad \chi = 1 \text{ for } |y| \le \epsilon,$$

we may construct a family of "localized plane waves"

$$e_\eta(y) = \chi(y)\exp(\mathrm{i}\langle\eta, y\rangle/\hbar), \qquad (y, \eta) \in D(2\epsilon, 2\epsilon),$$

such that the function $\mathcal{U}_k\psi_k(y)$ can be Fourier expanded into

$$\mathcal{U}_k\psi_k = (2\pi\hbar)^{-d/2}\int_{|\eta|\le 2\epsilon} e_\eta\hat{\psi}_k(\eta)\,\mathrm{d}\eta + \mathcal{O}_{L^2}(\hbar^\infty),$$

where $\hat{\psi}_k = \mathcal{F}_\hbar\psi_k$ is the $\hbar$-Fourier transform of $\mathcal{U}_k\psi_k(y)$. Each state e_η, $|\eta| \le 2\epsilon$, is a Lagrangian state associated with the "horizontal" Lagrangian leave $\Lambda_\eta = \{(y, \eta), |y| \le 2\epsilon\}$.

The change of coordinates (4.2) brings the energy layer $\{|p - \frac{1}{2}| \le \hbar^{1-\delta}\}$ into the slice $\{|\eta_0| \le \hbar^{1-\delta}\}$. As a result, the states ψ_k which are sharply localized in energy are easy to characterize.

Lemma 4.2. *Assume that for some integer $m \leq C_\delta |\log \hbar|$ the state ψ_k satisfies*

$$\chi^{(m)}\big(P(\hbar)\big)\psi_k = \psi_k + \mathcal{O}_{L^2}(\hbar^\infty). \tag{4.3}$$

In that case, the state ψ_k can be decomposed into

$$\psi_k = (2\pi\hbar)^{-d/2} \int_{|\eta_0| \leq e^{m\delta}\hbar^{1-\delta}, |\eta'| \leq 2\epsilon} \mathcal{U}_k^* e_\eta \hat{\psi}_k(\eta)\, d\eta + \mathcal{O}_{L^2}(\hbar^\infty). \tag{4.4}$$

Proof. The assumption (4.3) and the microlocalization of $\mathcal{U}_k\psi_k$ inside $D(2\epsilon, 2\epsilon)$ imply that its Fourier transform $\hat{\psi}_k$ satisfies $\hat{\psi}_k(\eta) = \mathcal{O}\big((\hbar/\langle\eta\rangle)^\infty\big)$ for η outside the strip $\{|\eta_0| \leq e^{m\delta}\hbar^{1-\delta}, |\eta'| \leq 2\epsilon\}$. □

Our aim is to prove (3.20). We consider a sequence $\boldsymbol{\alpha}$ of length n, and an arbitrary $\Psi \in L^2(X)$. We then apply the above decomposition to the case $k = \alpha_0$ and the state $\psi_{\alpha_0} = \Pi_{\alpha_0}\chi^{(n)}\big(P(\hbar)\big)\Psi$. By construction, this state satisfies the energy localization (4.3) if we take $m \geq n+1$.

4.2. Evolution of individual Lagrangian states. Our strategy will consist in controlling the states $\Pi_{\boldsymbol{\alpha}}\mathcal{U}_{\alpha_0}^* e_\eta$ individually. For each $|\eta| \leq 2\epsilon$, the state

$$\psi^0 \stackrel{\text{def}}{=} \mathcal{U}_{\alpha_0}^* e_\eta \in L^2(X) \tag{4.5}$$

is a Lagrangian state associated with a certain Lagrangian leaf Λ^0 which belongs to some unstable γ_1-cone in $\widetilde{E}_{\alpha_0}$. The operator $\Pi_{\boldsymbol{\alpha}}$ is a succession of evolutions along the Schrödinger flow (U) and truncations by quasiprojectors Π_{α_i}. Each quasiprojector $\Pi_k = \mathrm{Op}_\hbar(\widetilde{\pi}_k)$ is a pseudodifferential operator, which transforms a Lagrangian state associated with some Lagrangian leaf Λ, into another Lagrangian state on the same Λ, by modifying its symbol. This modification will generally reduce the L^2 norm of the state. In turn, the propagator U is a unitary Fourier Integral Operator associated with the map g^1, which transforms a Lagrangian state on Λ into a Lagrangian state on $g(\Lambda)$, keeping the L^2-norm unchanged.

More precisely, the operator $\Pi_{\alpha_i}U$ acts as follows on Lagrangian states.

Proposition 4.3. *Consider a Lagrangian leaf $\Lambda^0 \subset \widetilde{E}_{\alpha_0}$ in some unstable γ_1-cone, and a Lagrangian state $\psi^0 \in L^2(X)$ localized on this leaf, of the form*

$$\mathcal{U}_{\alpha_0}\psi^0(y) = a^0(y)e^{iS^0(y)/\hbar}, \qquad a^0 \in C_0^\infty\big(D(\epsilon)\big).$$

Then, the state $\psi^1 = \Pi_{\alpha_1}U\psi^0$ is a Lagrangian state associated with the leaf $\Lambda^1 = g(\Lambda^0)$. It can be expressed (in the coordinates attached to $\widetilde{E}_{\alpha_1}$) as

$$\mathcal{U}_{\alpha_1}\psi^1(y) = a^1(y,\hbar)e^{iS^1(y)/\hbar},$$

where S^1 is a generating function for Λ^1. The symbol $a^1(y,\hbar)$ admits an expansion

$$a^1(y,\hbar) = \sum_{j=0}^{L-1} \hbar^j a_j^1(y) + \hbar^L r_l(y,\hbar). \tag{4.6}$$

The inverse flow $g^{-1}_{\restriction\Lambda^1} : \Lambda^1 \to \Lambda^0$ can be expressed in the coordinates y attached respectively to $\widetilde{E}_{\alpha_1}$ and $\widetilde{E}_{\alpha_0}$, through a map

$$y^1 \in \pi\Lambda^1 \subset D(2\epsilon) \mapsto y^0 = \pi g^{-1}\big(y^1, dS^1(y^1)\big) \in D(2\epsilon). \tag{4.7}$$

