ON ATTRACTORS AND ITS BASINS FOR GENERIC DIFFEOMORPHISMS

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ABSTRACT. We show that attractors for $C^1$-generic diffeomorphisms are topologically ergodic. We also show that the Lebesgue measure of the union of the basins of the attractors varies semicontinuously, thus implying continuity in a residual subset.

1. INTRODUCTION

One of the key notions in the theory of dynamical systems is the notion of attractors. Since by definition, it has the asymptotic information of a large set of orbits, called its basin which contains always an open set. It is well known that if the attractor is hyperbolic then asymptotically the behavior of an orbit in its basin is governed by the dynamics of one orbit inside it (a shadowing property).

In particular, to know that essentially every orbit is attracted by one attractor and that the set of attractors is finite (and possibly hyperbolic) implies that the dynamics of the system is nicely described by the attractors. For instance, this leads Palis to conjecture in [15] that “There is a dense set $D$ of dynamics such that any element of $D$ has finitely many attractors whose union of basins of attraction has total probability”.

Actually, many efforts by mathematicians were made to understand this objects: attractors and its basins, not only for finite dimensional dynamics, but also for PDE’s (infinite dimensional dynamical systems). See for instance the works [17] or [10].

In the other hand, to understand properties of all of the set of dynamical systems is a difficult task and it is more reasonable to try to understand a large part of the set of dynamical systems. This reasoning leads to the theory of generic dynamics. Since the $C^r$ topology turns the space of diffeomorphisms a Baire space, it is natural to show that some properties holds for a residual subset of the space of diffeomorphisms, i.e. a countable intersection of open and dense subsets, since this will show the presence of this property for a dense subset and this property could be used to show another property in another residual subset. Indeed, the intersection of two residual subsets is also a residual subset. Usually, we say that a property holds for generic diffeomorphisms if it holds in a residual subset.

Let us give precise definitions involving attractors and its basins and state the main contributions to the study of them that appears in this note for generic diffeomorphisms.

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Definition 1. A compact and $f$-invariant subset $\Lambda$ is an attracting set if there exists an open set $U$, called the local basin of $\Lambda$, such that:

$$f(U) \subset U \text{ and } \Lambda = \bigcap_{n \geq 0} f^n(U).$$

The basin of $\Lambda$ is defined as:

$$B(\Lambda) = \bigcup_{n \geq 0} f^{-n}(U).$$

An attracting set $\Lambda$ is an attractor if it is transitive.

We are interested in the structure of attractors for generic diffeomorphisms. As mentioned before, a residual subset of $\text{Diff}^r(M)$ is a countable intersection of open and dense subsets in the $C^r$-topology for $r \geq 0$ (if $r = 0$ then we are actually dealing with homeomorphisms). By Baire’s theorem, any residual subset of $\text{Diff}^r(M)$ is a dense set. In particular, we will say that a property holds for $C^r$-generic diffeomorphisms, if there exist a residual $R$ of $\text{Diff}^r(M)$ such that the property holds for every $f \in R$.

In [1], Abdenur shows that attractors of $C^1$-generic diffeomorphisms are persistence. In fact, he obtain an inner structure of these attractors, showing that they are homoclinic classes.

Definition 2. Let $p$ be a periodic point then the homoclinic class of $p$ with respect to $f$ is:

$$H(p, f) = \overline{W^s(p) \cap W^u(p)}.$$

Theorem 3 (Abdenur). There exists a residual subset $R$ of $\text{Diff}^1(M)$ such that if $\Lambda$ is an attractor of $f \in R$ then there exist a neighborhood $U$ of $f$ and a neighborhood $V$ of $\Lambda$ such that if $g \in U \cap R$ then $\Lambda(g) = \bigcap_{n \in \mathbb{N}} g^n(V)$ is an attractor. Moreover, $\Lambda(g)$ is an homoclinic class and if $g_k \to f$ is a sequence inside $U \cap R$ then $\Lambda(g_k) \to \Lambda$ in the Hausdorff metric.

