A uniformization theorem
for closed convex polyhedra
in Euclidean 3-space.

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ABSTRACT. We introduce a notion of discrete-conformal equivalence of closed convex polyhedra in Euclidean 3-space. Using this notion, we prove a uniformization theorem for closed convex polyhedra in Euclidean 3-space.

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INTRODUCTION

In this paper, we introduce an equivalence relation on the class of closed convex polyhedra in the Euclidean 3-space $\mathbb{E}^3$. This equivalence relation has the property that, if $P$ and $Q$ are two convex polyhedra inscribed in the unit sphere, then $P$ is equivalent to $Q$ if and only if there exists a Möbius transformation on the sphere that maps the vertex set of $P$ to the vertex set of $Q$. This property suggests this equivalence relation as a concept of discrete conformality.

Inspired by Riemann’s mapping theorem and the more general uniformization theorem of Poincaré and Koebe, we prove a uniformization theorem for closed convex polyhedra in Euclidean 3-space in the following sense.

Theorem: (Uniformization of polyhedra) Every closed convex polyhedron in $\mathbb{E}^3$ is discrete-conformally equivalent to a closed convex polyhedron inscribed in the unit sphere. This polyhedron is unique up to Möbius transformations on the sphere.

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In a special case, we further characterize the equivalence relation by simple transformations on the vertices of the polyhedra. More specifically, if two polyhedra \( P \) and \( Q \) share a common Delaunay triangulation \( T \) (to be defined below), then \( P \) and \( Q \) are conformally equivalent if and only if there exists a real valued function \( u_T \) on the vertices of \( P \) such that, for every edge \( ij \) in the Delaunay triangulation between vertices \( i \) and \( j \), its length in \( Q \) is related to its length in \( P \) by

\[
l_Q(ij) = l_P(ij) e^{\frac{1}{2}(u_T(i)+u_T(j))}.
\]

We conjecture more generally that \( P \) and \( Q \) are discrete-conformally equivalent if and only if there exists a finite sequence of closed convex polyhedra \( P = P_1, P_2, \ldots, P_{n-1}, P_n = Q \) such that, for \( k = 1, \ldots, n-1 \) the polyhedra \( P_k \) and \( P_{k+1} \) share a common Delaunay triangulation \( T_k \) and there exists a real valued function \( u_{T_k} \) on the vertices of \( P_k \) with the following property. For every edge \( ij \) in the Delaunay triangulation between vertices \( i \) and \( j \), its length in \( P_{k+1} \) is related to its length in \( P_k \) by

\[
l_{P_{k+1}}(ij) = l_{P_k}(ij) e^{\frac{1}{2}(u_{T_k}(i)+u_{T_k}(j))}.
\]

This work arose out of a general interest in understanding the relationship between different concepts of discrete conformality that have been developed in the last decades.

The concept of discrete conformality by a vertex scaling as above, first appeared in a paper by Luo in 2004 [1]. Luo introduces a discrete scalar curvature on polyhedral surfaces and describes a discrete analog of Yamabe flow in this setting. A polyhedral surface is a pair \((S, T)\) of a surface \(S\) and a triangulation \(T\) of \(S\), together with a positive real valued function \(\rho\) on the set of edges of \(T\) such that the edge lengths of any triangle in \(T\) define an isometric Euclidean triangle. The function \(\rho\) is called a polyhedral metric on \((S, T)\).

Given a polyhedral metric \(\rho\) on \((S, T)\), let \(u\) be a real valued function defined on the vertex set of \((S, T)\), Luo defines a discrete-conformal change of \(\rho\) by the vertex scaling

\[
u * \rho(vv') = \rho(vv') e^{\frac{1}{2}(u(v)+u(v'))}
\]

on edges of \(T\). If \(u*\rho\) defines a polyhedral metric, we say that \(\rho\) and \(u*\rho\) are discrete-conformally equivalent.

The first hint that the concept of conformality could make sense also in a discrete setting appeared in the theory of circle packings in the 1980’s. A circle packing is a connected collection of circles in the plane whose interiors are disjoint. A classical result in this area is Koebe’s circle packing theorem [2].

**Theorem: (Koebe)** For every connected simple planar graph \(G\) there is a circle packing in the plane whose intersection graph is \(G\).

The intersection graph of a circle packing is the graph having a vertex for each circle, and an edge for every pair of circles that are tangent. Let \(S\) be an oriented surface, i.e. a connected topological 2-manifold, with
a metric. Given a collection \( C = \{c_v\} \) of circles in \( S \) and a simplicial 2-complex \( K \) triangulating an oriented surface, the pair \((C, K)\) is said to be a \textit{circle packing for a simplicial 2-complex} \( K \), denoted \( C_K \), if

1. for each vertex \( v \) in \( K \) there exists exactly one circle \( c_v \) in \( C \) and vice versa,
2. if \( \langle u, v \rangle \) is an edge of \( K \), then the two circles \( c_u \) and \( c_v \) form a tangent pair and
3. if \( \langle u, v, w \rangle \) forms a positively oriented face of \( K \), then the three circles \( c_u, c_v \) and \( c_w \) form a positively oriented tangent triple in \( S \).

We say that an abstract simplicial 2-complex \( K \) is a combinatorial closed disc if it triangulates a topological closed disc; that is, \( K \) is finite, simply connected, and has nonempty boundary.

For circle packings for a combinatorial closed disk \( K \), Thurston observed the following rigidity property.

**Theorem:** (Thurston) Let \( K \) be a combinatorial closed disc. Then there exists a univalent circle packing \( C_K \), i.e. the interior of the circles are disjoint, in the unit disk such that every boundary circle is a horocycle. This circle packing is unique up to Möbius transformations of the disk.

An \( \epsilon \) hexagonal circle packing of the plane is a circle packing with all circles of radius \( \epsilon \) associated to a tiling of the plane into equilateral triangles of edge length \( 2\epsilon \) and all vertices of degree 6. Let \( C_\epsilon \) be a portion of an \( \epsilon \) hexagonal circle packing of the plane cut out by a domain in \( \mathbb{C} \). The above theorem gives a “map” \( f_\epsilon \) that associates with \( C_\epsilon \) a “maximal” circle packing in the unit disk that is unique up to Möbius transformations.

Thurston’s observation proved very fruitful. In a 1985 lecture on the occasion of the proof of the Bieberbach conjecture, William Thurston conjectured that the above “map” converges to the Riemann map when \( \epsilon \) goes to zero [3]. In 1987 Rodin and Sullivan proved this conjecture, interpreting the map \( f_\epsilon \) as a map between the centers of circles [4].

Thurston’s observation led to an increased interest in circle packings in the late 80’s. Significant contributions and generalizations of the concept of circle packings were in particular made by his student, Oded Schramm.

Schramm is certainly best known for his invention of the Schramm-Loewner Evolution (SLE), and his subsequent collaboration with Lawler and Werner. Schramm’s early work however was around Koebe’s theorem and Thurston’s discrete version of the Riemann mapping theorem. The transition from circle packing to SLE was through a sequence of papers concerning probability on graphs, many of them written jointly together with Benjamini. An interesting short survey on Schramm’s work was written by Rohde [5].

One of Schramm’s last works concerned analogs of Koebe’s theorem in higher dimensions. Koebe’s Theorem says that a graph can be circle packed in the plane \( \mathbb{R}^2 \) if and only if it is planar. Which graphs can be sphere packed in \( \mathbb{R}^d, d > 2 \)? Using tools from non-linear potential theory, Schramm and Benjamini prove that the lattice grid \( \mathbb{Z}^{d+1} \) or the 3-regular tree \( \times \mathbb{Z} \) cannot be quasi-sphere packed in \( \mathbb{R}^d \), for all \( d \) [6]. In first approximation, a \textit{quasi-sphere packing} is a packing of domains, the
ratio between the inner radius and the outer radius of each domain, is uniformly bounded over the elements of the packing.

Inspired from Benjamini and Schramm, Pansu proposed a concept of large scale conformal maps between metric spaces [7]. Roughly speaking, a map between metric spaces is large scale conformal if it maps every packing by sufficiently large balls to a collection of large quasi-balls which can be split into the union of boundedly many packings. This notion allows to transfer some techniques and results from conformal geometry to discrete spaces like finitely generated groups.

Another theory on conformal changes of metrics inspired from circle packings was developed by Lee for graphs [8]. Consider a locally finite, connected graph $G$. A conformal metric on $G$ is a map $r : V(G) \to \mathbb{R}_{>0}$. This metric endows $G$ with a graph distance as follows: Given an edge $uv$ in $E(G)$, let

$$\text{len}_r(uv) := \frac{1}{2}(r(u) + r(v)).$$

This induces a length

$$\text{len}_r(\gamma) := \sum_{k>0} \text{len}_r(v_kv_{k+1})$$

for every path $\gamma = \{v_1, v_2, \ldots\}$ in $G$. Given any pair of vertices $u$ and $v$ in $G$, the distance $\text{dist}_r(u, v)$ between $u$ and $v$ is defined as the infimum of the length of all paths between $u$ and $v$ in $G$. This endows $G$ with a path metric $\text{dist}_r$.

One realizes that in the concept of discrete conformality coming from circle packings, the metric is changed by modifying the radii of circles. Hence, the metric is represented in form of an addition of two radii. In the other concept of discrete conformality coming from Yamabe flow, the metric is changed multiplicatively.

Closely related to circle packings is the concept of circle patterns. Let $G$ be an immersed connected planar graph in the plane, and let $w : E(G) \to (0, \pi)$ be a weight on the edges $E(G)$ such that for all edges incident to a face $f$ of $G$ we have

$$\sum_{\text{e incident to } f} w(e) = 2\pi.$$

An immersed planar circle pattern in the plane with adjacency graph $G$ and intersection angles $w$ is a collection of circles for each vertex, such that the following conditions hold.

1. For each edge $uv$ in $E(G)$, the two circles associated to $u, v$ in $V(G)$ intersect with exterior intersection angle $w(uv)$.
2. The circles corresponding to the vertices adjacent to the same face of $G$ intersect in a single point.
3. Consider a counterclockwise cyclic order of the intersection points from (2) on the circle corresponding to an interior vertex $v$ of $G$. This order agrees with the counterclockwise cyclic order of the cycle of faces of $G$ adjacent to $v$. 
Notice that for each circle packing for a simplicial 2-complex $K$ there is an associated orthogonal circle pattern. Simply add a circle for each triangular face which passes through the three touching points.

Circle patterns on the sphere are closely related to ideal polyhedra.

**Theorem:** Let $\Sigma$ be a cellular decomposition of the sphere. Let $w : \Sigma^{(1)} \to (0,\pi)$ be a weight on the edges of $\Sigma$ such that for all edges incident to a face $f$ of $\Sigma$ we have
\[
\sum_{e \text{ incident to } f} w(e) = 2\pi,
\]
and for every simple circuit $e_1, \ldots, e_k$ of edges in $\Sigma$ that does not bound a single face of $\Sigma$ we have
\[
\sum_i w(e_i) > 2\pi.
\]
Then there exists an immersed planar circle pattern $C_\Sigma$ in the sphere with adjacency graph $\Sigma^{(1)}$ and intersection angles $w$. This circle pattern is unique up to Möbius transformations on the sphere.

We may interpret the sphere as the ideal boundary of the hyperbolic space $\mathbb{H}^3$ in the Poincaré model. If we carve out all hyperbolic half-planes defined by the circles in $C_\Sigma$ on the ideal boundary of $\mathbb{H}^3$, we obtain an ideal convex polyhedron $P_{C_\Sigma}$ in $\mathbb{H}^3$ with the dihedral angle at an edge $e$ of $P_{C_\Sigma}$ given by $w(e)$.

In that sense, the above theorem is nothing but a reformulation of Igor Rivin’s celebrated theorem on the characterization of ideal polyhedra in hyperbolic 3-space [9]. This theorem can be stated as follows.

Suppose that a convex ideal polyhedron $P$ in $\mathbb{H}^3$ is given. Let $P^*$ denote the dual polyhedron of $P$, i.e. the abstract polyhedron that has a vertex for every face of $P$ and two vertices in $P^*$ are connected by an edge $e^*$ if and only if the corresponding faces in $P$ are share an edge $e$. Assign to each edge $e^*$ of $P^*$ a weight $w(e^*)$ equal to the exterior dihedral angle at the corresponding edge $e$ in $P$. Then the following result holds:

**Theorem:** (Rivin) The dual polyhedron $P^*$ of a convex ideal polyhedron $P$ in $\mathbb{H}^3$ satisfies the following conditions:
\begin{enumerate}
\item $0 < w(e^*) < \pi$ for all edges $e^*$ of $P^*$.
\item If the edges $e_1^*, e_2^*, \ldots, e_k^*$ form the boundary of a face of $P^*$, then $w(e_1^*) + w(e_2^*) + \cdots + w(e_k^*) = 2\pi$.
\item If $e_1^*, e_2^*, \ldots, e_k^*$ form a simple circuit which does not bound a face of $P^*$, then $w(e_1^*), w(e_2^*), \ldots, w(e_k^*) > 2\pi$.
\end{enumerate}

Conversely, any abstract polyhedron $P^*$ with weighted edges satisfying the conditions 1 - 3 is the dual polyhedron of a convex polyhedron $P$ with the exterior dihedral angles equal to the weights.

A convex polyhedron with prescribed dihedral angles is also determined uniquely up to ambient isometry, this is shown by Rivin using a variational principle in [10].
Rivin’s characterization of ideal convex polyhedra in hyperbolic 3-space is a generalization of Andreev’s theorem [11]. In fact, Thurston was led to study circle packings during his program on the geometrization conjecture. Andreev’s theorem is an essential ingredient in his proof of the hyperbolization theorem [12] [13] [14].

The full geometrization conjecture of Thurston was proven by Perelman in 2003 using Ricci flow with surgery [15] [16] [17]. The analog of Ricci flow for scalar curvature is the Yamabe flow, whose discretization is the setting in which the other concept of discrete conformality by vertex scaling first appeared.

A hint that the concept of discrete conformality by vertex scaling and the concept of discrete conformality associated to circle packings are related, appears in a paper by Bobenko, Pinkall and Springborn [18]. In their paper they address the following question: Given a polyhedral surface \((S, T, \rho)\) with \(N\) vertices and a set of complete angles \((\theta_1, \ldots, \theta_N)\) (i.e. the sum of angles around vertices), satisfying some necessary conditions, does there exist a conformal factor \(u\) such that \(u \ast \rho\) is a polyhedral metric and has complete angle \(\theta_i\) at each vertex? Bobenko, Pinkall and Springborn give a partial answer using a variational principle. Their functional is closely related to a family of functionals developed within the theory of circle packings and circle patterns. To this family belongs for example the functional of Rivin introduced in his paper on “Euclidean structures on simplicial surfaces and hyperbolic volume” [10] and the functional of Colin de Verdi`ere that gives an existence and uniqueness proof of circle packings [19].