Then, the principal symbol $a_0^1(x)$ *in* (4.6) *reads*

$$a_0^1(y^1) = e^{i\beta^1} a^0(y^0) \Big| \det \frac{\partial y^0}{\partial y^1} \Big|^{1/2} \widetilde{\pi}_{\alpha_1}\big(y^1, dS^1(y^1)\big), \tag{4.8}$$

with β^1 *a constant phase. The higher-order symbols* a_j^1 *and the remainder* r_l *satisfy the following bounds, for any* $l \in \mathbb{N}$:

$$\|a_j^1\|_{C^l} \le C_{l,j} \|a^0\|_{C^{l+2j}}, \qquad 0 \le j \le L-1, \tag{4.9}$$

$$\|r_l(\cdot, \hbar)\|_{C^l} \le C_{l,L} \|a^0\|_{C^{l+2L+d}}, \tag{4.10}$$

$$r_l = \mathcal{O}\left(\left(\frac{\hbar}{\hbar + \mathrm{dist}(\bullet, \pi \widetilde{E}_{\alpha_1})} \right)^{\infty} \right) \text{ outside } \pi \widetilde{E}_{\alpha_1}.$$

The constants $C_{l,j}$ *depend on the Lagrangian* Λ^0.

The proximity of Λ^0 from the unstable manifold $\Lambda_{\alpha_0} \stackrel{\text{def}}{=} W^{u0}(\rho_{\alpha_0})$ has another consequence. The map $y^1 \mapsto y^0$ in (4.7), projection of $g^{-1}_{\restriction \Lambda^1}$ to the coordinates y, is close to the projection of $g^{-1}_{\restriction g(\Lambda_{\alpha_0})}$. For this reason, the Jacobian $\det(\partial y^0/\partial y^1)$ appearing in (4.13) is close to the Jacobian of $g^{-1}_{\restriction g(\Lambda_{\alpha_0})}$. Using our coarse-grained Jacobians (3.16), we find

$$\Big| \det \frac{\partial y^0}{\partial y^1} \Big| = J^u(\alpha_0)^{-1} \big(1 + \mathcal{O}(\gamma_1, \epsilon^{\gamma})\big), \tag{4.11}$$

where $\gamma \in (0, 1]$ depends on the Hölder regularity of the unstable foliation.

4.3. n-steps evolution. The above proposition describes the 1-step evolution $\Pi_{\alpha_i} U$. We need to apply many ($n \sim \log(1/\hbar)$) similar steps. To control these many steps uniformly with respect to n, we first need to analyze the evolution of the Lagrangian leaf Λ^0 through the classical evolution corresponding to the operator $\Pi_{\boldsymbol{\alpha}}$: for $i = 0, \ldots, n-1$, the leaf Λ^{i+1} is obtained by truncating Λ^i on $\operatorname{supp} \widetilde{\pi}_{\alpha_i} \subset \widetilde{E}_{\alpha_i}$, and then evolving this truncated leaf through g^1. We will assume that the sequence $\boldsymbol{\alpha}$ is *admissible*, in the sense that Λ^i is nonempty for all $i = 1, \ldots, n$. Then, the Anosov structure of the geodesic flow induces the following *inclination lemma* [18], which describes the leaves Λ^i in the adapted coordinates (y, η) on $\widetilde{E}_{\alpha_i}$:

Lemma 4.4. *Assume the Lagrangian leave* $\Lambda^0 \subset \widetilde{E}_{\alpha_0}$ *belongs to a certain unstable* γ_1*-cone, as defined in Definition* 4.1. *Then, the Lagrangians* $\Lambda^i \subset \widetilde{E}_{\alpha_i}$, $i = 1, \ldots, n$, *also belong to the corresponding unstable* γ_1*-cones*:

$$\Lambda^i \subset \big\{ \big(y, dS^i(y)\big), |y| \le \epsilon \big\}, \qquad \sup_{|y| \le \epsilon} \|d^2 S^i(y)\| \le \gamma_1.$$

We also have a uniform control on the higher derivatives of the generating functions S^i. *For any* $l > 1$, *there exists* $\gamma_l > 0$ *such that, assuming* dS^0 *is in the* γ_1*-cone and satisfies* $\|dS^0\|_{C^l} \le \gamma_l$, *then the evolved Lagrangians also satisfy*

$$\|dS^i\|_{C^l} \le \gamma_l, \qquad i = 1, \ldots, n.$$

The above lemma shows that the evolution $\Lambda^i \mapsto \Lambda^{i+1}$ remains uniformly under control at long times. Putting together the lemma with Proposition 4.3, we get the following

Proposition 4.5. *For $\hbar < \hbar_0$ and $n \leq C \log \hbar^{-1}$, take any sequence $\boldsymbol{\alpha} = \alpha_0 \cdots \alpha_n$ of length $n+1$. Consider a Lagrangian leaf $\Lambda^0 \subset \widetilde{E}_{\alpha_0} \cap \mathcal{E}_{\eta_0}$ in some unstable γ_1-cone, and an associated Lagrangian state $\psi^0 \in L^2(X)$ localized on this leaf, of the form*

$$\mathcal{U}_{\alpha_0} \psi^0(y) = a^0(y) \mathrm{e}^{\mathrm{i} S^0(y)/\hbar}, \qquad a^0 \in C_c^\infty\big(D(\epsilon)\big).$$

We are interested in the evolved state

$$\psi^n \stackrel{\mathrm{def}}{=} \Pi_{\alpha_n} U \Pi_{\alpha_{n-1}} U \cdots \Pi_{\alpha_1} U \psi^0.$$

Then,

(i) *If the manifold Λ^n (obtained from Λ^0 from the classical evolution) is empty, then $\|\psi^n\|_{L^2} = \mathcal{O}(\hbar^\infty)$.*

(ii) *Otherwise, ψ^n is a Lagrangian state associated with Λ^n. It reads*

$$\mathcal{U}_{\alpha_n} \psi^n(y) = a^n(y, \hbar) \mathrm{e}^{\mathrm{i} S^n(y)/\hbar} + \mathcal{O}_{\mathcal{S}}(\hbar^\infty),$$

where the symbol $a^n(\bullet, \hbar)$ is supported in $D(\epsilon)$ and satisfies the bound

$$\|a^n(\bullet, \hbar)\|_{C^0(D(\epsilon))} \leq C J(\alpha_0 \cdots \alpha_{n-1})^{-1/2} \|a^0\|_{C^0}.$$

As a consequence, we obtain the L^2 estimate

$$\|\psi^n\|_{L^2(X)} \leq C J(\alpha_0 \cdots \alpha_{n-1})^{-1/2} \|a^0\|_{C^0}. \tag{4.12}$$

The constant C is uniform when the function S^0 generating Λ^0 remains in a bounded set in the C^∞ topology.

