Q-process and genealogy

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Outline

1. Constant size population models
   - Introduction to the lookdown model
   - A constant size population model with non-fixation
2. Branching population models
   - Branching population models process with immigration
   - An inhomogeneous branching population with non-extinction

The Moran model (1958)

Let $\{1, 2, \ldots, n\}$ be the individuals in a finite population of size n.

1. Initially, each individual is either black or blue, and the k black types are randomly distributed among the n individuals.
2. For every *ordered* couple of individuals (i, j), i reproduces over j at the times of independent Poisson processes with unit rate, and i gives its type to j.

We are interested in the process $X^n(t) = \sum_{1 \leqslant i \leqslant n} \mathbf{1}_{\{\text{individual i black at t}\}}$

Question: Does one type fixate? What about the distribution of this fixation time?

Graphical representation

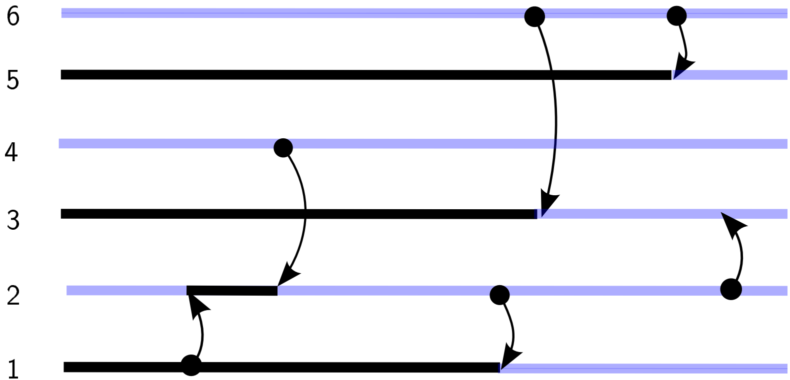

Figure: The blue type fixates the whole population

The lookdown particle system, [Donnelly Kurtz 96]

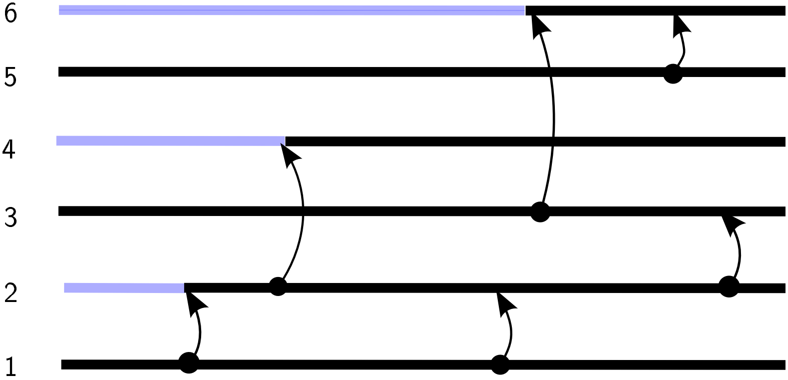

Figure: The black type fixates the whole population

A basic population model

In these two models, the Markov chain

$$X^n(t) = \sum_{1 \leqslant i \leqslant n} \mathbf{1}_{\{\text{individual i black at t}\}}$$

has the same law. From the second model:

$$\lim_{t \to \infty} X^n(t) = n \, \mathbf{1}_{\{\text{individual 1 black at 0 }\}}.$$

Therefore,

$$\lim_{t \to \infty} X^n(t) = \begin{cases} n & \text{with probability } k/n \\ 0 & \text{with probability } (n-k)/n \end{cases}$$

The modified lookdown particle system, [Donnelly Kurtz 99]

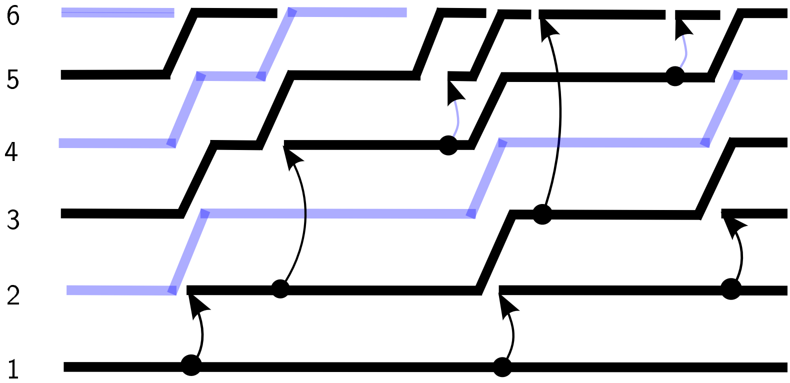

Figure: The black type (will) fixate the whole population

A basic population model

From the third model:

$$\inf\{t \geqslant 0, X^n(t) \in \{0, n\}\} = \sum_{i=L(0)}^{n} \mathbf{e}_i,$$

where:

