$L^p$-cohomology of symmetric spaces

P. Pansu

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A topological space $\rightarrow$ cohomology

A manifold $\rightarrow$ de Rham cohomology
Definition

Let $M$ be a Riemannian manifold. Let $p > 1$. $L^p$-cohomology of $M$ is the cohomology of the complex of $L^p$-differential forms on $M$ whose exterior differentials are $L^p$ as well,

$$H^{k, p} = \text{closed } k\text{-forms in } L^p / d((k-1)\text{-forms in } L^p),$$

$$R^{k, p} = \text{closed } k\text{-forms in } L^p / \text{closure of } d((k-1)\text{-forms in } L^p),$$

$$T^{k, p} = \text{closure of } d((k-1)\text{-forms in } L^p) / d((k-1)\text{-forms in } L^p).$$

$R^{k, p}$ is called the reduced cohomology. $T^{k, p}$ is called the torsion.
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What is it?

topological space \rightarrow \text{cohomology}

manifold \rightarrow \text{de Rham cohomology}

metric space \rightarrow \text{cohomology with decay condition}

Riemannian manifold \rightarrow \text{de Rham cohomology with decay condition}

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Here $H^{0,p} = 0 = H^{2,p}$ for all $p$. 
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Using conformal invariance, switch from hyperbolic metric to euclidean metric on the disk \( D \).

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which is Sobolev space $H^{1/2}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.
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More generally, for $p > 1$, $T^{1,p} = 0$ and $H^{1,p}$ is equal to the Besov space $B^{1/p}_{p,p}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.
Example: the real line $\mathbb{R}$

$H^0, p = 0.$
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since every function in $L^p(\mathbb{R})$ can be approximated in $L^p$ with derivatives of compactly supported functions. Therefore $H^{1,p}$ is only torsion.
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Indeed, the 1-form $\frac{dt}{t}$ (cut off near the origin) is in $L^p$ for all $p > 1$ but it is not the differential of a function in $L^p$. 
What are our favourite spaces?


- In this talk: manifolds with large ends, e.g. symmetric spaces themselves. $L^p$-cohomology is related to analytic features of a compactification (compare A. Koranyi’s lectures).
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cohomology → continuous maps
$L^p$-cohomology → uniform maps.

**Definition**

A map $f : X \rightarrow Y$ between metric spaces is uniform if $d(f(x), f(x'))$ is bounded from above in terms of $d(x, x')$ only.
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**Examples**
The obvious map $\mathbb{Z} \to \mathbb{R}$ is uniform. Any homomorphism between groups (with left invariant metrics) is uniform. The parametrization of a cusp by a punctured disk is not uniform.
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The obvious map \( \mathbb{Z} \rightarrow \mathbb{R} \) is uniform. Any homomorphism between groups (with left invariant metrics) is uniform. The parametrization of a cusp by a punctured disk is not uniform.

**Proposition**

Among contractible Riemannian manifolds admitting a cocompact isometric group action, \( L^p \)-cohomology is natural under uniform maps.
$L^p$-cohomology can be discretized.

In conclusion, $L^p$-cohomology is a tool to investigate discrete groups.

It shares nearly all properties of usual cohomology.

Nevertheless, it is not easy to calculate it.

In the case of cocompact lattices in Lie groups, it can probably be computed by analytic means.

In this talk, we explain 3 applications of $L^p$-cohomology to negatively curved Riemannian manifolds and groups.

1. Hopf's conjecture about Euler characteristic
2. Cannon conjecture on groups with boundary a 2-sphere
3. Curvature pinching
$L^p$-cohomology can be discretized. It makes sense for discrete groups, and cannot see any difference between a cocompact lattice in a semi-simple Lie group $G$, the Lie group $G$ itself or the Riemannian symmetric space $G/K$. 

In conclusion, $L^p$-cohomology is a tool to investigate discrete groups. It shares nearly all properties of usual cohomology. Nevertheless, it is not easy to calculate it. In the case of cocompact lattices in Lie groups, it can probably be computed by analytic means. In this talk, we explain 3 applications of $L^p$-cohomology to negatively curved Riemannian manifolds and groups.