This observation raises the question if both concepts of discrete conformality are just two sides of the same story. The uniformization theory of convex polyhedra in Euclidean 3-space aims to shed some light on this.

Let \(P_{\text{ideal}}\) be the space of ideal convex polyhedra in \(\mathbb{H}^3\), this space is equivalent to the space of convex Euclidean polyhedra inscribed in the unit sphere if \(\mathbb{H}^3\) is the Klein model of the hyperbolic 3-space. Hence, let us interpret \(P_{\text{ideal}}\) as the space of convex Euclidean polyhedra inscribed in the unit sphere. Let \(C\) be the set of circle patterns covering the unit sphere. To every circle pattern \(C\) in \(C\) corresponds a unique convex Euclidean polyhedron \(P_C\) in \(P_{\text{ideal}}\) by cutting off all half-planes defined by the circles in \(C\). Conversely, every convex polyhedron inscribed in the unit sphere corresponds to a unique circle pattern covering the unit sphere.

As a consequence of the results presented in this paper, we obtain the following theorem.

**Theorem:** Let \(C_1\) and \(C_2\) be two circle patterns in \(C\) and let \(P_{C_1}\) and \(P_{C_2}\) be the corresponding convex polyhedra inscribed in the unit sphere. There exists a M"obius transformation \(f\) on the sphere mapping the circle pattern \(C_1\) onto the circle pattern \(C_2\) if and only if \(P_{C_1}\) and \(P_{C_2}\) share a common Delaunay triangulation \(T\), and there exists a function \(u_T\) defined on the vertex set of \(P_{C_1}\) with the following property. For every edge \(ij\) in the Delaunay triangulation between vertices \(i\) and \(j\), its length in \(P_{C_2}\) is related to its length in \(P_{C_1}\) by

\[
l_{P_{C_2}}(ij) = l_{P_{C_1}}(ij) e^{\frac{1}{2} (u_T(i) + u_T(j))}.
\]
Moreover, \( f \) and \( u \tau \) are related by \( u \tau = \log |df|_V \), where \( V \) is the vertex set of \( P_{C_1} \).

This theorem not only relates the above mentioned concepts of discrete conformality to each other, but also suggests the introduction of a third variant of discrete conformality, namely \textit{discrete Möbius geometry}, which we discuss in the last chapter of this work.

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**ALEXANDROV’S THEORY ON CLOSED CONVEX POLYHEDRA**

We will consider closed convex polyhedra in Euclidean 3-space \( \mathbb{E}^3 \) and hyperbolic 3-space \( \mathbb{H}^3 \). A \textit{closed convex polyhedron} in \( \mathbb{E}^3 \) or \( \mathbb{H}^3 \) is the convex hull of a finite set of points in \( \mathbb{E}^3 \) or \( \mathbb{H}^3 \). This definition includes doubly-covered closed convex polygons. By a \textit{closed polygon} we mean any domain in \( \mathbb{E}^2 \) or \( \mathbb{H}^2 \) that is bounded by finitely many geodesic line segments.

The boundary of a closed convex polyhedron is composed of finitely many closed convex polygons in the respective 2-dimensional space. In the following, we will not explicitly stipulate that the polyhedron under consideration is closed and convex.

The polygons bounding a polyhedron are the \textit{faces} of the polyhedron. The sides and vertices of the faces of a polyhedron are the \textit{edges} and \textit{vertices} of the polyhedron.

In the same manner one could define the \textit{vertices} of a polyhedron \( P \) as the minimal number of points, whose convex hull agrees with \( P \).

A \textit{convex polyhedron with vertices at infinity} in \( \mathbb{H}^3 \) is the convex hull of a finite set of points, some of them lying on the ideal boundary of \( \mathbb{H}^3 \). A convex polyhedron with all vertices on the ideal boundary is called an \textit{ideal convex polyhedron}.

**Polyhedral surface.** A \textit{polyhedral surface} is a surface \( S \) together with a flat cone metric \( \rho \) on \( S \) that has finitely many cone points. A \textit{cone point} is a point \( v \) in \( S \) that admits a circle centered at \( v \) with circumference less than \( 2\pi r \), where \( r \) is its radius. A \textit{marking} \( \mu \) of a surface \( S \) is a labelling of a finite set of points in \( S \). We denote a marked surface by \( S_\mu \).

Given two points \( x \) and \( y \) on the boundary of a Euclidean or hyperbolic polyhedron \( P \), there exists a polygonal path from \( x \) to \( y \) on the boundary.

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of $P$. The infimum of the lengths of polygonal paths from $x$ to $y$ defines a distance $\rho_P(x, y)$ between them. To distinguish whether we are dealing with Euclidean or hyperbolic geometry, we denote the hyperbolic distance by $\hat{\rho}_P(x, y)$. This construction associates with every Euclidean polyhedron $P$ a marked Euclidean polyhedral surface $(S_\mu, \rho_P)$ homeomorphic to the sphere. The marking $\mu$ is inherited from the vertices of $P$. Analogously, with every hyperbolic polyhedron $P$ this construction associates a marked hyperbolic polyhedral surface $(S_\mu, \hat{\rho}_P)$ homeomorphic to the sphere.

An ideal polyhedral surface is a complete hyperbolic surface of finite area, homeomorphic to the $N$ times punctured sphere. We denote a surface homeomorphic to the $N$ times punctured sphere by $S_N$. Analogously, every ideal polyhedron $P$ gives rise to a marked ideal polyhedral surface $(S_\mu, \hat{\rho}_P)$.

The complete angle at a point $x$ in a polyhedral surface $(S, \rho)$ or $(S, \hat{\rho})$ is the number

$$\lim_{\epsilon \to 0} \frac{C_\epsilon(x)}{\epsilon},$$

where $C_\epsilon$ is the circumference of a circle of radius $\epsilon$ at $x$. The notion of a complete angle is an intrinsic property of the polyhedral surface.

Let $\theta$ be the complete angle at a point $x$, the difference $2\pi - \theta$ is the curvature at $x$. A polyhedral surface that has a positive curvature at every point is said to be a polyhedral surface of positive curvature.

A polyhedral surface arising as the boundary of a convex polyhedron has a positive curvature everywhere. Conversely, does every polyhedral surface of positive curvature arise from a convex polyhedron in $E^3$ or $H^3$?

An affirmative answer was given by Alexandrov in the 1940’s. In fact, Alexandrov showed that every polyhedral surface of positive curvature defines a unique polyhedron in $E^3$ or $H^3$ up to congruence [20].

Around 2000 Igor Rivin showed that every ideal polyhedral surface defines a unique ideal polyhedron up to congruence [21].

We will give a complete exposition of Alexandrov’s and Rivin’s beautiful theory below.

**Development.** A development is a finite collection of closed polygons in $E^2$ or $H^2$ together with a set of rules for “gluing” them together along their edges. The rule for gluing satisfies the following conditions:

1. The correspondence of “gluing” two segments is an isometry.
2. It is possible to pass from each polygon to any other polygon by traversing polygons with glued sides.
3. Each side of every polygon is glued to exactly one side of another polygon.

The sides and vertices of the polygons within a development are the edges and vertices of the development, where identified sides and vertices are considered the same. We denote a development by $R$. 

Every development defines an underlying marked polyhedral surface, which we denote by \((S_\mu, \rho_R)\). The marking \(\mu\) labels the points on the surface \(S\) corresponding to the vertices of \(R\). In other words, a development is a polyhedral surface plus a subdivision into geodesic polygons.

Several developments can define the same polyhedral surface. One may think of a development as a “coordinate representation” of a polyhedral surface. Different cuttings of a polyhedral surface into polygons correspond to different coordinate representations of the same polyhedral surface.

Two developments \(R\) and \(R'\) can be obtained from each other by cutting and gluing if the polygons in \(R\) can be cut into polygons and glued along edges such that we obtain the development \(R'\). One observes:

**Theorem:** Two developments \(R\) and \(R'\) are related by cutting and gluing if and only if \((S_\mu, \rho_R)\) and \((S_\mu, \rho_R')\) are isometric by an isometry homotopic to the identity.

We will use the above ideas to turn the space of closed polyhedral surfaces into a manifold by “cutting” polyhedral surfaces into triangles. Those representations will turn out to be convenient coordinate charts for our space.

Every convex polyhedron is naturally associated with a development. The face development of a polyhedron \(P\) is the development \(R_P\) whose polygons are the faces of the polyhedron \(P\).

**A rigidity property of convex polyhedra.** It is a fundamental result of rigidity theory that convex polyhedra in \(\mathbb{E}^3\) or \(\mathbb{H}^3\) with congruent corresponding faces must be congruent to each other. This result is attributed to Augustin Cauchy who published this result in 1813 [22]. His proof is widely considered as one of the most elegant arguments of geometry.

Cauchy’s argument is based on the following combinatorial observation.

**Lemma:** (Cauchy) Suppose that some edges of a closed convex polyhedron are labeled by either a plus or minus signs. Sign changes may occur between labeled edges around a vertex. It is impossible to have at least four sign changes at every vertex.

A slightly stronger statement is this purely topological observation.

**Lemma:** Suppose a “net of edges” is given on a surface homeomorphic to the sphere, i.e., suppose that finitely many edges (each of which is homeomorphic to a straight line segment) are given, and these edges are pairwise disjoint, except possibly at their endpoints, the “vertices of the net”. Assume further that none of the regions separated by the net of the sphere is bounded by only two edges. Assign pluses and minuses to the edges of the net. Let \(V\) be the total number of vertices and \(N\) the total number of sign changes at all vertices. Then

\[ N \leq 4V - 8. \]
By a net we mean an arbitrary finite collection of simple (i.e., not self-intersecting) polygonal lines on a polyhedral surface with no common points except possibly endpoints. Each polygonal line is called an edge of the net and endpoints of edges are the vertices of the net. A net may divide the polyhedral surface into several regions, i.e., points that can be joined with one another by a polygonal line not intersecting the net. A net may also consist of several disjoint parts. A part that cannot be further decomposed is called a connected component of the net.

In the proof of the stronger Cauchy lemma we make use of the following generalized Euler Theorem. A detailed proof can be found in Alexandrovs book on page 61 [20].

**Theorem: (The Generalized Euler Theorem)** Given a net on the boundary of a closed convex polyhedron, let $v$ be the number of vertices, $e$ the number of edges, $c$ the number of connected components, and $f$ the number of regions into which the net divides the polyhedron. Then

$$v - e + f = c + 1.$$ 

In particular, if the net is connected, then $v - e + f = 2$.

**Proof of the stronger Cauchy lemma:** $N$ is the total number of sign changes as one moves around the vertices of the net. We observe that counting the number of sign changes as one moves around each of the regions separated on the surface by the edges of the net gives the same number $N$.

Indeed, orient the surface and start going around the regions of the surface in the direction prescribed by this orientation. Suppose we are going along the contour of a region with vertices $A, B, C, D,$ and $E$. Assume we have passed the edge $AB$ in the order $A$ to $B$. Now, we are standing at the vertex $B$ and are about to pass the edge $BC$ in the order $B$ to $C$. The edge $BC$ follows the edge $AB$ as we go around the region in the same order as in the order given by moving around $B$. Thus, the sign change from $AB$ to $BC$ is counted in both cases. We proceed moving along the contour of the region until we arrive at the edge from where we started. If two edges $AB$ and $BC$ separate two distinct regions, then in one region they are passed from $AB$ to $BC$ and in the other from $BC$ to $AB$, which is the same as moving around the vertex $B$. Nets may have edges that do not separate two distinct regions. The region may have a free endpoint in its interior or an edge that bounds the same region on both sides. Every such edge will be passed twice when going along the region: first in one direction, then, in the opposite direction. If $D$ is a free endpoint of the edge $DE$, then when going along the contour, after passing the edge $DE$ in the direction $E$ to $D$, we again pass the edge $DE$ in the opposite direction. This is the same as moving around the vertex $D$.

When counting the edges of a region, we count edges of this region twice if they do no separate two distinct regions from another. Notice that the number of sign changes when going along the contour of a region cannot be bigger than the number $n$ of its edges. Furthermore, this number of
sign changes is always even, since when completing a full cycle, we return to the initial sign.

Let $F_n$ be the total number of regions with $n$ edges. From the above follows that the total number $N$ of sign changes has the upper bound

$$N \leq 2F_3 + 4F_4 + 4F_5 + \ldots .$$

Let $V$, $E$, and $F$ be the number of vertices, edges and regions in a net. By the generalized Euler formula

$$V - E + F \geq 2.$$ 

This is equivalent to

$$4V - 8 \geq 4E - 4F.$$ 

Since each edge either belong to two regions or is counted twice, we have

$$2E = \sum_n nF_n.$$ 

The total number $F$ of regions is

$$F = \sum_n F_n.$$ 

Substituting both equations in the inequality above gives

$$4V - 8 \geq \sum_n 2(n - 2)F_n = 2F_3 + 4F_4 + 6F_5 + \ldots .$$

The right-hand-side is clearly bigger than the total number $N$ of sign changes. Hence,

$$4V - 8 \geq N.$$ 

To use Cauchy’s lemma to prove the rigidity result of convex polyhedra in $\mathbb{E}^3$ or $\mathbb{H}^3$, requires some preparation.

The link of a vertex of a polyhedron in $\mathbb{E}^3$ or $\mathbb{H}^3$ is the spherical polygon obtained by intersecting an infinitesimal sphere centered at the vertex with the polyhedron, and rescaling so that the sphere has radius 1. The edge lengths in the link are precisely the face angles at the vertex. The link of a convex polyhedron is a convex polygon.

For a vertex at the boundary at infinity, there exists a one-parameter family of horospheres centered at that vertex. A sufficiently small horosphere intersects with the polyhedron in a Euclidean polygon, this polygon we call the link of the vertex at the boundary at infinity. Notice that the link in this case is only defined up to similarity. We may normalize the edge length of the link by setting the length of the longest side equal to 1.

The following characterization of convex spherical polygons is shown in detail by Alexandrov in his book on convex polyhedra, pp. 155 – 157 [20].

**Theorem:** If none of the angles of a spherical polygon exceeds $\pi$, then the polygon is convex.

**Lemma:** Let $P$ and $Q$ be two convex polygons with the same number of vertices $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$. Suppose that the following conditions are satisfied:
1. all their corresponding sides except \( p_0p_1 \) and \( q_0q_1 \) are of equal length, i.e.,

\[ |p_1p_2| = |q_1q_2|, \ldots, |p_{n-1}p_n| = |q_{n-1}q_n|. \]

2. the angles between these sides in \( P \) do not exceed those in \( Q \), i.e.,

\[ \angle p_2 \leq \angle q_2, \ldots, \angle p_{n-1} \leq \angle q_{n-1}, \]

and the strict inequality holds at least once.

Then the “exceptional” side of the first polygon is less than the “exceptional” side of the second, i.e.,

\[ |p_n p_1| < |q_n q_1|. \]

**Proof:** The proof goes by induction on the number of vertices of the polygons.

For triangles the lemma reduces to the well known result that if two triangles \( p_1p_2p_3 \) and \( q_1q_2q_3 \) have two sides of equal lengths, let’s say \( |p_1p_2| = |q_1q_2| \) and \( |p_2p_3| = |q_2q_3| \), and the angles between these two sides differ, then the third side is shorter in that triangle where the angle is smaller.