Proof. This proposition is proved in [3, Lemma 3.5], but we will rather use the notations of a similar result valid in a more general setup in [23, Proposition 4.1]. The strategy consists in a tedious but straightforward bookkeeping of the properties of the symbols

$$a^i(y, \hbar) = \sum_{j=0}^{L-1} \hbar^j a^i_j(y) + r^i_l(y, \hbar)$$

associated with the intermediate states $\mathcal{U}_{\alpha_i} \psi^i(y)$. Namely, one manages to control the C^l norms of the symbols and of the remainder, using the equations (4.13), (4.9), (4.10).

The principal symbol a^n_0 is given by the explicit formula

$$a^n_0(y^n) = \mathrm{e}^{\mathrm{i} \sum_{i=1}^n \beta^n} a^0(y^0) \prod_{i=1}^n \left| \det \frac{\partial y^{i-1}}{\partial y^i} \right|^{1/2} \widetilde{\pi}_{\alpha_i}\big(y^i, \mathrm{d}S^i(y^i)\big), \tag{4.13}$$

where each $y^{i-1} = \pi g_{S^i}^{-1}\big(y^i, \mathrm{d}S^i(y^i)\big)$ is the coordinate of the $(i-1)$th iterate of the point $\big(y^0, \mathrm{d}S^0(y^0)\big) \in \Lambda^0$. This formula shows that a^n_0 results from a transport of the amplitude (or half-density) a^0 through the flow, and a multiplication by successive factors $|\widetilde{\pi}_{\alpha_i}| \leq 1 + \mathcal{O}(\hbar)$. From this expression and (4.11) we directly get the C^0 bound

$$\|a^n_0\|_{C^0(D(\epsilon))} \leq C J(\alpha_0 \cdots \alpha_{n-1})^{-1/2} \|a^0\|_{C^0}. \tag{4.14}$$

The derivatives $\partial^l a^n_0 / \partial (y^n)^l$ are computed by applying the Leibnitz rule to the product (4.13) (which leads to $\mathcal{O}(n^l)$ terms), and then the chain rule $\partial f(y^i)/\partial y^n =$

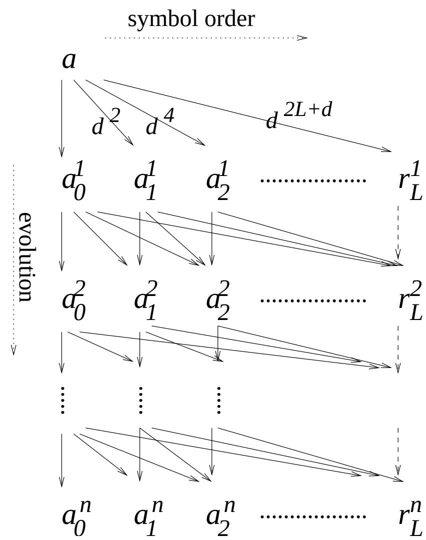

FIGURE 1. Each symbol a^i_j is linked to its direct "descendents." Vertical arrows represent operators of transport+multiplication, while oblique arrows include a certain number of differentiations, as given in (4.9), (4.10).

$(\partial f(y^i)/\partial y^i)(\partial y^i/\partial y^n)$. Using the fact that the Jacobian matrices $\partial y^i/\partial y^n$ are uniformly bounded, one obtains the bounds

$$\|a_0^n\|_{C^l} \leq C_l\, n^l\, J(\alpha_0 \cdots \alpha_{n-1})^{-1/2} \|a^0\|_{C^l}, \qquad l \geq 0.$$

The higher-order symbol a^i_j, $1 \leq j \leq L-1$, is obtained by the same transport-and-multiplication of the symbol a^{i-1}_j, but also by transporting and differentiating $2(j-j')$ times the symbols $a^{i-1}_{j'}$, $j' \leq j$. This procedure is sketched in Figure 1. On this figure, each symbol a^n_j results from the sum of $\mathcal{O}(n^j)$ paths starting from a^0, each path consisting in a succession of "long" vertical evolutions, and j "oblique" evolutions, involving altogether $2j$ differentiations performed at various stages. We have seen above that each differentiation leads, through Leibnitz's rule, to a factor $\mathcal{O}(n)$. Taking into account the number of paths, we obtain the bounds

$$\|a_j^n\|_{C^0} \leq C_j n^{3j} J(\alpha_0 \cdots \alpha_{n-1})^{-1/2} \|a^0\|_{C^{2j}}, \qquad 1 \leq j \leq L-1,$$

and l differentiations of the symbol a^n_j provide, for the same reasons as above, an additional factor n^l:

$$\|a_j^n\|_{C^l} \leq C_{j,l} n^{l+3j} J(\alpha_0 \cdots \alpha_{n-1})^{-1/2} \|a^0\|_{C^{l+2j}}, \qquad 1 \leq j \leq L-1.$$

At each stage, one also gets a remainder r^i_l, which results from the symbols a^{i-1}_j through transport, multiplication and differentiation, as well as from the *unitary* evolution of the previous remainder r^{i-1}_l (dashed vertical arrow). Taking the above bounds into account, one easily obtains the L^2 bound

$$\|r_l^n\|_{L^2} \leq C_l \|a^0\|_{C^{2L+d}}.$$

This last estimate shows that the full symbol $a^n(\bullet,\hbar)$ is dominated by the principal symbol a_0^n, so that the estimate (4.14) also applies to $\|a^n(\bullet,\hbar)\|_{C^0}$, and hence to $\|a^n(\bullet,\hbar)\|_{L^2} = \|\psi^n\|_{L^2} + \mathcal{O}(\hbar^\infty)$. □