In the next section we will define precisely the Hausdorff metric.

It is natural to ask if more information about the inner structure of these attractors can be obtained. In the topological theory of dynamical systems the notion of topologically mixing appears. We say that an invariant compact set $\Lambda$ is topologically mixing with respect to $f$ if for any non-empty open sets $U$ and $V$ there exists $n_0$ such that if $n \geq n_0$ then $f^n(U) \cap V \neq \emptyset$. So, we could ask if these attractors are topologically mixing.

One of the purposes of this note is to show an intermediate property of generic attractors. Given two non-empty open sets $U$ and $V$, we define the set of times that the the orbit of $U$ visit $V$.

$$N(U, V) = \{i \geq 1; f^i(U) \cap V \neq \emptyset \}.$$ 

Observe that transitiveness is equivalent to say that $N(U, V) \neq \emptyset$, and topologically mixing says that there exists an interval $(n_0, \infty)$ of natural numbers contained in $N(U, V)$.

1During the preparation of this article, Abdenur reported to us that he and Crovisier have obtained topological mixing for generic transitive diffeomorphisms, as well as more generally a decomposition into disjoint compacts sets each one of which is topologically mixing (for an iterate of the diffeomorphism) for isolated transitive sets for generic diffeomorphisms, see [2].
Inspired by the analogous notions in the ergodic theory we have the following notion.

**Definition 4.** An invariant and compact subset \( \Lambda \) of \( f \) is topologically ergodic if for every two non-empty open sets \( U, V \subset \Lambda \) we have
\[
\limsup_{n \to \infty} \frac{|N(U, V) \cap \{1, \ldots, n\}|}{n} > 0,
\]
If the whole manifold is topologically ergodic then we say that \( f \) itself is topologically ergodic.

We observe that by Birkhoff’s theorem, if \( f \) preserves an invariant measure which gives positive mass to open sets and it is ergodic then \( f \) is topologically ergodic. This kind of property were studied by some authors in many different context. See, for instance, [19] or [9].

Our first result is that

**Theorem 5.** \( C^1 \)-Generically if \( \Lambda \) is an attractor for \( f \) then \( f|_{\Lambda} \) is topologically ergodic.

As a corollary we obtain the following.

**Corollary 6.** \( C^1 \)-Generically if a diffeomorphism \( f \) is transitive then \( f \) is topologically ergodic.

We recall that the subset of diffeomorphims which preserves the Lebesgue measure induced by the Riemannian metric is also a Baire space when endowed with the \( C^1 \) topology. We denote this subset as \( Diff^1_m(M) \), where \( m \) denotes the Lebesgue measure. The methods can be used also in this context obtaining the following corollary.

**Corollary 7.** \( C^1 \)-Generically in \( Diff^1_m(M) \) any diffeomorphism is topologically ergodic.

We observe that this supports the following conjecture, credited to A. Katok.

**Conjecture 8** (Problem 2 of [3]). The Lebesgue measure of a \( C^1 \)-generic diffeomorphism in \( Diff^1_m(M) \) is ergodic.

Now, we observe that in Abdenur’s result, the local basin of the attractors is preserved. Many authors studied the topological behaviour of the basins through pertubations, in different contexts, see Hurley [12] for attracting sets, or Carballo and Morales [6] for Lyapunov stable sets. However, a natural question is how the basins varies in the measure theoretical sense. This lead us to define a map as follows. If \( f \) is a diffeomorphism then \( m(B(f)) \) denotes the measure of the union of the basins of the attractors of \( f \). This generates a map \( \Phi : Diff^1(M) \to [0, +\infty) \), defined as \( \Phi(f) := m(B(f)) \). The context of our next result deals with the behaviour of this map.

**Theorem 9.** There exists a residual subset \( R \) such that \( \Phi|_R \) is lower semicontinuous.

The definition of semi-continuity will be given in section 2. However, we remark that as a consequence of the theorem, we have a residual subset formed by continuity points of the map \( \Phi \).
This paper is organized as following. In section 2, we present some tools that will be used. In section 3, we prove theorem 5 and its corollaries. In section 4, we prove theorem 9. Finally, in section 5, we give some remarks related to the methods used in the previous sections and give elementary proofs of some folklore results using them.