Suppose the theorem is true for \((n-1)\)-gons. We prove that it is then also true for \( n \)-gons. Given two \( n \)-gons \( P \) and \( Q \) satisfying the conditions of the theorem, we distinguish two cases. Either (1) all angles at \( p_2, \ldots, p_{n-1} \) are strictly smaller than the angles at \( q_2, \ldots, q_{n-1} \) or (2) some of those angles are equal.

Consider the second case. Let \( p_k \) and \( q_k \) be corresponding vertices at which the angles are equal. Cut off the triangles \( p_{k-1}p_kp_{k+1} \) and \( q_{k-1}q_kq_{k+1} \) from \( P \) and \( Q \) respectively. The triangles \( p_{k-1}p_kp_{k+1} \) and \( q_{k-1}q_kq_{k+1} \) are congruent since two sides and one angle are equal. Hence, the remaining \((n-1)\)-gons clearly satisfy the assumptions of the theorem and therefore

\[ |p_1p_{n-1}| < |q_1q_{n-1}|. \]

Now, assume that all angles at \( p_2, p_3, \ldots, p_{n-1} \) of the polygon \( P \) are strictly smaller than the corresponding angles of the polygon \( Q \). Let \( p_k \) be a point of the polygon \( P \) not lying on the prolongation of the side \( p_1p_n \). Construct the triangle \( T = p_kp_1p_n \). This way, the polygon \( P \) is cut into three parts, two polygons \( P_1 \) and \( P_2 \) and the triangle \( T \).

Transform the triangle \( T \) continuously such that the angle at \( p_k \) increases while the lengths of the segments \( p_kp_1 \) and \( p_kp_n \) remain the same. By the elementary result for triangles, we have that \( |p_1p_n| < |p_1'p_n'| \), where \( p'_1 \) and \( p'_n \) are the points corresponding to \( p_1 \) and \( p_n \) after the transformation.

Transform \( T \) in such a way that the angle at \( p'_k \) agrees with the angle at \( q_k \).

If the transformed polygon \( P' \) is convex, then \( P' \) satisfies the assumptions of the second case above. Hence,

\[ |p'_1p'_n| < |q_1q_n|. \]
In particular,
\[ |p_1 p_n| < |q_1 q_n|, \]
which was to be shown.

However, the polygon \( P' \) may indeed fail to be convex. Interestingly, Cauchy missed to see this in his proof in 1813. The correction of this part of the proof is attributed to Ernst Steinitz (1871 – 1928) and Isaac Jacob Schönbürg (1903 – 1990).

Notice that increasing the angle at \( p_k \) may indeed also increase the angle at \( p_1 \) and \( p_n \). If an angle at \( p_1 \) or \( p_n \) exceeds \( \pi \) before \( p_k \) reaches the size of the angle at \( q_k \), then \( P' \) is not convex. In this case we may stop the transformation process as soon as the angle at \( p_1 \) or \( p_n \) reaches \( \pi \).

Assume that the angle at \( p_1 \) reaches \( \pi \) before the angle at \( p_n \) and before the angle at \( p_k \) reaches the size of the angle at \( q_k \). Then the transformed polygon \( P'' \) is convex and the segments \( p''_1p'_2 \) and \( p''_1p'_n \) form a single segment in \( P'' \). We can construct the triangle \( p''_2p'_k p'_n \) and begin to increase the length of the segment \( p''_2p'_n \) such that the angle at \( p'_k \) increases. The vertex \( p''_2 \) will be the point on the transformed segment \( p''_2p'_n \) such that \( |p''_2p'_2| = |p_1p_2| \).

As previously, during the transformation, the angle at \( p'_k \) increases while the angles at \( p'_2 \) and \( p'_n \) increase or decrease. There are three different possibilities of what may happen.

1. The angle at \( p'_k \) can be increased to the angle at \( q_k \) without violating the convexity and such that the angle at \( p'_2 \) stays smaller than the angle at \( q_2 \).
2. The angle at \( p'_2 \) increases and reaches the size of the angle at \( q_2 \) before \( p'_k \) reaches the size of the angle at \( q_k \).
3. The angle at \( p'_n \) reaches \( \pi \).

In the first two cases, there exists an angle in the convex polygon \( P'' \) that agrees with a corresponding angle in \( Q \). Hence, we can conclude the proof as we did previously.

In the third case, we again stop the transformation as soon as the angle at \( p'_n \) reaches \( \pi \). In this case, the segments \( p''_2p''_1 \), \( p''_1p''_n \) and \( p''_n p''_{n-1} \) form a single segment and we can construct the triangle \( p''_2p''_k p''_{n-1} \). When transforming this triangle such that the angle at \( p''_k \) increases, no violation of convexity can occur, since one of the angles at \( p''_2 \), \( p''_k \) and \( p''_{n-1} \) will reach the size of the corresponding angle in \( Q \). Finally we may use the arguments above to conclude that
\[ |p_1 p_n| < |q_1 q_n|. \]

**Lemma:** If two combinatorially equivalent links have corresponding sides of equal length but not all the corresponding angles are equal, then there are at least four sign changes in the differences between the corresponding angles as we go around the links.

**Proof:** Let \( \delta_i = \angle p_i - \angle q_i \); if not all \( \delta_i \) vanish, then there must exist a sign change. Indeed, suppose \( \angle p_2 \leq \angle q_2, \ldots, \angle p_{n-1} \leq \angle q_{n-1} \) with at least
one strict inequality, then by the theorem above $|p_1p_n| < |q_1q_n|$, which contradicts the assumption.

Suppose there exists a sign change, then the number of sign changes when travelling around the polygons is even. Hence, it suffices to show that there cannot be exactly two sign changes. Assume we have exactly two sign changes; number the vertices such that the angles at $p_1, \ldots, p_m$ are greater than the angles at $q_1, \ldots, q_m$, and the angles at $p_{m+1}, \ldots, p_n$ are less than the angles at $q_{m+1}, \ldots, q_n$. Bisect the sides $p_1p_n, p_mp_{m+1}, q_1q_n$ and $q_mq_{m+1}$ by adding points $i, j, k$ and $l$ respectively and polygonal lines $ij$ and $kl$ between them. This gives us two pairs of polygons both satisfying the assumptions of the theorem above. Applying the lemma to the first pair $p_1p_2 \ldots p_mp_{m+1}$ and $q_1q_2 \ldots q_mkl$ we conclude that $|ij| < |kl|$. Applying the lemma to the second pair, we conclude that $|kl| < |ij|$, leading to the desired contradiction.

The previous lemma can be reformulated in a more convenient form.

**Lemma:** If two convex polyhedral angles, distinct from dihedral angles and possibly degenerate, have corresponding planar angles of equal measure while not all of their dihedral angles are equal, then there are at least four sign changes in the differences between the corresponding dihedral angles as we move around the vertices.

**Proof:** If the two polyhedral angles are non-degenerate, then the associated links satisfy the assumptions of the previous lemma. Hence, there are at least four sign changes in the differences between corresponding dihedral angles as we move around the vertices.

Assume that at least one of the polyhedral angles is degenerate. Let $V_P$ be a degenerate polyhedral angle and denote the other by $V_Q$. Let $p_1$ and $p_2$ be the two edges of $V_P$ at which the dihedral angles are equal to zero. If in $V_Q$ the dihedral angle at the edge $q_1$ corresponding to $p_1$ in $V_P$ is zero, then so is the dihedral angle at $q_2$. Hence, the polyhedral angles $V_P$ and $V_Q$ are congruent.

Assume that the dihedral angle at $q_1$ is different from zero. Then also the dihedral angle at $q_2$ is different from zero. In particular, the dihedral angles at $q_1$ and $q_2$ are bigger than the dihedral angles at $p_1$ and $p_2$. The sum of the planar angles between $q_1$ and $q_2$ is the same from both sides. Hence, there exists an edge between $q_1$ and $q_2$ on both sides with dihedral angle smaller than $\pi$. However, the corresponding edges in $V_P$ have dihedral angle equal to $\pi$ since $V_P$ is degenerate. Thus, there are at least four sign changes in the difference between the corresponding dihedral angles as we move around the vertices.

Let us return to Cauchy’s rigidity theorem. As previously, we include among convex polyhedra doubly-covered convex polygons. For the application of this theorem, it will turn out to be convenient to admit the partition of faces of the polyhedron into finitely many smaller polygons, each of which is counted as a “face”. We may distinguish two sorts of edges and vertices: “genuine” and “fictitious”. A fictitious edge is an edge at which the dihedral angle equals $\pi$. A fictitious vertex is a vertex
At which the polyhedral angle is in fact a dihedral angle, in particular a plane when the vertex lies inside a genuine face of the polyhedron.

We say that two convex polyhedra have the same structure if there exists a one-to-one correspondence between the faces, edges and vertices that preserves the incidence relation.

**Theorem: (Cauchy)** If two closed convex polyhedra in $\mathbb{R}^3$ or $\mathbb{H}^3$ have the same structure and corresponding planar angles on corresponding faces are equal, then the dihedral angles at the corresponding edges are also equal.

**Proof:** Let $P$ and $Q$ be two polyhedra satisfying the conditions of the theorem. Assign either a plus or minus sign to the edges of $P$ at which the dihedral angle is larger or smaller than the corresponding dihedral angle of $Q$. If the corresponding dihedral angles agree, no sign is assigned to the edge.

Given two corresponding vertices $p$ and $q$, then either both are genuine or both are fictitious, since the curvatures at $p$ and $q$ are equal.

If $p$ and $q$ are genuine vertices, then by the preceding lemma either all dihedral angles are equal, or there are at least four sign changes as we move around $p$.

If $p$ and $q$ are fictitious vertices, we may split the discussion in two cases.

1. There are genuine edges incident to both $p$ and $q$, but to a genuine edge incident to $p$ corresponds a genuine edge incident to $q$.

2. For at least one of the vertices there are either no genuine incident edges, or all genuine incident edges correspond to each other.

Consider the first case. Recall that a polyhedral angle at a fictitious vertex is in fact a dihedral angle. Hence, if both $p$ and $q$ admit a genuine edge, they both admit precisely two genuine edges which are the prolongation of each other. Let $e_1$ and $e_3$ be the genuine edges incident to $p$ and $l_2$ and $l_4$ be the genuine edges incident to $q$, numbered in such a way that the ordered list $e_1, e_2, e_3, e_4$ corresponds to traversing $p$ in anti-clockwise direction. By assumption, the edges $e_2, e_4$ and $l_1, e_3, l_3$ corresponding to $l_2, l_4$ and $e_1, e_3$, respectively, are fictitious. A genuine edge has dihedral angle less than $\pi$ and a fictitious edge has dihedral angle equal to $\pi$. Therefore, the signs of the difference between the dihedral angles at $e_1, e_2, e_3$ and $e_4$ are $-, +, -, +$, which yields precisely four sign changes.

The second case may be further split up into

1. neither $p$ nor $q$ have incident genuine edges;

2. only one of the vertices has genuine edges;

3. both $p$ and $q$ have genuine edges which correspond to each other.

If all edges incident to $p$ and $q$ are fictitious, all dihedral angles are equal to $\pi$. Hence, there does not occur any labeling around $p$ and the vertices $p$ and $q$ and all edges incident to these vertices can be removed.

Assume that $p$ has a genuine edge $e_1$. Since $p$ is in fact dihedral, $e_1$ admits a prolongation $e_2$ and $p$ has precisely two genuine edges. Let $l_1$ and $l_2$ be the corresponding edges incident to $q$. If $q$ has any genuine edge, then $l_1$ and $l_2$ are precisely its genuine edges. All other edges incident to $p$ and $q$
are fictitious and can be removed. The edges $l_1$ and $l_2$ are prolongations of each other, since the planar angles between them agree with the planar angles between the edges $e_1$ and $e_2$ which are $\pi$. The dihedral angles at $l_1$ and $l_2$ are equal. Therefore, the labels attached to $e_1$ and $e_2$ agree. We may remove the vertices $p$ and $q$ by assigning the edge $e_1 \cup e_2$ to the edge $l_1 \cup l_2$.

We conclude that all fictitious vertices can be removed. Hence, if there exist sign assignments to the edges of the polyhedron $P$ as above, there would exist at least four sign changes at every vertex contained in some labeled edge. This is impossible by Cauchy’s lemma. Hence there are no labeled edges at all, which means that the corresponding dihedral angles of $P$ and $Q$ are equal.

Cauchy’s theorem can be used straightforwardly to obtain an analog rigidity statement for convex polyhedra in hyperbolic three-space with some or all vertices at the boundary at infinity. This result is due to Hodgson and Rivin [23]. Henceforth, we distinguish two types of vertices, vertices inside the hyperbolic three-space and vertices on the ideal boundary.

**Theorem:** (Hodgson, Rivin) A convex polyhedron in $\mathbb{H}^3$ with some or all vertices at the boundary at infinity is determined up to congruence by the type of its vertices and the edge length of the link of its vertices.

**Proof:** Let $P$ and $P'$ be two convex polyhedra in $\mathbb{H}^3$ with some or all vertices at the boundary at infinity, whose corresponding links have equal edge lengths. Apply Cauchy’s lemma to each pair of links. We conclude that all angles in corresponding links are equal. Hence, $P$ and $P'$ have corresponding equal dihedral angles.

This implies that $P$ and $P'$ have congruent ends corresponding to their vertices at the ideal boundary. If we truncate the ends of $P$ and $P'$ along suitably chosen corresponding planes, we obtain two compact convex polyhedra $Q$ and $Q'$ with equal face angles. By Cauchy’s theorem, $Q$ and $Q'$ are congruent. Hence, $P$ and $P'$ are also congruent since they are obtained from $Q$ and $Q'$ by attaching congruent ends.

Cauchy’s Theorem may be formulated in a stronger form as follows [20].

**Theorem:** (Aleksandrov) Every isometry $\varphi$ from the boundary of a closed convex polyhedron $P$ in $\mathbb{R}^3$ or $\mathbb{H}^3$ onto the boundary of another closed convex polyhedron $Q$ can be realized as a motion or a motion and a reflection, i.e. there is a motion, or a motion followed by a reflection, which takes each point of the boundary of $P$ to its image under the mapping $\varphi$.

This stronger form of Cauchy’s Theorem resulted from the work of Aleksandrov and was published in the 1940’s.

An analogous statement can be given for convex polyhedra in $\mathbb{H}^3$ with some or all vertices at the boundary at infinity [23].

**Theorem:** (Hodgson, Rivin) Every isometry $\varphi$ from the boundary of a convex polyhedron $P$ in $\mathbb{H}^3$ with some or all vertices at the boundary at infinity onto the boundary of another convex polyhedron $Q$ can be realized as a motion or a motion and a reflection, i.e. there is a motion, or a
motion followed by a reflection, which takes each point of the boundary of $P$ to its image under the mapping $\varphi$.