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Remark

- Compact 2-dimensional negatively curved manifolds have negative Euler characteristic.
- 2m-dimensional compact hyperbolic manifolds have Euler characteristic proportional to $(-1)^m$.
- This generalizes to all compact negatively curved locally symmetric spaces (Gauss-Bonnet).
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Conjecture

(H. Hopf). If $M$ is 2$m$-dimensional compact negatively curved, then $(-1)^m \chi(M) > 0$. 
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(H. Hopf). If $M$ is $2m$-dimensional compact negatively curved, then $(-1)^m \chi(M) > 0$.

Theorem

(M. Gromov, 1991). This is true provided $M$ also admits a Kähler metric.
Role of the Kähler condition

**Definition**

A Riemannian manifold $M$ is Kähler if it admits a parallel complex structure.
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Corollary
Let $M^{2m}$ be a complete Kähler manifold with Kähler form $\omega$. Then wedging with $\omega$ maps $L^2$-harmonic forms to $L^2$-harmonic forms, and this induces an injection in reduced $L^2$-cohomology $R^{k,2}(M) \rightarrow R^{k+2}(M)$ for all $k < m$. 
Role of negative curvature

Proposition

(M. Gromov). Let \( \tilde{M} \) be a complete simply connected negatively curved Riemannian manifold. Let \( k \geq 2 \).

- (Coning of cycles). Every \( k-1 \)-cycle \( z \) spans a \( k \)-chain \( c \) with \( \text{vol}(c) \leq \text{const.} \ \text{vol}(z) \).
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- **(Coning of forms).** Every closed bounded differential $k$-form $\alpha$ on $\tilde{M}$ is the differential of a bounded $(k-1)$-form $\beta$ with $\|\beta\|_{L^\infty} \leq \text{const.} \|\alpha\|_{L^\infty}$.
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Corollary

Assume \( \tilde{M}^{2m} \) covers both a compact Kähler manifold and a compact negatively curved Riemannian manifold. Then \( R^{k,2}(\tilde{M}) = 0 \) for all \( k \neq m \).
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Assume $\tilde{M}^{2m}$ covers both a compact Kähler manifold and a compact negatively curved Riemannian manifold. Then $R^{k,2}(\tilde{M}) = 0$ for all $k \neq m$. Furthermore, $T^{*,2}(\tilde{M}) = 0$. 
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Corollary

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Proof. Lift Kähler form to universal cover $\tilde{M}$. Write $\omega = db$ with $b$ bounded.
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**Proof.** Lift Kähler form to universal cover \( \tilde{M} \). Write \( \omega = db \) with \( b \) bounded. Let \( k < m \). For \( \alpha \) a closed \( k \)-form in \( L^2 \),

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\omega \wedge \alpha = d(b \wedge \alpha) \quad \text{and} \quad b \wedge \alpha \in L^2,
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\[ \omega \wedge \alpha = d(b \wedge \alpha) \quad \text{and} \quad b \wedge \alpha \in L^2, \]
thus \( \omega \wedge \alpha = 0 \) in \( R^{k+2,2}(\tilde{M}) \). If \( \alpha \) is harmonic, conclude that \( \alpha = 0 \) in \( R^{k,2}(\tilde{M}) \).
Let $\tilde{M}$ cover a compact manifold $M$. If nonzero, $R^{k,2}(\tilde{M})$ is infinite dimensional.
Let \( \tilde{M} \) cover a compact manifold \( M \). If nonzero, \( R^{k,2}(\tilde{M}) \) is infinite dimensional. Nevertheless, M. Atiyah defined a von Neumann dimension

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**Examples**

*(W. Lück)*. If $M$ admits a tower of finite degree $d_j$ normal coverings $M_j$ such that $\bigcap_j \pi_1(M_j) = \{1\}$, then

$$b^{k,2}(M) = \lim_{j \to \infty} \frac{b^k(M_j, \mathbb{R})}{d_j}.$$
Let $\tilde{M}$ cover a compact manifold $M$. If nonzero, $R_k,2(\tilde{M})$ is infinite dimensional. Nevertheless, M. Atiyah defined a von Neumann dimension

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**Proposition**

Let $\tilde{M}$ cover a compact manifold $M$. Then

$$\chi(M) = \sum_k (-1)^k b_k,2(M).$$
Proposition

(Relative index theorem, M. Gromov-B. Lawson). Let $\tilde{M}$ be a simply connected nonpositively curved Riemannian manifold. Then there exists $k$ such that $H^{k,2}(\tilde{M}) \neq 0$. 

