Coarse median preserving RAAG automorphisms, after Fioravanti

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$+$ *Adrien's explanations!*

standard generators of the group A_Γ = vertices of the graph Γ.

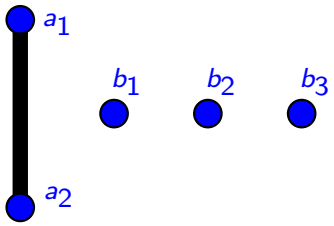
Theorem (Servatius, Laurence)

The automorphism group of a RAAG is generated by:

1. **graph automorphisms**, *acting by permuting standard generators,*
2. **inversions**: *one standard generator is mapped to its inverse, the others are unchanged,*
3. **partial conjugations**: *pick a vertex v, remove its star; pick a union P of connected components of the resulting graph; define an automorphism conjugating by v the vertices of P and fixing all other vertices,*
4. **transvections**: *pick two distinct vertices v and w, assuming that $\mathrm{link}(v) \subset \mathrm{star}(w)$; define an automorphism mapping v to $vw^{\pm 1}$ or to $w^{\pm 1}v$ and fixing all other vertices.*

Definition (Untwisted automorphisms, Day, CSV)

Say a transvection is a **twist** if the vertices v and w are adjacent, a **fold** otherwise. Say a RAAG automorphism is **untwisted** (Day says **long range**) if it admits an expression as a product of Laurence generators that does not contain twists.



Examples

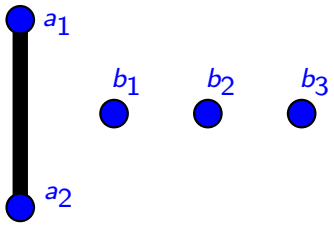
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Fioravanti's theorem

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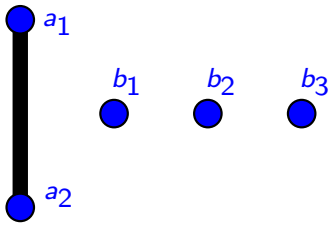
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An automorphism of a RAAG preserves the coarse median structure $\iff$ *it is untwisted.*

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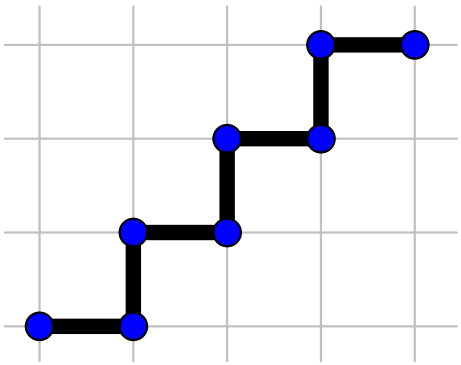


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Theorem (E. Fioravanti 2024)

An automorphism of a RAAG preserves the coarse median structure $\iff$ it is untwisted.



The transvection $a_1 \to a_1 a_2$, others fixed, does not preserve the coarse median structure. It maps a_1, whose axis is median convex, to $a_1 a_2$, whose axis α is very far from convex (the set of its median points $\mu(\alpha, \alpha, X)$ is a whole plane).

Scheme of proof of $\Longrightarrow$

The proof of $\Longrightarrow$, i.e. that coarse median structure preserving automorphisms are untwisted, follows Laurence's proof, with the following additional input:

1. An element g is label-irreducible $\Longleftrightarrow$ it generates a convex-cocompact subgroup (Lemma 3.11 (2) in Fioravanti).
2. A subgroup is convex-cocompact $\Longleftrightarrow$ it is quasiconvex, a notion which is defined for coarse median spaces (Haglund 2008).

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Explanation of terms

In $\mathbb{Z}^2 \star F_3$, the element $a_1 a_2$ (translation by $(1,1)$) has a nonhyperbolic behaviour: the convex hull of any of its axes is a whole plane. More generally, label-irreducibility has to do with the possibility of dividing the set of graph vertices occurring in a group element into two disjoint and commuting subsets, in which case the element can be written as a commuting product.

