A SIMPLE PROOF OF WIENER'S LEMMA

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The celebrated Wiener lemma [4] states as follows.

Theorem 1. *Denote by W the space of continuous functions f on $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ such that*

$$\sum_{k\in\mathbb{Z}} |\hat{f}(k)| < \infty \ ,$$

where

$$\hat{f}(k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \mathrm{e}^{-ikx}\, dx \ .$$

Assume $f \in W$ and $f(x) \neq 0$ for every $x \in \mathbb{T}$. Then $1/f \in W$.

A famous proof of this result is a consequence of Gelfand's theory of C^* algebras — see *e.g.* [2]. Another proof, directly identifying the spectral radius of elements of W, is due to Beurling in [1]. More recently, a direct proof was given in Sjöstrand [3] in connection with pseudo-differential calculus. In this short note, I would like to give a very elementary proof, which was suggested to me by discussions with G. Lebeau about twenty five years ago.This proof is in the same spirit as Beurling's proof.

We start with a couple of standard lemmas.

Lemma 1. *Let us endow W with the norm*

$$\|f\|_W := \sum_{k\in\mathbb{Z}} |\hat{f}(k)| < \infty \ .$$

Then

$$\|fg\|_W \leq \|f\|_W \|g\|_W$$

and W is a Banach algebra, where the trigonometric polynomials are dense.

Indeed, the inequality is just the convolution inequality on $\ell^1(\mathbb{Z})$. Completeness of W follows from the completeness of $C(\mathbb{T})$ and of $\ell^1(\mathbb{Z})$, and from the inequality

$$\|f\|_{L^\infty} \leq \|f\|_W \ .$$

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I am grateful to M. Zworski who encouraged me to write this note, and to M. Christ for pointing the paper [1].

Finally, the density of trigonometric polynomials in W is just the density of finitely supported sequences in $\ell^1(\mathbb{Z})$.

Lemma 2. *Every C^2 function on $\mathbb{T}$ belongs to W, with the estimate*

$$\|g\|_W \le \frac{\pi^2}{3}\|g''\|_{L^\infty} + \|g\|_{L^\infty} \ .$$

Indeed, by a simple integration by parts,

$$|\hat{g}(k)| \le \frac{1}{k^2}\|g''\|_{L^\infty}, k \neq 0 \ .$$

Let us prove Theorem 1. Let $f \in W$ satisfying $f(x) \neq 0$. Set

$$m := \min_{x\in\mathbb{T}} |f(x)| > 0 \ .$$

Applying Lemma 1, we can choose a trigonometric polynomial g such that

$$\|f - g\|_W \le \frac{m}{3} \ .$$

By the triangle inequality, we infer

$$|g(x)| \ge \frac{2m}{3} \ , \tag{1}$$

and in particular

$$\left|\frac{g(x) - f(x)}{g(x)}\right| \le \frac{1}{2} \ .$$

Next we observe that

$$\frac{1}{f} = \frac{1}{g}\frac{1}{1-\frac{g-f}{g}} = \frac{1}{g}\sum_{n=0}^{\infty}\left(\frac{g-f}{g}\right)^n ,$$

where the series in the right hand side converges in $C(\mathbb{T})$. Since $1/g \in C^2(\mathbb{T}) \subset W$ by Lemma 2, and since W is an algebra by Lemma 1, the proof will be complete if we manage to prove that the convergence of this series lies in W. Since W is a Banach space, it is enough to prove that the series of norms in W is convergent. Using Lemma 1, Lemma 2 and inequality (1), we obtain

$$\begin{aligned}
\left\|\left(\frac{g-f}{g}\right)^n\right\| &\le \|g-f\|_W^n \left\|\frac{1}{g^n}\right\|_W \\
&\le \left(\frac{m}{3}\right)^n \left(\frac{\pi^2}{3}\left\|\left(\frac{1}{g^n}\right)''\right\|_{L^\infty} + \left\|\frac{1}{g^n}\right\|_{L^\infty}\right) \\
&\le \left(\frac{m}{3}\right)^n \left(\frac{\pi^2}{3}\left[\frac{n\|g''\|_{L^\infty}}{(2m/3)^{n+1}} + \frac{n(n+1)\|g'\|_{L^\infty}^2}{(2m/3)^{n+2}}\right] + \frac{1}{(2m/3)^n}\right) \\
&\le C(g)(1+n^2)2^{-n} \ .
\end{aligned}$$

This completes the proof.

Notice that the same proof leads to the following more general result.

Theorem 2. *Let X be a compact manifold, and W be a Banach algebra of functions on X, continuously contained in $C(X)$, and including $C^\infty(X)$ as a dense subspace. Then W enjoys the Wiener property : if $f \in W$ satisfies $f(x) \neq 0$ for every $x \in X$, then $1/f \in W$.*

Indeed, the equivalent of Lemma 2, namely that there exists an integer k such that

$$\forall f \in C^\infty(X) \ , \ \|f\|_W \leq C\|f\|_{C^k(X)} \ ,$$

is a consequence of the closed graph theorem applied to the inclusion of the Fréchet space $C^\infty(X)$ into the Banach space W.

References

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