Proof of Gromov’s theorem.

Assume $M$ is compact and admits both a negatively curved metric and a Kähler metric. Then all $b_{k,2}(M)$ vanish except $b_{m,2}(M)$, which is nonzero, thus $(-1)^m \chi(M) > 0$.

In conclusion, we have used ▶ Lefschetz mechanism, L^2-Betti numbers. ▶ Vanishing of L^∞-cohomology. ▶ Cup-product $H^k,2 \otimes H^2,\infty \to H^{k+2,\infty}$. 

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- Lefschetz mechanism, $L^2$-Betti numbers.
- Vanishing of $L^\infty$-cohomology.
- Cup-product $H^{k,2} \otimes H^{2,\infty} \to H^{k+2,\infty}$. 
1. Hopf’s conjecture about Euler characteristic
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3. Curvature pinching
Remark

In negatively curved manifolds, triangles are thin.
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Examples
- A free group is hyperbolic, its ideal boundary is totally disconnected.
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(M. Gromov 1986). Say a finitely generated group is hyperbolic if its triangles are thin. A hyperbolic group has a functorial ideal boundary.

Examples
- A free group is hyperbolic, its ideal boundary is totally disconnected.
- The fundamental group of a compact negatively curved \(n\)-manifold is hyperbolic, its ideal boundary is an \(n - 1\)-sphere.
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Conjecture

(J. Cannon). Let $\Gamma$ be a hyperbolic group whose ideal boundary is a 2-sphere. Then $\Gamma$ is virtually a cocompact lattice in $\text{PSL}(2, \mathbb{C})$. 
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Strategy (B. Kleiner). Prove that this infimum is achieved. Then prove that dimension minimizing metrics are Riemannian if boundary is 2-dimensional. Then apply a result of D. Sullivan (1978): *every uniformly quasiconformal group of the standard 2-sphere is conjugate to a subgroup of $PSL(2, \mathbb{C})$.*
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Theorem
(S. Keith-T. Laakso, M. Bonk-B. Kleiner 2005). Let $\Gamma$ be a hyperbolic group whose ideal boundary is a 2-sphere. If conformal dimension is achieved, then $\Gamma$ is virtually a cocompact lattice in PSL(2, C).
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(Same people + M. Bourdon-H. Pajot 2003). Let $\Gamma$ be a hyperbolic group. Then $L^p$-dimension is less than or equal to conformal dimension. If conformal dimension is achieved, then $L^p$-dimension and conformal dimension coincide.

Examples
(M. Bourdon-H. Pajot). There exist hyperbolic groups for which conformal dimension $> 2 \geq L^p$-dimension. For such groups, conformal dimension cannot be achieved.

In conclusion, we have used
- Mayer-Vietoris and $L^2$-Betti numbers.
- Expression of $H^{1,p}$ as a function space on the ideal boundary.
1. Hopf’s conjecture about Euler characteristic
2. Cannon conjecture on groups with boundary a 2-sphere
3. Curvature pinching
Remark

Rank one symmetric spaces are hyperbolic spaces over the reals $H^n_{\mathbb{R}}$, the complex numbers $H^m_{\mathbb{C}}$, the quaternions $H^m_{\mathbb{H}}$, and the octonions $H^2_{\mathbb{O}}$. Real hyperbolic space has sectional curvature $-1$. Other rank one symmetric spaces are $-\frac{1}{4}$-pinched, i.e. their sectional curvature ranges between $-1$ and $-\frac{1}{4}$. 

Definition

Define the optimal pinching $\delta(G)$ of a discrete (or Lie) group $G$ as the least $\delta > -\frac{1}{4}$ such that $G$ is bi-uniformly equivalent to a $\delta$-pinched Riemannian manifold.

Conjecture

The optimal pinching of $SU(m,1)$, $Sp(m,1)$ ($m \geq 2$) and $F_{-20}^4$ is $-\frac{1}{4}$. 