In the following, we return to the usual usage of the notion of a face of a polyhedron as its genuine face without further subdivision.

Let $\varphi$ be an isometry from the face development $R_P$ of a closed convex polyhedron $P$ onto the face development $R_Q$ of a closed convex polyhedron $Q$. The image of the 1-skeleton of $R_P$ in $R_Q$ induces a partition of the polygons in $R_Q$. Analogously, the pre-image of the 1-skeleton of $R_Q$ in $R_P$ induces a partition of the polygons in $R_P$.

**Lemma:** The partition induced by $\varphi$ cuts $R_P$ and $R_Q$ into collections of convex polygons. $\varphi$ induces a correspondence between those polygons that preserves the incidence relation such that corresponding polygons are congruent.

**Proof:** Let $P_1$ be a polygon in $R_P$ and pick a polygon $Q_1$ in $R_Q$ that is partly covered by $\varphi(P_1)$. We like to show that $Q_1 \cap \varphi(P_1)$ is a convex polygon.

Let $q_1$ and $q_2$ be two points in $Q_1 \cap \varphi(P_1)$ and let $p_1$ and $p_2$ be their pre-image in $P_1$ via $\varphi$, respectively. Since $P_1$ is convex, the segment $p_1p_2$ lies in $P_1$; hence, its image $\varphi(p_1p_2)$ lies in $\varphi(P_1)$. Since $\varphi$ is an isometry and $q_1$ and $q_2$ lie on the same polygon $Q_1$, $\varphi(p_1p_2)$ agrees with the segment $q_1q_2$. Thus, $q_1q_2$ lies in $Q_1 \cap \varphi(P_1)$ and $Q_1 \cap \varphi(P_1)$ is convex.

The image of edges of $R_P$ are shortest arcs in $R_Q$. Shortest arcs in $R_Q$ are polygonal lines with at most one segment in each polygon in $R_Q$. Hence, $Q_1 \cap \varphi(P_1)$ is cut out from $Q_1$ by finitely many straight line segments and therefore forms a bounded polygon.

The isometry $\varphi$ is cellular with respect to the partition of $R_P$ and $R_Q$ induced by $\varphi$. Indeed, every element of the partition of $R_P$ will be of the form $P_1 \cap \varphi^{-1}(Q_1)$ for some polygons $P_1$ and $Q_1$ in $R_P$ and $R_Q$ respectively. This polygon is mapped to the element $\varphi(P_1) \cap Q_1$ of the partition of $R_Q$.

**Proof of Alexandrov’s stronger Cauchy Theorem:** Given an isometry $\varphi$ from the boundary of a closed convex polyhedron $P$ onto the boundary of a closed convex polyhedron $Q$, $\varphi$ is cellular with respect to its induced partition of $R_P$ and $R_Q$ in polygons as described above. The corresponding polygons are congruent and in particular have equal angles. We can apply Cauchy’s Lemma as discussed above to conclude that all polyhedral angles along their edges must be equal.

Let $P_1$ be a polygon in the partition of $R_P$ induced by $\varphi$ and let $Q_1$ be the corresponding polygon in the partition of $R_Q$. Since $P_1$ and $Q_1$ are congruent, there exists a motion $\psi$ that assigns to every point of the boundary of $P$ corresponding to $P_1$ its image under $\varphi$ in $Q$. If the image of $P$ under this motion and $Q$ lie on opposite sides of the plane determined by $Q_1$, reflect $P$ along this plane.

Since all dihedral angles along edges are equal, the image under $\psi$ of an adjacent polygon $P_2$ of $P_1$ in the partition of $R_P$ lies in the plane determined by the corresponding polygon $Q_2$ in the partition of $R_Q$. Since
ψ(P_1) and Q_1 agree on all edges, ψ(P_2) and Q_2 clearly agree on the edge shared with Q_1. ψ(P_2) also lies on the same side of this edge as Q_2, hence ψ(P_2) and Q_2 agree.

Since the partition of R_P is connected, ϕ can be realized by a motion or a motion and a reflection.

**Isometric embedding of polyhedral surfaces.** We now return to the previously stated question: Does every polyhedral surface of positive curvature in E^3 or H^3 arise as the boundary of a convex polyhedron? Alexandrov gave the following answer [20]:

**Theorem: (Alexandrov)** Let (S,ρ) be a polyhedral surface with N cone points of strictly positive curvature, homeomorphic to the sphere. Then (S,ρ) can be realized as the boundary of a closed convex polyhedron P_N with N vertices. This polyhedron is unique up to congruence.

We suggested to think of developments of polyhedral surfaces as coordinate representations of the polyhedral surface. In this section we will make this more precise by cutting polyhedral surfaces into triangles. From this we will establish that the space of polyhedra with N vertices up to congruence and the space of polyhedral surfaces of positive curvature with N cone points are homeomorphic.

Let T PL(N) be the space of polyhedral metrics on the sphere with N marked points. The set of marked points is required to contain all points on the polyhedral surface of non-zero curvature. The polyhedral metrics ρ in T PL(N) are considered up to isometry homotopic to the identity. A triangulation of a marked sphere S_µ is a triangulation T of the sphere with vertices the N marked points on the sphere. A geodesic triangulation of a marked polyhedral surface T PL(N) is a triangulation of S_µ whose edges are minimizing geodesics.

Let T be a triangulation of the marked sphere S_µ with N marked points and E(T) the set of edges associated to T. Let E(T)^+ be the set of positive real valued functions on E(T), satisfying

$$x(e_i) + x(e_j) > x(e_k)$$

for every triangle ijk ∈ T with edges e_i, e_j and e_k. For a function x ∈ E(T)^+ let ρ_x be a polyhedral metric on the sphere constructed by isometrically gluing Euclidean triangles ijk ∈ T of edge lengths x(e_i), x(e_j), x(e_k) along corresponding edges. This provides an injective map

$$\iota : E(T)^+ \to T_{PL}(N)$$

mapping x to ρ_x.

Let U_T be the image of \iota in T PL(N) and θ_T the inverse of \iota on U_T. The pair (U_T, θ_T) is a length coordinate chart on T PL(N).

**Theorem:** The set of length coordinate charts

$$\{(U_T, \theta_T) \mid T \text{ triangulation of } S_\mu\}$$
forms a real-analytic atlas on $T_{PL}(N)$.

**Proof:** Every polyhedral metric on the sphere with $N$ points of non-zero curvature admits a geometric triangulation, hence $\bigcup_T U_T$ covers $T_{PL}(N)$. If two triangulations $T$ and $T'$ are related by a diagonal flip along an edge $e$, then the transition function $\vartheta_T \vartheta_T^{-1}$ is the identity in all components except for the component associated to $e$. Let $ABC$ and $ADC$ be the triangles in $T$ adjacent to $e$. The length of the diagonally switched edge is

$$\sqrt{x_{AB}^2 + x_{DC}^2 - 2x_{AB}x_{DC}\cos(\arccos(x_{AC}x_{AB} - x_{BC}^2/4x_{AC}x_{AB}) + \arccos(x_{AC}x_{DC} - x_{AD}^2/4x_{AC}x_{DC})))}.$$ 

Hence, the transition function $\vartheta_T \vartheta_T^{-1}$ is real analytic.

Given two arbitrary triangulations of $S^\mu$, there exists a sequence of triangulations of $S^\mu$, each related to the next by a single diagonal flip along an edge. Hence, all transition functions are real analytic.

The set of closed simply-connected marked polyhedral surfaces with $N$ marked points of strictly positive curvature $T_{PL}^{con}(N)$ forms a subset of $T_{PL}(N)$. For some coordinate chart $\vartheta_T$ of $T_{PL}(N)$ let $\sum_j \alpha_{ij}$ be the complete angle at $i$, where $j$ enumerates the angles at the $i$-th vertex. The complete angle is a continuous function of the edge length. The set $T_{PL}^{con}(N)$ is determined by the inequalities

$$\sum_j \alpha_{ij} < 2\pi \quad (i = 1, \ldots, N).$$

Hence, $T_{PL}^{con}(N)$ is an open submanifold of $T_{PL}(N)$.

**Lemma:** If $N > 3$, then the manifold $T_{PL}^{con}(N)$ is a proper subset of $T_{PL}(N)$ and consequently has a topological boundary in $T_{PL}(N)$.

**Proof:** We first prove that in any triangulation of an element in $T_{PL}(N)$ with $N > 3$, there are at least three triangles touching at a single vertex. In fact, we will show that, if at most two triangles touch at each vertex, then $N$ is at most three.

Let $T$ be a geodesic triangulation of $(S^\mu, \rho)$, such that at most two triangles touch at each vertex. Let $ABC$ be a triangle in $T$. Suppose that some triangle $ABD$ is glued to the side $AB$ of the triangle $ABC$. There are two triangles touching at the vertices $A$ and $B$. If there are no other triangles touching at $A$ and $B$, then the sides $AC$ and $BC$ must be glued to $AD$ and $BC$ respectively. If $AC$ were glued to $BC$, then $A$ and $B$ would be identified, which is impossible by our convention of triangulation. Hence, $(S^\mu, \rho)$ is a doubly covered triangle.

Let $T$ be an abstract triangulation of the sphere with more than three vertices. Construct a polyhedral surface $(S^\mu, \rho)$ by assigning length 1 to all edges of $T$. By the argument above, there exists a vertex $A$ of $T$ where at least three triangles touch. Construct a sequence of polyhedral surfaces $\{(S^\mu, \rho_t)\}_{t \in [1, 2]}$ by taking $\rho_t(e) = 1/t$ for every edge $e$ adjacent to $A$ and constant equal to 1 otherwise. Notice that the angle at $A$ in every triangle increases to $\pi$ as $t$ approaches 2. Since there are at least three triangles touching at $A$, there exists a $t' \in [1, 2]$ such that the sum of the angles at
A is greater than or equal to $2\pi$. Such a development is not contained in \( T_{PL}^{con} \), which was to be proven.

**Theorem:** (Alexandrov) Let \((S, \rho)\) be a polyhedral surface with \(N\) cone points of strictly positive curvature, homeomorphic to the sphere. Then \((S, \rho)\) can be realized as the boundary of a closed convex polyhedron \(P_N\) with \(N\) vertices. This polyhedron is unique up to congruence.

**Outline of the proof.** Let \(\mathcal{P}^N\) be the space of closed convex polyhedra with \(N\) vertices in \(\mathbb{R}^3\), parametrized by the positions of their vertices. Three vertices are sent by an isometry to the origin, the positive x-axis and the half-plane \(y > 0\) of the \(xy\)-plane, respectively. If the polyhedron does not degenerate into a doubly-covered polygon, then a fourth point not contained in the \(xy\)-plane is mapped into the half-space \(z > 0\) by reflecting along the \(z = 0\) plane if needed. This eliminates the action of the isometry group of \(\mathbb{R}^3\). There are \(3N\) variable coordinates, however three vertices are constant in three, two and one coordinates respectively. Therefore, we have \(3N - 6\) variable coordinates and \(\mathcal{P}^N\) is a \(3N - 6\) dimensional manifold.

The boundary of every closed convex polyhedron with \(N\) vertices can be viewed as a polyhedral surface with \(N\) cone points of strictly positive curvature homeomorphic to the sphere. Formally this gives a map \(g : \mathcal{P}^N \to T_{PL}^{con}(N)\). In the following we show that \(g\) is a (1) continuous, (2) injective and (3) closed map and (4) that every connected component of \(T_{PL}^{con}(N)\) admits a preimage in \(\mathcal{P}^N\).

\(\mathcal{P}^N\) and \(T_{PL}^{con}(N)\) are manifolds of equal dimension, by (1) and (2) and the invariance of domain principle of Brouwer, \(g\) is an open map. Since \(g\) is also closed, we conclude together with (4) that \(g\) is a homeomorphism from \(\mathcal{P}^N\) onto \(T_{PL}^{con}(N)\).

**Remark:** The fact that a polyhedron in \(\mathbb{R}^3\) is determined by the geometry of its surface, is particular to polyhedra in three dimensional space. A polygon is not at all determined by the length of its edges. Also, in higher dimensions such a correspondence does not hold in general. The dependends of the theory on the dimension reveals itself in the usage Brouwer’s invariance of domain principle. That the dimension of the space of closed convex polyhedra with \(N\) vertices has the same dimension as the space of polyhedral surfaces with \(N\) cone points of strictly positive curvature, is particular to \(\mathbb{R}^3\).

A *triangulation* of a polyhedron \(P\) with \(N\) vertices is a geometric triangulation of its polyhedral surface \((S_\mu, \rho_P)\).

**Theorem:** The map \(g\) is continuous.

**Proof:** Let \(T\) be a triangulation of a closed convex polyhedron \(P\). There exists an \(\epsilon > 0\) such that, if the vertices of some polyhedron \(Q\) are at distances less than \(\epsilon\) from the vertices in \(T\) (with exactly one vertex of \(Q\) corresponding to a vertex of \(T\)), then there is a unique triangulation of \(Q\) close to \(T\) that has the same structure. The continuity of \(g\) follows immediately.

**Theorem:** The map \(g\) is injective.
Proof: The injectivity follows from the above proven theorem that, if there exists an isometry \( \varphi \) between the boundary of two closed convex polyhedra \( P \) and \( Q \), i.e. their associated marked polyhedral surfaces \( (S_\mu, \rho_P) \) and \( (S_\nu, \rho_Q) \) are isometric, then there exists a motion or a motion and a reflection from \( P \) to \( Q \) realizing \( \varphi \). That is to say, \( P \) and \( Q \) are equal in \( \mathcal{P}^N \) since we eliminated the action of the isometry group of \( \mathbb{R}^3 \).

Theorem: The image of \( g \) is closed.

Proof: Let \( \rho_1, \ldots, \rho_i, \ldots \) be a sequence of metrics in \( \mathcal{T}^{\mathbb{R}^3}_\infty(N) \), converging to a metric \( \rho \). For each \( i \) let \( P_i \) be a closed convex polyhedron in \( \mathcal{P}^N \) such that \( g(P_i) = \rho_i \). There exists a closed convex polyhedron \( P_\infty \) in \( \mathcal{P}^N \) such that \( g(P_\infty) = \rho \).

Indeed, since the sequence of polyhedral surfaces associated with \( \{P_i\}_{i \geq 1} \) converges, the distances between vertices of polyhedra in \( \{P_i\}_{i \geq 1} \) are uniformly bounded. Hence, the polyhedra are contained in some ball centered at the origin.

Pick a subsequence \( \{\rho_{i_k}\}_{k \geq 1} \) such that all \( \rho_{i_k} \) admit a triangulation \( T_{i_k} \) of the same combinatorics. The sequence of triangulations \( \{T_{i_k}\}_{k \geq 1} \) converges to a unique triangulation \( T \) of \( \rho \). Enumerate the vertices of the triangulations such that the sequence of vertices associated with \( i \) converges to the \( i \)-th vertex in \( T \). This enumeration gives \( N \) sequences in \( \mathbb{R}^3 \) associated with the vertices of the polyhedra. Pick a subsequence \( \{P_{i_k}\}_{k \geq 1} \) such that those sequences converge. Let \( P_\infty \) be the boundary of the convex hull of the \( N \) limit points.

