FILLING HORIZONTAL CURRENTS IN HEISENBERG BALLS

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ABSTRACT. R. Young's filling inequality for low-dimensional cycles in Heisenberg groups is cast in the language of horizontal currents: below the middle dimension, every closed horizontal current spans a horizontal current with the expected mass control. In addition, a relative version of the filling inequality is obtained. The main tools are an additive and stochastic version of Federer-Fleming's deformation theorem in Riemannian manifolds, and a generalization of Balogh-Fässler-Platis' stretch contactomorphism.

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1. Introduction

1.1. **Filling problems.** In Euclidean n-space, the isoperimetric inequality states that the volume of a bounded domain is sharply controlled by the volume of its boundary, round balls achieving the limiting case. Here is a fancier version of this theorem, in terms of integral currents: every closed integral $n-1$-current S bounds an integral n-current T such that

$$\mathcal{M}(T) \leq c_n \, \mathcal{M}(S)^{\frac{n}{n-1}}, \tag{1}$$

with equality for round spheres. Note that the isoperimetric inequality is equivalent to a $(\frac{n}{n-1}, 1)$-Sobolev inequality.

The isoperimetric inequality has a local version, equivalent to the $(\frac{n}{n-1}, 1)$-Poincaré inequality (up to a small loss in the constant): in a closed ball B, let S be an integral $n-1$-current whose boundary ∂S is supported in the boundary sphere ∂B. Then S, together with some integral $n-1$-current S' supported in ∂B bounds an integral n-current T such that

$$\mathcal{M}(T) \leq c'_n \, \min\{\mathcal{M}(S)^{n/(n-1)}, \mathcal{M}(B)^{1/n} \, \mathcal{M}(S)\}.$$

Filling inequalities generalize the isoperimetric inequality when hypersurfaces are replaced with objects of intermediate dimensions. In a Riemannian manifold V, given a closed integral k-current S which is 0 in the homology group $H_k(V;\mathbb{Z})$, which mass bound can one give on an integral $k+1$ current T such that $\partial T = S$?

For Euclidean spaces, the answer is due to F. Almgren, [2]. The sharp bound is

$$\mathcal{M}(T) \leq c_{k+1}\, \mathcal{M}(S)^{\frac{k+1}{k}},$$

with the same sequence of constants c_k encountered in inequality (1). Indeed, equality holds for round k-spheres in flat $k+1$-planes.

Filling inequalities are central in Riemannian geometry, see [10]. When V is the universal covering space of a compact Riemannian manifold with fundamental group Γ, filling inequalities in V are quasiisometry invariants of Γ, up to the usual equivalence. The case $k = 1$ is especially interesting, since it is related to the hardness of the word problem for presentations of Γ. For recent examples of filling estimates with significant consequences in geometric group theory, see [11] and [5].

1.2. **Horizontal fillings.** In this paper, we are concerned with Heisenberg groups $\mathbb{H}^n$ endowed with their left-invariant Riemannian metrics, see Section 2.1. We view the filling problem as a local, analytical problem, and we shall not attempt to obtain sharp constants.

The first issue is what should play the role of integral currents. We intend to exploit the existence of dilations in Heisenberg groups. These are group automorphisms $\mathfrak{s}_\lambda$, $\lambda > 0$, which behave pretty much like Euclidean homotheties. Except that volumes of smooth k-dimensional submanifolds W do not scale exactly by the expected factor λ^k. This happens only if W is *horizontal*, i.e. the left-invariant contact form θ restricts to 0 to it. This suggests considering *horizontal integral currents*. Note that in bounded parts of $\mathbb{H}^n$, left-invariant Riemannian distances are equivalent to a Euclidean distance in exponential coordinates, so the notion of integral current is the familiar one, from [7].

Definition 1.1. *A* horizontal integral current *in $\mathbb{H}^n$ is an integral current T with compact support, satisfying the extra conditions*

$$T \llcorner \theta = 0, \quad \partial T \llcorner \theta = 0.$$

Such currents exist only in dimensions k up to n. Indeed, these conditions imply that $T \llcorner d\theta = 0$, so the tangent spaces of T are a.e. isotropic for $d\theta$, hence have dimension $\leq n$. So the horizontal filling problem is well-posed only when $k \leq n-1$.

Theorem 1.2. *Let $1 \leq k \leq n-1$. There exists a constant $C = C(n,k)$ with the following property. For every closed horizontal integral k-current S in $\mathbb{H}^n$, there exists a horizontal integral $(k+1)$-current T such that*

(1) $\partial T = S$;
(2) $\mathcal{M}(T) \leq C\, \mathcal{M}(S)^{(k+1)/k}$.

Here is a local version of the horizontal filling inequality. It is a bit weak since a loss on the domain is tolerated (but this is not a surprise; existing Poincaré inequalities for differential forms also suffer from losses on domains). Cygan-Koranyi balls are defined in Section 2.1.

Theorem 1.3. *Let $1 \leq k \leq n-1$ and $0 < \lambda < \mu < 1$. There exists a constant $C = C(n,k,\lambda,\mu)$ with the following property. Let U be a Cygan-Koranyi r-ball in Heisenberg group $\mathbb{H}^n$. For every horizontal integral k-current S in $\mathbb{H}^n$ such that ∂S is supported in the complement of U, there exists a horizontal integral k-current S' and a horizontal integral $(k+1)$-current T such that*

1. *T is supported in μU and S' is supported in $\mu U \setminus \lambda U$;*
2. *$\partial T - S - S'$ is supported in the complement of μU ;*
3. *$\mathcal{M}(T) \leq C\,\Phi(\mathcal{M}(S))$, where*
$$\Phi(t) = \min\{t^{(k+1)/k}, r\,t\};$$
4. *$\mathcal{M}(S') \leq C\,\mathcal{M}(S)$, $\mathcal{M}(\partial S') \leq \frac{C}{r}\,\mathcal{M}(S)$.*

1.3. **Additive versions.** We expect these results to serve as building blocks for a general horizontal filling inequality in contact manifolds. Therefore, we provide more handy versions of both filling results, where the filling $S \mapsto T$ is provided by an additive operator $T = \rho(S)$. Additive means that, when closed, a sum of integral currents $S_1 + S_2$ is filled by $\rho(S_1) + \rho(S_2)$. The price to pay is that ρ depends on a random parameter, it becomes a random chain homotopy of the chain complex of horizontal integral currents.

Theorem 1.4. *The group I^h of horizontal integral currents of $\mathbb{H}^n$ is randomly null-homotopic in dimensions up to $n-1$, with homogeneous mass bounds in expectation.*

Namely, there exists a constant $C = C(n)$ and random maps $\rho_k : I^h_k \to I^h_{k+1}$, for $0 \leq k \leq n-1$, such that

1. *ρ is additive: for every horizontal integral currents S_1, S_2, $\rho(S_1 + S_2) = \rho(S_1) + \rho(S_2)$;*
2. *$\rho\partial + \partial\rho$ is the identity map on $I^h_{\leq n-1}$;*
3. *for every horizontal integral k-current S on $\mathbb{H}^n$, $1 \leq k \leq n-1$,*
$$\mathbb{E}(\mathcal{M}(\rho(S))) \leq C\,\mathcal{M}(S)^{(k+1)/k}.$$
4. *The random map ρ commutes with the Heisenberg dilation $\mathfrak{s}_2$.*

Here is the relative version.

Theorem 1.5. *Let U be an open Cygan-Koranyi r-ball in Heisenberg group $\mathbb{H}^n$. The group $I^h(U)$ of horizontal integral currents on U is randomly null-homotopic in dimensions up to $n-1$, with homogeneous mass bounds in expectation.*

Namely, for $0 < \lambda < 1$*, there exist a constant* $C = C(n, \lambda)$ *and random maps* $\rho_{k,U} : I^h_k \to I^h_{k+1}(U)$*, for* $0 \leq k \leq n-1$*, such that*

(1) ρ_U *is additive;*
(2) *the random operator* $\rho_U \partial + \partial \rho_U$ *restricts to the identity on* U*,* $(\rho_U \partial + \partial \rho_U) \llcorner U = Id \llcorner U$ *in dimensions up to* $n-1$*;*
(3) *for every horizontal integral* k*-current* S *on* $\mathbb{H}^n$*,* $0 \leq k \leq n-1$*,*

$$\mathbb{E}(\mathcal{M}(\rho_U(S))) \leq C\, r\, \mathcal{M}(S),$$

and if $1 \leq k \leq n-1$*,*

$$\mathcal{M}(\rho_U(S)) \leq C\, \mathcal{M}(S)^{(k+1)/k};$$

(4) *if in addition* S *has support in the complement of* U*, then* $\rho_U(S)$ *has support in the complement of* λU*.*

Question 1.6. *In dimension* 0*, the map* ρ *can be viewed as a stochastic process (see Remark 6.4). Is it worth being studied?*

1.4. **Method.** Theorem 1.2 follows almost immediately from R. Young's filling inequality for discrete Heisenberg groups, [15]. Indeed, R. Young proves a version of Theorem 1.2 for Lipschitz chains in $\mathbb{H}^n$. By density of Lipschitz chains, rescaling and compactness, Theorem 1.2 follows. This observation appears in [13, Theorem A.5], details can be found in Section 2.

Similarly, Theorem 1.3 follows from a relative filling inequality for Lipschitz chains, Theorem 2.2. The proof of Theorem 2.2 follows closely R. Young's method, with some adaptation to the relative setting.

Recall first Federer-Fleming's proof of the Euclidean filling inequality (with unsharp constant). Fixing a scale $\epsilon > 0$, they use the family of translates of a grid of mesh ϵ. They deform every closed integral k-current S to a finite sum P of k-simplices of the grid. Deforming means expressing $S = P + \partial Q$ where both P and Q have masses controlled by $\mathcal{M}(S)$. When ϵ is chosen of the order of $\mathcal{M}(S)^{1/k}$, the bound on $\mathcal{M}(P)$ makes it smaller that the mass of a k-simplex of the grid. This implies that $P = 0$, thus $S = \partial Q$ with a scale invariant control on mass, i.e. $\mathcal{M}(Q) = O(\mathcal{M}(S)^{(k+1)/k})$.

In $\mathbb{H}^n$, one can still use periodic triangulations τ as grids, and a deformation theorem à la Federer-Fleming works. However, one must use Heisenberg dilatations $(\mathfrak{s}_t)_{t>0}$, which unfortunately do not enlarge masses of k-chains exactly by t^k, but rather by t^{k+1}, unless they are horizontal. So the deformation à la Federer-Fleming needs be followed by a map that turns simplices of τ into horizontal chains. This introduces some error, hence can be done only at a small scale. So R. Young considers all deformations P_i of rescalings of S consecutively, and fills in the differences $P_{i+1} - P_i$.

The new features in the relative setting are the following. We must construct two chains, S' that turns S into a closed chain, and T which fills in $S + S'$. The Deformation Theorem and Young's iteration produce both, see Section 4. We arrange that S' avoids a small ball, see Section 4.5. When $\mathcal{M}(S)$ is large, this makes the iteration stop earlier, and gives the linear bound involving the diameter r. However, the chain S' merely avoids a relatively small ball. Thus we use a generalization of Balogh, Fässler and Platis' stretch map, [3]. This is a one parameter group of contactomorphisms which stretch that small ball to a large ball, see Section 5.

A comment on the deformation theorem is in order. We use an additive version of it, Theorem 3.8, which has independent interest. In Euclidean space, it is a mere restatement of an intermediate step of the classical construction, a step where algebraic properties still hold, which facilitates applications. It seems to have appeared for the first time in [14]. We observe that the extension to Riemannian manifolds is fairly easy. The harmless drawback is that it becomes stochastic: the additive filling procedure depends on a random choice. The final fillings, i.e. Theorems 1.4 and 1.5, are thus additive, and stochastic.

Horizontal currents may be replaced with Ambrosio-Kirchheim metric currents (with respect to the Cygan-Koranyi distance) or with Rumin currents, as shown in the companion paper [8].