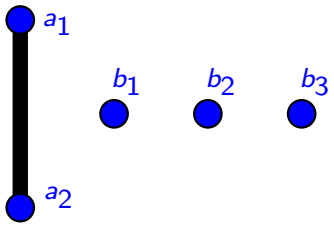
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A group element g is **label-irreducible** if the subgraph spanned by $\Gamma(g)$ is not a join (i.e. the opposite graph is connected).



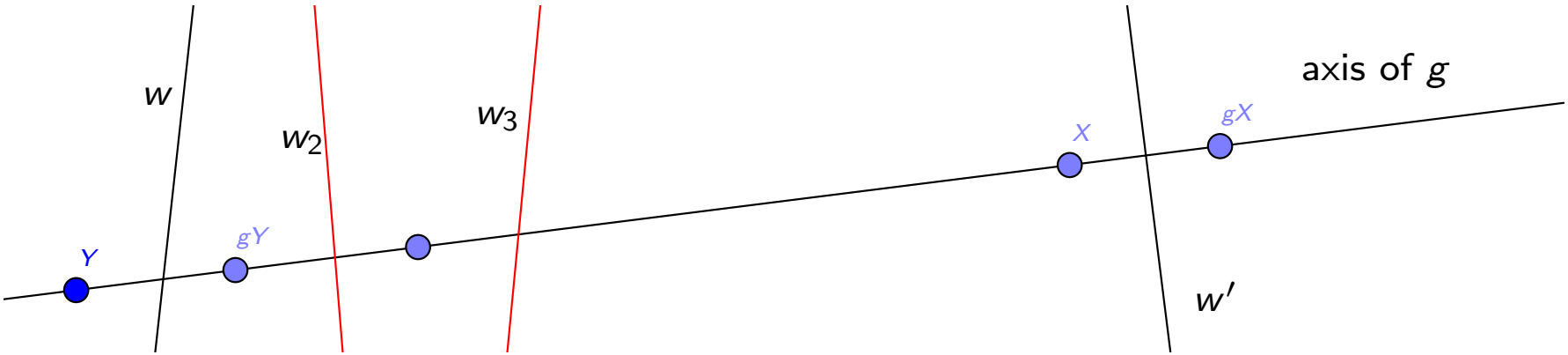
In $\mathbb{Z}^2 \star F_3$, all elements are label-irreducible except conjugates of elements of $\mathbb{Z}^2 \star \{e\}$ with both coordinates nonzero.

An automorphism ϕ is **simple** if for each standard generator v, $\phi(v)$ is label-irreducible and $v \in \Gamma(\phi(v))$.

Proof of the characterization of label-irreducibility in terms of quasiconvexity I

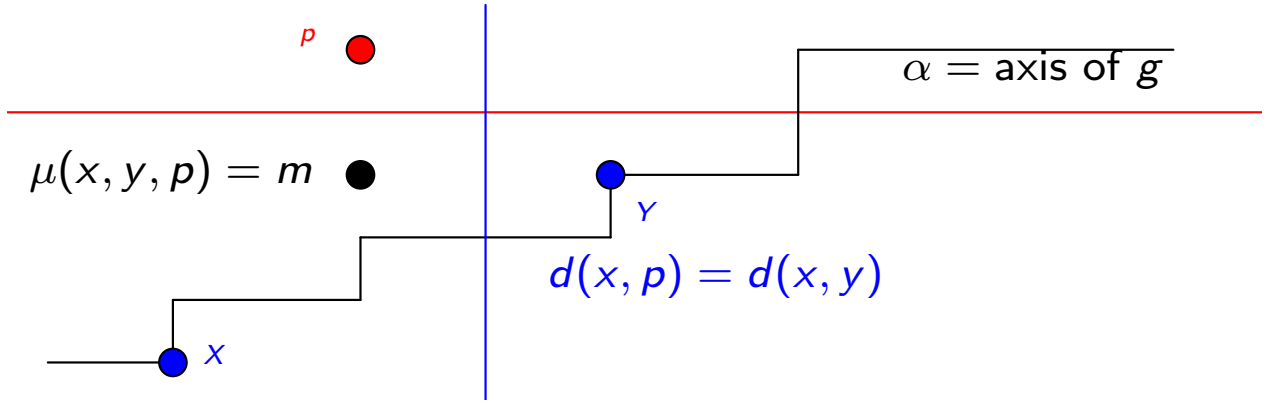
When g is label-irreducible, in order to collect all labels in $\Gamma(g)$, it suffices to consider hyperplanes separating w from $g^{4r}w$, for an arbitrary hyperplane w crossing the axes of g.