For notational simplicity we replace \( \{P_i\}_{i \geq 1} \) with the sequence \( \{P_{i_k}\}_{i \geq 1} \). The enumeration of vertices in \( \{T_i\}_{i \geq 1} \) induces an enumeration of edges in \( \{T_i\}_{i \geq 1} \). Given \( T_i \), the map \( g \) induces a triangulation \( T_i' \) of \( P_i \). We say that \( T_i \) is plotted on \( P_i \). Let \( n_i \) denote the \( n \)-th edge in \( T_i' \). We will make use of the following statement, proved as a lemma below.

There exists an integer \( C \), such that for every \( i \) the number of intersecting points of \( n_i \) with edges of \( P_i \) is bounded by \( C \), i.e.,

\[
\sup_i \# \{ n_i \cap P_i^{(0)} \} \leq C.
\]

If for \( \{i_k\}_{k \geq 1} \) there exist infinitely many polyhedra in \( \{P_i\}_{i \geq 1} \) such that \( 1 \) is an edge of \( P_i \), take this infinite subset \( \{P_{i_k}\}_{k \geq 1} \). Since the edges of \( \{P_{i_k}\}_{k \geq 1} \) converge, \( 1_{i_k} \) converges. Continue with \( \{2_{i_k}\}_{k \geq 1} \). Otherwise, since the number of intersecting points of \( 1 \) with \( P_i \) is uniformly bounded in \( i \), there exists a subsequence \( \{P_{i_k}\}_{k \geq 1} \) such that each edge \( 1_{i_k} \) has the same number of intersecting points with \( P_{i_k} \) and all intersecting points converge. The segments between adjacent intersecting points converge as well. In particular, \( 1_{i_k} \) converges. Continue with \( \{2_{i_k}\}_{k \geq 1} \). Define \( T' \coloneqq \lim_k T_{i_k} \), by continuity of \( g \), \( T = g(T') \). Hence, \( P_\infty \) is a closed convex polyhedron in \( \mathcal{P}^N \) such that \( g(P_\infty) = \rho \).

Lemma: There exists an integer \( C \), such that for every \( i \) the number of intersecting points of \( n_i \) with edges of \( P_i \) is bounded by \( C \), i.e.,

\[
\sup_i \# \{ n_i \cap P_i^{(0)} \} \leq C.
\]
Proof: We will prove that there exists an integer $C_0$ uniform in $i$, such that if an edge of $P_i$ does not coincide with any edge in $T'_i$, then the number of intersections of this edge with all edges in $T'_i$ is smaller than $C_0$. We will conclude from this the lemma above.

Let $\{l_{P_i}(e) | e \text{ edge of } P_i, i \geq 1\}$ be the set of edge lengths in $\{P_i\}_{i \geq 1}$ and let $\cup_{i \geq 1} \cup_{ABC \in T'_i} \{h_A, h_B, h_C\}$ be the set of altitudes of triangles of $\{T'_i\}_{i \geq 1}$ in $\{P_i\}_{i \geq 1}$. Let $L := \sup \{l_{P_i}(e) | e \text{ edge of } P_i, i \geq 1\}$ and $h := \inf \{l_{i \geq 1} \cup_{ABC \in T'_i} \{h_A, h_B, h_C\} \}$. Notice that $h > 0$ since $\{P_i\}_{i \geq 1}$ converges. Let $m \in \mathbb{Z}_{>0}$ be the maximum of the degree of vertices in $T'_i$, this maximum exists since all triangulations in $\{T'_i\}_{i \geq 1}$ have the same combinatorics.

Assume there exists an edge $e$ in a polyhedron $P_k \in \{P_i\}_{i \geq 1}$ such that the number of intersections with the edges of $T'_k$ is

$$C_0 > \frac{2Lm}{h} + m.$$ 

The edge $e$ of the polyhedron is partitioned by the $C_0$ intersecting points into $C_0 + 1$ segments. We will show that among these segments there are at least $m$ successive segments, each of length less than $h/2$.

If that were not the case, then in every collection of $m$ successive segments there would exist a segment of length greater than or equal to $h/2$. The number of collections of $m$ successive segments among all $C_0 + 1$ segments is equal to $\left[ \frac{C_0 + 1}{m} \right]$, i.e. the integer part of the fraction $\frac{C_0 + 1}{m}$. Hence, the length of the edge $e$ would be greater than or equal to $\frac{C_0 + 1}{m} \cdot \frac{h}{2}$. This is impossible, since

$$\left[ \frac{C_0 + 1}{m} \right] \frac{h}{2} > \left[ \frac{C_0}{m} \right] \frac{h}{2} > \left[ \frac{2L}{h} + 1 \right] \frac{h}{2} > L.$$

Set

$$C_0 := \frac{2Lm}{h} + m.$$ 

Let $EF$ be the first segment of $m$ successive segments of length less than $h/2$, cut out of $e$ by a triangle $ABC$ in $T'_i$. Drop a perpendicular $FG$ from $F$ to $AB$ and a perpendicular $CH$ from $C$ to $AB$. The triangles $AGF$ and $ACH$ are similar. Since $FG \leq EF < h/2 \leq \frac{1}{3}CH$, we have $FA < \frac{1}{3}AC$. Hence, point $F$ is closer to $A$ than to $C$. A similar argument holds for point $E$.

Leaving the triangle $ABC$ the edge $e$ of $P_k$ enters a neighboring triangle $ACD$ of $T'_i$ where it again has a segment of length less than $h/2$. Hence, in this triangle also the segment passes closer to point $A$ than to the other points. Since we have $m$ such segments, while at most $m$ triangles touch at $A$, it follows that the edge $e$ of $P_k$ goes around $A$ and returns to the side $AB$ (even intersects it again). This is impossible, since $A$ is a vertex of $P_k$ and therefore must be incident with an edge joining $A$ with another vertex of $P_k$. According to the above, this edge must then intersect $e$, which is impossible, since edges meet only at vertices.

We have thus proved that the number of intersections of an arbitrary edge of any polyhedron $P_k$ in $\{P_i\}_{i \geq 1}$ with the edges of the triangulation $T'_i$
The total number of intersecting points in a polyhedron $P$ in $\{P_1\}_{i \geq 1}$ does not exceed $C_0$. The total number of intersecting points in a polyhedron $P$ in $\{P_1\}_{i \geq 1}$ is therefore at most $C_0 C_1$. Hence, for each edge in $T_k$, there are at most $C_0 C_1$ intersections with edges of $P_k$. Set $C = C_0 C_1$.

We say that a marked polyhedral surface $(S_\mu, \rho)$ is realizable if there exists a polyhedron $P$ whose boundary surface $(S_\mu, \rho_P)$ is isometric to $(S_\mu, \rho)$. 

**Theorem:** Every connected component of $T_{PL}^{con}(N)$ contains a realizable polyhedral surface.

**Proof:** Notice that $N$ is at least 3, since a polyhedral surface homeomorphic to the sphere has total curvature equal to $4\pi$, while the curvature at each vertex is less than $2\pi$. Given a polyhedral surface in $T_{PL}^{con}(3)$, let $A, B, C$ be the vertices of the polyhedral surface and connect them by shortest arcs. This splits the topological sphere into two triangles, each of which contains no interior point with cone angle other than $2\pi$. Therefore, the triangles can be developed on the plane (a detailed exposition of this fact is given by Alexandrov, p. 79). Superposing them so that the corresponding sides coincide, we obtain a doubly-covered triangle that realizes the polyhedral surface in $T_{PL}^{con}(3)$. Hence, every polyhedral surface in $T_{PL}^{con}(3)$ is realizable as a doubly-covered triangle. We will show that if every polyhedral surface with less than $N$ vertices is realizable, then also every polyhedral surface in $T_{PL}^{con}(N)$ is realizable.

The manifold $T_{PL}^{con}(N)$ is an open submanifold of the manifold of polyhedral surfaces $T_{PL}(N)$. The topological boundary of $T_{PL}^{con}(N)$ in $T_{PL}(N)$ consists of polyhedral surfaces with cone angles less than or equal to 2\pi and some cone angle equal to 2\pi. Every polyhedral surface in $\partial T_{PL}^{con}(N)$ is isometric to a polyhedral surface in $T_{PL}^{con}(N')$ for $N' < N$, which is realizable by the induction hypothesis. Hence, $\partial T_{PL}^{con}(N)$ is realizable.

Let $C \subset T_{PL}^{con}(N)$ be a connected component. We will see in the lemma below that $\partial C$ contains a point that admits a neighborhood devoid of any points of any other connected component of $T_{PL}^{con}(N)$. Let $(S_\mu, \rho)$ be such a polyhedral surface in $\partial C$.

Since $(S_\mu, \rho)$ is realizable, there exists a convex polyhedron $P$ such that $g(P) = \rho$. However, not all marked points on $S_\mu$ are vertices of $P$, since the cone angle at some of the marked points equals 2\pi. Let $A_1, \ldots, A_l$ be points on $P$ that correspond to the remaining marked points on $S_\mu$. These points either lie in the interior of a face of $P$ or on an edge of $P$. Let $A_{l+1}, \ldots, A_N$ be the vertices of $P$. Move the points $A_1, \ldots, A_l$ away from the polyhedron $P$ for a sufficiently small distance. Let $Q$ be the convex hull of the moved points $A_1, \ldots, A_l$ and $A_{l+1}, \ldots, A_N$. Now $Q$ is a convex polyhedron with $N$ vertices close to the points $A_1, \ldots, A_l, A_{l+1}, \ldots, A_N$ on the boundary of $P$. In particular, the complete angles at the points $A_1, \ldots, A_l, A_{l+1}, \ldots, A_N$ of $(S_\mu, \rho_Q)$ are less than $2\pi$ and $g(Q)$ belongs to $T_{PL}^{con}(N)$. Since $g(Q)$ is close to $g(P)$, and close to $g(P)$, there are no elements of $T_{PL}^{con}(N)$ except for those in $C$, $g(Q)$ belongs to $C$. Thus, each connected component of $T_{PL}^{con}(N)$ contains a realizable polyhedral surface.

**Lemma:** For $N > 3$, the boundary of every connected component of
\( T_{PL}^{\text{con}}(N) \) contains a point that admits a neighborhood devoid of any points of any other connected component of \( T_{PL}^{\text{con}}(N) \).

Proof: Recall that given a coordinate chart \( \vartheta_{\tau} \) on \( T_{PL}(N) \), the submanifold \( T_{PL}^{\text{con}}(N) \) is determined by the inequalities

\[
\sum_j \alpha_{ij} < 2\pi \quad (i = 1, 2, \ldots, N),
\]

with \( \alpha_{ij} \) denoting the \( j \)-th angle at the \( i \)-th vertex. Hence, the topological boundary of \( T_{PL}^{\text{con}}(N) \) in \( T_{PL}(N) \) is composed of pieces of \( N \) surfaces \( F_1, \ldots, F_N \) determined by the equations

\[
\sum_j \alpha_{ij} = 2\pi \quad (i = 1, 2, \ldots, N).
\]

Let \( C \subset T_{PL}^{\text{con}}(N) \) be a connected component of \( T_{PL}^{\text{con}}(N) \). Let \( \rho \in \partial C \) be a point on the boundary of \( C \) that belongs to the least number of surfaces \( F_1, \ldots, F_N \), say \( F_{l_1}, \ldots, F_{l_k} \). The corresponding polyhedral surface \( (S_\mu, \rho) \) has cone angles equal to \( 2\pi \) at the marked points \( A_1, \ldots, A_l \) and cone angles less than \( 2\pi \) at the marked points \( A_{l+1}, \ldots, A_N \).

Let \( \vartheta_{\tau} \) be a coordinate chart of \( T_{PL}(N) \) containing \( \rho \). There exists an \( \epsilon > 0 \) such that all polyhedral surfaces in the \( \epsilon \)-ball \( B_\epsilon(\vartheta_{\tau}(\rho)) \) have cone angles less than \( 2\pi \) at \( A_{l+1}, \ldots, A_N \). In other words, \( \vartheta_{\tau}^{-1}(B_\epsilon(\vartheta_{\tau}(\rho))) \) contains no points of any surface \( F_i \) other than \( F_{l_1}, \ldots, F_{l_k} \).

Consider the triangles in \( \mathcal{T} \) containing the vertex \( A_1 \). Let \( e \) be an edge of one of them subtended by \( A_1 \). Notice that the cone angle at \( A_1 \) is an increasing function in the length \( \vartheta_{\tau}(\rho)_e \) of \( e \). Label the edges in \( \mathcal{T} \) such that \( e \) is the first edge. Let \( \delta_1, \delta_2 > 0 \) be small enough such that the disks

\[
\{ \vartheta_{\tau}(\rho)_e - \delta_1 \} \times B_{\delta_2}((\vartheta_{\tau}(\rho)_{e_2}, \ldots, \vartheta_{\tau}(\rho)_{e_k}))
\]

and

\[
\{ \vartheta_{\tau}(\rho)_e + \delta_1 \} \times B_{\delta_2}((\vartheta_{\tau}(\rho)_{e_2}, \ldots, \vartheta_{\tau}(\rho)_{e_k}))
\]

are contained in \( B_\epsilon(\vartheta_{\tau}(\rho)) \), and all polyhedral surfaces in the first disk have cone angles smaller than \( 2\pi \) at \( A_1 \), and all polyhedral surfaces in the second disk have cone angles greater than \( 2\pi \) at \( A_1 \).

Let \( \rho_1 \) be a point in the first disk and \( \rho_2 \) its translation along the \( e \) axis in the second disk. Since the cone angle is increasing monotonically along the \( e \) axis when moving \( \rho_1 \) to \( \rho_2 \), there exists a unique point where the cone angle equals \( 2\pi \). Hence, the face \( F_1 \) divides the cylinder

\[
\vartheta_{\tau}^{-1}([\vartheta_{\tau}(\rho)_e - \delta_1, \vartheta_{\tau}(\rho)_e + \delta_1] \times B_{\delta_2}((\vartheta_{\tau}(\rho)_{e_2}, \ldots, \vartheta_{\tau}(\rho)_{e_k}))
\]

into two pieces \( V_1 \) and \( V_2 \), where all polyhedral surfaces in \( V_2 \) have cone angles greater than \( 2\pi \) at \( A_1 \) and all polyhedral surfaces in \( V_1 \) have cone angles less than \( 2\pi \) at \( A_1 \).