We now conclude the proof of Proposition 3.6. We use the decomposition (4.4) of the state ψ_{α_0}, and apply Proposition 4.5 to each state $\psi^0 = \mathcal{U}_{\alpha_0}^* e_\eta$, $|\eta| \leq 2\epsilon$. Notice that the manifolds Λ_η remain in a bounded cone in C^∞: a generating function for Λ_η is simply $S_\eta(y) = \langle \eta, y\rangle$. Therefore, the constant C in the estimate (4.12)) is uniform with respect to η. The triangle inequality then implies the bound

$$\|\Pi_{\alpha_n} U \Pi_{\alpha_{n-1}} \cdots \Pi_{\alpha_1} U \psi_{\alpha_0}\| \leq C\hbar^{-d/2} J(\alpha_0 \cdots \alpha_{n-1})^{-1/2} \int |\hat{\psi}_{\alpha_0}(\eta)|\, \mathrm{d}\eta + \mathcal{O}_{L^2}(\hbar^\infty).$$

The right-hand side contains the L^1 bound for the Fourier transform $\hat{\psi}_{\alpha_0}(\eta)$. Since this function is $\mathcal{O}(\hbar^\infty)$ outside the set $\{|\eta_0| \leq \mathrm{e}^{m\delta}\hbar^{1-\delta}, |\eta'| \leq 2\epsilon\}$, the Cauchy–Schwarz inequality leads to

$$\|\hat{\psi}_{\alpha_0}\|_{L^1} \leq (C\epsilon \mathrm{e}^{m\delta}\hbar^{1-\delta})^{1/2}\, \|\hat{\psi}_{\alpha_0}\|_{L^2} + \mathcal{O}_{L^2}(\hbar^\infty).$$

Since $\|\hat{\psi}_{\alpha_0}\|_{L^2} = \|\psi_{\alpha_0}\|_{L^2} \leq 1 + \mathcal{O}(\hbar^\infty)$, we obtain (3.20). □

5. Extension to Anosov toral diffeomorphisms

In this section we will give the few modifications necessary to prove Theorem 1.7(ii) dealing with Anosov toral diffeomorphisms. Although the strategy of proof is exactly the same as for the Laplacian eigenstates, the quantum setting is slightly different, and not so well-known as the one used above. One advantage of dealing with maps instead of flows is that we will not need any sharp energy cutoff: the torus will take the place of the energy shell $\mathcal{E}$, so there won't be any need to localize ourselves on a submanifold.

Let us mention that the bound (1.9) is not a new result in the case of *linear* hyperbolic symplectomorphisms on the 2-dimensional torus: as explained in Section 1.4, it is a consequence of Brooks's more precise result (Theorem 1.9). For linear symplectomorphisms in higher dimension, an improvement of the lower bound (1.9) has been recently obtained by G. Rivière [26].

5.1. Quantum mechanics on a torus phase space. We briefly recall the properties of quantum mechanics associated with the torus phase space. $\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ is equipped with the symplectic form $\omega = \sum_{i=1}^d \mathrm{d}\xi_i \wedge \mathrm{d}x_i$. Quantum states are defined as the distributions $\psi \in \mathcal{S}'(\mathbb{R}^d)$ which are $\mathbb{Z}^d$-periodic, and the $\hbar$-transforms of which are also $\mathbb{Z}^d$-periodic:

(5.1)
$$\forall n \in \mathbb{Z}^d, \forall x \in \mathbb{R}^d, \quad \psi(x+n) = \psi(x), \qquad \forall \xi \in \mathbb{R}^d, \quad (\mathcal{F}_\hbar\psi)(\xi+n) = (\mathcal{F}_\hbar\psi)(\xi).$$

A simple calculation shows that such distributions can be nontrivial iff

$$\hbar = \hbar_N \stackrel{\text{def}}{=} (2\pi N)^{-1} \qquad \text{for some integer } N > 0.$$

Such values of $\hbar$ are called *admissible*. From now on we will only consider admissible values of $\hbar$. The distributions (5.1) then form a N^d-dimensional subspace of $\mathcal{S}'(\mathbb{R}^d)$, (denoted by $\mathcal{H}_N$), which is spanned by the following basis of "Dirac combs":

$$e_j(x) = N^{-d/2} \sum_{\nu \in \mathbb{Z}^d} \delta\Big(x - \frac{j}{N} - \nu\Big), \qquad j \in (\mathbb{Z}/N\mathbb{Z})^d \simeq \{0,\dots,N-1\}^d. \tag{5.2}$$

It is natural to equip $\mathcal{H}_N$ with the Hermitian norm $\|\bullet\|_{\mathcal{H}_N}$ for which the basis $\{e_j, j \in (\mathbb{Z}/N\mathbb{Z})^d\}$ is orthonormal. One can construct the space $\mathcal{H}_N$ by "projecting" states $\psi \in \mathcal{S}(\mathbb{R}^d)$:

$$\forall \psi \in \mathcal{S}(\mathbb{R}^d), \qquad \Pi_N \psi(x) \stackrel{\text{def}}{=} \sum_{\mu,\nu \in \mathbb{Z}^d} \psi(x-\nu) \mathrm{e}^{2\mathrm{i}\pi\langle \mu, x\rangle}$$

belongs to $\mathcal{H}_N$, and the map Π_N is surjective. In general, there is no obvious link between the norms $\|\psi\|_{L^2}$ and $\|\Pi_N\psi\|_{\mathcal{H}_N}$. Yet, imposing some localization for ψ, we get the following relation:

Lemma 5.1. *Assume that $\psi \in \mathcal{S}(\mathbb{R}^d)$ is microlocalized inside a set $E \subset \mathbb{R}^{2d}$ of diameter$< \frac{1}{2}$. Then, its projection $\Pi_N \psi \in \mathcal{H}_N$ satisfies*

$$\|\Pi_N \psi\|_{\mathcal{H}_N} = \|\psi\|_{L^2} + \mathcal{O}(\hbar^\infty).$$

Let us now describe observables on $\mathbb{T}^{2d}$. Any smooth function on $\mathbb{T}^{2d}$ is also a $\mathbb{Z}^{2d}$-periodic function on $\mathbb{R}^{2d}$. It is natural to introduce symbol classes

$$S^k_\nu(\mathbb{T}^{2d}) = \{f(\hbar) \in C^\infty(\mathbb{T}^{2d}), |\partial_x^\alpha \partial_\xi^\beta f(\hbar)| \leq C_{\alpha,\beta} \hbar^{-k-\nu|\alpha+\beta|}\}, \qquad \nu \in [0, \tfrac{1}{2}), k \in \mathbb{R}$$

(due to periodicity, we cannot have growth or decay in the variable ξ as in the classes (2.8)).