2. From chains to horizontal currents

2.1. **Heisenberg groups.** The n-th Heisenberg group $\mathbb{H}^n$ is a central extension of the abelian group $\mathbb{C}^n$ by $\mathbb{R}$, its Lie algebra $\mathfrak{h}$ splits as

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2,$$

with $\dim(\mathfrak{h}_1) = 2n$, $\dim(\mathfrak{h}_2) = 1$ and a Lie bracket $\mathfrak{h}_1 \to \mathfrak{h}_2$ given by the standard symplectic form of $\mathbb{C}^n$. Therefore $\mathbb{H}^n$ can be viewed as a group multiplication on $\mathbb{C}^n \times \mathbb{R}$,

$$(z,t) \star (z',t') = (z + z', t + t' + 2\Im(z\bar{z}')).$$

Heisenberg groups admits group automorphisms $\mathfrak{s}_\lambda$, $\lambda > 0$, called *dilations*, defined on the Lie algebra by

$$\mathfrak{s}_\lambda = \lambda Id \oplus \lambda^2 Id.$$

There exist left-invariant distances which are homogeneous of degree 1 under these dilations. For instance, the Cygan-Koranyi distance defined in coordinates by

$$r(e,(z,t)) := \sqrt[4]{|z|^2+t^2}.$$

They constitute rough but convenient approximations of the left-invariant Riemannian distance, at large scales.

2.2. **Lipschitz chains.** By a scaling, compactness and convergence argument, Theorems 1.2 and 1.3 follow from the version of the same statement with currents replaced with Lipschitz chains which are not required to be horizontal. The Lipschitz chain version of the filling inequality for a family of groups that includes Heisenberg groups is due to R. Young. His paper [15] has been our main source of inspiration. Nevertheless, the Lipschitz chain filling inequality we need is slightly different, since it is provided by an additive operator on chains.

Definition 2.1. *A* polyhedron *of dimension k is a compact convex subset with nonempty interior in Euclidean k-space, which is the convex hull of finitely many points.*

A Lipschitz k-chain in a Riemannian manifold V is a finite sum of the form

$$\sum_i a_i(\phi_i)_{\#}[\![P_i]\!]$$

where, for each i, $a_i \in \mathbb{Z}$, P_i is a k-dimensional polyhedron, $\phi_i : P_i \to V$ is a Lipschitz map and $[\![P_i]\!]$ denotes the (Federer-Fleming) current of integration on P_i.

A Lipschitz chain is horizontal *if the maps ϕ_i satisfy $\phi^*\theta = 0$ a.e. This implies that $\phi^* d\theta = 0$ a.e.*

Hence Lipschitz chains are particular cases of integral currents and horizontal Lipschitz chains are particular cases of horizontal integral currents

The group of Lipschitz k-chains in $\mathbb{H}^n$ (resp. with support in a set $\mathcal{U}$) is denoted by L_k (resp. $L_k(\mathcal{U})$). The subgroups of horizontal chains are denoted by L^h_k (resp. $L^h_k(\mathcal{U})$).

2.3. **From Lipschitz chains to horizontal currents.**

Theorem 2.2. *For every n and $\lambda < 1$, there exist $\nu = \nu(n) > 1$ and $C = C(n, \lambda) > 0$ with the following significance. Let U be a Cygan-Koranyi r-ball in Heisenberg group $\mathbb{H}^n$, $r \geq 1$. There exists a random map $R : L^h_{\leq n-1} \to L_{\leq n}(\nu U)$, mapping each L^h_k to $L_{k+1}(\nu U)$, such that*

(1) *R is additive;*
(2) *the random operator $R\partial + \partial R$ restricts to the identity on U, $(R\partial + \partial R) \llcorner U = Id$ in dimensions up to $n-1$;*
(3) *for every horizontal Lipschitz k-chain S in $\mathbb{H}^n$, $0 \leq k \leq n-1$,*

$$\mathbb{E}(\mathcal{M}(R(S))) \leq C\left(\mathcal{M}(S) + \Phi_1(\mathcal{M}(S), \mathcal{M}(\partial S))\right);$$

if $2 \leq k \leq n-1$ (or $k = 1$ if $\partial S = 0$), then for every $\eta > 1$, with probability $\geq 1 - \frac{1}{\eta}$,

$$\mathcal{M}(R(S)) \leq C\left(\mathcal{M}(S) + \Phi_2(\eta\mathcal{M}(S), \eta\mathcal{M}(\partial S))\right),$$

where

$$\Phi_1(a,b) = ra + r^2 b, \quad \Phi_2(a,b) = a^{(k+1)/k} + b^{(k+1)/(k-1)};$$

(4) *if in addition S has support in the complement of U, then $R(S)$ has support in the complement of λU.*

We state the $r = +\infty$ case separately.

Theorem 2.3. *For every n, there exist $\nu = \nu(n) > 1$ and $C = C(n) > 0$ with the following significance. There exists a random map $R : L^h_{\cdot \le n-1} \to L_{\cdot \le n}$, mapping each L^h_k to L_{k+1}, such that*

(1) *R is additive;*
(2) *the random operator $R\partial + \partial R$ is the identity in dimensions up to $n-1$.*

For every horizontal Lipschitz k-chain S in $\mathbb{H}^n$, $2 \le k \le n-1$ (or $k = 1$ if $\partial S = 0$), and for every $\eta > 1$, with probability $\ge 1 - \frac{1}{\eta}$,

(3) *the mass of $R(S)$ is bounded by*

$$\mathcal{M}(R(S)) \le C\,(\mathcal{M}(S) + \Phi_2(\eta\mathcal{M}(S), \eta\mathcal{M}(\partial S))),$$

where

$$\Phi_2(a,b) = a^{(k+1)/k} + b^{(k+1)/(k-1)};$$

(4) *the support of $R(S)$ is contained in the Cygan-Koranyi Δ-neighborhood of $\operatorname{supp}(S)$, with*

$$\Delta \le C\,((\eta\mathcal{M}(S))^{1/k} + (\eta\mathcal{M}(\partial S))^{1/(k-1)}).$$

Question 2.4. *Can one upgrade the estimate "in probability" of Theorem 2.3 to an estimate "in expectation"*

$$\forall S \in L^h_{\cdot \le n-1}, \quad \mathbb{E}(\mathcal{M}(R(S))) \le C\,(\mathcal{M}(S) + \Phi_2(\mathcal{M}(S), \mathcal{M}(\partial S)))?$$

2.4. **Riemannian properties of dilations.** The dilations $(\mathfrak{s}_t)$ are not homothetic for the left-invariant Riemannian metric, they are even more contracting if $t < 1$, and this will help. Here, $\mathcal{M}$ denotes the Riemannian mass (supremum of values on differential forms with Riemannian norm a.e. ≤ 1), and F the flat norm

$$F(T) = \inf\{\mathcal{M}(S) + \mathcal{M}(R)\,;\, T = S + \partial R\}.$$

Lemma 2.5. *Let $t < 1$. Then*

(1) *$Lip(\mathfrak{s}_t) \le t$.*
(2) *Let u denote the Cygan-Koranyi distance to the origin. Then, on $\{u \ge 1\}$, u is Lipschitz with respect to the left-invariant Riemannian metric.*

(3) *For every k-dimensional current S,*

$$\mathcal{M}((\mathfrak{s}_t)_\# S) \leq t^k \mathcal{M}(S), \quad F((\mathfrak{s}_t)_\# S) \leq t^k F(S).$$

Proof. 1. Let $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}$ be the homomorphism that collapses the 1-dimensional center. Let g_e denote the pull-back by π of the Euclidean metric. Then $(\mathfrak{s}_t)^* g_e = t^2 g_e$. Since the Riemannian metric is $g = g_e + \theta^2$,

$$(\mathfrak{s}_t)^* g = t^2 g_e + t^4 \theta^2 \leq t^2 g_t,$$

since $t < 1$. So $Lip(\mathfrak{s}_t) \leq t$.

2. Since u is smooth away from the origin e, $Lip(u)$ is bounded, say, by c_1, on $\{1 \leq u \leq 2\}$. On $\{2^j \leq u \leq 2_{j+1}\}$,

$$u = 2^j u \circ \mathfrak{s}_{2^{-j}},$$

so

$$Lip(u_{|\{2^j \leq u \leq 2_{j+1}\}}) \leq 2^j Lip(u_{|\{1 \leq u \leq 2\}}) Lip(\mathfrak{s}_{2^{-j}}) \leq c_1.$$

3. The general fact $\mathcal{M}(f_\# T) \leq Lip(f)^k \mathcal{M}(T)$ yields, for $t < 1$,

$$\mathcal{M}((\mathfrak{s}_t)_\# T) \leq t^k \mathcal{M}(T).$$

If $T = S + \partial R$ with $\mathcal{M}(S) + \mathcal{M}(R) = F(T)$, $(\mathfrak{s}_t)_\# T = (\mathfrak{s}_t)_\# S + \partial (\mathfrak{s}_t)_\# R$, with

$$\mathcal{M}((\mathfrak{s}_t)_\# S) \leq t^k \mathcal{M}(S), \quad \mathcal{M}((\mathfrak{s}_t)_\# R) \leq t^{k+1} \mathcal{M}(R) \leq t^k \mathcal{M}(R),$$

since $t < 1$, so

$$F((\mathfrak{s}_t)_\# T) \leq \mathcal{M}((\mathfrak{s}_t)_\# S) + \mathcal{M}((\mathfrak{s}_t)_\# T) \leq t^k (\mathcal{M}(S) + \mathcal{M}(T)) = t^k F(T).$$

□

2.5. **Clipping.** Let S be a horizontal curent. Note that Theorem 1.3 does not make any assumption on the mass of ∂S, whereas Theorem 2.2 requires such an assumption. To achieve it, one clips S by a concentric ball μU in order to achieve a control of boundary mass by $\mathcal{M}(S)$.

Lemma 2.6. *Let $\mu < 1$. Let U be a Cygan-Koranyi r-ball. Assume that $\mu R \geq 1$. There exists a constant $C_1 = C_1(\mu)$ with the following property. Let S be a horizontal integral k-current in $\mathbb{H}^n$ such that ∂S is supported in the complement of U. There exists $\lambda' \in (\mu, 1)$ such that $S'' := S \llcorner \lambda' U$ satisfies $\mathcal{M}(\partial S'') \leq \tilde{\Phi}(\mathcal{M}(S))$, for*

$$\tilde{\Phi}(t) = C_1 \frac{t}{r}.$$

Proof. We use a Riemannian version of the Slicing Theorem (which is well known in Euclidean space, [7] p. 395 or [12], Chapter IV formula (4)). For any Lipschitz continuous function $u : \mathbb{R}^{2n+1} \to \mathbb{R}$ we have

$$\int_a^b \mathcal{M}(< S, u, x+ >)\, d\mathcal{L}^1(x) \leq \mathrm{Lip}(u)\mathcal{M}(S)(\{a < x < b\}),$$

where masses and Lip are Riemannian and

$$< S, u, x+ >= \partial(S \llcorner \{u \leq x\}) - (\partial S) \llcorner \{u \leq x\}.$$

Let us choose for u the Cygan-Koranyi distance to the origin, $a = \mu r$ and $b = r$. By assumption, if $x \leq r$, $(\partial S) \llcorner \{u \leq x\} = 0$. Thus

$$\begin{aligned}\int_{\mu r}^r \mathcal{M}(\partial(S \llcorner \{xU\}))\, d\mathcal{L}^1(x)& \\ \leq c_1\, \mathcal{M}(S) \llcorner \{\mu r < x < r\}) \leq c_1\, \mathcal{M}(S),&\end{aligned}$$

where c_1 is the upper bound for the Riemannian Lipschitz constant of u given by Lemma 2.5.

Therefore there exists $\lambda' \in [\mu, 1]$ such that

$$r(1-\mu)\mathcal{M}(\partial(S \llcorner \lambda' U)) \leq c_1\, \mathcal{M}(S),$$

and the assertion follows with $C_1 = \frac{c_1}{1-\mu}$.