The polyhedral surface \( \rho \) lies on the boundary \( \partial C \) of the connected component \( C \) of \( T_{PL}^{\text{con}}(N) \). Clearly, \( C \cap V_1 \) is non-empty. We will show that in fact \( V_1 \subset C \).
Assume the contrary. Let $\rho_1$ be a polyhedral surface in $V_1$ not belonging to $C$, and let $\rho_2$ be a polyhedral surface in $V_1$ that belongs to $C$. $\rho_1$ and $\rho_2$ project to polyhedral surfaces $\rho'_1$ and $\rho'_2$ in the disk 

$$\{\vartheta(\rho)_e - \delta_1\} \times B_{\delta_2}(\vartheta(\rho)_2, \ldots, \vartheta(\rho)_E).$$

We obtain a continuous polygonal line $L = \vartheta(\rho_1)\vartheta(\rho'_1)\vartheta(\rho'_2)\vartheta(\rho'_2)\vartheta(\rho_2)$ in $V_1$ that joins $\rho_1$ to $\rho_2$. Since $\rho_2$ belongs to $C$ and $\rho_1$ does not, this line must intersect the boundary $\partial C$.

Being a boundary point of $T_{PL}^{\infty}(N)$, the intersecting point of the line $L$ with $\partial C$ must lie on some surface $F_i$. Since line $L$ stays in $V_1$, it cannot lie on the face $F_i$. Line $L$ also lies in the ball $B_i(\vartheta(\rho))$ and no point in $B_i(\vartheta(\rho))$ lies on a surface other than $F_1, \ldots, F_i$. Hence, the intersecting point of $L$ with $\partial C$ lies on at most $l - 1$ surfaces. However, $\rho$ was a point on $\partial C$ that intersects the minimal number of surfaces $F_i$, which was $l$.

We come to a contradiction, which shows that $V_1$ is entirely contained in the connected component $C$ of $T_{PL}^{\infty}(N)$.

Since $V_2$ does not contain any point of $T_{PL}^{\infty}(N)$ at all, the cylinder

$$\vartheta^{-1}_T([\vartheta(\rho)_e - \delta_1, \vartheta(\rho)_e + \delta_1] \times B_{\delta_2}(\vartheta(\rho)_2, \ldots, \vartheta(\rho)_E))$$

is the required neighborhood of $\rho$.

**RIVIN’S THEORY ON IDEAL CONVEX POLYHEdra**

**Isometric embedding of ideal polyhedral surfaces.** Does every ideal polyhedral surface arise from the boundary of an ideal hyperbolic polyhedron? Rivin gave the following answer [21]:

**Theorem: (Rivin)** Let $(S_N, \hat{\rho})$ be an ideal polyhedral surface. Then $(S_N, \hat{\rho})$ can be isometrically embedded in $\mathbb{H}^3$ as the boundary of a convex polyhedron $P$ with all vertices on the sphere at infinity.

The proof needs some specific techniques related to the fact that we are dealing with geodesics between ideal points. Nevertheless, the proof follows essentially the same philosophy as Alexandrov’s.

**Proof: (Outline)** Let $P_{\text{ideal}}^N$ be the space of convex ideal polyhedra with $N$ vertices in $\mathbb{H}^3$, this space is parametrized by the positions of their vertices on the sphere at infinity. Three of their vertices are fixed at 0, 1, and $\infty$. This eliminates the action of the isometry group of $\mathbb{H}^3$. There are $2N$ variable coordinates, however three vertices are fixed. Therefore, we have $2N - 6$ variable coordinates and $P_{\text{ideal}}^N$ is a $2N - 6$ dimensional manifold.

Let $S_\mu$ be a surface homeomorphic to the $N$-times punctured sphere, together with a marking $\mu$, that is, a labelling of the punctures.

Let $T(N)$ be the set of hyperbolic metrics $\hat{\rho}$ on the surface $S_\mu$, such that the hyperbolic surface $(S_\mu, \hat{\rho})$ is complete and of finite volume. Two hyperbolic metric metrics $\hat{\rho}_1$ and $\hat{\rho}_2$ in $T(N)$ are identified if $(S_\mu, \hat{\rho}_1)$ and $(S_\mu, \hat{\rho}_2)$ are isometric by an isometry homotopic to the identity.

The set $T(N)$ is parametrized by shears along the edges of a geodesic triangulation of $S_\mu$. This notion measures the shift between two abutting ideal triangles and will be introduced below. We will see that this
The parametrization turns $T(N)$ into a $2N - 6$ dimensional connected manifold.

The boundary of every convex ideal polyhedron with $N$ vertices can be viewed as a complete hyperbolic surface of finite area, homeomorphic to the $N$ times punctured sphere. Formally this gives a map $\mathcal{h} : \mathcal{P}_\text{ideal}^N \rightarrow T(N)$. In the following, we show that $\mathcal{h}$ is a (1) continuous, (2) injective and (3) closed map.

$\mathcal{P}_\text{ideal}^N$ and $T(N)$ are manifolds of equal dimension, by (1) and (2) and the invariance of domain principle of Brouwer, $\mathcal{h}$ is an open map. Since $\mathcal{h}$ is also closed, we conclude, together with the fact that $T(N)$ is connected, that $\mathcal{h}$ is a homeomorphism from $\mathcal{P}_\text{ideal}^N$ onto $T(N)$.

We will need some more notions of hyperbolic geometry to make the above precise.

Let $ABC$ be an ideal triangle in $H^2$. Let $h_A$ be a horocycle centered at $A$, define $D_{ABC}(h_A)$ to be the length of the arc of $h_A$ cut out by the triangle $ABC$. The difference in size between arcs of two horocycles $h_A$ and $h'_A$ cut out by $ABC$ gives information on the distance between the arcs. More precisely:

**Lemma:** Let $h_A$ and $h'_A$ be two horocycles at $A$. The hyperbolic distance between $h_A$ and $h'_A$ is equal to $|\log(D_{ABC}(h_A))/\log(D_{ABC}(h'_A))|$.

**Proof:** Let $ABC$ be the triangle $A = \infty$, $B = 0$ and $C = 1$ in the upper half-space model. The horocycles $h_A$ and $h'_A$ are horizontal lines through $i/y$ and $i/y'$, respectively. Hence, the length of the arcs of $h_A$ and $h'_A$ cut out by $ABC$ is $1/y$ and $1/y'$ respectively and the distance between $h_A$ and $h'_A$ is $|\log(y/y')|$. Two ideal triangles $ABC$ and $ADC$ can slide with respect to each other along the common side $AC$. For any choice of horocycle $h_A$, the number $f_{AC} := \log(D_{ABC}(h_A))/D_{ADC}(h_A))$ measures the shear between the triangles $ABC$ and $ADC$ along $AC$. The shear $f_{AC}$ does not depend on which of the vertices $A$ or $C$ is taken as the center of the horocycles.

Intuitively, two triangles $ABC$ and $ADC$ are joined along $AC$ without a shear, if for any horocycle at $A$ the arcs cut out by $ABC$ and $ADC$ have the same “distance” to $A$.

The cross-ratio of four points $z_1, z_2, z_3, z_4$ in the complex plane is the number

$$[z_1, z_2, z_3, z_4] := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}.$$

The notion of cross-ratio of four points and shear between two triangles are related.

**Lemma:** The shear between two triangles $ABC$ and $ABD$ is equal to the log of the absolute value of the cross-ratio $[C, B, D, A]$.

**Proof:** Let $ABC$ be the triangle $A = \infty$, $B = 1$ and $C = 0$. In this case, the shear between $ABC$ and $ABD$ is $\log|D|$.

A **triangulation** of an ideal polyhedral surface is a triangulation whose vertices are at the cusps of the hyperbolic surface. A **geometric triangulation**
of an ideal polyhedral surface is a triangulation of the ideal polyhedral surface whose edges are geodesics. Let $T$ be a geometric triangulation of an ideal polyhedral surface $(S_\mu, \hat{\rho})$ with $N$ cusps. To each edge of $T$, associate the shear of the two abutting triangles of $T$. This information determines the geometry of $(S_\mu, \hat{\rho})$ completely. Conversely, an assignment of real numbers to the edges of $T$ specifies a complete hyperbolic structure on $S_\mu$ if and only if the shears around any cusp add up to zero. Hence the set $T(N)$ is naturally parametrized by $\mathbb{R}^{|E(T)|-N}$. According to the Euler formula, $|E(T)| - N = 2N - 6$, so the dimension of this space depends only on the number of cusps.

**Lemma:** Any triangulation of a complete hyperbolic surface with cusps can be straightened to a geodesic triangulation.

**Proof:** We need to show that, if $A, B, C$ and $D$ are cusps of a complete hyperbolic surface such that $A$ and $B$ are connected by a path $\gamma_1$ in $T$ and $C$ and $D$ are connected by a path $\gamma_2$ in $T$, then the corresponding geodesics also do not intersect.

The path $\gamma_1$ and $\gamma_2$ do not intersect in $S_N$ if and only if their lifts to the universal cover $\mathbb{H}^2$ of $S_N$ do not intersect. However, if two paths between the ideal boundary of $\mathbb{H}^2$ do not intersect, then the corresponding minimizing geodesics do not intersect either.

The **shear coordinate system** on $T(N)$ corresponding to the triangulation $T$ of $(S_\mu, \hat{\rho})$ is the map $\varphi_T : T(N) \to \mathbb{R}^{2N-6}$, associating a particular metric with its shear along the straightened edges of $T$.

Notice that $\varphi_T : T(N) \to \mathbb{R}^{2N-6}$ is a homeomorphism. Given a point $x$ in $\mathbb{R}^{2N-6}$, we can compute the remaining $N$ shears from the condition that shears must add up to zero around vertices. Hence, $T(N)$ is connected.

**Theorem:** The set of shear coordinate charts $\{(T(N), \varphi_T) \mid T \text{ triangulation of } S_\mu\}$ forms a real analytic atlas on $T(N)$. This turns $T(N)$ into a $2N - 6$ dimensional, connected, real analytic manifold.

**Proof:** Let $T$ and $T'$ be two triangulations of $S_\mu$ related by flipping an edge $e$. Let $ABC$ and $ADC$ be the triangles in $T$ adjacent to $e$. Flipping the diagonal corresponds to permuting the arguments of the cross-ratio. A permutation of the cross-ratio is a fractional linear transformation of the cross-ratio itself. Hence the transition function $\varphi_T \varphi_T^{-1}$ is real analytic.

Two arbitrary triangulations of $S_\mu$ can be obtained from each other by a sequence of edge flips. Hence, the set of shear coordinate charts forms a real analytic atlas.

We already introduced the map $\mathcal{h} : \mathcal{P}_N^{\text{ideal}} \to T(N)$ that formalizes the operation of viewing the boundary of a convex ideal polyhedron with $N$ vertices as a complete hyperbolic surface up to isometry homotopic to the identity. In the following, we show that $\mathcal{h}$ is a (1) continuous, (2) injective and (3) closed map.

For the next theorems we will view the ideal hyperbolic polyhedron $P$ in the upper half-space model. $P$ is the intersection of the half-spaces defined
by its faces. We think of the ideal boundary as the Riemann sphere $\mathbb{C}$.
Through an isometry of $\mathbb{H}^3$ and re-labelling, we can transform $P$ in such a way that the face $f_1$ rises above the real axis and the vertices $v_1$, $v_2$ and $v_3$ are at 0, 1 and $\infty$, respectively. Furthermore, we assume that the polyhedron $P$ lies above the half-plane $\text{Im}(z) \geq 0$.

The polyhedron $P$ casts a shadow on the ideal boundary of $\mathbb{H}^3$ under the orthogonal projection. The edges of $P$ are mapped to straight line segments and the faces of $P$ to convex polygons in $\mathbb{C}$.

This gives a tessellation in $\mathbb{C}$ of the set of vertices $\{v_1, v_2, \ldots, v_N\}\{v_3\}$ of $P$ in $\mathbb{C}$. In fact, every triangulation of this tessellation, is a Delaunay triangulation. A Delaunay triangulation of a finite set of points $V$ in $\mathbb{C}$ is a triangulation of the convex hull of $V$ into triangles such that no point in $V$ is inside the circumcircle of any other triangle. We call the tessellation in $\mathbb{C}$ induced from $P$, the Delaunay tessellation induced by $P$.

Every triangulation of the Delaunay tessellation induced by $P$ corresponds to a triangulation of the boundary of $P$, we call such a triangulation a Delaunay triangulation of the convex ideal polyhedron $P$.

Let $T$ be a Delaunay triangulation of $P$. Let $ABC$ and $ADC$ be two abutting triangles in $T$. If $A, B, C$ and $D$ are transformed by way of hyperbolic isometry in such a way that $A$ is mapped to $\infty$, $B$ to 1, $C$ to 0 and $D$ to $z \in \mathbb{C}$, then the cross-ratio

$$\frac{B - A}{B - C} : \frac{D - A}{D - C}$$

equals $z$. Hence, the following theorem follows from the above discussion of shears in the Poincaré model.

**Theorem:** The shear between two abutting triangles $ABC$ and $ADC$ in $T$ is given by the logarithm of the absolute value of the complex cross-ratio

$$\left| \frac{B - A}{B - C} : \frac{D - A}{D - C} \right|$$

Using the continuity of cross-ratios, we may now prove that the map $g$ is continuous.

**Theorem:** The map $\mathcal{H}$ is continuous.

**Proof:** If the Delaunay tessellation of an ideal hyperbolic polyhedron $P$ is already a triangulation, then any small perturbation of the vertices of $P$ does not change the combinatorics of the Delaunay tessellation. In this case, the continuity of $\mathcal{H}$ follows directly from the continuity of cross ratios.

If the Delaunay tessellation of $P$ is not a triangulation, then any small perturbation of the vertices of $P$ does change the combinatorics of the Delaunay tessellation in $\mathbb{C}$. However, in both conditions there exists a uniform $\epsilon > 0$ such that, if a point $D$ is closer than $\epsilon$ to the circumcircle of a triangle $ABC$, then $A, B, C$ and $D$ are co-circular.

In other words, for a sufficiently small $\epsilon$, the Delaunay tessellation of the perturbed polyhedron $P'$ is combinatorially equivalent to the Delaunay tessellation of $P$ with some diagonals added to the non-triangular faces.
This shows that, for a sufficiently small perturbation of $P$, there always exists a coordinate system in which $h$ is clearly continuous. Since all transition maps are continuous, we conclude that $h$ is continuous.

**Theorem:** The map $h$ is injective.

**Proof:** If $h(P)$ equals $h(Q)$, then $P$ and $Q$ are isometric according to the Cauchy–Rivin rigidity theorem. Hence $P$ and $Q$ agree in $T_{\text{ideal}}^N$.

**Theorem:** The map $h$ is closed.

**Proof:** Let $\hat{\rho}_R, \ldots, \hat{\rho}_P, \ldots$ be a sequence of hyperbolic metrics on a surface $S_\mu$ homeomorphic to the $N$-times punctured sphere and converging to a metric $\hat{\rho}$ such that $h(P_\lambda) = \hat{\rho}_P_\lambda$.