- $L(0) = \inf\{i \geqslant 1, \{\text{black, blue}\} \subset \text{the types of } \{1, 2, \ldots, i\} \text{ at time } 0\}$ is a random variable with law:

$$\mathbb{P}(L(0) = \ell) = \frac{\binom{n-k}{\ell-1}k + \binom{k}{\ell-1}(n-k)}{\binom{n}{\ell}\ell}.$$

- the $(\mathbf{e}_i, 2 \leqslant i \leqslant n)$ are independent exponential random variables with parameter $2\binom{i}{2} = i(i-1)$.

We now switch to *infinite* constant size population.

A more general process, [Bertoin Le Gall 03]

A pure jump Generalized Fleming Viot (GFV) process $X_t \in [0,1]$ has generator:

$$f(x) \to \quad x \int_{(0,1]} \nu(dy)\, [f(x(1-y)+y) - f(x)] \\ +(1-x) \int_{(0,1]} \nu(dy)\, [f(x(1-y)) - f(x)].$$

At rate $\nu(dy)$, a reproduction event with size y affects the population, currently at state x.

- with probability x, black type reproduces.
- with probability $1-x$, blue type reproduces.
- the past population x is rescaled by a factor $(1-y)$.

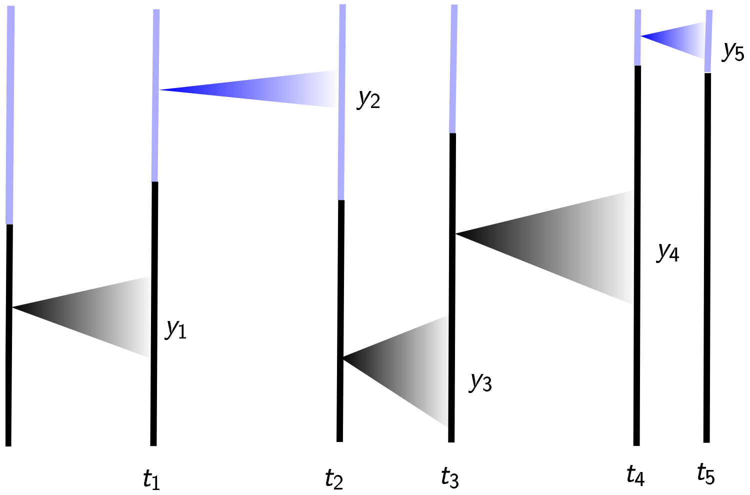

Figure: The dynamic of the GFV process

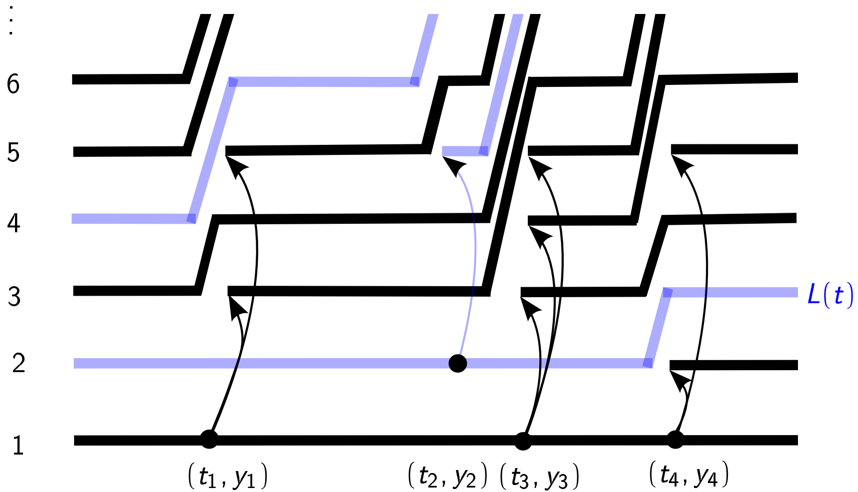

Figure: The associated lookdown process

We define the first level at which the two types are encountered:

$$L(t) = \inf\{i \geqslant 1, \{\text{black, blue}\} \subset \text{the types of } \{1, 2, \dots, i\} \text{ at time t }\},$$

which forms a Markov chain in continuous time, started at ℓ under $\mathbb{P}_\ell$. Notice that

$$\{T > t\} = \{L(t) < \infty\} \text{ a.s.}$$

Under the condition:

$$\mathbb{P}_3(L(t) < \infty)/\mathbb{P}_2(L(t) < \infty) \to 0 \text{ as } t \to \infty, \quad (1)$$

the fixation time $T = \inf\{t > 0, X_t \in \{0, 1\}\}$ is a.s. finite, and:

$$\begin{aligned} \mathbb{P}(L(t) < \infty) &= \sum_{k \geqslant 2} \mathbb{P}(L(s) = k)\mathbb{P}_k(L(t-s) < \infty) \\ &\sim \mathbb{P}(L(s) = 2)\mathbb{P}_2(L(t-s) < \infty) \text{ as } t \to \infty. \end{aligned}$$

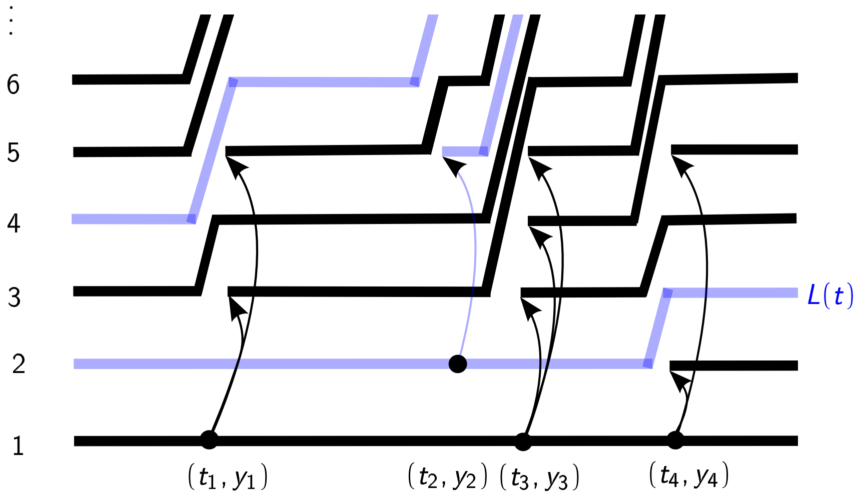

Figure: The lookdown process conditioned on non-fixation

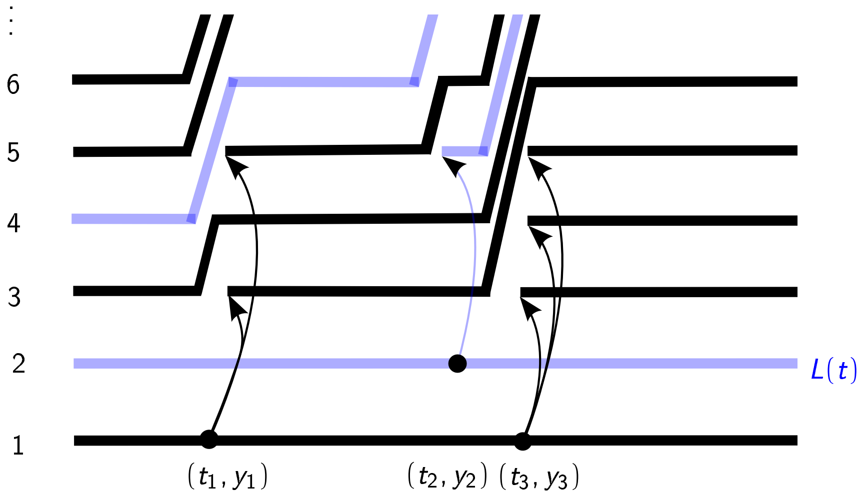

Figure: The lookdown process conditioned on non-fixation

Set $r_2 = \int_{(0,1]} \nu(dy) y^2$. The process

$$\begin{aligned} h(t, X_t) &= X_t (1 - X_t)\, e^{r_2 t} \\ &= \frac{1}{2} \mathbb{P}(L(t) = 2 \mid X_{[0,t]})\, e^{r_2 t} \end{aligned}$$

defines a non-negative martingale. We therefore consistently define a new process by setting:

$$\mathbb{P}(X^h \in A) = \mathbb{E}\left(\frac{h(t, X_t)}{h(0, X_0)}, A\right)$$

for any event $A \in \sigma(X_s, s \leqslant t)$.