□

2.6. **Proof of Theorem 1.3 from Theorem 2.2.** From now on, S is replaced with the clipped current $S \llcorner \lambda' U$ provided by Lemma 2.6. It satisfies the mass bound $\mathcal{M}(\partial S) \leq \frac{C_1}{R}\, \mathcal{M}(S)$. We apply the dilation $\mathfrak{s}_{1/\lambda'}$ and get a current with support in U and boundary supported in ∂U, still denoted by S. Therefore, in the sequel, $\mu = 1$. For all $t > 1$ we set $S^t := (\mathfrak{s}_t)_\# S$. Since S is horizontal, $\mathcal{M}(S^t) = t^k \mathcal{M}(S)$, $\mathcal{M}(\partial S^t) = t^{k-1}\mathcal{M}(\partial S)$. By the Deformation Theorem 3.8 with parameter $\epsilon(t)$ tending to 0 as t tends to $+\infty$, we can make a random choice that achieves at least the expected mass bound and write

$$S^t = S^t_{\mathrm{Lip}} + V + \partial W,$$

where $S^t_{\mathrm{Lip}} = P(S^t)$, $V = Q(\partial S^t)$, $W = Q(S^t)$ are integral Lipschitz chains,

$$\mathcal{M}(V) + \mathcal{M}(W) \leq C\, \epsilon(t)(\mathcal{M}(S^t) + \mathcal{M}(\partial S^t)), \tag{2}$$

$$\mathcal{M}(S^t_{\mathrm{Lip}}) \leq C\, (\mathcal{M}(S^t) + \epsilon(t)\mathcal{M}(\partial S^t)), \tag{3}$$

$$\mathcal{M}(\partial S^t_{\mathrm{Lip}}) \leq C\, \mathcal{M}(\partial S^t). \tag{4}$$

In particular, the flat norm

$$\mathcal{F}(S^t - S^t_{\mathrm{Lip}}) \leq C\, \epsilon(t)(t^k \mathcal{M}(S) + t^{k-1}\mathcal{M}(\partial S)). \tag{5}$$

Also, $\operatorname{supp} S^t_{\mathrm{Lip}}$ is contained in a neighborhood of S^t and $\operatorname{supp}(\partial S^t_{\mathrm{Lip}})$ is contained in a neighborhood of ∂S^t.

We apply Theorem 2.2 to S^t_{Lip}, picking at random a filling that achieves the mass bound in probability. We obtain a k-chain $(\tilde{S}^t_{\mathrm{Lip}})$ and a $(k+1)$-chain T^t_{Lip} such that

i) T^t_{Lip} is supported in $(t+1)U$, $\tilde{S}^t_{\mathrm{Lip}}$ is supported in $(t+1)U \setminus (\lambda t - 1)U$;
ii) $\partial T^t_{\mathrm{Lip}} - S^t_{\mathrm{Lip}} - \tilde{S}^t_{\mathrm{Lip}}$ has support outside $(t-1)U$;
iii) $\mathcal{M}(T^t_{\mathrm{Lip}}) \leq \Phi_2(\mathcal{M}(S^t_{\mathrm{Lip}}))$;
iv) $\mathcal{M}(\tilde{S}^t_{\mathrm{Lip}}) \leq C\,\mathcal{M}(S^t_{\mathrm{Lip}})$, $\mathcal{M}(\partial \tilde{S}^t_{\mathrm{Lip}}) \leq C\,\mathcal{M}(\partial S^t_{\mathrm{Lip}})$.

By (5) we have

$$\mathcal{F}((\partial T^t - S^t - \tilde{S}^t_{\mathrm{Lip}}) \llcorner (t-1)U) \leq C\,\epsilon(t)(t^k \mathcal{M}(S) + t^{k-1}\mathcal{M}(\partial S)). \tag{6}$$

By iii) above and (2),

$$M(T^t_{\mathrm{Lip}}) \leq \Phi_2(\mathcal{M}(S^t_{\mathrm{Lip}})) \leq \Phi_2(C\,(\mathcal{M}(S^t) + \mathcal{M}(\partial S^t))) = O(t^{k+1}).$$

By iv) and (2),

$$M((\tilde{S}^t_{\mathrm{Lip}}) \leq C\,M(S^t_{\mathrm{Lip}}) \leq C\,(\mathcal{M}(S^t) + \mathcal{M}(\partial S^t)) = O(t^k).$$

By i), iv) and (2),

$$M(\partial(T^t_{\mathrm{Lip}}) \llcorner (t-1)U) \leq (\mathcal{M}(S^t_{\mathrm{Lip}}) + \mathcal{M}(\tilde{S}^t_{\mathrm{Lip}})) \leq C\,\mathcal{M}(S^t_{\mathrm{Lip}}) = O(t^k).$$

Finally, by i), iv) and (2)

$$M(\partial \tilde{S}^t_{\mathrm{Lip}})) \leq C\,M(\partial S^t_{\mathrm{Lip}}) \leq C\,M(\partial S^t) = O(t^{k-1}).$$

According to Lemma 2.5 (3),

$$\mathcal{M}((\mathfrak{s}_{1/t})_\# T^t_{\mathrm{Lip}}), \quad \mathcal{M}(\partial(\mathfrak{s}_{1/t})_\# T^t_{\mathrm{Lip}} \llcorner \frac{t-1}{t} U),$$
$$\mathcal{M}((\mathfrak{s}_{1/t})_\# \tilde{S}^t_{\mathrm{Lip}}) \quad \text{and} \quad \mathcal{M}(\partial(\mathfrak{s}_{1/t})_\# \tilde{S}^t_{\mathrm{Lip}})$$

are bounded for $t > 1$ and these currents are supported in a fixed compact subset of $\mathbb{H}^n$. Therefore, by the Compactness Theorem of [7], 4.2.17 or [12], Chapter V, Theorem 5.5, we can assume that

$$(\mathfrak{s}_{1/t})_\# T^t_{\mathrm{Lip}} \llcorner U \to T \quad \text{and} \quad (\mathfrak{s}_{1/t})_\# \tilde{S}^t_{\mathrm{Lip}} \to S'$$

in the flat norm topology. It follows from (6) and Lemma 2.5 that

$$\mathcal{F}(((\mathfrak{s}_{1/t})_\# \partial T^t - S - (\mathfrak{s}_{1/t})_\# \tilde{S}^t_{\mathrm{Lip}}) \llcorner \frac{t-1}{t} U) = O(\epsilon(t)) = o(1).$$

and hence

$$(\partial T - S + \tilde{S}) \llcorner U = 0.$$

On the other hand T is horizontal. Indeed, every covector η in the Lie algebra $\mathfrak{h}^n$ splits orthogonally as $\eta = \eta_V + \eta_H$, with η_V a multiple of θ and η_H homogeneous of degree k under dilations. Let ω be a test k-form. Then

$$\|(\mathfrak{s}_{1/t})^{\#}\omega_H\|_\infty = t^{-k}\|\omega_H\|_\infty \leq t^{-k}\|\omega\|_\infty.$$

Then

$$\begin{aligned}\langle((\mathfrak{s}_{1/t})_{\#}T^t_{\mathrm{Lip}})\llcorner\theta|\omega\rangle &= \langle(\mathfrak{s}_{1/t})_{\#}T^t_{\mathrm{Lip}}|\theta\wedge\omega_H\rangle = \langle T^t_{\mathrm{Lip}}|(\mathfrak{s}_{1/t})^{\#}(\theta\wedge\omega_H)\rangle\\ &= \langle T^t_{\mathrm{Lip}}|t^{-2}\theta\wedge t^{-k}(t^k\mathfrak{s}_{1/t})^{\#}\omega_H)\rangle\\ &\leq t^{-2-k}\mathcal{M}(T^t_{\mathrm{Lip}})\|t^k\mathfrak{s}^{\#}_{1/t}\omega_H\|_\infty\\ &\leq t^{-2-k}O(t^{k+1})\|\omega\|_\infty.\end{aligned}$$

If follows that

$$\mathcal{M}(((\mathfrak{s}_{1/t})_{\#}T^t_{\mathrm{Lip}})\llcorner\theta) = O(t^{-1})$$

tends to 0. *A fortiori*, $((\mathfrak{s}_{1/t})_{\#}T^t_{\mathrm{Lip}})\llcorner\theta$ tends to 0 weakly, so $T\llcorner\theta = 0$. Similarly, if ω is a test $k-1$-form,

$$\begin{aligned}\langle(\partial(\mathfrak{s}_{1/t})_{\#}T^t_{\mathrm{Lip}})\llcorner U\llcorner\theta|\omega\rangle &= \langle(\partial T^t_{\mathrm{Lip}})\llcorner U|t^{-2}\theta\wedge t^{-(k-1)}(t^{k-1}\mathfrak{s}_{1/t})^{\#}\omega_H)\rangle\\ &\leq t^{-1-k}\mathcal{M}((\partial T^t_{\mathrm{Lip}})\llcorner U)\|t^{k-1}(\mathfrak{s}_{1/t})^{\#}\omega_H\|_\infty\\ &\leq t^{-1-k}O(t^k)\|\omega\|_\infty\end{aligned}$$

tends to 0, so $\partial T\llcorner\theta = 0$.

$$\begin{aligned}\langle(\mathfrak{s}_{1/t})_{\#}\tilde{S}^t_{\mathrm{Lip}}\llcorner\theta|\omega\rangle &= \langle\tilde{S}^t_{\mathrm{Lip}}|t^{-2}\theta\wedge t^{-(k-1)}(t^{k-1}\mathfrak{s}_{1/t})^{\#}\omega_H)\rangle\\ &\leq t^{-1-k}\mathcal{M}(\tilde{S}^t_{\mathrm{Lip}})\|t^{k-1}\mathfrak{s}^{\#}_{1/t}\omega_H\|_\infty\\ &\leq t^{-1-k}O(t^k)\|\omega\|_\infty\end{aligned}$$

tends to 0, so $S'\llcorner\theta = 0$.

The information on the support of S' follows by semi-continuity of the support.

The mass estimates on T, S' and $\partial S'$ follow by lower semi-continuity of mass, thus proving Theorem 1.3.

2.7. **Proof of Theorem 1.2.** Theorem 1.2 is a special case of Theorem 1.3. It also follows from R. Young's filling inequality and scaling, compactness and convergence arguments very similar to those of the previous paragraph.

3. The random deformation theorem

3.1. Central projections in Euclidean convex polyhedra.

Definition 3.1. *Let* Π *be a compact convex polyhedron.*

- *The* inradius *of* Π *is the largest radius of a ball contained in it.*

- *The* aspect ratio *of* Π *is*

$$A(\Pi) := \frac{\text{diameter}(\Pi)}{\text{inradius}(\Pi)}.$$

- *The* padded interior $\ddot{\Pi}$ *is the set of points of* Π *sitting at distance at least* $\frac{1}{2}\text{inradius}(\Pi)$ *from the boundary of* Π.
- *The uniform probability measure is used on* $\ddot{\Pi}$.
- *For* $\omega \in \ddot{\Pi}$, *let* $\pi^\omega : \Pi \setminus \{\omega\} \to \partial\Pi$ *denote the radial projection to the boundary.*

Lemma 3.2. *Let* Π *be a convex Euclidean polyhedron of aspect ratio* A. *For every* k*-dimensional rectifiable set* $S \subset \Pi$,

$$\mathbb{E}_\omega(\mathcal{H}^k(\pi^\omega(S))) \leq (2k+1)(2A)^{2k}\mathcal{H}^k(S).$$

Proof. Let $D = \text{diameter}(\Pi)$ and $i = \text{inradius}(\Pi)$. If x, x' are two infinitesimally nearby points, their projections y, y' satisfy

$$|y - y'| \leq \frac{D^2}{d(\omega, \partial\Pi)} \frac{|x - x'|}{|x - \omega|}.$$

If $\omega \in \ddot{\Pi}$, then $d(\omega, \partial\Pi) \geq \frac{i}{2}$, so $\frac{D^2}{d(\omega,\partial\Pi)} \leq 2i\,A^2$. Thus the local Lipschitz constant of the projection satisfies

$$Lip_x(\pi^\omega) \leq 2i\,A^2|x - \omega|^{-1}.$$

The pointwise Jacobian J_x being estimated by Lip_x^k, the area formula (Section 3.2.3 in [7]) implies

$$\mathcal{H}^k(\pi^\omega(S)) \leq (2i)^k A^{2k} \int_S |x - \omega|^{-k}\, d\mathcal{H}^k(x).$$

Next we integrate with respect to ω,

$$\int_{\ddot{\Pi}} \mathcal{H}^k(\pi^\omega(S))\, d\omega \leq (2i)^k A^{2k} \int_S \int_{\ddot{\Pi}} |x - \omega|^{-k}\, d\omega\, d\mathcal{H}^k(x).$$