Choose a subsequence $\{P_{\lambda_k}\}$ such that all polyhedra have the same combinatorics. The vertices and faces of the polyhedra are labelled such that $v_1(P_{\lambda_k}) = 0$, $v_3(P_{\lambda_k}) = 1$, $v_3(P_{\lambda_k}) = \infty$ and $f_1(P_{\lambda_k})$ rises above the real axis. Since the Riemann sphere $\bar{\mathbb{C}}$ is compact, there exists a limiting tessellation $T$. If $T$ is non-degenerate, then by the continuity of $h$ it follows that there exists $P_\infty$ such that $h(P_\infty) = \hat{\rho}$.

We will show that $T$ is always non-degenerate. Suppose it is not, then there exist two abutting triangles $ABC$ and $ADC$ in $T$ such that $ADC$ is collapsed and $ABC$ is not. Such two triangles exist, since the face $f_1$ is not collapsing. Map $A$ to $\infty$, $B$ to $1$ and $C$ to $0$. Then the shear between the two triangles equals $\log |D|$. Since $\hat{\rho}$ is a non-degenerate metric, $|D|$ must stay away from $0$ and $\infty$. But this means that $ADC$ is non-degenerate after all.

**A CONFORMAL EQUIVALENCE RELATION FOR CONVEX POLYHEDRA**

**Discrete conformity of convex polyhedra.** A Delaunay triangulation of a development $R$ is a Delaunay triangulation of every polygon in $R$. The following lemma is a classical property of Delaunay triangulations in the plane. A proof can be found in Aurenhammer’s book on Voronoi diagrams [24].

**Lemma:** If a finite set of points in the plane admits two Delaunay triangulations, then there exists a sequence of Delaunay triangulations between them, such that each is related to the next by a diagonal switch.

Hence, if a Euclidean development $R$ admits two distinct Delaunay triangulations, then they differ by a finite number of diagonal switches between two abutting triangles within a polygon in $R$ that share the same circumcircle.

Every Euclidean development $R$ has a unique set of circumspheres attached to its vertices, by taking the circumcircles of a Delaunay triangulation of $R$. A Euclidean triangle with its circumcircle can be viewed as an ideal hyperbolic triangle in the Klein model. This construction does not depend on the chosen Delaunay triangulation and associates with every Euclidean development $R$ with $N$ vertices a marked ideal polyhedral surface $(S_\mu, \hat{\rho}_R)$ with a cusp for each vertex of the development. Indeed, by the following
Theorem: Let $R$ be a Euclidean development with $N$ vertices and $T$ a Delaunay triangulation of $R$. Let $ijk$ and $ijl$ be two triangles in $T$ abutting along the edge $ij$. The hyperbolic structure $\tilde{\rho}_R$ on $S_\mu$ is the unique complete hyperbolic structure on $S_\mu$ with shear

$$\log \frac{\partial_T(\rho_R)_{il}}{\partial_T(\rho_R)_{ik}} : \frac{\partial_T(\rho_R)_{jl}}{\partial_T(\rho_R)_{jk}}$$

along the edge $ij$ of $T$.

Proof: The map associating every Euclidean development $R$ with a hyperbolic surface with cusps $(S_\mu, \tilde{\rho}_R)$, can be described in upper half-space model as follows. Consider $C$ as the sphere at infinity of the hyperbolic 3-space $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$. Let $ijk$ and $ijl$ be two abutting triangles in $R$. Embed $ijk \cup ijl$ into the sphere at infinity by an isometry $f$. The hyperbolic metric $\tilde{\rho}_R$ on $ijk \cup ijl$ is the hyperbolic metric of the ideal hyperbolic triangles in $\mathbb{H}^3$, having the same vertices as $ijk$ and $ijl$, glued by the same isometry $f$, considered as a hyperbolic motion of $\mathbb{H}^3$.

The sheaf of $(S_\mu, \tilde{\rho}_R)$ along the edge $ij$ is the logarithm of the absolute value of the complex cross-ratio of the four vertices $z_i, z_j, z_k$ and $z_l$ of the triangles $ijk$ and $ijl$ in $C$. Clearly,

$$\log \frac{z_i - z_l}{z_i - z_k} : \frac{z_j - z_l}{z_j - z_k} = \log \frac{\partial_T(\rho_R)_{il}}{\partial_T(\rho_R)_{ik}} : \frac{\partial_T(\rho_R)_{jl}}{\partial_T(\rho_R)_{jk}}.$$

A Delaunay triangulation of a convex polyhedron $P$ is a triangulation of its boundary coming from a Delaunay triangulation of its face development $R_P$.

Note: A Delaunay triangulation of an ideal convex polyhedron $P$ in $\mathbb{H}^3$, is a Delaunay triangulation of a convex polyhedron if $P$ is viewed as a Euclidean convex polyhedron inscribed in the sphere.

Given a convex Euclidean polyhedron $P$. We associate with $P$ the marked ideal polyhedral surface $(S_\mu, \tilde{\rho}_P)$ coming from the face development of $P$. In the following, we denote $(S_\mu, \tilde{\rho}_P)$ by $(S_\mu, \hat{\rho}_P)$. Formally we obtain a function $f : P^N \to T(N)$ mapping $P$ to $(S_\mu, \hat{\rho}_P)$.

Definition: Two closed convex polyhedra $P$ and $Q$ with $N$ vertices are discrete-conformally equivalent if and only if $(S_\mu, \hat{\rho}_P)$ and $(S_\mu, \hat{\rho}_Q)$ are isometric.

Theorem: Let $P$ and $Q$ be two convex polyhedra inscribed in the unit sphere that are discrete-conformally equivalent. Then there exists a Möbius transformation on the sphere that maps the vertex set of $P$ to the vertex set of $Q$.

Proof: If $P$ is inscribed in the unit sphere, then the association $P \to (S_\mu, \hat{\rho}_P)$ defined above is nothing but interpreting $P$ as a convex ideal polyhedron in the Klein model and moving to the boundary. Hence, if $P$ and $Q$ are discrete-conformally equivalent, there exists a hyperbolic isometry from $P$ onto $Q$. According to the rigidity theory of Cauchy,
Alexandrov and Rivin, this isometry can be realized as a motion or a motion and a reflection in $H^3$. Equally, there exists a Möbius transformation on the sphere mapping the vertex set of $P$ to the vertex set of $Q$.

The above rigidity theorem allows us to classify Euclidean polyhedra up to discrete conformality.

**Theorem: (Uniformization)** Every closed convex polyhedron in $E^3$ is discrete-conformally equivalent to a closed convex polyhedron inscribed in the unit sphere. This polyhedron is unique up to Möbius transformations on the sphere.

The uniqueness part was proven above. The existence follows from Rivin’s isometric embedding of ideal polyhedra in hyperbolic 3-space.

**Theorem:** Given a convex polyhedron $P$ in $E^3$, there exists a convex polyhedron $Q$ inscribed in the unit sphere that is discrete-conformally equivalent to $P$.

**Proof:** Let $(S_\mu, \hat{\rho}_P)$ be the marked ideal polyhedral surface associated with $P$. According to Rivin’s isometric embedding theorem, $(S_\mu, \hat{\rho}_P)$ can be isometrically embedded in $H^3$ as the boundary of a convex hyperbolic polyhedron $Q$ with all vertices on the sphere at infinity. The polyhedron $Q$ may be interpreted as a convex Euclidean polyhedron inscribed in the sphere if viewed in the Klein model. This interpretation is just the inverse of the map $Q \to (S_\mu, \hat{\rho}_Q)$. Hence, $(S_\mu, \hat{\rho}_P)$ is isometric to $(S_\mu, \hat{\rho}_Q)$ and $P$ and $Q$ are discrete-conformally equivalent.

**Characterization of discrete conformality.** The notion of discrete conformality passes through hyperbolic geometry. In the following we characterize discrete conformality of Euclidean polyhedra that share a common Delaunay triangulation by elementary transformations on vertices.

**Theorem:** Let $P$ and $Q$ be two polyhedra that share a common Delaunay triangulation $T$, then $P$ and $Q$ are discrete-conformally equivalent if and only if there exists a real valued function $u_T$ on the vertices of $P$ so that if, $e$ is an edge in $T$ between the vertices $i$ and $j$, then the length $l_P(e)$ and $l_Q(e)$ of $e$ in $P$ and $Q$ are related by

$$l_Q(e) = l_P(e) e^{\frac{1}{2}(u_T(i)+u_T(j))}.$$ 

We will first give an alternative description of the function $f : \mathcal{P}^N \to T(N)$ using Penner’s theory on decorated Teichmüller spaces [25]. To shorten notation, we call $u_T$ a conformal factor and write $u_T \ast P = Q$ if $P$ and $Q$ are related by the conformal factor $u_T$ as above.

A **decorated ideal triangle** is an ideal triangle $ABC$ together with a choice of horocycles $h_A$, $h_B$ and $h_C$. The **Penner distance** between two distinct horocycles $h_A$ and $h_B$ is

$$d_{AB}^P := t^P(h_A, h_B) := e^{\lambda_{AB}/2},$$
where \( \lambda_{AB} := \lambda(h_A, h_B) \) is the signed distance between two distinct horocycles.

Two decorated ideal triangles \((ABC, h_A, h_B, h_C)\) and \((ADC, h_A, h_D, h_C)\) can be glued along the edge \(AC\) by an isometry preserving the horocycles \(h_A\) and \(h_C\).

Recall that \( S_\mu \) is a surface homeomorphic to the 2-sphere with \( N \) punctures, together with its marking. A decorated hyperbolic metric on \( S_\mu \) is a complete finite area hyperbolic metric \( \hat{\rho} \) on \( S_\mu \), together with horoballs \( h_i \) at every cusp \( i \). Two decorated hyperbolic metrics on \( S_\mu \) are equivalent if there exists an isometry homotopic to the identity between them that preserves the horoballs.

Let \( T_D(N) \) be the set of equivalence classes of decorated hyperbolic metrics on \( S_\mu \). Let \( T(N) \) be the set of complete, finite volume hyperbolic structures on \( S_\mu \) as introduced above. The mapping

\[
T_D(N) \to T(N) \times \mathbb{R}^\infty_{>0},
\]

where \( w_i \) is the sum of the lengths \( D_{ijk}(h_i) \) of horoarcs cut out by the ideal triangles at \( i \).

Theorem: The set of Penner coordinate charts

\[
\{(U_T, \varphi_T) | T \text{ triangulation of } S_\mu \}
\]

forms a real analytic atlas on \( T_D(N) \). This turns \( T_D(N) \) into an \(|E|\) dimensional, real analytic manifold.

Proof: For any two triangulations of \( S_\mu \) there exists a sequence of triangulations of \( S_\mu \) between them, such that each is related to the next by a diagonal switch along an edge. Hence, it is enough to show that the transition function \( \varphi_T \varphi_T^{-1} \) is real analytic if \( T \) and \( T' \) are related by a diagonal switch.

In this case,

\[
\varphi_T \varphi_T^{-1}(x_0, x_1, \ldots, x_n) = \left( \frac{x_1 x_3 + x_2 x_4}{x_0}, x_1, x_2, \ldots, x_n \right),
\]

which is real analytic.

Let \( p : T_D(N) \to T(N) \) be the projection, mapping \( (\hat{\rho}, \{h_i\}^N_i=1) \) to \( \hat{\rho} \), and let \( g : P^N \to T_{PL}^\text{core}(N) \) be Alexandrov’s homeomorphism. We aim for an
Proof: Suppose classes of triangulations of $S$. The function $\rho$ of the associated polyhedron $P$ metrics $h$ between the horospheres $T$. $F$ launay triangulations of $P$. Then $\rho$ is obtained from $T$ by a diagonal switch. In particular $F_T(\rho_P) = F_T(\rho_P)$ follows from $F_T(\rho_P) = F_{T_{i+1}}(\rho_P)$ for $i = 1, 2, \ldots, n - 1$. Hence, assume that $T'$ is obtained from $T$ by a diagonal switch at an edge $e$.

Let $\vartheta_T(\rho_P) = (x_0, x_1, \ldots, x_n)$. Since both $T$ and $T'$ are Delaunay triangulations of $P$, the triangles abutting at $e$ share a common circumcircle. In this case the transition function is of the form

$$\vartheta_T \vartheta_T^{-1}(x_0, x_1, \ldots, x_n) = \left(\frac{x_1 x_3 + x_2 x_4}{x_0}, x_1, x_2, \ldots, x_n\right).$$

On the other hand, according to Penner [25] the $\lambda$-lengths satisfy the Ptolemy relation for decorated ideal triangles. Hence,

$$\varphi_T \varphi_T^{-1}(x_0, x_1, \ldots, x_n) = \left(\frac{x_1 x_3 + x_2 x_4}{x_0}, x_1, x_2, \ldots, x_n\right).$$

This shows,

$$\vartheta_T \vartheta_T^{-1}(x_0, x_1, \ldots, x_n) = \varphi_T \varphi_T^{-1}(x_0, x_1, \ldots, x_n),$$

which is

$$F_T(\rho_P) = \varphi_T^{-1} \vartheta_T(\rho_P) = \varphi_T^{-1} \vartheta_T(\rho_P) = F_T(\rho_P).$$

In fact, the function $F$ maps $\rho_P$ to $(\hat{\rho}_P, \{w_i\}_{i=1}^N)$ where $\hat{\rho}_P$ is the image of $P$ under $f$. To see this, we will study the shear coordinates of the underlying ideal polyhedron surface of a decorated ideal polyhedral surface $(\hat{\rho}, \{w_i\}_{i=1}^N)$ if $(\hat{\rho}, \{w_i\}_{i=1}^N)$ is given in Penner coordinates.

Theorem: Let $(\hat{\rho}, w) \in T_D(N)$ and let $\varphi_T$ be a coordinate chart containing $(\hat{\rho}, w)$, then the shear coordinate between two abutting triangles $ij$ and $ik$ in $T$ of $\hat{\rho}$ is given by

$$\log \frac{\varphi_T((\hat{\rho}, w))_{ij}}{\varphi_T((\hat{\rho}, w))_{ik}} = \frac{\varphi_T((\hat{\rho}, w))_{ij}}{\varphi_T((\hat{\rho}, w))_{jk}}.$$

Proof: Recall that $\varphi_T((\hat{\rho}, w))_{ij} = e^{\lambda_{ij}/2}$, where $\lambda_{ij}$ is the signed distance between the horospheres $h_i$ and $h_k$. Hence,

$$\log \frac{\varphi_T((\hat{\rho}, w))_{ij}}{\varphi_T((\hat{\rho}, w))_{ik}} = \frac{1}{2}(\lambda_{ij} - \lambda_{ij} + \lambda_{jk} - \lambda_{ki}).$$
Let us focus first only on the decorated triangle \( ijk \). The axis of symmetry through the point \( i \) of the ideal triangle \( ijk \) splits the signed distance \( \lambda_{jk} \) between the horocycles \( h_i \) and \( h_i \) into the sum of two numbers \( p^i_{jk} \) and \( p^k_{ji} \), being the signed distance between the base point of the axis of symmetry and the horocycle \( h_k \) and \( h_j \), respectively. Doing the same for \( \lambda_{ij} \) and \( \lambda_{ki} \) gives \( \lambda_{ij} = p^i_{jk} + p^k_{ji} \), \( \lambda_{jk} = p^k_{ji} + p^i_{jk} \) and \( \lambda_{ki} = p^k_{ij} + p^i_{jk} \). Solving for \( p^k_{ji} \) gives
\[
p^k_{ji} = \frac{1}{2}(\lambda_{ij} + \lambda_{jk} - \lambda_{ki}).
\]
Doing the same for the triangle \( ilj \) gives
\[
p^k_{il} = \frac{1}{2}(\lambda_{ij} + \lambda_{jl} - \lambda_{il}).
\]
Hence,
\[
\frac{\varphi_T((\hat{\rho}, w))_{il}}{\varphi_T((\hat{\rho}, w))_{ik}} : \frac{\varphi_T((\hat{\rho}, w))_{jk}}{\varphi_T((\hat{\rho}, w))_{jl}} = p^i_{kl} - p^k_{il}.
\]
But the right-hand-side is nothing but the shear between the two triangles \( ijk \) and \( ilj \).

