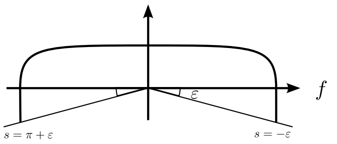
Theorem 1 ([Gay06, Wen])**.** *Any strong symplectic filling of a contact 3–manifold with positive Giroux torsion could be enlarged to a weak filling of an overtwisted manifold.*

On $\mathbb{T}^2 \times \mathbb{R}$ with coordinate (θ, φ, s), let ξ_G be the contact structure defined by $\lambda_G := \sin(s)d\theta + \cos(s)d\varphi$. Let $G_{2\pi} := \mathbb{T}^2 \times [-\pi, \pi]$. By definition [Gir00], a contact manifold (V, ξ) has positive Giroux torsion if there is a contact embedding of $(G_{2\pi}, \xi_G)$ in the interior of V. Using that characteristic foliations determine the germ of a contact structure near a surface, such an embedding can be extended to a contact embedding of $G_{2\pi+\varepsilon} := \mathbb{T}^2 \times [-\pi - \varepsilon, \pi + \varepsilon]$ for some positive ε. This ε is fixed throughout the paper.

We now describe a surgery turning (V, ξ) into the overtwisted contact manifold mentionned in the theorem. Let η be a positive number smaller than $\sqrt{\varepsilon}$. We denote by D the open disk with radius η around the origin in $\mathbb{R}^2$. We denote by $\dot{D}$ the punctured disk $D \setminus \{0\}$. We use polar coordinates (r, θ) on $\mathbb{R}^2$. We also consider $\Sigma := \mathbb{S}^1 \times [0, \pi]$ so that $G_\pi := \mathbb{S}^1 \times \Sigma$ is half of $G_{2\pi}$. Let Ψ from $\dot{D} \times \partial\Sigma$ to $G_{2\pi+\varepsilon} \setminus G_\pi$ be the diffeomorphism defined by $\Psi(r, \theta, \varphi, s) = (\theta, \varphi, s \pm r^2)$ where the sign is positive when $s = \pi$ and negative when $s = 0$. All $\pm$ in this paper refer to this convention. The surgered manifold is $V' := (V \setminus G_\pi) \cup_\Psi (D \times \partial\Sigma)$. The contact structure $\Psi^*\lambda_G = \sin(s \pm r^2)d\theta + \cos(s \pm r^2)d\varphi$ is equivalent, when r goes to zero, to $\mp d\varphi \pm r^2 d\theta$ so that it extends smoothly to a contact form on the whole V'. Note that neither V' nor its contact structure ξ' depend on the choice of η. The contact structure ξ' is overtwisted because $G_{2\pi} \setminus G_\pi$ gets compactified to a Lutz tube.

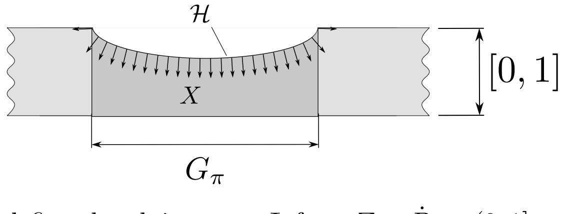
In the following, it will be convenient to view the standard Giroux domain from a slightly more flexible perspective, rescaling the contact form and reparametrizing the interval $[-\pi - \varepsilon, \pi + \varepsilon]$. This leads to contact forms $u(s)d\theta + f(s)d\varphi$ where $s \mapsto [f, u]$ is a path in $\mathbb{R}^2/\mathbb{R}^*_+$ homotopic to the standard one $[\cos, \sin]$ with fixed end-points and satisfying the contact condition $\delta := fu' - f'u > 0$. We impose that the restriction of this path to $[-\varepsilon, \pi + \varepsilon]$ has the same end-points that $[\cos, \sin]$ and still has $f' < 0$ wherever u is positive but $f' = 0$ and $u' = \pm 1$ elsewhere.



Any strong filling would have a collar $(0, 1] \times V$ with symplectic form $\omega = d(t\lambda)$ where t is the coordinate in $(0, 1]$. In $(0, 1] \times G_\pi \subset (0, 1] \times V$ we consider the hypersurface $\mathcal{H} = \{t = h(s)\}$ where $h = 1 - \sin(s)/2$.