We claim that among all open sets U of fixed volume v, the integral

$$\int_U |x - \omega|^{-k}\, d\omega$$

is maximal for a ball B centered at x. Indeed, mapping $U \setminus B$ to $B \setminus U$ in a measure preserving manner puts every point closer to x so it can only increase the integral. Then one can compute the integral using polar

coordinates $(r,\theta)\mapsto\omega=x+r\theta$, $[0,R]\times\Sigma\to B$,

$$\begin{aligned}\int_B|x-\omega|^{-k}\,d\omega&=\int_\Sigma\Big(\int_0^R r^{n-1-k}\,dr\Big)\,d\theta\\&=\frac{R^{n-k}}{n-k}\int_\Sigma d\theta\\&=R^{-k}\frac{n}{n-k}\int_\Sigma\Big(\int_0^R r^{n-1}\,dr\Big)\,d\theta\\&=R^{-k}\frac{n}{n-k}v,\end{aligned}$$

where $n=\dim(\Pi)$. Since $\ddot{\Pi}$ contains a ball of radius $\frac{i}{2}$, $R\geq\frac{i}{2}$. Hence

$$\mathbb{E}_\omega(\mathcal{H}^k(\pi^\omega(S)))\leq 2^{2k}A^{2k}\frac{n}{n-k}\mathcal{H}^k(S).$$

Finally, if $k=0$, $\frac{n}{n-k}\leq 1$. Otherwise,

$$\frac{n}{k(n-k)}=\frac{1}{n-k}+\frac{1}{k}\leq 2.$$

In both cases, $\frac{n}{n-k}\leq 2k+1$. □

3.2. **Filling projected currents.** Let Π be a compact convex Euclidean polyhedron of aspect ratio A. Let S be a k-dimensional Lipschitz chain or integral current with support in Π. Lemma 3.2 provides a expected mass bound on the projection $\pi^\omega_\# S$. Next we fill the difference $S-\pi^\omega_\# S$ with a random integral current or Lipschitz chain T with controlled expected mass. Let D denote the diameter of Π. Picking $\omega\in\ddot{\Pi}$ at random, the map

$$\begin{aligned}h^\omega:&(\Pi\setminus\{\omega\})\times[0,D]\to\Pi,\\&(x,t)\mapsto(1-\frac{t}{D})\pi^\omega(x)+\frac{t}{D}x,\end{aligned}$$

has local Lipschitz constant

$$Lip_{(x,t)}(h^\omega)\leq Lip_x(\pi^\omega)+\frac{1}{D}|\pi^\omega(x)-x|\leq Lip_x(\pi^\omega)+1.$$

So one can define a random filling as follows.

Notation 3.3. *If S is a Lipschitz chain (resp. integral current), let $T(S)$ be the random Lipschitz chain (resp. integral current) defined by*

$$T^\omega(S)=h^\omega_\#(S\times[0,D]).$$

It satisfies

$$\begin{aligned}\partial T^\omega(S)&=h^\omega_\#(S\times\{D\}-S\times\{0\}-\partial S\times[0,D])\\&=S-\pi^\omega_\# S-T^\omega(\partial S).\end{aligned}\tag{7}$$

Integrating the local Lipschitz constant estimate yields

Lemma 3.4. *Let Π be a compact convex Euclidean polyhedron of aspect ratio A and diameter D. Let $S \subset \Pi$ be a k-dimensional rectifiable subset and $T^\omega(S) = h^\omega(S \times [0, D])$. Then*

$$\mathbb{E}(\mathcal{H}^{k+1}(T(S))) \leq ((2k+1)(2A)^{2k} + 1) D\, \mathcal{H}^k(S).$$

3.3. **Convex cell complexes.**

Definition 3.5. *A* convex cell complex *is a metric space together with a covering by closed sets F, called* cells*, taken in a collection $\mathcal{F}$, such that*

1. *Each $F \in \mathcal{F}$ comes with an isometry $\iota_F : \Pi_F \to F$ from a convex Euclidean polyhedron.*
2. *For every $F \in \mathcal{F}$, the image by ι_F of each face of Π_F belongs to $\mathcal{F}$. These subsets will be called faces of F.*
3. *For every $F, F' \in \mathcal{F}$, $F \cap F'$ is either empty or a face of F.*

The aspect ratio *of a convex cell complex is the supremum of the aspect ratios of the polyhedra Π_F, $F \in \mathcal{F}$.*

3.4. **Cellulations.** A possible reference for this section is [4].

Definition 3.6. *Let V be a Riemannian manifold, possibly with boundary. Let $A, L > 0$. An (A, L)-*cellulation *of V is the data of a convex cell complex X of aspect ratio $\leq A$ and an L-biLipschitz homeomorphism κ of X to V.*

Proposition 3.7. *Let V be Riemannian manifold, possibly with boundary, admitting a triangulation with finitely many isometry types of simplices. There exist $A(V)$, $L(V)$ such that for every $\epsilon > 0$, V admits an (A, L)-cellulation where all cells have diameters $\leq \epsilon$.*

Proof. We use the folklore decomposition of the standard d-simplex into $d+1$ cubes, see for instance [9]

For $x \in \mathbb{R}^d$, let $x \cdot 1 = \sum_{i=1}^d x_i$. Let $\Delta^d = \{x \in \mathbb{R}^{d+1}_+ ;\, x \cdot 1 = 1\}$. Let us define $\chi : [0,1]^d \to \mathbb{R}^d$ by

$$\chi(t) = \frac{1}{1 + t \cdot 1} t,$$

and $\Xi_0 : [0,1]^d \to \Delta^d$ by

$$\Xi_0(t) = (1 - \chi(t) \cdot 1, \chi(t)).$$

Then Ξ_0 is a diffeomorphism onto its image which is characterized by inequalities,

$$\Xi_0([0,1]^d) = \{(\omega, \dots, x_d) \in \Delta^d ;\, \forall i = 1, \dots, d,\, x_i \leq \omega\}.$$

Indeed, the inequality $t_i \leq 1$ implies $\Xi_0(t)_i \leq \frac{1}{1+\chi(t)\cdot 1} = \Xi_0(t)_0$. Conversely, for $x \in \Delta^d$, let $t = \frac{x_i}{\omega}$. If $\forall i = 1, \dots, d$, $x_i \leq \omega$ then $t \in [0,1]^d$

and $\Xi_0(t) = x$. By circular permutation of coordinates, one gets maps $\Xi_i : [0,1]^d \to \Delta^d$ whose images cover Δ^d and intersect along faces, i.e.

$$\Xi_i([0,1]^d) \cap \Xi_j([0,1]^d) = \Xi_i([0,1]^{d'}),$$

$d' < d$, is the image of one of the faces of $[0,1]^d$. This is the requested decomposition into cubes.

Starting with a triangulation τ with finitely many isometry types of simplices, we decompose each simplex into $n+1$ cubes of aspect ratio $A = 2\sqrt{n}$, getting a cellulation with some Lipschitz constant L. Cutting these unit cubes into subcubes of side $\frac{1}{m}$ provides a cellulation by cubes with the same aspect ratio A and Lipschitz constant L, but with diameters $\sim \frac{\sqrt{n}}{m}$. □

3.5. **Statement of the random deformation theorem.** The following completes a theorem appearing in [6], and [1]. Like B. White's version [14], it should be viewed as variant of the classical deformation theorem where a map deforming integral currents into chains of a cubical subdivision of $\mathbb{R}^n$ is replaced with a random map, i.e. a random variable with values in the space of maps from the group I of integral currents to the group I_κ of chains of a cellulation. We borrow B. White's improvement on the expected mass bound on the cellular chain component P.

Theorem 3.8. *Let V be a Riemannian manifold, possibly with boundary, of dimension n. Let κ be an (A, L)-cellulation of V with cells of diameters $\leq \epsilon$. Then the group I of integral currents in V and the group I_κ of chains of the cellulation κ are randomly chain homotopic, with uniform mass bounds in expectation.*
Namely, there exists a constant $C = C(n, A, L)$ and random self-maps P and $Q : I \to I$ such that

(1) $\partial P = P\partial$;
(2) $P + Q\partial + \partial Q$ *is the identity map ;*
(3) P *and* Q *are additive: for every integral currents* S, S', $P(S+S') = P(S) + P(S')$, $Q(S+S') = Q(S) + Q(S')$.
(4) P *is a projector onto the subgroup* $I_\kappa \subset I$*:* $P(I) \subset I_\kappa$ *and* $P \circ P = P$;
(5) *For every integral current* S *in* V,

$$\mathbb{E}(\mathcal{M}(P(S))) \leq C\,\mathcal{M}(S), \quad \mathbb{E}(\mathcal{M}(Q(S))) \leq C\epsilon\,\mathcal{M}(S).$$

Furthermore, $P(S), Q(S)$ have support in a $C\epsilon$-neighborhood of the support of S.

Finally, if S is a Lipschitz (resp. C^1) chain, so are $P(S)$ and $Q(S)$.

3.6. **Proof of the random deformation theorem.** Since the problem is invariant under biLipschitz homeomorphisms, one merely needs handle convex cell complexes.

3.6.1. *Radial projections in convex cell complexes.* When passing from a single polyhedron Π to a convex cell complex X, instead of picking uniformly at random a single point $\omega \in \ddot{\Pi}$, one picks uniformly at random a point $\omega(F)$ of $\ddot{F}$ for each face F of X. So, from now on, the randomness lies in the choice of a map ω from the set of faces $\mathcal{F}$ of X to X. Let $\pi^\omega : X \setminus \omega(\mathcal{F}^{top}) \to X^{(n-1)}$, the $n-1$-skeleton of X, denote the map which on each top dimensional face F is given by the radial projection $\pi^{\omega(F)}$. Let

$$h^\omega(X \setminus \omega(\mathcal{F}^{top})) \times [0, \epsilon] \to X$$

be the face-wise linear homotopy from π^ω to the identity, depicted on Figure 1. This defines random maps $\pi_\#$ and T on currents in X. The estimates of Corollary 3.2 and Lemma 3.4 piece together into expected mass estimates for the random operators $\pi_\#$ and T on currents.

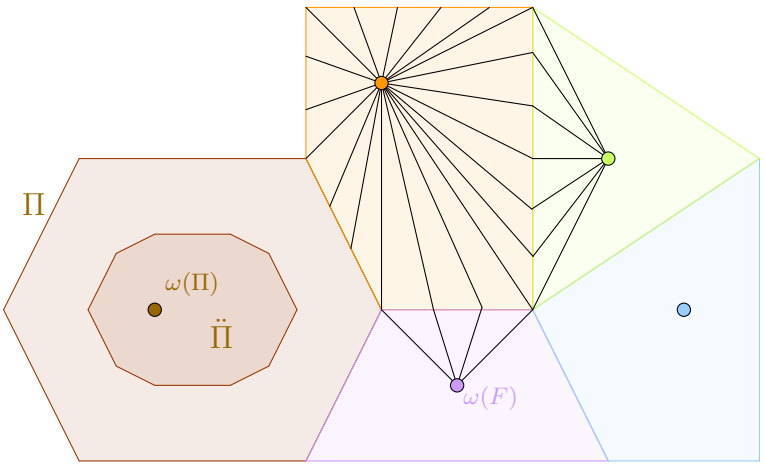


FIGURE 1. The cell-complex version of the projection π^ω

Lemma 3.9. *Let S be a k-dimensional integral current in X. Then*

$$\mathbb{E}(\mathcal{M}(\pi_\# S)) \leq C(k, A)\, \mathcal{M}(S),$$
$$\mathbb{E}(\mathcal{M}(T(S))) \leq C(k, A)\, \epsilon\, \mathcal{M}(S).$$

Note that $\pi_\#$ and T are additive on integral currents.