Indeed, w separates a point y of an axis from gy. Let $x = g^{2r}y$. Let w' separate x from gx. Being a connected subgraph of the opposite graph Γ^o, $\Gamma(g)^o$ has bounded diameter ($\leq 2r$). Whence a bounded sequence of nonconnected labels and walls w_i from w to w', with w_i separating $g^i y$ from $g^{i+1}y$, all crossed by the axis, such that w_i and w_{i+1} are not transverse. So each w_i separates previous walls from later walls, showing that w and w' are not transverse. Similarly, w' is not transverse to $g^{2r}w$, so w' separates w from $g^{4r}w$.



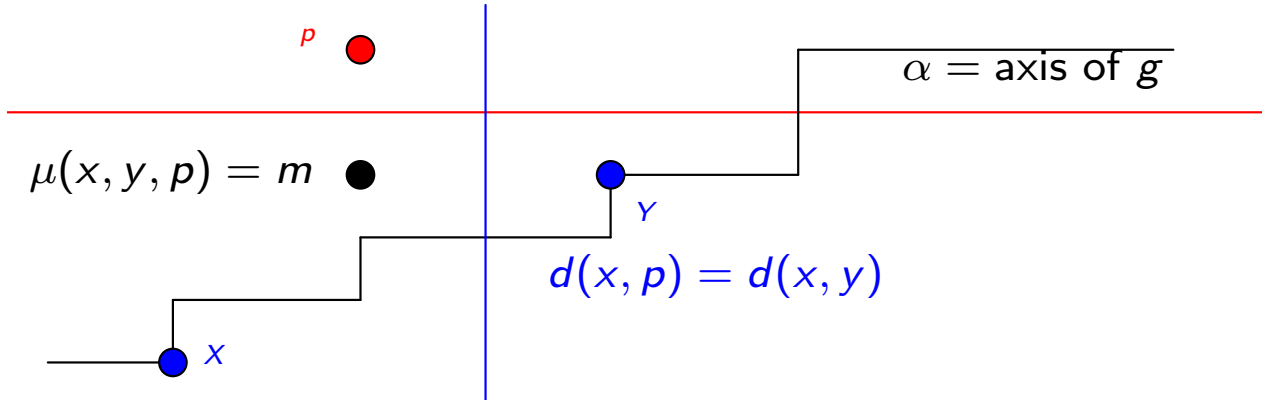
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Indeed, p roughly splits the axis α of g into a left and a right side. Pick $x \in \alpha$ far on the left. On the right of x, pick $y \in \alpha$ such that $d(x, y) = d(x, p)$. Assume that $d(p, y)$ is too large. Then $m = \mu(x, y, p)$ satisfies $d(m, y) = d(p, m) = \frac{1}{2}d(p, y)$ is too large as well. By construction, half-spaces separating p from m are transverse from those separating m from y. Indeed, their complements all intersect α along right half-lines, and thus pairwise intersect along right half-lines on α. Both sets of half-spaces, mod g^{4r}, have cardinality bounded. By the previous argument, each contains all labels in $\Gamma(g)$. So the same label appears in each. This contradicts transversality.

Thus the cyclic group $\langle g \rangle$ acts cocompactly on this hull, and on its essential core which is a restriction quotient of the hull.

Completion of the proof of $\Longrightarrow$

Conversely, if g is non label-irreducible, the essential core splits as a nontrivial product, the $\langle g \rangle$ action cannot be cocompact.

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An example of Laurence's algorithm.