Let X be the vector field on $(0, 1] \times G_{2\pi+\varepsilon}$ which is ω–dual to $-d\theta$. One has $X = X^t\,\partial_t + X^s\,\partial_s$ with $X^t = f'/\delta$ and $X^s = -f/(t\delta)$. Our constraints on (f, u) imply that X is transverse to $\mathcal{H}$ and coincides with $\pm\frac{1}{t}\partial_s$ near its boundary.

We now discard the epigraph $\{t \geq h\}$ and glue in $\Sigma \times \dot{D}$ using the flow φ^X of X starting along $\mathcal{H}$ —naturally identified with Σ— at time r^2.



In formulas, we define the gluing map Ψ from $\Sigma \times \dot{D}$ to $(0, 1] \times V$ by

$$\Psi(s, \varphi, re^{i\theta}) = \varphi^X_{r^2}\big(s, \varphi, \theta, h(s)\big).$$

In order for this to make sense we choose η small enough to ensure that the flow does not run out of $(0,1] \times G_{2\pi+\varepsilon}$ before time η^2. Because of the form of X near $\partial\mathcal{H}$, this map extends the gluing map Ψ used to define the surgery. In particular we expressed the surgery as resulting from the attachment of the "handle" $\Sigma \times D$ to $(0,1] \times V$. We now want to equip this handle with a symplectic form.

Lemma 2. *The gluing map Ψ from $\Sigma \times \dot{D}$ to $(0,1] \times V$ pulls ω back to*

$$\Psi^*\omega = \omega_D + d(hu) \wedge d\theta + \Omega_0$$

where $\omega_D := -2r\, dr \wedge d\theta$ and Ω_0 is a symplectic form on Σ.

Proof. One has $\Psi = \Phi \circ P$ where P from $\Sigma \times \dot{D}$ to $\Sigma \times [0,\eta^2] \times \mathbb{S}^1$ is defined by $P(\sigma, re^{i\theta}) = (\sigma, r^2, \theta)$ and $\Phi(\sigma,\tau,\theta) := \varphi^X_\tau\big(\sigma,\theta,h(\sigma)\big)$. The identification of Σ with $\mathcal{H}$ pulls ω back to $d(h\lambda)$. Since $\iota_X\omega = -d\theta$ and the flow of ${\varphi^X_\tau}_*\partial_\tau = X$ preserves ω, we have $\Phi^*\omega = -d\tau \wedge d\theta + d(h\lambda)$. So we can set $\Omega_0 = d(hf\,d\varphi)$ which is symplectic on Σ. □

We now modify $\Psi^*\omega$ away from $\Sigma \times \partial D$ to extend it to a symplectic form on $\Sigma \times D$. Let ρ_1 and ρ_2 be non-negative functions on $[0,\eta]$. We set:

$$\tilde{\omega} := \rho_1\omega_D + d(\rho_2 hu) \wedge d\theta + \Omega_0 = g\omega_D + \rho_2 d(hu) \wedge d\theta + \Omega_0 \text{ with } g := \rho_1 - \frac{hu\rho_2'}{2r}.$$

We choose $\rho_1(r) = \rho_2(r) = 1$ for r close to η so that $\tilde{\omega}$ extends $\Psi^*\omega$. Near 0, we choose ρ_1 to be constant and ρ_2 to be quadratic so that $\tilde{\omega}$ makes sense near the center of D. One has $\tilde{\omega}^2 = 2g\,\omega_D \wedge \Omega_0$. Since Ω_0 is symplectic on Σ, the extension $\tilde{\omega}$ is symplectic as soon as g is positive. This condition is easily arranged by choosing ρ_1 sufficiently large away from $r = \eta$.

Because hu is constant on $\partial\Sigma$, $\tilde{\omega}$ restricts as $g\omega_D$ on the part $\partial\Sigma \times D$ of V' which does not come from V. The kernel of ω_D is spanned by ∂_φ so the contact structure described above on V' is weakly filled by $\tilde{\omega}$.

References

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