3.6.2. *Construction of the filling homotopy.* The construction is performed by induction on codimension, i.e. if n is the dimension of the ambient convex cell complex X and $k \leq n$, one defines P on k-currents and Q on $k-1$-currents, for larger and larger $\ell = n-k$, satisfying $1 = P + \partial Q + Q\partial$ and

$$\mathbb{E}(\mathcal{M}(P(S))) \leq C(k,A)^{\ell+1}\, \mathcal{M}(S),$$
$$\mathbb{E}(\mathcal{M}(Q(S))) \leq C(k,A)^{\ell+1}\, \epsilon\, \mathcal{M}(S).$$

3.6.3. *Top dimensional currents.* Let $\ell = 0$. On $n-1$-dimensional currents, let

$$Q := T$$

be the map initially defined in Notation 3.3 for a single polyhedron and generalized to convex cell complexes in Subsection 3.6.1. By Lemma 3.9, when the diameters of the faces of X are $\leq \epsilon$, for every $n-1$-dimensional integral current S,

$$\mathbb{E}(\mathcal{M}(Q(S))) \leq C(k,A)\epsilon\, \mathcal{M}(S).$$

For n-dimensional chains or currents S,

$$T^\omega(\partial S) = h^\omega_\#(\partial S \times [0,D]).$$

By Equation (7), this current satisfies

$$\partial T^\omega(\partial S) = \partial S - \pi^\omega_\# \partial S,$$

since $\partial \circ \partial = 0$. The top dimensional current $S - T^\omega(\partial S)$ has its boundary $\pi^\omega_\# \partial S$ supported in the $n-1$-skeleton $X^{(n-1)}$. Therefore its restriction to each top dimensional face F is an integral multiple of the chain F (or of the current of integration on F). Thus one can interpret

$$P^\omega(S) := S - T^\omega(\partial S)$$

as a cellular chain of the cell complex X, i.e. an element of the subgroup I_κ. By construction, P and Q are additive and satisfy

$$S = P(S) + Q(\partial S).$$

The top dimensional current S can be viewed as an integer valued L^1 function Θ on the union of top dimensional faces. In each such face F, if $\omega(F)$ is not a point of density of Θ, $P(S \llcorner F) = 0$. Otherwise $P(S \llcorner F)$ is either 0 or $\Theta(\omega(F))[F]$, where $[F]$ denotes the chain associated to F. So

$$\mathbb{E}(\mathcal{M}(P(S \llcorner F))) \leq \frac{\mathcal{M}([F])}{vol(\ddot{F})} \int_F |\Theta| \leq 2^{\dim(F)} \mathcal{M}(S \llcorner F).$$

Thus

$$\mathbb{E}(\mathcal{M}(P(S))) \leq 2^n \mathcal{M}(S).$$

Finally, if $S \in I_\kappa$, ∂S is supported in $X^{(n-1)}$, where $\pi_\#$ is the identity, $T = 0$, so $P(S) = S$.

3.6.4. *General case.* Observing that the random map $\pi_\#$ decreases the codimension of currents by one unit, we define inductively

$$P = P \circ \pi_\# S, \quad Q = T + Q \circ \pi_\#,$$

where, at each step, the randomness has an independent origin. These are additive operators, and P equals identity on I_κ. On convex cell complexes of dimension n, aspect ratio A and faces of diameters $\leq \epsilon$, by iterating Lemma 3.2 $n - k - 1$ times, one gets, on codimension ℓ currents,

$$\mathbb{E}(\mathcal{M}(P(S))) \leq C(k, A)^\ell \times C(k, A)\, \mathcal{M}(S). \tag{8}$$

On the other hand, on codimension ℓ currents,

$$Q = \sum_{i=0}^{\ell-1} T \circ (\pi_\#)^i.$$

The last term dominates, so

$$\mathbb{E}(\mathcal{M}(Q(S))) \leq C(k, A)\epsilon \times C(k, A)^\ell\, \mathcal{M}(S).$$

Let us show by induction on codimension ℓ that $1 = P + \partial Q + Q\partial$ on currents of codimension ℓ. This holds, by construction, when $\ell = 0$. If S is a k-dimensional integral current in X of dimension n, $\pi^\omega(S)$ is a k-dimensional current in $X^{(n-1)}$, of codimension $\ell - 1$, so, by the induction hypothesis, the random currents $P(\pi^\omega_\# S)$ and $Q(\pi^\omega_\# S)$ satisfy

$$\pi^\omega_\# S = P(\pi^\omega_\# S) + \partial Q(\pi^\omega_\# S) + Q(\partial \pi^\omega_\# S).$$

On the other hand, by Equation (7),

$$\partial T^\omega(S) = S - \pi^\omega_\# S - T^\omega(\partial S),$$

so

$$\begin{aligned} S &= P(\pi^\omega_\# S) + \partial T^\omega(S) + \partial Q(\pi^\omega_\# S) + T^\omega(\partial S) + Q(\pi^\omega_\# \partial S) \\ &= P(S) + \partial Q(S) + Q(\partial S), \end{aligned}$$

since we have defined $P = P \circ \pi^\omega_\#$ and $Q = T^\omega + Q \circ \pi^\omega_\#$.

This completes the proof of Theorem 3.8.

4. Relative fillings

4.1. **Young's method.** The proof of Theorem 2.2 mimics R. Young's argument for the absolute filling inequality, in the special case of Heisenberg groups, [15].

A major ingredient is a map ϕ defined and horizontal on the n-skeleton of a periodic triangulation τ of $\mathbb{H}^n$. Such a map is constructed in Section 4 of [15], after Lemma 4.5 in that paper. Young then uses a variant, a map ψ defined and horizontal on the n-skeleton of a periodic triangulation of $\mathbb{H}^n \times [0,1]$, restricting to ϕ on $\mathbb{H}^n \times \{0\}$ and to $\phi \circ \mathfrak{s}_2$ on $\mathbb{H}^n \times \{1\}$, where $(\mathfrak{s}_t)_{t>0}$ denote the Heisenberg dilations.

Applying the deformation theorem 3.8, the given Lipschitz chain S is first deformed to a simplicial chain $P(S)$ of the triangulation τ and then pushed by ϕ to a horizontal Lipschitz chain $P_\phi(S)$.

The same procedure can be applied using the inflated triangulations $\mathfrak{s}_{2^i}(\tau)$, producing large horizontal chains $P_i(S)$. For i large, $P_i(S)$ vanishes, since its simplices would be too large in comparison with the mass of S.

There remains to deform each $P_i(S)$ into the next one, $P_{i+1}(S)$. This is performed by applying Theorem 3.8 and the horizontal map ψ on $\mathbb{H}^n \times [0,1]$. The fact that $P_i(S)$ and the fillings $R_i(S)$ of $P_{i+1}(S) - P_i(S)$ are horizontal allows for an optimal estimate on their masses.

Here are the new features in the relative setting.

- When $\partial S \neq 0$, Theorem 3.8 produces two filling chains instead of one, the deformation being written
$$S - P_\phi(S) = \partial Q_\phi(S) + Q_\phi(\partial S),$$
and at later steps,
$$P_{i+1}(S) - P_i(S) = \partial R_i(S) + R_i(\partial S).$$
- A second mechanism can stop the iteration. One can arrange that $P_i(S)$ does not intersect a ball of radius $2^i\epsilon$, for some $\epsilon > 0$. When this radius exceeds R, one may stop. This allows for the double estimate in the statement of Theorem 2.2.
- The radius ϵ is obtained in Section 4.5 in a nonconstructive manner. Fortunately, this flaw can be corrected using the stretch maps constructed in Section 5.

4.2. **First step: fine grained approximation.** Now be embark for the proof of Theorem 2.2.

The periodic triangulation ([15], Lemma 4.4) is constructed as follows. Let us fix a discrete cocompact subgroup $\Gamma < \mathbb{H}^n$ such that $\mathfrak{s}_2(\Gamma) \subset \Gamma$. Let τ be a triangulation of $\mathbb{H}^n$ obtained by lifting a finite triangulation of the compact manifold $\mathbb{H}^n/\Gamma$. Let $\phi : \tau \to \mathbb{H}^n$ be a map which is uniformly

close to the identity (i.e. $D := \sup_{x\in\tau} d(x,\phi(x)) < +\infty$) and horizontal on simplices of τ of dimension $\leq n$. Such a map is constructed in [15], Lemma 4.5.

Applying Theorem 3.8, one first deforms S into a random chain $P(S)$ of the triangulation τ with two error terms $Q(S)$ and $Q(\partial S)$ (which depends only on ∂S, hence is supported near the boundary of U). Then one pushes P through ϕ, getting a random horizontal Lipschitz chain $P_\phi(S)$. Similarly, one pushes Q and one adds up the current associated to a homotopy between identity and ϕ, to get random Lipschitz chains $Q_\phi(S)$ and $Q_\phi(\partial S)$ such that

$$S = P_\phi(S) + \partial Q_\phi(S) + Q_\phi(\partial S),$$

with a linear control on expected masses,

$$\mathbb{E}(\mathcal{M}(P_\phi(S))) \leq C_2\mathcal{M}(S), \quad \mathbb{E}(\mathcal{M}(Q_\phi(S))) \leq C_2\mathcal{M}(S),$$
$$\mathbb{E}(\mathcal{M}(Q_\phi(\partial S))) \leq C_2\mathcal{M}(\partial S),$$

where C_2 depends on the constant provided by Theorem 3.8 and $Lip(\phi)$, as in [15], Lemma 3.1.

4.3. **Coarse grained approximations.** One applies the same therapy to the dilated chains, getting chains

$$P_i = (\mathfrak{s}_{2^i})_\# P_\phi((\mathfrak{s}_{2^{-i}})_\# S).$$

The dilations $\mathfrak{s}_u$ with $u \leq 1$ contract all masses of k dimensional chains by at least u^k (Lemma 2.5). Therefore

$$\mathbb{E}(\mathcal{M}(P_\phi((\mathfrak{s}_{2^{-i}})_\# S))) \leq C_2 2^{-ki}\,\mathcal{M}(S),$$
$$\mathbb{E}(\mathcal{M}(\partial P_\phi((\mathfrak{s}_{2^{-i}})_\# S))) \leq C_2 2^{-(k-1)i}\,\mathcal{M}(\partial S).$$

Since $P_\phi((\mathfrak{s}_{2^{-i}})_\# S)$ is horizontal, the dilation $\mathfrak{s}_{2^i}$ multiplies its mass exactly by 2^{ki}, leading to

$$\mathbb{E}(\mathcal{M}(P_i)) \leq C_2\,\mathcal{M}(S), \quad \mathbb{E}(\mathcal{M}(\partial P_i)) \leq C_2\,\mathcal{M}(\partial S), \tag{9}$$

compare [15], Lemma 3.2.

4.4. **Interpolation.** Lifting a triangulation τ' on $(\mathbb{H}^n/\Gamma)\times[0,1]$ inducing the triangulations τ and $\mathfrak{s}_2(\tau)$, on $(\mathbb{H}^n/\Gamma)\times\{0\}$ and $(\mathbb{H}^n/\Gamma)\times\{1\}$ respectively and using a horizontal approximation ψ of the projection to the first factor, one fills $P_{i+1} - P_i$, writing

$$P_{i+1} - P_i = \partial R_i(S) + R'_i(\partial S).$$

$R_i(S)$ and $R'_i(\partial S)$ are obtained as follows (Lemma 3.3 in [15]). Let $\tau_0 = \tau$ and $\tau_1 = \mathfrak{s}_2(\tau)$. We apply Theorem 3.8 to both triangulations and use

indices 0 and 1 to distinguish the resulting chains. Consider the following random Lipschitz chains in $\mathbb{H}^n \times [0,1]$,

$$X(S) := -Q_1(S) + S \times [0,1] + Q_0(S),$$
$$X_i := X((\mathfrak{s}_{2^{-i}})_\# S), \quad X_i' := X((\mathfrak{s}_{2^{-i}})_\# \partial S).$$

Figures 2 and 3 may help the reader figure out what is going on. They feature a manifold (a square) together with two different cubulations (with one cube on the right and 4 cubes on the left). For readability reasons, the product with an interval is not cubulated, but decomposed into a single cell with 9 faces. The chain S is the red arc. Figure 2 features the chains $X_0 = -Q_1(S) + S \times [0,1] + Q_0(S)$ in brown and $X_0' = -Q_1(\partial S) + \partial S \times [0,1] + Q_0(\partial S)$ in green. The chains $P_0(S)$ and $P_1(S)$, in blue, cover part of the boundary of X_0.

Figure 3 features the chains $P(X_0)$ (in orange with its boundary in blue and gray) and $P(X_0')$ (in gray). One can see that $\partial P(X_0) + P(X_0') = P_1(S) - P_0(S)$.

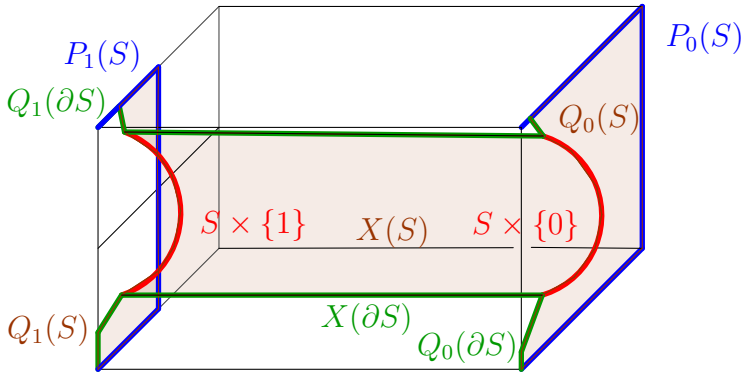


FIGURE 2. The chains X_0 and X_0'

Back to the proof of Theorem 2.2.