2. Definitions and Some Tools

In the whole paper, we will denote the $f$-orbit of a point $p$ by $O_f(p)$. If there are no danger of confusion, we will drop the dependence of the diffeomorphism.

**Definition 10.** The Hausdorff metric on $F(M)$ is given by

$$d_H(A, B) = \max(d_A(B), d_B(A)) \quad \forall A, B \in F(M)$$

Where $d_A(B) = \max_{b \in B} \{ \min_{a \in A}(d(a, b)) \}$, and $d$ is the metric induced by the Riemannian structure of $M$.

Thus, we can talk about continuity of maps $\Phi : (X, d) \to F(M)$, where $(X, d)$ is a metric space. Moreover, we have the notion of lower semicontinuity: let $(X, d)$ be a metric space and $\varphi : X \to F(M)$. We say that $\varphi$ is lower semicontinuous at $y \in X$ if $y_n \to y$ implies

$$d_{\varphi(y_n)}(\varphi(y)) \to 0$$

The following result is a well-known result in basic topology and can be found in the book [13].

**Lemma 11.** If $\Gamma : X \to F(M)$ is a lower semicontinuous map and $(X, d)$ is a Baire space, then there exists a residual subset $\mathcal{R}$ such that every point of $\mathcal{R}$ is a continuity point of $\Gamma$.

Now, we also give the notion of lower semicontinuity of a function. If $(X, d)$ is a metric spaces and $\varphi : X \to \mathbb{R}$. We say that $\varphi$ is lower semicontinuous at $x_0 \in X$ if

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

Now, we recall the notion of Lyapunov stability.

**Definition 12.** An invariant and compact subset $A \subset M$ is called Lyapunov stable if given $U$ an open neighborhood of $A$ there exists another neighborhood $V$ of $A$ such that

$$f^n(V) \subset U, \forall n \in \mathbb{N}.$$ 

The following result can be found in [8].

**Lemma 13** (Lemma 3.4 of [8]). If $f$ is a $C^1$-generic diffeomorphism then $W^s(\overline{O(p)})$ is Lyapunov stable for $f$.

We have also the following

**Theorem 14** (Theorem A of [14]). There exists a residual subset $R \subset \text{Diff}^1(M)$ such that if $g \in R$ then the set $S = \{ x \in M; \omega(x) \text{ is Lyapunov stable } \}$ is a residual subset of $M$.

Finally, we recall the connecting lemma of Hayashi [11] that is one of the most useful technique in the $C^1$-generic theory of dynamical systems. The formulation that we give here is taken from the paper [18].
Theorem 15 (Connecting Lemma). Let \( f \in Diff^1(M) \) and \( z \) a non periodic point of \( f \). Given a neighborhood \( U \) of \( f \). There exist \( \rho > 1, L \in N \) and \( \delta_0 > 0 \) such that if \( 0 < \delta < \delta_0 \) and

\[
p, q \notin \Delta(\delta) := \bigcup_{n=1}^{L} (f^{-n}(B(z, \delta))),
\]

if there exist \( a > L \), such that \( f^{a}(p) \in B(z,\delta/2) \), and \( b \geq 0 \), such that \( f^{-b}(q) \in B(z,\delta/p) \). Then there exists \( g \in U \) such that \( g \) is a future \( g \)-iterate of \( p \) and \( g \equiv f \) outside \( \Delta(\delta) \).

3. Proof of theorem 5

First we recall a well known lemma.

Lemma 16. \( C^1 \)-Generically if \( \Lambda \) is an attractor then for every periodic point \( p \in \Lambda \) we have that

\[
\overline{W^u(O(p))} = \Lambda.
\]

Proof. By lemma 13, we have that if \( p \in Per(f) \cap \Lambda \) then \( \overline{W^u(O(p))} \) is Lyapunov stable, since \( \Lambda \) is transitive, then \( \Lambda \subset \overline{W^u(O(p))} \). The other inclusion follows directly since \( \Lambda \) is an attractor.