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If \( M^n \) is simply connected and \( \delta \)-pinched for some \( \delta \in [-1,0) \), then

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This is sharp. For instance, consider the semidirect product $G = \mathbb{R}^3 \rtimes_\alpha \mathbb{R}$ where $\alpha = \text{diag}(1,1,2)$.

- It admits a $-\frac{1}{4}$-pinched left-invariant Riemannian metric, therefore $\delta(G) \leq -\frac{1}{4}$.
- It has $T^{2,p}(G) \neq 0$ for $2 < p \leq 4$. This implies that $\delta(G) = -\frac{1}{4}$. 

Remark
Complex hyperbolic plane $\mathbb{H}^2_C$ is isometric to $G' = \text{Heis} \rtimes_\alpha \mathbb{R}$ where $\alpha = \text{diag}(1,1,2)$ and $\text{Heis}$ denotes the Heisenberg group. Therefore it is very close to $G$. 

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**Theorem**

$T^{2,p}(H^2_{\mathbb{C}}) = 0$ for $2 < p < 4$. 

Proof of torsion comparison theorem

Use the gradient vectorfield $\xi$ of a Busemann function and its flow $\phi_t$, whose derivative is controlled by sectional curvature. For $\alpha$ a closed $k$-form in $L^p$,

$$\phi_t^*\alpha = \alpha + d\left(\int_0^t \phi_s^* \iota_\xi \alpha \, ds\right)$$

has a limit as $t \to +\infty$ under the assumptions of the theorem. This boundary value map injects $H^{k,p}$ into a function space of closed forms on the ideal boundary, showing that $H^{k,p}$ is Hausdorff.
Proofs of vanishing theorems

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Proof of torsion vanishing for $H^2_C$
For $p \notin \{4/3, 2, 4\}$, differential forms $\alpha$ on $H^2_C$ split into components $\alpha_+$ and $\alpha_+$ which are contracted (resp. expanded) by $\phi_t$. Then

$$B_t : \alpha \mapsto \int_0^t \phi_s^* \iota_\xi \alpha_+ \, ds + \int_{-t}^0 \phi_s^* \iota_\xi \alpha_- \, ds$$

converges as $t \to +\infty$ to a bounded operator $B$ on $L^p$. $P = 1 - dB - Bd$ retracts the $L^p$ de Rham complex onto a complex of differential forms on $Heis^3$ with missing components and weakly regular coefficients. If $2 < p < 4$, this complex is nonzero in degrees 1 and 2, but it is so small that its cohomology can be shown to be Hausdorff.
Use Poincaré duality. Let $p' = p/p - 1$ denote the conjugate exponent. In order to prove that a closed $k$-form $\alpha$ is nonzero in cohomology, it suffices to construct a sequence $\psi_j$ of $(n - k)$-forms such that $\|d\psi_j\|_{L^{p'}}$ tends to zero but $\int \alpha \wedge \psi_j$ does not tend to zero.
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In conclusion, we have used

- Poincaré duality.
- A deformation retraction of space onto a subspace, with controlled effect on the $L^p$-norms of forms. For certain ranges of $p$, this provides a boundary value.
Non-vanishing of torsion

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\begin{itemize}
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\end{itemize}

Conjecture
\begin{itemize}
  \item For rank 1 symmetric spaces, \( T^{k,p} = 0 \) except for at most 1 value of \( p \) in each degree.
  \item For higher rank symmetric spaces, \( H^{k,p} = 0 \) for \( k < \text{rank} \), \( T^{k,p} = 0 \) for \( k = \text{rank} \).
  \item For \( k = \text{rank} \), \( R^{k,p} \neq 0 \) for \( p \) large, and \( R^{k,p} \) is a function space on the maximal boundary.
  \item For each \( p > 1 \), there exists \( k \) such that \( H^{k,p} \neq 0 \).
\end{itemize}
$L^p$-cohomology of $H^2_C$

- degree 0: 
  - exponent $p = 1, 4/3, 2, 4$
- degree 1: 
- degree 2: 
- degree 3: 
- degree 4: 

- non vanishing reduced cohomology
- non vanishing torsion
$L^p$-cohomology of $G$

$L^p$ cohomology of $(\mathbb{R}^4, g_1)$

- **Exponent $p$:** $1$  $4/3$  $2$  $4$
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- **Non-vanishing reduced cohomology**
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