Theorem (H. 12)

If the condition (1) holds, then:

$$\mathbb{P}(X \in A | T > t) \to \mathbb{P}(X^h \in A), \text{ as } t \to \infty.$$

The dynamic of the process X^h

Proposition (H. 12)

The generator of X^h is:

$$
\begin{aligned}
f(x) \to & \int_{(0,1]} \nu(dy) y(1-y) \{ f(x(1-y)+y) - f(x) \} \\
& + \int_{(0,1]} \nu(dy) y(1-y) \{ f(x(1-y)) - f(x) \} \\
& + x \int_{(0,1]} \nu(dy)(1-y)^2 \{ f(x(1-y)+y) - f(x) \} \\
& + (1-x) \int_{(0,1]} \nu(dy)(1-y)^2 \{ f(x(1-y)) - f(x) \}
\end{aligned}
$$

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A continuous state branching process, [Jirina58,Lamperti67]

- A (critical) pure jump continuous state branching process (CB) $Y_t \in [0, \infty)$ has generator:

$$f(x) \to x \int_{(0,\infty)} \pi(dy) \, [f(x+y) - f(x) - yf'(x)]$$

- At rate $x\pi(dy)$, a reproduction event with size y affects a population with size x; y is simply added to the size of the (non-constrained) population.
- Due to the branching property, a family of CB processes $(Y_t(x), t \geqslant 0, x \geqslant 0)$ starting from $x \geqslant 0$ may be constructed.

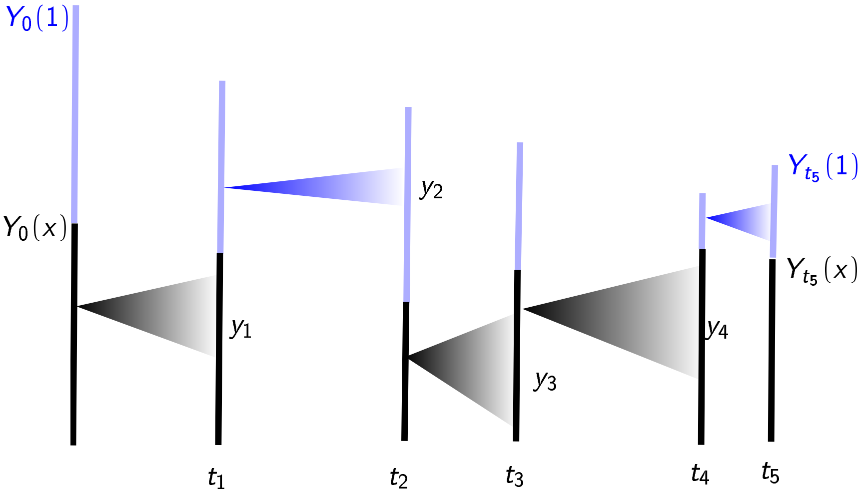

Figure: The pure jump CB process, and its *ratio* process $Y_t(x)/Y_t(1)$

First similarities between the ratio process of a CB and a GFV

- The ratio $Y_t(x)/Y_t(1) \in [0, 1]$ is a pure jump martingale, with absorbing points $\{0, 1\}$.
- At the time of a reproduction event, the father is chosen to be blue or black according to the respective proportions of blue and black types.

A link with GFV processes

How to cook up from the ratio process a GFV process?

1. A reproduction of size y arises with rate $z\pi(dy)$ in the total population with size z.
2. Therefore, for the ratio, a reproduction event of size r arises at rate

$$z\phi_z^{\star}(\pi)(dr) \text{ with } \phi_z(y) = \frac{y}{y+z} \in (0,1).$$

3. There exists $\lambda : (0,\infty) \to (0,\infty)$ and a measure ν on $(0,1)$ such that:

$$\phi_z^{\star}(\pi)(dr) = \lambda(z)\nu(dr),$$

iff π belongs to the family of stable measures, $\pi(dy) = y^{-1-\alpha}dy$ for some $0 < \alpha < 2$, in which case $\nu(dr) = r^{-2}\text{Beta}(2-\alpha,\alpha)(dr)$ and $\lambda(z) = z^{-\alpha}$, see [Birkner & al 05].