In general, the chains X_i and X_i' satisfy

$$\partial P(X_i) + P(X_i') = P_1((\mathfrak{s}_{2^{-i}})_\# S) - P_0((\mathfrak{s}_{2^{-i}})_\# S),$$

and

$$\mathbb{E}(\mathcal{M}(X_i)) \leq (2C_3 + 1)\mathcal{M}((\mathfrak{s}_{2^{-i}})_\# S) = (2C_3 + 1)2^{-ki}\,\mathcal{M}(S),$$

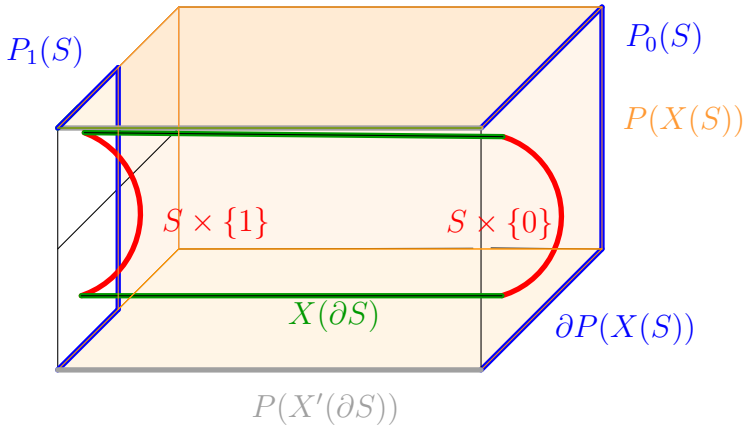


FIGURE 3. The chains $P(X_0)$ and $P(X_0')$

$$\begin{aligned}\mathbb{E}(\mathcal{M}(\partial X_i)) &= \mathcal{M}(P_1((\mathfrak{s}_{2^{-i}})_\# S) - P_0((\mathfrak{s}_{2^{-i}})_\# S) - X_i') \\ &\leq 2C_3 2^{-ki}\,\mathcal{M}(S) + (2C_3+1)2^{-(k-1)i}\,\mathcal{M}(\partial S),\end{aligned}$$

so

$$\mathbb{E}(\mathcal{M}(X_i) + \mathcal{M}(\partial X_i)) \leq (4C_3+1)(2^{-ki}\,\mathcal{M}(S) + 2^{-(k-1)i}\,\mathcal{M}(\partial S)).$$

Here, C_3 depends only on C_2. We do not need estimate $\mathcal{M}(X_i')$ and $\mathcal{M}(\partial X_i')$ since X_i' is obtained by the same procedure, applied to ∂S instead of S.

We let

$$R_i(S) = (\mathfrak{s}_{2^i})_\# P_\psi(X_i), \quad R_i(\partial S) = (\mathfrak{s}_{2^i})_\# P_\psi(X_i').$$

Using the properties of the random map P (additivity, commutation with ∂ and $P \circ P = P$),

$$\begin{aligned}\partial R_i(S) + R_i(\partial S) &= (\mathfrak{s}_{2^i})_\# \psi_\#(\partial P(X_i) + P(X_i')) \\ &= (\mathfrak{s}_{2^i})_\# \psi_\#(P_1((\mathfrak{s}_{2^{-i}})_\# S) - P_0((\mathfrak{s}_{2^{-i}})_\# S)) \\ &= P_{i+1} - P_i.\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathbb{E}(\mathcal{M}(P_\psi(X_i))) &\leq C_2\, Lip(\psi)^k\, \mathbb{E}(\mathcal{M}(X_i)) \\ &\leq C_4\, 2^{-ki}\,\mathcal{M}(S),\end{aligned}$$

where C_4 depends on C_2, C_3 and $Lip(\psi)$.

$R_i(S)$ is a horizontal chain, whose mass rescales exactly by $2^{(k+1)i}$, hence

$$\begin{aligned}\mathbb{E}(\mathcal{M}(R_i(S))) &\leq C_4\, 2^{(k+1)i} 2^{-ki} \mathcal{M}(S) \\ &= C_4\, 2^i \mathcal{M}(S).\end{aligned}$$

This parallels Lemma 3.4 of [15].

Let

$$T_j = Q_0(S) - \sum_{i=0}^{j} R_i(S) \quad \text{and} \quad S'_j = Q_0(\partial S) - \sum_{i=0}^{j} R'_i(\partial S).$$

By construction,

$$\begin{aligned} \partial T_j + S'_j &= Q_0(S) + Q_0(\partial S) - \sum_{i=0}^{j} (R_i(S) + R'_i(\partial S)) \\ &= S + P_j. \end{aligned}$$

Since the bounds on masses of summands in T_j grow exponentially, the sum can be estimated by its last term,

$$\mathbb{E}(\mathcal{M}(T_j)) \leq \mathbb{E}(\mathcal{M}(Q_0(S))) + 2C_4\, 2^j \mathcal{M}(S) \tag{10}$$

$$\leq 4C_4(1 + 2^j)\mathcal{M}(S). \tag{11}$$

4.5. **Avoiding concentric balls.** We next show that the chains $R_i(S)$ avoid larger and larger concentric Cygan-Koranyi balls.

Since the horizontal map $\psi : \tau'/\Gamma \to \mathbb{H}^n/\Gamma$ is smooth, its image has empty interior in $\mathbb{H}^n/\Gamma$. So does the lift $\psi(\tau') \subset \mathbb{H}^n$. Moving $\psi(\tau')$ by a small left translation, we can assume that it does not hit the origin. Let $B(\epsilon)$ be a Koranyi ball centered at the origin that does not intersect $\phi(\tau)$. Then $R_i(S) = (\mathfrak{s}_{2^i})_{\#} P_\phi(X_i)$ does not intersect $B(2^j\epsilon)$.

In addition, $R_i(S)$ lies not far from the support of S: $Q_0(S)$ lies at distance at most δ apart, where $\delta = D + D'$, with $D =$ maximal displacement of points by ψ and $D' =$ maximal diameter of a simplex of τ. Thus $R_i(S)$ lies at most $2^i\delta$ apart, so the support of T_j is contained in the (Cygan-Koranyi) tubular Δ_j-neighborhood of the support of S, for

$$\Delta_j \leq 2^{j+1}\delta. \tag{12}$$

Let us draw a second consequence, needed to establish item 2.2(4) of Theorem 2.2.

Lemma 4.1. *There exists a constant $\lambda_0 = \lambda_0(n) > 0$ such that, if S has support outside U, $R(S)$ has support outside $\lambda_0 U$.*

Proof. Assume that S has support outside $U = B(r)$, $Q_0(S)$ has support outside the $r - \delta$-ball $B(r - \delta) = (1 - \frac{\delta}{r})U$. Similarly, since $\mathfrak{s}_{2^{-i}}(S)$ avoids $B(2^{-i}r)$, $R_i(S)$ avoids $B(2^i(2^{-i}r - \delta)) = B(r - 2^i\delta)$. On the other hand, $R_i(S)$ is contained in $\mathfrak{s}_{2^i}(\psi(\tau'))$, hence avoids $B(2^i\epsilon)$. Since

$$\min_{i\in\mathbb{N}} \max\{r - 2^i\delta, 2^i\epsilon\} \geq \frac{\epsilon r}{\delta},$$

we see that $R(S)$ avoids $\frac{\epsilon}{\delta}U$. So we can take $\lambda_0 = \frac{\epsilon}{\delta}$. □

4.6. **Stopping time, I.** The process stabilizes in restriction to U when R_i escapes U, which happens before i reaches the first value j_1 such that 2^{j_1} exceeds $\frac{r}{\epsilon}$, where ϵ has been determined in the previous section. For every $j \leq j_1$,

$$\mathbb{E}(\mathcal{M}(T_j)) \leq 4C_4\,(1+2^j)\mathcal{M}(S) \leq C_5\, r\, \mathcal{M}(S),$$

where C_5 depends on C_2, C_4 and ϵ.

4.7. **Truncation.** By construction, for $j > j_1$, $R_j(S)$ has support in the complement of U. For each $i \in \mathbb{N}$, let us truncate all chains of the form $\phi_\#(\kappa)$, with κ a chain of the triangulation $\mathfrak{s}_{2^i}(\tau)$, by keeping only simplices which intersect U. The truncated chains have support in νU where $\nu = 1 + \frac{D''}{\epsilon}$, and D'' is the maximum diameter of simplices of $\phi_\#(\tau)$. Let $\tilde{R}_i(S)$ denote the truncation of $R_i(S)$. Then the operator $\tilde{R}_i$ is additive and vanishes when $i > j_1$. It follows that the random operator between Lipschitz chains

$$S \mapsto R(S) := Q_0(S) - \sum_{i=0}^{\infty} \tilde{R}_i(S)$$

is well defined and additive. Furthermore, the expected mass estimate for T_{j_1} implies expected mass estimates on $R(S)$ and $R(\partial S)$,

$$\mathbb{E}(\mathcal{M}(R(S))) \leq C_5\, r\, \mathcal{M}(S), \quad \mathbb{E}(\mathcal{M}(R(\partial S))) \leq C_5\, r \mathcal{M}(\partial S).$$

The identity $\partial T_j + S'_j = S + P_j$ implies that

$$\partial R(S) + R(\partial S) = S$$

holds up to Lipschitz chains with support in $\nu U \setminus U$. In particular, it holds in the sense of integral currents restricted to U.

4.8. **Stopping time, II.** When $k \geq 1$, there is a second cause that may lead the process to stop. Let μ be a lower bound on the masses of the simplices of $\phi(\tau)$. Let j_2 be the (random) last time when $2^{kj}\mu \leq \mathcal{M}(P_j(S))$. By Hölder's inequality, since $k \geq 1$,

$$\mathbb{E}(2^{j_2}) \leq (\mathbb{E}(2^{kj_2}))^{1/k} \leq (\frac{C_2}{\mu}\mathcal{M}(S))^{1/k}. \tag{13}$$

For $j > j_2$, $P_j(S)$ is an integral combination of simplices of $\mathfrak{s}_{2^j} \circ \phi(\tau)$, its mass is smaller than the mass of any of these simplices, so $P_j(S) = 0$ and the process stops. So

$$R(S) = T_{\min\{j_1, j_2\}},$$

$$\mathbb{E}(\mathcal{M}(R(S))) \leq C_5\,\mathcal{M}(S)\,\mathbb{E}(2^{j_2}) \leq C_5\,\mathcal{M}(S)\,(\frac{C_2}{\mu}\mathcal{M}(S))^{1/k}$$
$$\leq C_6\,\mathcal{M}(S)^{(k+1)/k},$$

where C_6 depends only on C_2, C_5 and μ.

4.9. **Final steps.** When $r = +\infty$, the truncation is unneeded, thus $\partial R(S) + R(\partial S) = Id$ holds without restriction. Only the second stopping mechanism is at work, the number of steps needed is bounded only in terms of $\log \mathcal{M}(S)$, by (13). Using Estimates (12) and (13), with high probability, the support of $R(S)$ is contained in the Δ-neighborhood of $\mathrm{supp}(S)$, with

$$\Delta \leq C_7\,\mathcal{M}(S))^{1/k},$$

where C_7 depends on C_6, μ and δ.

This completes the proof of Theorem 2.3.

When $r < +\infty$, to complete the proof of Theorem 2.2, there remains to improve the noneffective factor λ_0 obtained in Lemma 4.1 to an arbitrary given $\lambda < 1$. This will be done in the next section.

5. Using a stretch map

Imitating Balogh-Fässler-Platis (who did it for $n = 1$), we construct a one-parameter group of contact transformations $(f_u)_{u>0}$ of $\mathbb{H}^n$ mapping Koranyi spheres centered at the origin to concentric Koranyi spheres, while raising there radii R to R^u. Like these authors, we shall denote one-parameter groups multiplicatively: $f_u \circ f_v = f_{uv}$.

Let (M, θ) be a contact manifold. A vectorfield X generates a one-parameter group of contactomorphisms if and only if the Lie derivative $\mathcal{L}_X\theta$ is collinear to θ at every point. Using Cartan's formula, this means that

$$d(\iota_X\theta) = -\iota_X d\theta \mod \theta.$$

Let T denote the Reeb vectorfield, defined by

$$\iota_T\theta = 1, \quad \iota_T d\theta = 0.$$

We see that a vectorfield $X = H\,T + X_c$, where X_c is horizontal, is contact if and only if X_c is the symplectic dual of the horizontal differential $d_cH = dH_{|\mathrm{Ker}(\theta)}$. Thus every smooth one-parameter group of contact transformations is *driven* by a unique smooth function H, called its *Hamiltonian*.