Now we prove theorem 5.

Proof. By Abdenur’s result, we have that \( \Lambda \) is an homoclinic class. Thus it is transitive and it has dense periodic points. In the following, we will work with the topology relative to \( \Lambda \).

Let \( U, V \subset \Lambda \) be two open sets. By denseness of periodic points, there exist \( p \in Per(f) \cap U \) and let \( L \) be the period of \( p \). Since \( \overline{W^u(O(p))} = \Lambda \) then there exists \( j \in \{0, 1, ..., L-1\} \) such that \( W^u(f^j(p)) \cap V \neq \emptyset \), in fact we may assume that there is a disk \( B \) of dimension \( \dim(E^u(p)) \) in this intersection.

Moreover, there is an open neighborhood \( T \) of \( f^j(p) \) such that \( f^{-j}(T) \subset U \). Let \( k_0 \in N \) such that

\[
A := f^{-k_0 L}(B) \subset W^u_{loc}(f^j(p)) \subset T.
\]

Since \( A \subset W^w_{loc}(f^j(p)) \) then \( f^{-mL}(A) \subset T \) for every \( m \geq 0 \). Then for every \( k \geq k_0 \) we have that \( f^{kL}(f^{-}(k-k_0)L(A)) = B \subset V \), so we conclude that \( f^{kL}(T) \cap V \neq \emptyset \).

Now observe that since we have chosen \( T \) such that \( f^{-j}(T) \subset U \) then we have that \( f^{kL+j}(U) \cap V \neq \emptyset \). Then we have that \( \{j + kL; k \geq k_0\} \subset N(U, V) \).

Since for \( s \geq k_0 \), so

\[
\lim_{s \to \infty} \frac{\#\{j + kL; k \geq k_0\} \cap \{1, ..., j + sL\}}{j + sL} = \lim_{s \to \infty} \frac{s - k_0 + 1}{j + sL} = \frac{1}{L} > 0.
\]

Then we have that

\[
\limsup_{n \to \infty} \frac{\#N(U, V) \cap \{1, ..., n\}}{n} \geq \frac{1}{L} > 0.
\]

Then we conclude that \( f \) is topologically ergodic. \( \square \)

The proof of corollary 6 follows immediately from this proof. The proof of corollary 7, also follows from this result, since we can use the following theorem by Bonatti and Crovisier [4].
Theorem 17 (Bonatti-Crovisier). There exists a residual \( \mathcal{R} \subset \text{Diff}^1_m(M) \) such that any \( f \in \mathcal{R} \) is transitive and \( M \) is an homoclinic class.

Moreover, the analogous statement of lemma 16 applied to \( M \) also holds in the volume preserving case since it relies only on the connecting lemma, which is true in this case, see [18].

4. Proof of Theorem 9

First we observe that, since attractors are isolated, there are at most countable many of them. Let \( \Lambda_1, \ldots, \Lambda_i, \ldots \) be the attractors of a generic \( f \). Denote by \( B(\Lambda_1), \ldots, B(\Lambda_i), \ldots \) its basins.

We select \( \Lambda_1, \ldots, \Lambda_r \) such that

\[
\sum_{i=1}^r m(B(\Lambda_i)) \geq m(B(f)) - \varepsilon.
\]

There exists compact sets \( K_1, \ldots, K_r \) such that \( \Lambda_i \subseteq K_i \subseteq B(\Lambda_i) \) for \( i = 1, \ldots, r \) and such that

\[
m(B(\Lambda_i) - K_i) < \frac{\varepsilon}{r}.
\]