Given automorphism

$$\phi : \begin{matrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \end{matrix} \mapsto \begin{matrix} a_2 \\ a_1^{-1} \\ b_2 a_2 \\ b_3 \\ a_1^{-1} b_1 a_1 \end{matrix},$$

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postcompose to simplify as follows:

$$\phi : \begin{array}{cc} a_1 & a_2 \\ a_2 & a_1^{-1} \\ b_1 & b_2 a_2 \\ b_2 & b_3 \\ b_3 & a_1^{-1} b_1 a_1 \end{array} \overset{\phi}{\mapsto} \overset{\textit{permutation}}{\mapsto} \begin{array}{cc} a_1 & a_1 \\ a_2^{-1} & a_2 \\ b_1 a_1 & b_1 a_1 \\ b_2 & b_2 \\ a_2^{-1} b_3 a_2 & a_2 b_3 a_2^{-1} \end{array} \overset{\textit{inversion}}{\mapsto} \overset{\textit{fold}}{\mapsto} \begin{array}{cc} a_1 & a_1 \\ a_2 & a_2 \\ b_1 a_1 & b_1 \\ b_2 & b_2 \\ a_2 b_3 a_2^{-1} & a_2 b_3 a_2^{-1} \end{array}$$

which is a partial conjugation.

Scheme of the proof of $\Leftarrow$ I

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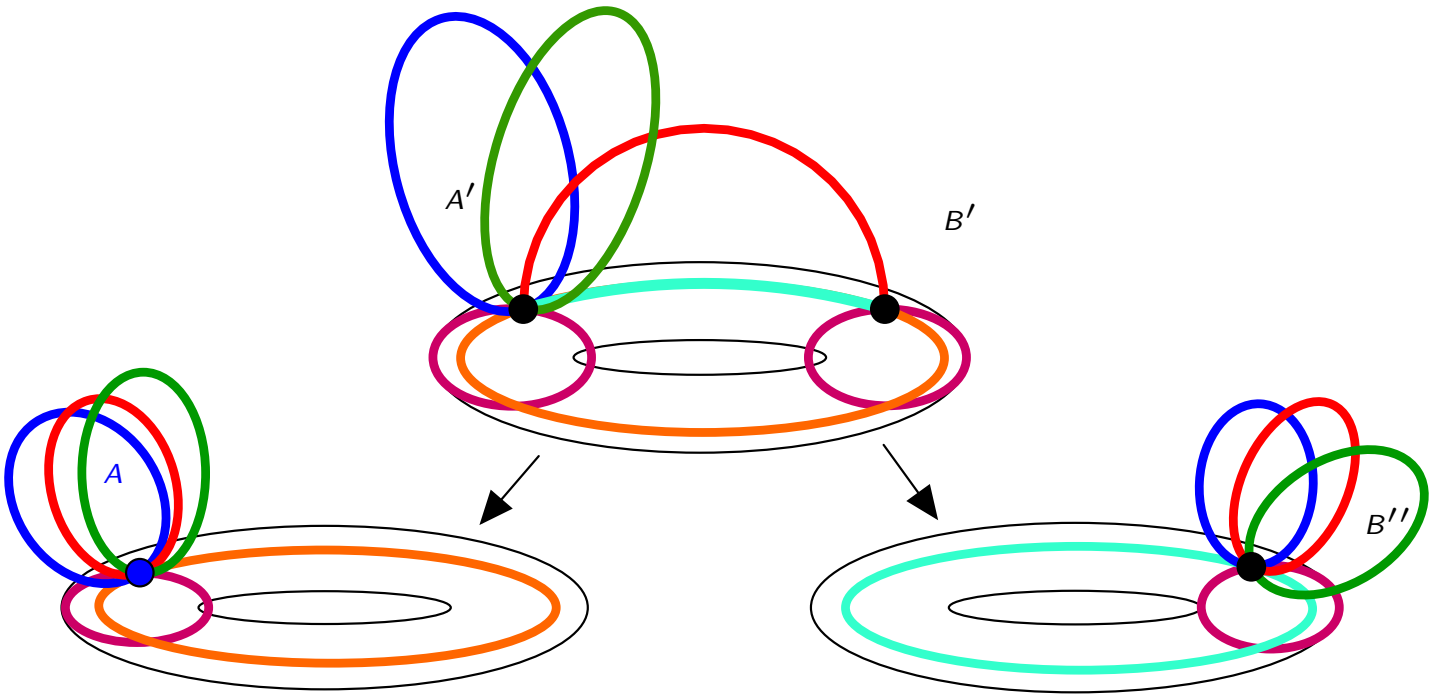
Every fold or partial conjugation ϕ can be embedded in a blow up, i.e. an action on a $CAT(0)$ cube complex Y with two restriction-quotient maps (i.e. collapsing walls): $Y \to X_\Gamma$, the CCC associated to the graph Γ, the first equivariant with respect to the standard action of A_Γ, and the other with the action twisted by ϕ.