For instance, in $\mathbb{H}^1$, let us use exponential coordinates (x, y, t), left-invariant vectorfields X, Y, T dual to left-invariant 1-forms dx, dy and $\theta = dt - \frac{1}{2}(xdy - ydx)$. Then $H = 2t$ drives the vectorfield

$$X_{2t} = 2X_t = xX + yY + 2tT,$$

which generates the one-parameter group of dilations. The Hamiltonian $H = s := \frac{1}{2}(x^2 + y^2)$ drives the vectorfield

$$X_s = yX - xY + sT,$$

which generates the one-parameter group of rotations around the center.

Lemma 5.1 (The Balogh-Fässler-Platis flow)**.** *In $\mathbb{H}^1$, the Hamiltonian driving Balogh-Fässler-Platis' one-parameter group of stretch maps is*

$$H = 2t \log r,$$

where $r = \sqrt[4]{4s^2 + t^2}$ is the Cygan-Koranyi distance to the origin.

Indeed, let us use Platis' coordinates (ξ, ψ, η) related to exponential coordinates (x, y, t) by

$$x + iy = r\sqrt{\cos(\psi)}e^{i\phi}, \quad t = -r^2 \sin(\psi),$$
$$\xi = 2 \log r, \quad \eta = \frac{1}{3}(\pi + \psi - 2\phi)$$

Balogh-Fässler-Platis express the contact form

$$\theta = -e^{\xi}(\sin(\psi)d\xi + 3\cos(\psi)d\eta),$$

and the maps

$$f_u(\xi, \psi, \eta) = (u\xi, \arctan(\frac{\tan(\psi)}{u}), \eta).$$

One computes the infinitesimal generator

$$X_{BFP} = \frac{\partial f_u}{\partial u}_{|u=1} = (\xi, -\cos(\psi)\sin(\psi), 0)$$

and

$$\theta(X_{BFP}) = -e^{\xi}\sin(\psi)\xi = -2r^2 \sin(\psi)\log(r) = 2t \log r.$$

Now we generalize this construction to higher dimensions. On $\mathbb{H}^n$, in exponential coordinates $(x_1, y_1, \ldots, x_n, y_n, t)$, the left-invariant contact form becomes

$$\theta := dt - \frac{1}{2}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i).$$

Again, here are two examples of contact flows. The Hamiltonian $H = 2t$ drives the vectorfield

$$2X_t = \sum_{i=1}^{n}(x_i X_i + y_i Y_i) + 2tT,$$

which generates the one-parameter group of dilations $(\mathfrak{s}_u)_{u>0}$ of $\mathbb{H}^n$.

The Hamiltonian $H = s := \frac{1}{2}\sum_i (x_i^2 + y_i^2)$ drives the vectorfield

$$X_s = \sum_{i=1}^{n} (y_i X_i - x_i Y_i) + sT$$

which generates a one-parameter group of rotations around the center of the group $\mathbb{H}^n$.

Lemma 5.2 (Flows of stretch maps on $\mathbb{H}^n$)**.** *On $\mathbb{H}^n$, let us define the Cygan-Koranyi distance to the origin by*

$$r = \sqrt[4]{4s^2 + t^2}, \quad \textit{where} \quad s := \frac{1}{2}\sum_{i=1}^{n} (x_i^2 + y_i^2).$$

Let χ be a C^1 function on some open interval $J \subset (0, +\infty)$. Let G be a function on $J \setminus \chi^{-1}(0)$ whose derivative is

$$G'(x) = \frac{1}{x\chi(x)}.$$

The Hamiltonian $2t\chi(r)$ drives a flow $(f_u)_{u>0}$ of smooth contactomorphisms of the shell $r^{-1}(J) = \{p \in \mathbb{H}^n \,;\, r(p) \in J\}$ that satisfies

$$\forall u > 0, \quad G \circ r \circ f_u = G \circ r + \log(u),$$

where $G \circ r$ is defined. In particular, if $\chi = \log$, on $\mathbb{H}^n \setminus \{e\}$,

$$\forall u > 0, \quad r \circ f_u = r^u.$$

Proof. As r is homogeneous of degree 1 under dilations, $r \circ \mathfrak{s}_u = ur$. Since $2X_t$ generates the flow $(\mathfrak{s}_u)$, by differentiation at $u = 1$, we get

$$2\mathcal{L}_{X_t} r = r.$$

The rotations generated by X_s preserve the Cygan-Koranyi distance to the origin, so

$$\mathcal{L}_{X_s} r = 0.$$

If a Hamiltonian is a function of t and s, then the horizontal component of the corresponding contact vectorfield is

$$X_{H,c} = \frac{\partial H}{\partial t} X_{t,c} + \frac{\partial H}{\partial s} X_{s,c}.$$

Hence

$$X_H = (H - t\frac{\partial H}{\partial t} - s\frac{\partial H}{\partial s})T + \frac{\partial H}{\partial t} X_t + \frac{\partial H}{\partial s} X_s.$$

Therefore, using $\mathcal{L}_T r = \frac{\partial r}{\partial t} = \frac{t}{2r^3}$, $\mathcal{L}_{X_t} r = \frac{r}{2}$ and $\mathcal{L}_{X_s} r = 0$,

$$\begin{aligned}\mathcal{L}_{X_H} r &= (H - t\frac{\partial H}{\partial t} - s\frac{\partial H}{\partial s})\mathcal{L}_T r + \frac{\partial H}{\partial t}\mathcal{L}_{X_t} r \\ &= \frac{1}{2r^3}\left(Ht + 4s^2\frac{\partial H}{\partial t} - st\frac{\partial H}{\partial s}\right).\end{aligned}$$

When $H = 2t\chi(r)$, for some auxiliary C^1 function χ,

$$\frac{\partial H}{\partial t} = 2\chi(r) + \frac{2t^2\chi'(t)}{2r^3}, \quad \frac{\partial H}{\partial s} = \frac{\partial H}{\partial r}\frac{\partial r}{\partial s} = \frac{8st\chi'(r)}{2r^3},$$

so

$$2r^3\mathcal{L}_{X_H} r = (2t^2 + 4s^2)\chi(r) + (8s^2t^2 - 8s^2t^2)\chi'(r) = 2r^4\chi(r),$$

i.e.

$$\mathcal{L}_{X_H} r = r\chi(r).$$

This is the infinitesimal form of the equation $G(r \circ f_u) = G(r) + \log(u)$. It implies the global form as follows. For every $p \in \mathbb{H}^n$, the curve $v \mapsto f_{e^v}(p)$ is an integral curve of the vectorfield X_H, so

$$\forall v \in \mathbb{R}, \quad \frac{\partial}{\partial v} f_{e^v}(p) = X_H(f_{e^v}(p)).$$

In particular,

$$\forall v \in \mathbb{R}, \quad \frac{\partial}{\partial v} r \circ f_{e^v}(p) = (\mathcal{L}_{X_H} r)(f_{e^v}(p)).$$

The smooth function $g : v \mapsto r \circ f_{e^v}(p)$ satisfies

$$\forall v \in \mathbb{R}, \quad \frac{\partial g}{\partial v} = g(v)\chi(g(v)),$$

which means that

$$\forall v \in \mathbb{R}, \quad \frac{\partial G(g(v))}{\partial v} = 1.$$

We get $G(g(v)) = G(r(p)) + v$, $G(r \circ f_u(p)) = G(r(p)) + \log(u)$. This is valid on each connected component of the shell $\{p \in \mathbb{H}^n\,;\, r(p) \in J$ and $\chi(r(p)) \neq 0\}$.

In particular, if $\chi = \log$, $J = (0, +\infty)$, $G = \log|\log|$, $\log|\log(r \circ f_u(p))| = \log|\log(r(p))| + \log(u)$, $|\log(r \circ f_u(p))| = u|\log(r(p))|$. By continuity, the sign of $\log(r \circ f_u(p))$ is constant in u, so $|\log(r \circ f_u(p)) = u\log(r(p))$ and finally

$$r \circ f_u(p) = e^{ur(p)} = r(p)^u,$$

both on $r^{-1}((0,1))$ and on $r^{-1}((1,+\infty))$. The flow being smooth on all of $r^{-1}((0,+\infty)) = \mathbb{H}^n \setminus \{0\}$, the announced identity holds everywhere on this set.

□

Remark 5.3. *Every smooth distance which is homogeneous under the dilations $\mathfrak{s}_u$ and invariant under the rotations ρ_u could replace Cygan-Koranyi's distance in the above arguments. For instance, only the lack of smoothness of the subRiemannian distance prevents us from using it.*

From now on, $\chi = \log$. The stretch maps f_u are continuous but not smooth at the origin. To remedy this, let us modify H near the origin. Given $\epsilon < 1$, let $h_\epsilon : \mathbb{R} \to \mathbb{R}$ be a smooth, even function such that $h_\epsilon = \log$ on $(\epsilon^4, +\infty)$. Let

$$^\epsilon H = \frac{1}{2} t\, h_\epsilon(r^4).$$

Then $^\epsilon H = H$ outside the Koranyi ball of radius ϵ, denoted by $B(\epsilon)$. Let $^\epsilon f_u$ denote the corresponding smooth contactomorphism flow. Along an integral curve of $-X_H$ starting in $B(1) \setminus B(\epsilon)$, the function r is nondecreasing, the curve does not enter $B(\epsilon)$. Thus this curve is an integral curve of $-X_{^\epsilon H}$ as well. This shows that $^\epsilon f_u = f_u$ on $B(1) \setminus B(\epsilon)$ when $u \leq 1$. In particular, when $u \leq 1$,

$$^\epsilon f_u(B(\epsilon)) = B(\epsilon^u) \quad \text{and} \quad {}^\epsilon f_u(\mathbb{H}^n \setminus B(\epsilon)) = \mathbb{H}^n \setminus B(\epsilon^u).$$

Let $U = B(R)$ be a Koranyi ball of radius R. Given $0 < \lambda < \mu$, there is a global contactomorphism f mapping U to itself and mapping $U \setminus \lambda U$ into $U \setminus \mu U$, such that, with respect to a subRiemannian or Koranyi metric, $Lip(f) + Lip(f^{-1}) \leq C_8$ which depends only on λ and μ. One can take $f = \mathfrak{s}_R \circ {}^\lambda f_u \circ \mathfrak{s}_{1/R}$ for $u = \frac{\log(\mu)}{\log(\lambda)}$. This allows to replace the parameter λ whose existence is asserted in Theorem 2.2 by an arbitrary given $\mu < 1$. Indeed, given a chain S, one applies Theorem 2.2 to $(f^{-1})_\# S$, whose mass and boundary mass have increased by a factor at most C_8^k. One gets a filling T_f and the accompanying S'_f. Then $T = f_\# T$, $S' = f_\# S'_f$ achieve the required filling of S, but S' is supported in the complement of μU, at the cost of increased masses by at most a factor of C_8^{2k+1}.

6. Proofs of Theorems 1.4 and 1.5

6.1. **Summary of previous sections.** We observe that the results of Section 4 extend to horizontal integral currents. Indeed, the random deformation theorem 3.8 applies to integral currents and produces chains $P(S)$ of the triangulation τ. Thus, mimicking the construction of Section 4, one obtains Lipschitz chains $P_i(S)$ and $R_i(S)$ contained in $\phi(\tau)$, with the same

expected mass bounds, and avoiding the ball ϵU. Therefore the arguments of Section 4 apply to horizontal integral currents without any change. So in Theorem 2.2 and Proposition 6.3, one can replace the group L^h of horizontal Lipschitz chains with the larger group I^h of horizontal integral currents. Instead of the truncation of Section 4.7, restriction to U suffices, leading to the following conclusion.