We are now prepared to prove the alternative description of the function \( f \).

**Theorem:** Let \( f : P \to T(N) \) be the map associating a convex polyhedron with the ideal polyhedral surface \((S_\mu, \hat{\rho}P)\), let \( g : P \to TPL(N) \) be Alexandrov’s homeomorphism and let \( p : T_p(N) \to T(N) \) be the projection on the underlying hyperbolic surface. Then,
\[
p \circ F \circ g = f.
\]

**Proof:** Let \( T \) be a Delaunay triangulation of \( P \). Let \( ijk \) and \( ilj \) be two triangles in \( T \) abutting along the edge \( ij \). The ideal polyhedral surface \( f(P) = \hat{\rho}P \) has shear coordinates
\[
\log \frac{\varphi_T((\hat{\rho}, w))_{ij}}{\varphi_T((\hat{\rho}, w))_{ik}} : \frac{\varphi_T((\hat{\rho}, w))_{jk}}{\varphi_T((\hat{\rho}, w))_{jl}} = p^i_{kl} - p^k_{il}.
\]
along the edge \( ij \) of \( T \). Let \( (\hat{\rho}, w) = F \circ g(P) \), the shear coordinates of the decorated ideal polyhedral surface \((\hat{\rho}, w)\) along the edge \( ij \) is
\[
\log \frac{\varphi_T((\hat{\rho}, w))_{ij}}{\varphi_T((\hat{\rho}, w))_{ik}} : \frac{\varphi_T((\hat{\rho}, w))_{jk}}{\varphi_T((\hat{\rho}, w))_{jl}}.
\]
But \( g(P) \) lies in \( DPL(T) \), hence \( (\hat{\rho}, w) = F_T(\rho_P) = \varphi_T^{-1} \varphi_T(\rho_P) \). Hence,
\[
\log \frac{\varphi_T((\hat{\rho}, w))_{ij}}{\varphi_T((\hat{\rho}, w))_{ik}} : \frac{\varphi_T((\hat{\rho}, w))_{jk}}{\varphi_T((\hat{\rho}, w))_{jl}} = \log \frac{\varphi_T(\rho_P)_{ij}}{\varphi_T(\rho_P)_{ik}} : \frac{\varphi_T(\rho_P)_{jk}}{\varphi_T(\rho_P)_{jl}},
\]
and \( \hat{\rho} = \hat{\rho}P \).

Let us return to the main theorem of this section.

**Theorem:** Let \( P \) and \( Q \) be two polyhedra that share a common Delaunay triangulation \( T \), then \( P \) and \( Q \) are discrete-conformally equivalent if and
only if there exists a real valued function \( u_\mathcal{T} \) on the vertices of \( P \) so that, if \( e \) is an edge in \( \mathcal{T} \) between the vertices \( i \) and \( j \), then the length \( l_P(e) \) and \( l_Q(e) \) of \( e \) in \( P \) and \( Q \) are related by

\[
l_Q(e) = l_P(e) \, e^{\frac{1}{2} (u_\mathcal{T}(i) + u_\mathcal{T}(j))}.
\]

**Proof:** We have seen that the hyperbolic structure \( \hat{\rho}_Q \) on \( S_\mu \) is the unique complete hyperbolic structure on \( S_\mu \) with shear

\[
\log \frac{\varphi_\mathcal{T}(\rho_Q)_{il}}{\varphi_\mathcal{T}(\rho_Q)_{ik}} = \frac{\varphi_\mathcal{T}(\rho_P)_{il}}{\varphi_\mathcal{T}(\rho_P)_{ik}}
\]

along the edge \( ij \) of \( \mathcal{T} \). If there exists a conformal factor \( u_\mathcal{T} \) such that \( Q = u_\mathcal{T} \ast P \), this number equals

\[
\log \frac{e^{\frac{1}{2} (u_\mathcal{T}(i) + u_\mathcal{T}(l))} \varphi_\mathcal{T}(\rho_P)_{il}}{e^{\frac{1}{2} (u_\mathcal{T}(j) + u_\mathcal{T}(k))} \varphi_\mathcal{T}(\rho_P)_{jk}} = \frac{e^{\frac{1}{2} (u_\mathcal{T}(i) + u_\mathcal{T}(l))} \varphi_\mathcal{T}(\rho_P)_{il}}{e^{\frac{1}{2} (u_\mathcal{T}(j) + u_\mathcal{T}(k))} \varphi_\mathcal{T}(\rho_P)_{jk}},
\]

which equals

\[
\log \frac{\varphi_\mathcal{T}(\rho_P)_{il}}{\varphi_\mathcal{T}(\rho_P)_{ik}} = \frac{\varphi_\mathcal{T}(\rho_P)_{il}}{\varphi_\mathcal{T}(\rho_P)_{ik}}.
\]

Hence, \((S_\mu, \hat{\rho}_P)\) is isometric to \((S_\mu, \hat{\rho}_Q)\).

If \( P \) and \( Q \) are discrete-conformally equivalent, i.e. \( f(P) = f(Q) \), then \( p \circ F \circ g(P) = p \circ F \circ g(Q) \). In other words, we obtain an ideal polyhedral surface with two decorations \( \{h^i_P\}_{i=1}^\infty \) and \( \{h^i_Q\}_{i=1}^\infty \) corresponding to \( P \) and \( Q \) respectively.

Let \( \lambda_P \rightarrow Q \) be the signed distance between the horoballs \( h^i_P \) and \( h^i_Q \) at the \( i \)-th cone point of the given ideal polyhedral surface, which is negative if and only if the horoball \( h^i_P \) is smaller than the horoball \( h^i_Q \). Given an edge \( ij \) of \( \mathcal{T} \), the signed distances between horoballs \( \lambda^i_{ij} = \lambda(h^i_P, h^j_Q) \) and \( \lambda^j_{ij} = \lambda(h^j_P, h^i_Q) \) are related by

\[
\lambda^i_{ij} = \lambda^j_{ij} + \lambda_P \rightarrow Q + \lambda_P \rightarrow Q.
\]

In particular,

\[
e^{\lambda^i_{ij}/2} = e^{\lambda^j_{ij}/2} e^{\frac{1}{2} (\lambda_P \rightarrow Q + \lambda_P \rightarrow Q)}.
\]

Since \( F \circ g(P) = \varphi^{-1}_\mathcal{T} \varphi_\mathcal{T}(\rho_P) \) as well as \( F \circ g(Q) = \varphi^{-1}_\mathcal{T} \varphi_\mathcal{T}(\rho_Q) \), we have \( e^{\lambda^i_{ij}/2} = l_P(ij) \) and \( e^{\lambda^j_{ij}/2} = l_Q(ij) \). Hence, if we define

\[
u_\mathcal{T}(i) := \lambda_P \rightarrow Q,
\]

for every vertex \( i = 1, \ldots, N \) of the polyhedron \( P \), then \( u_\mathcal{T} \) is a conformal factor satisfying \( u_\mathcal{T} \ast P = Q \).

**Concepts of discrete conformality and Möbius geometry.** Let \( \mathcal{P}_{\text{ideal}} \) be the space of ideal convex polyhedra in \( \mathbb{H}^3 \), this space is equivalent to the space of convex Euclidean polyhedra inscribed in the unit
Moreover, \( P \) set of \( \mathbb{H}^3 \) is the Klein model of the hyperbolic 3-space. Hence, let us interpret \( \mathcal{P}_{\text{ideal}} \) as the space of convex Euclidean polyhedra inscribed in the unit sphere. Let \( \mathcal{C} \) be the set of circle patterns covering the unit sphere. To every circle pattern \( C \) in \( \mathcal{C} \) corresponds a unique convex Euclidean polyhedron \( P_C \) in \( \mathcal{P}_{\text{ideal}} \) by cutting off all half-planes defined by the circles in \( C \). Conversely, every convex polyhedron inscribed in the unit sphere corresponds to a unique circle pattern covering the unit sphere.

**Theorem:** Let \( C_1 \) and \( C_2 \) be two circle patterns in \( \mathcal{C} \) and let \( P_{C_1} \) and \( P_{C_2} \) be the corresponding convex polyhedra inscribed in the unit sphere. There exists a Möbius transformation \( f \) on the sphere mapping the circle pattern \( C_1 \) onto the circle pattern \( C_2 \) if and only if \( P_{C_1} \) and \( P_{C_2} \) share a common Delaunay triangulation \( T \) and there exists a function \( u_T \) defined on the vertex set of \( P_{C_1} \) such that for every edge \( ij \) in the Delaunay triangulation between vertices \( i \) and \( j \), its length in \( P_{C_2} \) is related to its length in \( P_{C_1} \) by

\[
l_{P_{C_2}}(ij) = l_{P_{C_1}}(ij) e^{\frac{1}{2}(u_T(i) + u_T(j))}.
\]

Moreover, \( f \) and \( u_T \) are related by \( u_T = \log |df|_V \), where \( V \) is the vertex set of \( P_{C_1} \).

**Proof:** It only remains to prove the relation of \( u_T \) and the Möbius transformation \( f \). Let \( x \) and \( y \) be two distinct vertices of \( P_{C_1} \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences on the unit sphere converging to \( x \) and \( y \), respectively, but not containing the points \( x \) and \( y \). The Euclidean length cross-ratio is invariant under Möbius transformations on the sphere. Hence

\[
\frac{|x - x_n|}{|x - y_n|} : \frac{|y - x_n|}{|y - y_n|} = \frac{|f(x) - f(x_n)|}{|f(x) - f(y_n)|} : \frac{|f(y) - f(x_n)|}{|f(y) - f(y_n)|}.
\]

A rearrangement gives

\[
\frac{|f(x) - f(y_n)|}{|x - y_n|} \frac{|f(y) - f(x_n)|}{|y - x_n|} = \frac{|f(x) - f(x_n)|}{|x - x_n|} \frac{|f(y) - f(y_n)|}{|y - y_n|}.
\]

Taking the limit \( n \to \infty \) results in

\[
\frac{|f(x) - f(y)|^2}{|x - y|^2} = |df(x)| |df(y)|.
\]

This shows that \( P_{C_1} \) and \( P_{C_2} \) are discrete-conformally equivalent with \( u_T = \log |df|_V \).

As we mentioned in the introduction, this theorem suggests a third variant of discrete conformality, namely *discrete Möbius geometry*.

Roughly speaking, a Möbius structure on a set \( X \) is an equivalence class of metrics on \( X \), where two metrics are equivalent if they define the same crossratio. Let \( \mathcal{M} \) be a Möbius structure on a set \( X \). The pair \((X, \mathcal{M})\) is called a Möbius space.

If \( X \) is a strongly hyperbolic metric space, then its ideal boundary carries a natural Möbius structure as observed by Nica and Spakula.

Let \( X \) be a finite set, let \( df \) be the pull-back metric of the Euclidean distance on \( X \) induced by an embedding \( f \) of \( X \) into the sphere.
Theorem: The metric spaces \((X,d_{f_1})\) and \((X,d_{f_2})\) are Möbius equivalent if and only if the associated convex polyhedra \(P_{f_1}\) and \(P_{f_2}\) are discrete-conformally equivalent.

Proof: If \((X,d_{f_1})\) and \((X,d_{f_2})\) are Möbius equivalent, then \((S_\mu,\hat{P}_{f_1})\) is isometric to \((S_\mu,\hat{P}_{f_2})\), i.e. \(P_{f_1}\) and \(P_{f_2}\) are discrete-conformally equivalent. If \(P_{f_1}\) and \(P_{f_2}\) are discrete-conformally equivalent, then there exists a Möbius transformation on the sphere mapping the vertex set of \(P_{f_1}\) onto the vertex set of \(P_{f_2}\), hence \((X,d_{f_1})\) and \((X,d_{f_2})\) are Möbius equivalent.

Note: It follows from previous considerations that two metric spaces \((X,d_{f_1})\) and \((X,d_{f_2})\) with \(N\) points are already Möbius equivalent if only a small subset of \(2N - 6\) cross-ratios is preserved, namely the cross-ratios along edges within a quadrilateral of a common Delaunay triangulation of \(P_{f_1}\) and \(P_{f_2}\).

DIRECTIONS OF FURTHER RESEARCH

Characterization of discrete conformality. It would be more elegant to have a definition of discrete-conformal equivalence of convex polyhedra by elementary transformations on vertices. We conjecture that \(P\) and \(Q\) are discrete-conformally equivalent if and only if there exists a finite sequence of closed convex polyhedra \(P = P_1, P_2, \ldots, P_{n-1}, P_n = Q\) such that, for \(k = 1, \ldots, n - 1\) the polyhedra \(P_k\) and \(P_{k+1}\) share a common Delaunay triangulation \(T_k\) and there exists a real valued function \(u_{T_k}\) on the vertices of \(P_k\) with the following property. For every edge \(ij\) in the Delaunay triangulation between vertices \(i\) and \(j\), its length in \(P_{k+1}\) is related to its length in \(P_k\) by

\[
 l_{P_{k+1}}(ij) = l_{P_k}(ij) e^{\frac{1}{2}(u_{T_k}(i)+u_{T_k}(j))}.
\]

Difficulties arise from the fact that a Delaunay triangulation of a Euclidean convex polyhedron \(P\) is not a Delaunay triangulation of the associated marked polyhedral surface \((S_\mu,\hat{P})\). Hence, the function \(f\) can be discontinuous when passing from a “cell” \(D_{PL}(T)\) to another.

Variational principles. The uniformization theory of convex polyhedra may shed some light on the relationships between the different variational principles developed in the context of discrete conformality. Glickenstein suggested a formal framework in [26]. The natural appearance of real analytic cell decompositions in the work of Gu, Luo, Sun and Wu [27], may suggest the theory of moment maps as a general setting. According to Atiyah, Guillemin and Sternberg, the image of the moment map of a hamiltonian torus action on a compact connected symplectic manifold is always a polytope [28] [29].
References


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