Observables $f = (f(\hbar)) \in S^k_\nu(\mathbb{T}^{2d})$ can be Weyl-quantized as operators $\mathrm{Op}_\hbar(f)$ acting on $\mathcal{S}(\mathbb{R}^d)$, but also on $\mathcal{S}'(\mathbb{R}^d)$ by duality. We already know that any observable $f \in S^0_\nu(\mathbb{T}^{2d})$ satisfies

$$\|\mathrm{Op}_\hbar(f)\|_{L^2 \to L^2} \leq \|f\|_\infty + \mathcal{O}(\hbar^{1-2\nu}).$$

Proposition 5.2 ([6])**.** *Take $f \in C^\infty(\mathbb{T}^{2d})$. For any admissible $\hbar = \hbar_N$, the operator $\mathrm{Op}_\hbar(f)$ leaves invariant the subspace $\mathcal{H}_N \subset \mathcal{S}'(\mathbb{R}^d)$. Let us call $\mathrm{Op}_N(f)$ its restriction on $\mathcal{H}_N$. These two operators satisfy*

$$\|\mathrm{Op}_N(f)\|_{\mathcal{H}_N \to \mathcal{H}_N} \leq \|\mathrm{Op}_\hbar(f)\|_{L^2 \to L^2}. \tag{5.3}$$

This property allows to carry the pseudodifferential calculus on $\mathbb{R}^{2d}$ down to the torus. For instance, for any two observables $f, g \in S^0_0(\mathbb{T}^{2d})$, the product $\mathrm{Op}_N(f)\,\mathrm{Op}_N(g)$ is the restriction of $\mathrm{Op}_\hbar(f)\,\mathrm{Op}_\hbar(g) = \mathrm{Op}_\hbar(f \sharp g)$. One can check that $f \sharp g$ is a periodic function, and that each term in the expansion $f \sharp g = \sum_{j=0}^{L-1} \hbar^j (f \sharp g)_j + \hbar^L R_l(f, g, \hbar)$ (including the remainder) is also periodic. Hence, the estimates on $\|\mathrm{Op}_\hbar\big((f \sharp g)_j\big)\|_{L^2}$ and $\big\|\mathrm{Op}_\hbar\big(R_l(f,g,\hbar)\big)\big\|_{L^2}$ can be directly translated to estimates of their restrictions on $\mathcal{H}_N$.

5.2. Quantum maps on the torus. We now give a brief overview of what we mean by a quantum propagator associated with a smooth symplectic diffeomorphism $\kappa\colon \mathbb{T}^{2d} \circlearrowleft$. We will not try to provide a general recipe to "quantize" all possible κ, but only give some relevant examples.

(1) Consider the flow g^t generated by some Hamilton function $p \in C^\infty(\mathbb{T}^{2d})$. If we quantize p into $P(\hbar) = \mathrm{Op}_\hbar(p)$, then the quantum propagator quantizing g^t is naturally $U^t = e^{-itP(\hbar)/\hbar}$. Since $P(\hbar)$ leaves $\mathcal{H}_N$ invariant, so does the propagator. Hence, the map $\kappa \stackrel{\text{def}}{=} g^1$ is quantized by

$$U_N(\kappa) = \mathrm{e}^{-\mathrm{i}P(\hbar)/\hbar} \restriction \mathcal{H}_N = \exp\big(-\mathrm{i}2\pi N\,\mathrm{Op}_N(p)\big),$$

which is unitary on $\mathcal{H}_N$. The Egorov property (2.13) can be directly translated to the torus setting:

$$U_N(\kappa)^{-1}\operatorname{Op}_N(f)U_N(\kappa) = \operatorname{Op}_N(f\circ\kappa) + \mathcal{O}_{\mathcal{H}_N\to\mathcal{H}_N}(\hbar). \tag{5.4}$$

(2) If κ is a linear symplectomorphism of $\mathbb{T}^{2d}$ associated with the symplectic matrix $S_\kappa = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}(2d,\mathbb{Z})$, then it can be quantized as a metaplectic transformation $U_\hbar(\kappa)$, which acts unitarily on $L^2(\mathbb{R}^d)$. Provided S_κ satisfies some "checkerboard conditions" [6], the extension of $U_\hbar(\kappa)$ to $\mathcal{S}'(\mathbb{R}^d)$ leaves $\mathcal{H}_N$ invariant; its restriction $U_N(\kappa) = U_\hbar(\kappa)\restriction\mathcal{H}_N$ is unitary on $\mathcal{H}_N$. If S_κ has no eigenvalues on the unit circle (meaning that the matrix is hyperbolic), the map κ is Anosov. Such a map is often called a "generalized quantum cat map," by reference to Arnold's "cat map", a specific hyperbolic matrix in $SL(2,\mathbb{Z})$. It satisfies an exact Egorov property: $U_N(\kappa)^{-1}\operatorname{Op}_N(f)U_N(\kappa) = \operatorname{Op}_N(f\circ\kappa)$.

(3) Let us combine these two types of maps, namely a linear symplectomorphism κ_0 (satisfying the checkerboard condition), and the flow g^t generated by some $p\in C^\infty(\mathbb{T}^{2d})$, to get

$$\kappa = g^1\circ\kappa_0. \tag{5.5}$$

If S_{κ_0} is hyperbolic and p is small enough in the C^2 topology, then it is known that κ still has the Anosov property (and it is topologically conjugate with κ_0). The quantum propagator can be defined as

$$U_N(\kappa) \stackrel{\text{def}}{=} \exp\big(-\mathrm{i}2\pi N\operatorname{Op}_N(p)\big)\circ U_N(\kappa_0).$$

It obviously satisfies the Egorov property (5.4).