Proposition 6.1. *For every n and $\lambda < 1$, there exists $C = C(n, \lambda) > 0$ with the following significance. There exists a random map $R : I^h_{\leq n-1} \to I_{\leq n}$, mapping each I^h_k to I_{k+1}, such that*

a.1 *R is additive;*
a.2 *$R\partial + \partial R = Id$ in dimensions up to $n-1$;*
a.3 *for every horizontal integral k-current S in $\mathbb{H}^n$, $1 \leq k \leq n-1$ (or $k = 1$ if $\partial S = 0$),*

$$\mathbb{E}(\mathcal{M}(R(S))) \leq C\,(\mathcal{M}(S) + mcM(S)^{(k+1)/k}).$$

Let U be a Cygan-Koranyi r-ball and let $R_U = (f_\# \circ R \circ (f^{-1})_\#) \llcorner U$, for a suitable biLipschitz stretch map f. Then

b.1 *R_U is additive;*
b.2 *$(R_U\partial + \partial R_U) \llcorner U = Id \llcorner U$ in dimensions up to $n-1$;*
b.3 *for every horizontal integral k-current S in $\mathbb{H}^n$, $0 \leq k \leq n-1$,*

$$\mathbb{E}(\mathcal{M}(R_U(S))) \leq C\,(r+1)\,\mathcal{M}(S), \tag{14}$$

and if $1 \leq k \leq n-1$,

$$\mathbb{E}(\mathcal{M}(R_U(S))) \leq C\,(\mathcal{M}(S) + \mathcal{M}(S)^{(k+1)/k}), \tag{15}$$

b.4 *if in addition S has support in the complement of U, then $R_U(S)$ has support in the complement of λU.*

6.2. **Functional analytic setting.** Unlike in Section 2.6, we shall not rely on compactness, but show that rescaled versions of R converge in a space of random variables.

Consider the group $Hom(I^h, I)$ of homomorphisms of the group I^h of horizontal integral currents to the group of integral currents on $\mathbb{H}^n$. As norms on I^h and I, we use the mass $\mathcal{M}$. As norm on $Hom(I^h, I)$, we use

$$\|\rho\| = \sup\{\frac{\mathcal{M}(\rho(S))}{\mathcal{M}(S)}\,;\, S \in I^h\}.$$

Let us view random elements of $Hom(I^h, I)$ as measurable maps from some fixed probability space Ω to $Hom(I^h, I)$. Here, $\Omega = \prod_{F\in\mathcal{F}} \ddot{F}$ is the product of padded interiors (see Definition 3.1) of all faces of the infinite triangulation τ' of $\mathbb{H}^n \times [0,1]$, each with its uniform measure. According to Kolmogorov's extension theorem, a σ-algebra and a probability measure

$\mathbb{P}$ on Ω are well-defined. On the space $L^1(\Omega, Hom(I^h, I))$, we use the L^1 norm

$$\|\rho\|_1 = \mathbb{E}(\|\rho\|).$$

Lemma 6.2. *The space $L^1(\Omega, Hom(I^h, I))$ is complete.*

Proof. FIND A GOOD REFERENCE □

Proposition 6.3. *Let R be the random element of $Hom(I^h, I))$ provided by Proposition 6.1. For $j \in \mathbb{N}$, let*

$$\rho_j = (\mathfrak{s}_{2^{-j}})_\# \circ R \circ (\mathfrak{s}_{2^j})_\#.$$

Then $\rho_{j+\ell} - \rho_j$ belongs to $L^1(\Omega, Hom(I^h, I))$ and has a limit σ_j in this space, as ℓ tends to $+\infty$. It satisfies

$$\mathbb{E}(\mathcal{M}(\sigma_j(S))) \leq C\, 2^{-j} \mathcal{M}(S).$$

Proof. Recall the expression $R = Q_0 - \sum_{i=0}^\infty R_i$, where

$$R_i = (\mathfrak{s}_{2^i})_\# \circ R_0 \circ (\mathfrak{s}_{2^{-i}})_\#.$$

For S a horizontal Lipschitz k-chain, the bound

$$\mathbb{E}(\mathcal{M}(R_i(S))) \leq C\, 2^i \mathcal{M}(S),$$

holds for all $i \in \mathbb{Z}$. Indeed,

$$\begin{aligned}\mathbb{E}(\mathcal{M}(R_i(S))) &= 2^{i(k+1)} \mathbb{E}(\mathcal{M}(R_0((\mathfrak{s}_{2^{-i}})_\# S))) \\ &\leq C\, 2^{i(k+1)} \mathcal{M}((\mathfrak{s}_{2^{-i}})_\# S)) \\ &= C\, 2^i \mathcal{M}(S).\end{aligned}$$

One computes

$$\rho_j = (\mathfrak{s}_{2^{-j}})_\# \circ Q_0 \circ (\mathfrak{s}_{2^j})_\# - \sum_{i=-j}^\infty R_i,$$

and

$$\begin{aligned}\rho_{j+\ell} - \rho_j = (\mathfrak{s}_{2^{-j-\ell}})_\# \circ Q_0 \circ (\mathfrak{s}_{2^{j+\ell}})_\# - (\mathfrak{s}_{2^{-j}})_\# \circ Q_0 \circ (\mathfrak{s}_{2^j})_\# \\ - \sum_{i=-j-\ell}^{-j-1} R_i.\end{aligned}$$

Since

$$\mathbb{E}(M(Q_0(S))) \leq C\, \mathcal{M}(S),$$

and since $(\mathfrak{s}_{2^{-j}})_\#$ compresses masses of $k+1$-chains by at least $2^{-j(k+1)}$ (Lemma 2.5),

$$\mathbb{E}(\mathcal{M}(\rho_{j+\ell}(S)-\rho_j(S))) \le C\,(2^{-j-\ell}\mathcal{M}(S)+2^{-j}\mathcal{M}(S)+\sum_{i=-j-\ell}^{-j-1}2^i\mathcal{M}(S))$$
$$\le C\,2^{-j}\mathcal{M}(S).$$

A similar estimate holds for $\partial\circ\rho_j$, since $\partial\rho_j+\rho_j\partial = Id$.

This shows that the sequence $\rho_j-\rho_0$ is Cauchy, hence the limit

$$\sigma_j := \lim_{\ell\to+\infty}\rho_{j+\ell}-\rho_j \in L^1(\Omega, Hom(I^h, I))$$

exists for every $j\in\mathbb{N}$. By continuity, the expected mass estimate on $\rho_{j+\ell}(S)-\rho_j(S)$ passes to the limit as $\ell\to\infty$.

□

6.3. **Proof of Theorem 1.4.** Proposition 6.3 provides us with a random homomorphism $\rho = R+\sigma_0 = \lim_{\ell\to+\infty}\rho_\ell$ from horizontal Lipschitz chains to integral currents solving $\partial\rho+\rho\partial = Id$ almost surely. One can equally write $\rho=\rho_j+\sigma_j$ for every $j\in\mathbb{N}$. For each S,

$$((\mathfrak{s}_{2^{-j}})_\#\circ Q_0\circ(\mathfrak{s}_{2^j})_\#(S))\llcorner\theta \quad\text{and}\quad (\partial(\mathfrak{s}_{2^{-j}})_\#\circ Q_0\circ(\mathfrak{s}_{2^j})_\#(S))\llcorner\theta$$

tend to 0 weakly, as follows from the argument of Section 2.6. The R_i's being horizontal by construction, this suffices to show that $\rho(S)\llcorner\theta = 0$ and $(\partial\rho(S))\llcorner\theta = 0$, i.e. $\rho(S)$ is horizontal.

Since $(\mathfrak{s}_{1/2})_\#\circ\rho_j\circ(\mathfrak{s}_2)_\# = \rho_{j+1}$, ρ commutes with $(\mathfrak{s}_2)_\#$.

Using Propositions 6.1 and 6.3, for every horizontal integral current S, $\mathbb{E}(\mathcal{M}(\sigma_j))$ tends to 0,

$$\mathbb{E}(\mathcal{M}(\rho_j(S))) = \mathbb{E}(\mathcal{M}(R(S))) \le C\,(\mathcal{M}(S)+\Phi_2(\mathcal{M}(S))),$$

hence

$$\mathbb{E}(\mathcal{M}(\rho(S))) \le C\,(\mathcal{M}(S)+\Phi_2(\mathcal{M}(S))).$$

This completes the proof of Theorem 1.4.

6.4. **Proof of Theorem 1.5.** Let us denote by

$$\rho_{j,U} := (f_\#\circ\rho_j\circ(f^{-1})_\#)\llcorner U = (\mathfrak{s}_{2^{-j}})_\#\circ R_{2^jU}\circ(\mathfrak{s}_{2^j})_\#.$$

Then the random homomorphisms $\rho_{j,U}\in Hom(I^h, I(U))$ converge in L^1 to $\rho_U := (f_\#\circ\rho\circ(f^{-1})_\#)\llcorner U$. The limiting homomorphism $\rho_U\in Hom(I^h, I^h(U))$ satisfies

$$(\partial\rho_U+\rho_U\partial)\llcorner U = Id\llcorner U.$$

Let S be a horizontal integral current. The expected mass bound (14) for horizontal integral currents S', after restriction to the $2^j r$-ball and stretching,

$$\mathbb{E}(\mathcal{M}(R_{2^jU}(S'))) \leq C\,(\mathcal{M}(S') + (2^j r)\mathcal{M}(S')$$

implies

$$\mathbb{E}(\mathcal{M}(\rho_{j,U}(S))) \leq C\,2^{-j(k+1)}(2^{jk}\mathcal{M}(S) + (2^j r)2^{kj}\mathcal{M}(S),$$

i.e.

$$\mathbb{E}(\mathcal{M}(\rho_{j,U}(S))) \leq C\,(2^{-j}\mathcal{M}(S) + r\mathcal{M}(S)).$$

By continuity of $\mathbb{E}(\mathcal{M}(\rho(S)))$ as a function of ρ, we conclude that

$$\mathbb{E}(\mathcal{M}(\rho_U(S))) \leq C\,r\,\mathcal{M}(S).$$

If $\operatorname{supp} S$ avoids U, by Proposition 6.1, each $\rho_{j,U}(S)$ has support away from λU. By semi-continuity of the support, this persists in the limit. This completes the proof of Theorem 1.5.

Remark 6.4. *On the* 0*-dimensional map* ρ*.*

The first Heisenberg group $\mathbb{H}^1$ maps to its abelianization $\mathbb{H}^1/[\mathbb{H}^1, \mathbb{H}^1] = \mathbb{R}^2$. The image under this projection of the random map R is obtained by a similar construction as in $\mathbb{H}^1$. We describe next a simplified variant.

In the Euclidean plane, the triangulation τ is replaced with the integer grid. The triangulation τ' of $\mathbb{H}^n \times [0,1]$ is replaced with the cellulation of $\mathbb{R}^n \times [0,1]$ by polyhedra $2q \times [0,1]$ where q is an integer square, with the face $q \times \{0,1\}$ split into four integer squares, so that each cell comes with 9 faces, as shown on Figure 2. The map $\phi : \tau \to \mathbb{R}^n$ is the identity, the map $\psi : \tau' \to \mathbb{R}^n$ is the projection onto the first factor.

Given a 0-cycle $S = A - O$, with O the origin of $\mathbb{R}^2$ and $A = (x, y)$ a point with real coordinates, the random path $R(S)$ is made of edges of finer and finer dyadic grids. It depends on the dyadic expansions of coordinates x and y, and on randomness. Figure 4 shows, for two points A and B, two typical trajectories.

Taking horizontal lifts in $\mathbb{H}^1$ of these curves, conditioned to enclose equal areas, gives a rough idea of what the random map ρ can look like on 0-cycles of the form $A - O$.

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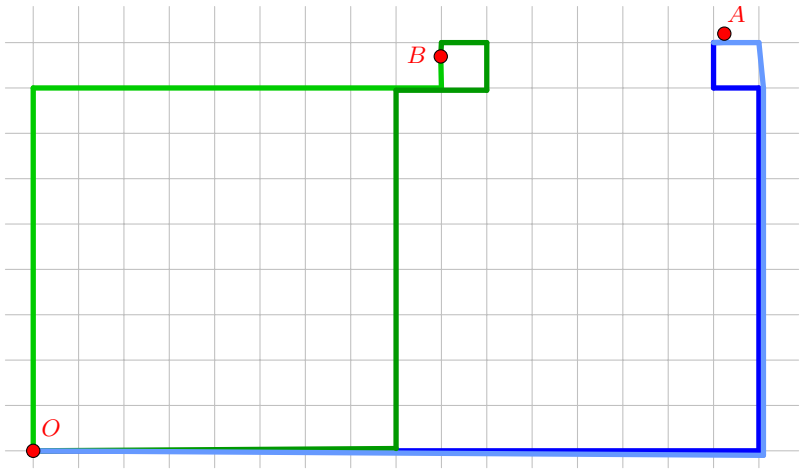


FIGURE 4. The Euclidean projection of the process ρ on 0-cycles

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