Université Paris-Saclay **M2 AAG**
Groups and geometry **2024**

Mid-term exam

Exercise 1. Let G be a real Lie group with Lie algebra $\mathfrak{g}$ and universal cover $\tilde{G}$.
Denote by $\mathrm{Aut}(\mathfrak{g})$ the group of bijective Lie algebra morphisms from $\mathfrak{g}$ to $\mathfrak{g}$.
Denote by $\mathrm{Aut}(G)$ the group of smooth automorphisms of G.

(1) Prove that $\mathrm{Aut}(\mathfrak{g})$ is an embedded Lie subgroup of $\mathrm{GL}(\mathfrak{g})$.
(2) Prove that, for every $\sigma \in \mathrm{Aut}(G)$, $T_1\sigma \in \mathrm{Aut}(\mathfrak{g})$.

From now on, assume G is *connected*.

(3) Prove that $T_1 : \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ is injective.
(4) Prove that $T_1 : \mathrm{Aut}(\tilde{G}) \to \mathrm{Aut}(\mathfrak{g})$ is an isomorphism.
In particular, this gives $\mathrm{Aut}(\tilde{G})$ a Lie group structure.
(5) We denote by $\alpha : \mathrm{Aut}(\tilde{G}) \times \tilde{G} \to \tilde{G}$ the map $(\varphi, g) \mapsto \varphi(g)$.
  (a) Prove that α is smooth near $\mathrm{Aut}(\tilde{G}) \times \{1\}$.
  (b) Prove that, for every $g \in \tilde{G}$, $\varphi \mapsto \varphi(g)$ is smooth.
  (c) Prove that α is smooth.
(6) Prove that every automorphism of G lifts to a unique automorphism of $\tilde{G}$, and that the lifting operation is a group morphism ι from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(\tilde{G})$.

We denote by N the kernel of the Lie covering map from $\tilde{G}$ to G.

(7) Prove that $\iota(\mathrm{Aut}(G))$ is the subgroup of $\mathrm{Aut}(\tilde{G})$ formed by transformations preserving N.
(8) Prove that $\iota(\mathrm{Aut}(G))$ is a closed subgroup of $\mathrm{Aut}(\tilde{G})$.
In particular this endows $\mathrm{Aut}(G)$ with a Lie group structure.
Prove that the action of $\mathrm{Aut}(G)$ on G is smooth.

Exercise 2. Let V be a finite-dimensional real vector space, and n its dimension. Fix an inner product of V. Denote by $\mathrm{O}(V)$ the subgroup of $\mathrm{GL}(V)$ formed by transformations preserving the inner product. This is a closed Lie subgroup of $\mathrm{GL}(V)$. For every k in $\{0, ..., n\}$, denote by $\mathrm{Gr}_k(V)$ the set of k-dimensional linear subspaces of V. The group $\mathrm{O}(V)$ acts on each $\mathrm{Gr}_k(V)$ by direct image.

(1) Prove that, for every k in $\{0, ..., n\}$, $\mathrm{O}(V)$ acts transitively on $\mathrm{Gr}_k(V)$ with closed stabilizers.
(2) From the previous question we get, for every k in $\{0, ..., n\}$ and every E in $\mathrm{Gr}_k(V)$, a smooth structure on $\mathrm{Gr}_k(V)$ coming from $\mathrm{O}(V)/\mathrm{O}(V)_E$. Prove that it does not depend on the choice of E.
(3) Prove that, for every k in $\{0, ..., n\}$, the map $E \mapsto E^\perp$ is a diffeomorphism from $\mathrm{Gr}_k(V)$ to $\mathrm{Gr}_{n-k}(V)$.