The long-time Egorov theorem (Proposition 3.4) can also be brought to the torus framework:

Proposition 5.3. *Choose $\epsilon>0$ small and $\nu\in\big((1-\epsilon)/2,\frac12\big)$. Take $f\in S^0_0(\mathbb{T}^{2d})$. Then, for any admissible $\hbar=\hbar_N$ and any time $n=n(\hbar)$ in the range $|n|\le T_{\epsilon,\hbar}$, we have*

$$U_N^{-n}\operatorname{Op}_N(f)U_N^n = \operatorname{Op}_N(\tilde f_n) + \mathcal{O}(\hbar^\infty), \qquad \textit{with } \tilde f_n\in S^0_\nu(\mathbb{T}^{2d}),\ \tilde f_n - f\circ\kappa^n\in S_\nu^{-(1+\epsilon)/2}(\mathbb{T}^{2d}). \tag{5.6}$$

5.3. Quantum partitions on the torus. Through the quantization $f\mapsto\operatorname{Op}_N(f)$, we can associate to any semiclassical sequence of normalized states $(u_N\in\mathcal{H}_N)_{N\to\infty}$ one or several semiclassical measures, as in Section 2.4. Starting from an Anosov map κ of the form (5.5), we consider sequences $(\psi_N\in\mathcal{H}_N)_{N\to\infty}$ where each ψ_N is an eigenstate of $U_N(\kappa)$. After possibly extracting a subsequence, this sequence admits a single semiclassical measure μ_{sc}, which is a probability κ-invariant measure on $\mathbb{T}^{2d}$. To analyze the KS entropy of this measure (defined as in Section 3.1 after replacing g^1 by κ), we set up a partition $\mathcal{P}=\bigsqcup_{k=1}^K E_k$ of $\mathbb{T}^{2d}$ of diameter ϵ, such that $\mu_{\mathrm{sc}}(\partial\mathcal{P})=0$. The partition $\mathcal{P}$ is smoothed into $\mathcal{P}_{\mathrm{sm}}=\{\pi_k\}$, where $\pi_k\in C^\infty(\mathbb{T}^{2d},[0,1])$, each π_k is supported near E_k, and $\sum_k\pi_k=1$. The quantum partition is defined similarly as in Section 3.2, except that $\Pi_k=\operatorname{Op}_N(\widetilde\pi_k)$, $\widetilde\pi_k\in S(\mathbb{T}^{2d})$ satisfy

$$\sum_{k=1}^K \Pi_k^2 = \mathrm{Id}_{\mathcal{H}_N} + \mathcal{O}(\hbar^\infty). \tag{5.7}$$

The refined quasiprojectors $\Pi_{\boldsymbol{\alpha}}$ are defined similarly as in (3.9), after setting $U = U_N(\kappa)$. The symbolic measures μ_N, $\tilde{\mu}_N$ associated with ψ_N are defined as in (3.14), (3.34).

5.4. Hyperbolic dispersive estimate. One needs to adapt the proof of the hyperbolic dispersion estimate (5.8) to the torus setting. We can prove the following

Proposition 5.4. *Fix a constant $C_1 \gg 1$. Then there exists $C > 0$ such that for any $N \geq 1$, any $0 \leq n \leq C_1 \log N$ and any sequence $\boldsymbol{\alpha}$ of length n, the following estimate holds*:

$$\|\Pi_{\boldsymbol{\alpha}}\| \leq C N^{d/2} J^u_n(\boldsymbol{\alpha})^{-1/2}. \tag{5.8}$$

Here $J^u_n(\boldsymbol{\alpha}) = \prod_{j=0}^{n-1} J^u(\alpha_j)$ is the coarse-grained unstable Jacobian of the map κ. Notice that the power of $N = (2\pi\hbar)^{-1}$ is $d/2$ instead of $(d - 1 + \delta)$ in (3.20).

Proof of the proposition. The proof of this estimates proceeds along the same lines as in Section 4, that is by explicitly computing the action of the operator $\Pi_{\boldsymbol{\alpha}}$ on an arbitrary normalized state $\Psi \in \mathcal{H}_N$. To do this, we first expand each localized piece $\psi = \Pi_k \Psi$, $k = 1, \ldots, K$, into an orthonormal basis $\{f_j, j \in (\mathbb{Z}/N\mathbb{Z})^d\}$ obtained from the original basis (5.2) through a well-chosen quantized linear symplectomorphism $U(\tilde{\kappa}_k)$. Let us recall a few facts about linear symplectomorphisms of the torus. Any $\tilde{\kappa} \in \mathrm{Sp}(2d, \mathbb{R})$ acting on $\mathbb{R}^{2d}$ maps a "position Lagrangian" $\Lambda^{\mathbb{R}}_{x_0} = \{x = x_0, \xi \in \mathbb{R}\}, x_0 \in \mathbb{R}^d$ into another "linear" Lagrangian[7]

$$\tilde{\Lambda}^{\mathbb{R}}_{x_0} = \tilde{\kappa}(\Lambda^{\mathbb{R}}_{x_0}) = \{\xi = DB^{-1}x + (C - DB^{-1}A)x_0,\ x \in \mathbb{R}^d\}. \tag{5.9}$$

Each position Lagrangian $\Lambda^{\mathbb{R}}_{x_0}$ is associated with a "position eigenstate" $e^{\mathbb{R}}_{x_0}(x) = (2\pi\hbar)^{d/2}\delta(x - x_0)$. The metaplectic operator $U_\hbar(\tilde{\kappa})$ maps this position eigenstate into the Lagrangian state

$$\begin{aligned} f^{\mathbb{R}}_{x_0} &\stackrel{\text{def}}{=} U_\hbar(\tilde{\kappa}) e^{\mathbb{R}}_{x_0}, \\ f^{\mathbb{R}}_{x_0}(x) &= \frac{1}{\sqrt{\det(B)}} \mathrm{e}^{(\mathrm{i}/2\hbar)(\langle DB^{-1}x, x\rangle - 2\langle x, B^{-1}x_0\rangle + \langle B^{-1}Ax_0, x_0\rangle)}, \end{aligned} \tag{5.10}$$

which is associated with the Lagrangian $\tilde{\Lambda}^{\mathbb{R}}_{x_0}$. If $\tilde{\kappa}$ has integer coefficients and satisfies the "checkerboard condition", then this construction can be brought down to the torus. Each basis state e_j in (5.2) is associated with the Lagrangian $\Lambda^{\mathbb{T}}_{j/N}$, the projection of $\Lambda^{\mathbb{R}}_{j/N}$ on $\mathbb{T}^{2d}$. The state $f_j \stackrel{\text{def}}{=} U_N(\tilde{\kappa}) e_j \in \mathcal{H}_N$ is a "Lagrangian state on the torus" associated with the projected Lagrangian $\tilde{\Lambda}^{\mathbb{T}}_{j/N} = \tilde{\kappa}(\Lambda^{\mathbb{T}}_{j/N})$.