The argument will use the fact that in Abdenur’s result, the generic attractors share the same local basin. Thus, there are local basins \( U_i \) of \( \Lambda_i \) such that these local basins persists in a \( C^1 \)-generic neighborhood of \( f \). Next we also observe that \( B(\Lambda_i) = \bigcup_{n \in \mathbb{N}} f^{-n}(U_i) \), then since \( K_i \subseteq B(\Lambda_i) \) is a compact set, we have that there is a natural number \( n_0 \in \mathbb{N} \) such that

\[
K_i \subseteq \bigcup_{j=0}^{n_0} f^{-j}(U_i).
\]

Since \( \bigcup_{j=0}^{n_0} f^{-j}(U_i) \) is open then for \( g \) \( C^1 \)-close to \( f \) we obtain that

\[
K_i \subseteq \bigcup_{j=0}^{n_0} g^{-j}(U_i).
\]

Thus, if \( g \) is generic and \( C^1 \)-close to \( f \) we have \( m(B(\Lambda_i(g))) \geq m(K_i) \).

Hence,

\[
m(B(g)) \geq \sum_{i=1}^r m(K_i) \geq m(B(f)) - \varepsilon.
\]

Then we proved the lower semicontinuity and the proof is now complete. \( \square \)

5. Remarks

We remark that the method used in the proof of theorem 9 could be used to prove the topological semi-continuity of the basins of generic attractors. However, in the \( C^1 \)-topology a stronger property can be obtained, which, together with the continuity given by the stable manifold theorem, quickly implies this semi-continuity in this topology.

Proposition 18. \( C^1 \)-Generically if a diffeomorphism has an attractor then there exists a periodic point inside the attractor such that the stable manifold of it is dense in the basin of attractor.
Proof. Let \( f \) be a \( C^1 \)-generic diffeomorphism with an attractor \( \Lambda \). Let \( U \) be its local basin, by theorem 3 we know that there exists a periodic point \( p \) such that \( \Lambda = H(p, f) \). Moreover, there exists a neighborhood \( U \) of \( f \) such that there exists a residual subset \( G \subset U \) such that if \( g \in G \) then \( \Lambda(g) = \bigcap_{n \geq 0} g^n(W) \) is an attractor for \( g \).

Using the stable manifold theorem [16], we have:

**Lemma 19.** For every \( N \in \mathbb{N} \), the maps

\[
\Phi_N : U(f) \to F(M),
\]

where

\[
\Phi_N(g) = W^s_N(p(g)),
\]

are continuous. In particular, the map \( \Phi : U(f) \to F(M) \),

\[
\Phi(g) = \bigcup_{N \in \mathbb{N}} W^s_N(p(g)),
\]

is continuous in a residual subset \( U(f) \).

This together with theorem 14 implies the following corollary.

**Corollary 20.** There exists a residual subset \( R_1 \subset U \) such that if \( g \in R_1 \) then there exists a residual subset \( P \subset B(\Lambda(g)) \) such that if \( x \in P \) then \( \omega(x) = \Lambda(g) \).

**Proof.** Let \( R \) and \( S \) given by theorem 14. Denote by \( R_1 := G \cap R \), and define \( P = B(\Lambda) \cap S \). Hence, if \( x \in P \) then \( x \in B(\Lambda) \) and \( \omega(x) \subset \Lambda \). However, since \( x \in P \) we know that \( \omega(x) \) is Lyapunov stable. By the previous remark, since \( \Lambda \) is transitive, we have that \( \Lambda \subset \omega(x) \). Thus \( \omega(x) = \Lambda \).

Now, we study the consequences of the continuity of \( \Phi \).

**Lemma 21.** If \( \Phi \) is continuous in \( g \in R_1 \) and \( S \) is the set given by theorem 14, then \( B(\Lambda(g)) \cap S \subset \Phi(g) \).

**Proof.** We prove the lemma by contradiction. If the Lemma does not hold, then there exists \( x \in B(\Lambda(g)) \cap S - \Phi(g) \). Let \( U \) be a neighborhood of \( \Phi(g) \) such that \( x \notin U \). By continuity there exists a neighborhood \( V \) of \( g \) such that, if \( h \in V \) then \( \Phi(h) \subset U \).