CB process with immigration, [Kawazu Watanabe 71]

- A (pure jump) CB process with immigration $Y_t \in [0, \infty]$ -called CBI process- has generator:

$$f(x) \to x \int_{(0,\infty)} \pi(dy) \left[f(x+y) - f(x) - yf'(x) \right] + \int_{(0,\infty)} \pi^0(dy) \left[f(x+y) - f(x) \right]$$

  At constant rate $\pi^0(dy)$, independently of the population size, additional immigration events with size y affects the population.

- Once again, a family $(Y_t(x), t \geqslant 0, x \geqslant 0)$ of such CBI processes can be constructed, $(Y_t(0), t \geqslant 0)$ counts the immigrants.

A GFV process with immigration, [Foucart 11]

- A (pure jump) GFV process with immigration $X_t \in [0,1]$ -called a GFVI process- has generator:

$$\begin{aligned} f(x) \to \quad & x \int_{(0,1]} \nu(dy) \, [f(x(1-y)+y) - f(x)] \\ & +(1-x) \int_{(0,1]} \nu(dy) \, [f(x(1-y)) - f(x)] \\ & + \int_{(0,1]} \nu^0(dy) \, [f(x(1-y)+y) - f(x)]. \end{aligned}$$

- At constant rate $\nu^0(dy)$, independently of x, an immigration event with size y affects the population.

Let Y be a CBI with reproduction and immigration measures:

$$\pi(dy) = y^{-1-\alpha}dy \text{ and } \pi^0(dy) = y^{-\alpha}dy$$

for $1 < \alpha < 2$. We also set:

$$C(t) = \int_0^t ds\ Y_s(1)^{1-\alpha}.$$

Theorem (Foucart, H., 12)

The process:

$$\left(\frac{Y_{C^{-1}(t)}(x)}{Y_{C^{-1}(t)}(1)}, t \geqslant 0 \right)$$

is a GFVI starting at x, with reproduction and immigration measures:

$$\nu(dr) = r^{-2}Beta(2-\alpha, \alpha)(dr) \text{ and } \nu^0(dr) = r^{-1}Beta(2-\alpha, \alpha-1)(dr).$$

Another presentation of branching processes

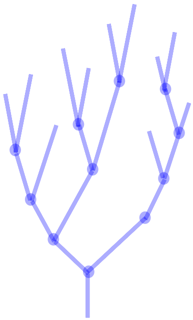

Figure: A (discrete) branching process, with one (blue) type

- Let Y_t be our branching process.

$$\mathbb{E}_x\left(e^{-\lambda Y_t}\right) = e^{-x u_t^\lambda}, \;\; t \geqslant 0,$$

- where u_t satisfies:

$$u_t^\lambda + \int_0^t ds \; \psi(u_{t-s}^\lambda) = \lambda,$$

- with the branching mechanism:

$$\psi(\lambda) = \int_{(0,\infty)} [e^{-\lambda y} - 1 + \lambda y] \pi(dy)$$

Another presentation of branching processes

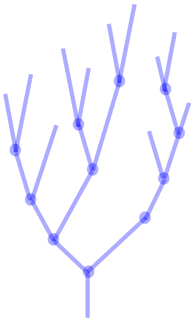

Figure: A (discrete) branching process, with one (blue) type

- Let Y_t be a branching process.

$$\mathbb{E}_x\left(e^{-\lambda Y_t}\right) = e^{-x u_t^\lambda}, \;\; t \geqslant 0,$$

- where u_t satisfies:

$$u_t^\lambda + \int_0^t ds\, \psi(u_{t-s}^\lambda) = \lambda,$$

- with the branching mechanism:

$$\psi(\lambda) = \int_{(0,\infty)} [e^{-\lambda y} - 1 + \lambda y]\pi(dy) + \alpha\lambda + \beta\lambda^2$$

Measure valued branching processes, [Dawson, Dynkin]