For each partition component E_k, we select an appropriate linear transformation $\tilde{\kappa}_k$. In each neighbourhood $\widetilde{E}_k$ of $\operatorname{supp} \pi_k$ (assumed to have diameter $\leq \epsilon$), we use an adapted coordinate chart $\{(y, \eta)\}$ as in Section 4.1. Using these coordinates, we consider the family of γ_1-cones defined in Definition 4.1.

Lemma 5.5. *Provided the diameter of $\widetilde{E}_k$ is small enough, we can choose the automorphism $\tilde{\kappa}_k$ such that each connected component of $\tilde{\Lambda}^{\mathbb{T}}_{x_0} \cap \widetilde{E}_k$ belongs to the unstable γ_1-cone.*

[7] We use the representation $S_{\tilde{\kappa}} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and assume for simplicity that B is nonsingular.

PROOF. The pieces of unstable manifolds inside $\widetilde{E}_k$ are then "almost flat" and "almost parallel," so they can be approached by a family of (local) parallel linear Lagrangians; the latter can be assumed to be rational, in the sense that they are parallel to $\tilde{\kappa}_k(\Lambda_0^{\mathbb{T}})$ for some well-chosen transformation $\tilde{\kappa}_k \in \mathrm{Sp}(2d, \mathbb{Z})$. □

For each $k = 1, \ldots, K$ we expand $\psi = \Pi_k \Psi$ in the orthonormal basis $\{f_j = U_N(\tilde{\kappa}_k) e_j, j \in (\mathbb{Z}/N\mathbb{Z})^d\}$. Using cutoffs $\mathbb{1}_{\widetilde{E}_k} \succ \pi_k^\sharp \succ \pi_k$ we can localize the Lagrangian states f_j inside $\widetilde{E}_k$:

$$\psi = \sum_{j \in (\mathbb{Z}/N\mathbb{Z})^d} \psi_j f_j = \sum_{j \in (\mathbb{Z}/N\mathbb{Z})^d} \psi_j \tilde{f}_j + \mathcal{O}(\hbar^\infty), \qquad \tilde{f}_j \stackrel{\text{def}}{=} \mathrm{Op}_N(\pi_k^\sharp) f_j.$$

We can thus proceed as in Section 4, namely compute separately each $\Pi_{\boldsymbol{\alpha}} \tilde{f}_j$. To be able to use Section 4, we will switch back from states in $\mathcal{H}_N$ to states in $\mathcal{S}(\mathbb{R}^d)$.

Lemma 5.6. *Each localized Lagrangian state $\tilde{f}_j = \mathrm{Op}_N(\pi_k^\sharp) f_j \in \mathcal{H}_N$ is equal (up to $\mathcal{O}_{\mathcal{H}_N}(\hbar^\infty)$) to the projection on $\mathcal{H}_N$ of finitely many Lagrangian states $\tilde{f}_{j,n}^{\mathbb{R}} \in \mathcal{S}(\mathbb{R}^d)$ microlocalized inside a single representative $\widetilde{E}_k^{\mathbb{R}}$ of $\widetilde{E}_k$ in $\mathbb{R}^{2d}$.*

PROOF. The localized Lagrangian state $\tilde{f}_j = \mathrm{Op}_N(\pi_k^\sharp) f_j$ is microlocalized on $\tilde{\Lambda}_{j/N}^{\mathbb{T}} \cap \widetilde{E}_k$, which is a finite collection of Lagrangian leaves. Each of these leaves is the projection on the torus of a leaf of the form $\tilde{\Lambda}_{j/N+n}^{\mathbb{R}} \cap \widetilde{E}_k^{\mathbb{R}}$, where $\tilde{\Lambda}_*^{\mathbb{R}}$ is given in (5.9), $n \in \mathbb{Z}^d$ and $\widetilde{E}_k^{\mathbb{R}}$ is a certain (arbitrary) representative of $\widetilde{E}_k$ in $\mathbb{R}^{2d}$. Accordingly, each $\tilde{f}_j$ can be split into a finite linear combination of Lagrangian states $\tilde{f}_{j,n} \in \mathcal{H}_N$ supported on these individual leaves. If we call $\pi_k^{\sharp\mathbb{R}} \in C_c^\infty(\mathbb{R}^{2d})$ the component of $\pi_k^\sharp$ supported on the representative $\widetilde{E}_k^{\mathbb{R}}$, then the Lagrangian state $\tilde{f}_{j,n}^{\mathbb{R}} = \mathrm{Op}_\hbar(\pi_k^{\sharp\mathbb{R}}) f_{j/N+n}^{\mathbb{R}}$ is microlocalized inside $\widetilde{E}_k^{\mathbb{R}}$. We claim that

$$\tilde{f}_{j,n} = \Pi_N \tilde{f}_{j,n}^{\mathbb{R}} + \mathcal{O}_{\mathcal{H}_N}(\hbar^\infty).$$

We will only give the proof in the case where f_j is a momentum state $\langle e_l, f_j \rangle = N^{-d/2} \mathrm{e}^{2\mathrm{i}\pi\langle l,j\rangle/N}$ associated with the momentum Lagrangian $\{\xi = \xi_j = j/N\}$, and the cutoff is the operator $\mathrm{Op}_N(\chi_1)\,\mathrm{Op}_N(\chi_2)$, where $\chi_1(x)$ (resp. $\chi_2(\xi)$) is obtained by periodizing $\chi_1^{\mathbb{R}} \in C_c^\infty\big((0,1)\big)$ (resp. $\chi_2^{\mathbb{R}} \in C_c^\infty\big((0,1)\big)$). In that case, the Lagrangian state $\tilde{f}_j = \mathrm{Op}_N(\chi_1)\,\mathrm{Op}_N(\chi_2) f_j$ admits the components $\langle e_l, \tilde{f}_j \rangle = N^{-d/2}\,\chi_1(l/N)\chi_2(j/N)\mathrm{e}^{2\mathrm{i}\pi\langle l,j\rangle/N}$. On the other hand, the state $\tilde{f}_j^{\mathbb{R}} = \mathrm{Op}_\hbar(\chi_1^{\mathbb{T}}) \times \mathrm{Op}_\hbar(\chi_2^{\mathbb{R}}) f_j^{\mathbb{R}}$ can be expressed as $\tilde{f}_j^{\mathbb{R}}(x) = \chi_1^{\mathbb{R}}(x)\,\chi_2^{\mathbb{R}}(\xi_j)\mathrm{e}^{\mathrm{i}\langle \xi_j, x\rangle/\hbar}$. The projection on $\mathcal{H}_N$ of that state gives