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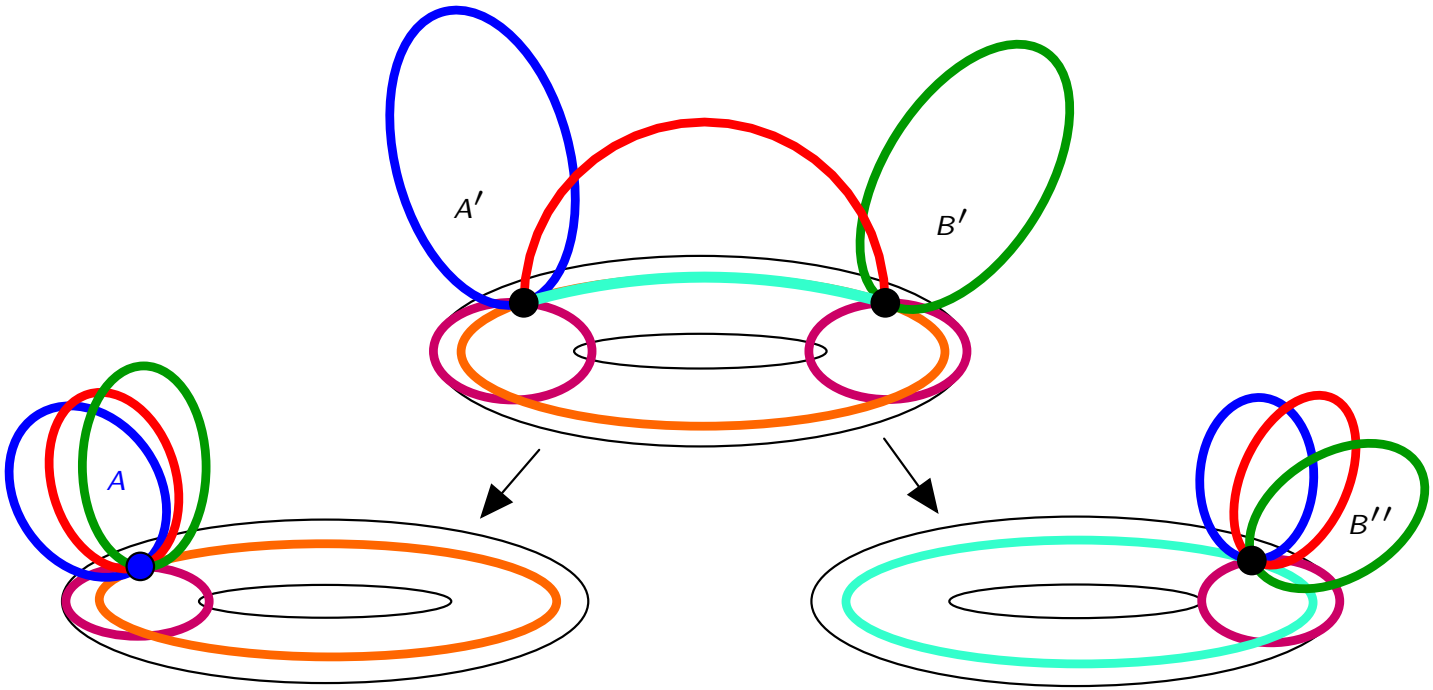
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Charney, Stambaugh and Vogtmann's blowup construction for the fold $b_1 \mapsto a_1 b_1$.



Blowup for the automorphism obtained by composing the flip $b_1 \mapsto a_1 b_1$ and the partial conjugation $b_3 \mapsto a_1^{-1} b_3 a_1$:



Since both blowup and restriction quotient preserve coarse median structures, folds and partial conjugations preserve the canonical coarse median structure of the universal cover X_Γ of the Salvetti complex.