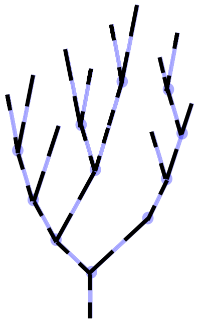

Figure: A (discrete) branching process, with black and blue types

- Let Y_t be an homogeneous measure-valued branching process.

$$\mathbb{E}_{\delta_x}\left(e^{-Y_t(f)}\right) = e^{-u_t^f(x)}, \quad t \geqslant 0,$$

- where u_t satisfies:

$$u_t^f(x) + \mathbb{E}_x\left(\int_0^t ds\, \psi(u_{t-s}^f(Z_s))\right) = \mathbb{E}_x(f(Z_t))$$

- with the branching mechanism:

$$\psi(\lambda) = \int_{(0,\infty)} [e^{-\lambda y} - 1 + \lambda y]\pi(dy) + \alpha\lambda + \beta\lambda^2$$

Measure valued branching processes, [Dawson, Dynkin]

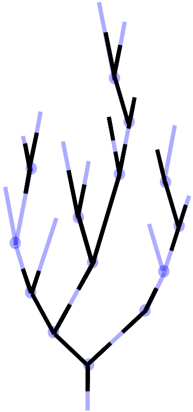

Figure: A (discrete) branching process, with black and blue types

- Let Y_t be an inhomogeneous measure-valued branching process.

$$\mathbb{E}_{\delta_x}\left(e^{-Y_t(f)}\right) = e^{-u_t^f(x)}, \quad t \geqslant 0,$$

- where u_t satisfies:

$$u_t^f(x) + \mathbb{E}_x\left(\int_0^t ds\, \psi(Z_s, u_{t-s}^f(Z_s))\right) = \mathbb{E}_x(f(Z_t))$$

- with the branching mechanism:

$$\psi(z,\lambda) = \alpha(z)\lambda + \beta(z)\lambda^2$$

Williams decomposition under $\mathbb{N}_x$

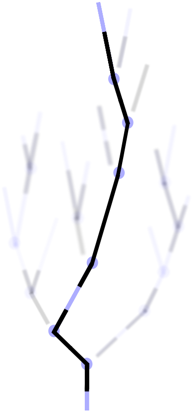

Figure: A branching process decomposed into a trunk and subtrees

- $\mathbb{N}_x$ denotes the canonical measure = "law" of the process started at an infinitesimal individual at x.
- We assume that the height $H_{max} = \inf\{t \geqslant 0, Y_t = 0\} \in [0, \infty]$ is a.e. finite:

$$\mathbb{N}_x(H_{max} = \infty) = 0.$$

- We define $\mathrm{P}_x^{(h)}$ by its Radon-Nikodym derivative w.r.t. P_x on $\mathcal{D}_t$, $0 \leqslant t \leqslant h$:

$$\frac{\partial_h v_{h-t}(Z_t)}{\partial_h v_h(x)} \, e^{-\int_0^t ds \ \partial_\lambda \psi(Z_s, v_{h-s}(Z_s))},$$

with $v_h(x) := \mathbb{N}_x(H_{max} > h)$,

Williams decomposition under $\mathbb{N}_x$

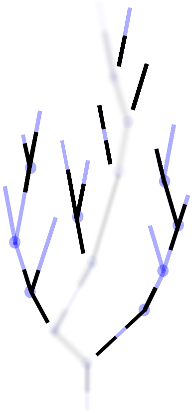

Figure: A branching process decomposed into a trunk and subtrees

- Conditionally on $(Z_s, 0 \leqslant s < h)$ with law $\mathrm{P}_x^{(h)}$, we define a Poisson point measure $\sum_{i \in \mathcal{I}} \delta_{(s_i, Y^i)}(ds, dY)$ with intensity

$$\mathbf{1}_{\{0 \leqslant s < h, H_{max}(Y)+s \leqslant h\}} ds \; 2\beta(Z_s) \mathbb{N}_{Z_s}(dY).$$

- Denote by $\mathbb{N}_x^{(h)}$ the law of $(\sum_{i \in \mathcal{I}} Y^i_{(t-s_i)^+}, 0 \leqslant t < h)$.

Theorem (Delmas, H., 12)

The following desintegration of the canonical measure holds:

$$\mathbb{N}_x = \int_{h>0} dh \, |\partial_h v_h(x)| \, \mathbb{N}_x^{(h)}.$$

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