$$\begin{aligned}
\Pi_N \tilde{f}_j^{\mathbb{T}}(x) &= \chi_2^{\mathbb{R}}(\xi_j) \sum_{\nu,\mu} \chi_1^{\mathbb{R}}(x-\nu) \mathrm{e}^{2\mathrm{i}\pi N\langle \xi_j+\mu, x\rangle} \\
&= \chi_2^{\mathbb{R}}(\xi_j)\chi_1^{\mathbb{R}}([x])\mathrm{e}^{2\mathrm{i}\pi N\langle \xi_j, x\rangle}\delta_{\mathbb{Z}^d}(Nx) \\
&= N^{-d/2}\chi_2^{\mathbb{R}}\left(\frac{j}{N}\right) \sum_{l \in (\mathbb{Z}/N\mathbb{Z})^d} e_l(x)\chi_1\left(\frac{l}{N}\right)\mathrm{e}^{2\mathrm{i}\pi\langle j,l\rangle} = \tilde{f}_j(x).
\end{aligned}$$ □

This lemma allows us to use our control of the evolution of the Lagrangian states $\tilde{f}^{\mathbb{R}} \stackrel{\text{def}}{=} \tilde{f}_{j,n}^{\mathbb{R}}$ through the sequence of operators $U_\hbar(\kappa)$ and $\Pi_k = \mathrm{Op}_\hbar(\widetilde{\pi}_k)$. We have to be a little careful when applying Propositions 4.3 and 4.5, because $\widetilde{\pi}_{\alpha_j}$ are now periodic symbols. Still, the Lagrangian state $\tilde{f}^{\mathbb{R}}$ is localized inside a single copy

$\widetilde{E}^{\mathbb{R}}_{\alpha_0}$ (of diameter$\leq \epsilon$), so its image $U_\hbar(\kappa)\tilde{f}^{\mathbb{R}}$ is a Lagrangian state microlocalized in a set of diameter $\mathcal{O}(\epsilon)$, which can intersect at most one copy $\widetilde{E}^{\mathbb{R}}_{\alpha_1}+n_1$. As a result, the state $\Pi_{\alpha_1}U_\hbar(\kappa)\tilde{f}^{\mathbb{R}}$ is also localized in this single copy, and is of the form of the state ψ_1 in Proposition 4.3. Importantly, the estimates we have on the remainder $r_l(\bullet,\hbar)$ in the expansion (4.6) show that $\mathrm{e}^{\mathrm{i}S_1/\hbar}r_l$ is microlocalized in the single copy $\widetilde{E}_{\alpha_1}+n_1$, and decays fast away from it. As a result, Lemma 5.1 implies that the projection on $\mathcal{H}_N$ of that remainder has a norm comparable with $\|r_l\|_{L^2(\mathbb{R}^d)}$.

Since the initial Lagrangian piece $\tilde{\Lambda}$ lies in some γ_1-unstable cone, we can iterate the evolution as in Proposition 4.5. At each step we get a Lagrangian state and some remainder $\mathrm{e}^{\mathrm{i}S_t(y)/\hbar}r_l^t(y,\hbar)$, which can be projected to $\mathcal{H}_N$ with a control on its norm. When we act on this remainder through operators $\Pi_{\alpha_j}U_n(\kappa)$, its norm can increase at most by a factor $\big(1+\mathcal{O}(\hbar^\infty)\big)$. Finally, we obtain a Lagrangian state microlocalized in some representative $\widetilde{E}^{\mathbb{R}}_{\alpha_n}+n_n$, and a sum of remainders. The projection in $\mathcal{H}_N$ of that state satisfies the same bound as (4.12):

$$\|\Pi_{\boldsymbol{\alpha}}\tilde{f}_j\|_{\mathcal{H}_N}\leq CJ^u(\boldsymbol{\alpha})^{-1/2}.$$

Summing over all the states $\tilde{f}_j$ and taking into account $\|\psi\|_{\mathcal{H}_N}=\sqrt{\sum_{i\in(\mathbb{Z}/N\mathbb{Z})^d}|\psi_i|^2}$ ≤ 1, we obtain (invoking Cauchy−Schwarz)

$$\|\Pi_{\boldsymbol{\alpha}}\Psi\|_{\mathcal{H}_N}\leq\sum_{j\in(\mathbb{Z}/N\mathbb{Z})^d}|\psi_j|CJ^u(\boldsymbol{\alpha})^{-1/2}\leq CN^{d/2}J^u(\boldsymbol{\alpha})^{-1/2}. \qquad \square$$

5.5. EUP and subadditivity. The Hyperbolic dispersive estimate can now be injected in some form of Entropic Uncertainty Principle as in Section 3.8. Since we don't need any energy cutoff, the setting we need is actually simpler than Proposition 3.12: we can take $S_{c_1}=S_{c_2}=\mathrm{Id}_{\mathcal{H}_N}$. In the application, we also take $\varepsilon=\hbar^L$ for any large $L>0$. We obtain a lower bound of the form (3.33) on the pressures associated with the symbolic measures $\mu_N,\tilde{\mu}_N$ and the Ehrenfest time $n=\lfloor 2T_{\epsilon,\hbar}\rfloor$, with a constant $C_{\mathrm{cone}}(\hbar)=C\hbar^{-d/2}$. Using the Egorov theorem up to $T_{\epsilon,\hbar}$, one shows that the quantum pressures satisfy an approximate subadditivity property similar with (3.37), which allows to prove

$$\frac{p_0^{n_0-1}(\mu_{\mathrm{sc}},v,\mathcal{P})}{n_0}\geq-\frac{d\lambda_{\max}}{2}+\mathcal{O}(\epsilon).$$

The rest of the proof is unchanged.

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Institut de Physique Théorique, CEA-Saclay, 91191 Gif-sur-Yvette, France
E-mail address: `snonnenmacher@cea.fr`