Université Paris-Saclay **M2 AAG**
Groups and geometry **2025**

Mid-term exam

Duration: 3 hours. **Allowed documents**: five A4 sheets of handwritten notes.

Exercise 1. Prove that every connected abelian real Lie group is isomorphic to $\mathbb{R}^k \times (\mathbb{R}/\mathbb{Z})^l$ for some natural numbers k and l.

Exercise 2. The Heisenberg group is

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

We saw in exercises sessions this is a Lie group. We set $\Gamma = H \cap GL_3(\mathbb{Z})$,

$$Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \middle| z \in \mathbb{R} \right\}.$$

and $Z_0 = Z \cap \Gamma$. Recall Z is the center of H. In the following, Γ acts on H by left translations and Z acts on H by right translations.

(1) Prove that there is a unique manifold structure on $\Gamma\backslash H$ such as the projection $\pi : H \to \Gamma\backslash H$ is a covering map.
(2) Is there a Lie group structure on $\Gamma\backslash H$ such that π is a Lie group morphism?
(3) Prove there is a unique Lie group structure on H/Z such that the projection $\pi_Z : H \to H/Z$ is a Lie group morphism.
(4) Prove that H/Z is isomorphic to $\mathbb{R}^2$ as a Lie group.
(5) Prove that the action of Z on H induces a smooth action of Z/Z_0 on $\Gamma\backslash H$. Here "induces" means we have a commutative diagram

$$\begin{array}{ccc} H \times Z & \longrightarrow & H \\ \downarrow{\scriptstyle \pi\times\pi_Z} & & \downarrow{\scriptstyle \pi} \\ (\Gamma\backslash H) \times (Z/Z_0) & \longrightarrow & \Gamma\backslash H \end{array}$$

where the top row is the action of Z on H.
(6) Prove there is unique manifold structure on $(\Gamma\backslash H)/(Z/\mathrm{Z}_0)$ such that the projection map $\pi_3 : (\Gamma\backslash H) \to (\Gamma\backslash H)/(Z/\mathrm{Z}_0)$ is a fiber bundle.

Exercise 3. Let G be a (real) *connected* Lie group, and denote by $\mathfrak{g}$ its Lie algebra. Let M be a (real) manifold. We denote by pr_G and pr_M the projections of $G \times M$ onto G and M respectively. A smooth action of $\mathfrak{g}$ on M is a Lie algebra morphism $X \mapsto X_M$ from $\mathfrak{g}$ to the opposite of the Lie algebra of vector fields on M such that the corresponding map $(X, m) \mapsto X_M(m)$ from $\mathfrak{g} \times M$ to TM is smooth. Opposite Lie algebra means the same vector space structure and bracket $[\cdot,\cdot]^{\mathrm{op}} = -[\cdot,\cdot]$.

We saw in exercises session that any smooth left action of G on M gives a Lie algebra action defined by

$$X_M(m) = \frac{d}{dt}_{|t=0} \exp(tX)m.$$

(1) Given a smooth action of G on M, prove that, for every $X \in \mathfrak{g}$, every $t \in \mathbb{R}$, and every $m \in M$, $\exp(tX)m = \varphi^t_{X_M}(m)$, where $\varphi^t_{X_M}$ is the flow of X_M at time t. In particular each X_M is a complete vector field.
(2) Prove that two actions of G on M that induce the same action of $\mathfrak{g}$ are equal.
(3) The group G acts on itself by left translations. The corresponding action of $\mathfrak{g}$ on G will be denoted by $X \mapsto X_G$. Prove that, for every X, X_G is the *right*-invariant vector field with value X at 1 (recall that a vector field Z on G is right-invariant if, for every $g \in G$, $Z(g) = T_1 R_g(Z(1))$).

Until the end of the exercise, we fix an action of $\mathfrak{g}$ on M and assume that G is *simply connected* and that, for every $X \in \mathfrak{g}$, X_M is *complete*. The final goal is to prove that this action of $\mathfrak{g}$ comes from a unique action of G. Together with the action of $\mathfrak{g}$ on G from Question 3, we get an action $X \mapsto X_{G\times M} := (X_G, X_M)$ of $\mathfrak{g}$ on $G \times M$. We fix an open neighborhood V of 0 in $\mathfrak{g}$ such that exp restricted to V is a diffeomorphism onto its image U in G.

(4) For every (g, m) in $G \times M$, we set $\Delta_{(g,m)} = \mathrm{im}(X \mapsto X_{G\times M}(g, m))$. Prove that, for every $m \in M$, there exists a unique maximal connected submanifold $L_m \subseteq G \times M$ containing $(1, m)$ and with tangent space $\Delta_{(g,m')}$ at every $(g, m') \in L_m$. Prove that L_m has the same dimension as G.
(5) For every $g \in G$, we denote by $\iota_g : M \to G \times M$ the map $m \mapsto (g, m)$. Prove that, for every $g \in G$ and $m \in M$, $\iota_g \pitchfork L_m$ and $\iota_g^*(L_m)$ is a 0-dimensional submanifold of M.
(6) Let g be an element of G. Let m be a point in M. Denote by π_m the restriction of pr_G to L_m. We define $\Phi_{(g,m)} : V \times \iota_g^*(L_m) \to G \times M$ by

$$\Phi_{(g,m)} : (X, m') \mapsto \varphi^1_{X_{G\times M}}(g, m').$$

Prove that $\Phi_{(g,m)}$ is well-defined, can be written as

$$(X, m') \mapsto (\exp(X)g,\ \varphi^1_{X_M}(m')),$$

is a diffeomorphism onto $\pi_m^*(Ug)$, and that

$$\begin{array}{ccc} V \times \iota_g^*(L_m) & \xrightarrow{\Phi_{(g,m)}} & \pi_m^*(Ug) \\ {\scriptstyle \exp\circ \mathrm{pr}_V}\downarrow & & \downarrow{\scriptstyle \pi_m} \\ U & \xrightarrow{R_g} & Ug \end{array}$$

commutes. (There are many things to check, so this is a longer question).
(7) Prove that π_m is a smooth covering map, and then that π_m is a diffeomorphism.
(8) For every $g \in G$ and $m \in M$, we set $gm = \mathrm{pr}_M(\pi_m^{-1}(g))$ (but we do not know this gives a smooth action yet). Prove that, for every $X \in V$ and every $m \in M$, $\exp(X)m = \varphi^1_{X_M}(m)$.
(9) Prove that, for every $X \in V$, every $g \in G$ and every $m \in M$, $\exp(X)(gm) = (\exp(X)g)m$.
(10) Conclude that $(g, m) \mapsto gm$ is indeed a smooth Lie group action of G on M whose associated Lie algebra action is the one we started with. (This is also a longer question).