Since \( x \in B(\Lambda(g)) \cap S \) then \( \omega(x) = \Lambda(g) \). Thus, there exists a sequence \( (m_n) \subset \mathbb{N} \) such that \( g^{m_n}(x) \to p(g) \) (where \( p(g) \) denotes the continuation of \( p \)). By Hartman-Grobman’s theorem, there exists another sequence \( (t_n) \subset \mathbb{N} \), such that

\[
g^{t_n}(x) \to q \in W^{s}_{loc,g}(p(g)) - \{p(g)\}.
\]

Let \( p > 1, L \in \mathbb{N} \) and \( \delta_0 > 0 \) given by the \( C^1 \)-Connecting Lemma applied to \( q \) and \( U \). Choose \( 0 < \delta < \delta_0 \) and \( V \) be a neighborhood of the orbit of \( p(g) \) such that

\[
p(g), x \notin \Delta(\delta) = \bigcup_{n=1}^{L} (g^{-n}(B(q, \delta)))
\]

and

\[
\bigcup_{n=1}^{L} (g^{-n}(B(q, \delta))) \cap V = \emptyset.
\]
Pick \( y \in B(q, \frac{\delta}{\rho}) \cap W^u_g(p(g)) \) and let \( z = g^k(y) \) such that \( z \in \left( W^u_g(p(g)) - \{p(g)\} \right) \cap V \). By definition, we have that \( g^{-k}(z) = y \in B(q, \frac{\delta}{\rho}) \). Using that \( g^\omega(x) \to q \) we obtain some \( n_0 > L \) such that

\[
g^{tn_0}(x) \in B(q, \frac{\delta}{\rho}).
\]

Applying the \( C^1 \)-Connecting Lemma, we obtain \( h \in V \) such that \( h \) is outside of \( \Delta(\delta) \) and \( x \) belongs to the \( h \)-negative orbit of \( z \). However, since \( z \in \left( W^u_g(p(g)) - \{p(g)\} \right) \cap V \), we obtain that the \( h \)-positive orbit of \( z \) belongs to \( V \). Thus

\[
z \in W^s_h(p(h))
\]

and

\[
x \in W^u_h(p(h)).
\]

This leads to a contradiction since \( h \in V \) and \( x \notin U \).

To finish the proof, we denote by \( R_2 \) the residual subset given by lemma 19 and defining \( R_3 = R_1 \cap R_2 \) we obtain that \( \Phi \) is continuous in \( R_3 \). Using the previous remark, we obtain that

\[
B(\Lambda(g)) \cap S \subset \Phi(p(g)),
\]

for every \( g \in R_3 \). The theorem follows now globalizing \( R_3 \).

There is another consequence of proposition 18. We say that a diffeomorphism is tame if it has only a finite number of homoclinic classes. In particular, by Abdenur’s result, generic tame diffeomorphisms have only a finite number of attractors. However, the union of its basins covers a large part of the manifold, as proved by Carballo and Morales in [7]

**Theorem 22** (Carballo-Morales). *If \( f \) is a \( C^1 \)-generic tame diffeomorphism then the union of the basins of its attractors is an open and dense subset of \( M \).*

Thus, this implies:

**Corollary 23.** *If \( f \) is a \( C^1 \)-generic tame diffeomorphism then the union of the stable manifolds of the hyperbolic periodic orbits is dense in \( M \).*

We observe that this result was proved in more general setting (partially hyperbolic diffeomorphisms with one-dimensional central bundle) by Bonatti, Gan and Wen in [5]. In particular, they obtain this corollary using stronger methods. However, the short proof given here only uses the connecting lemma. This is a particular case of Bonatti’s Conjecture, see [5]:

**Conjecture 24** (Bonatti’s Conjecture). *There exists a residual subset \( \mathcal{R} \subset Diff^1(M) \) such that, for any \( f \in \mathcal{R} \), the union of the stable manifolds of the hyperbolic periodic orbits is dense in \( M \).*
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