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Elementary theory of line bundles

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1 Hermitian line bundles

In this text, V is any smooth manifold. Recall that $\pi : L \to V$ is a line bundle if there exists an open cover $\mathfrak{U} = (U_i)$ of V and commuting diagrams:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\varphi_i} & U_i \times \mathbb{C} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U_i & \end{array}$$

such that, for all $U_{ij} = U_i \cap U_j$, the map $\varphi_i \circ \varphi_j^{-1} \colon U_{ij} \times \mathbb{C} \to U_{ij} \times \mathbb{C}$ can be written as $(x, v) \mapsto (x, g_{ij}(x)v)$ for some function $g_{ij} \colon U_{ij} \to \mathbb{C}^*$. Those functions are called transition functions and satisfy the cocycle condition:

$$\begin{aligned} g_{ij} &= g_{ji}^{-1} \quad \text{on } U_{ij} = U_i \cap U_j \\ g_{ij}g_{jk} &= g_{ik} \quad \text{on } U_{ijk} = U_i \cap U_j \cap U_k. \end{aligned} \tag{1}$$

Conversely, any set of functions (g_{ij}) satisfying this condition defines a unique line bundle over V. If they take value in $U(1) \subset \mathbb{C}^*$ then the canonical hermitian product on $\mathbb{C}$ gives rise to a well defined hermitian product on each fiber of π and L is called a hermitian line bundle. For such bundles, we will always use local trivialization φ_i which are unitary in fibers.

In each trivializing open set U_i, a section s is seen as $s_i = \mathrm{pr}_2 \circ \varphi_i \circ s \colon U_i \to \mathbb{C}$. For each U_{ij}, one then has $s_i = g_{ij} s_j$.

Example. On $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ one can use the covering $U_0 = \mathbb{C}$, $U_1 = \mathbb{C}^* \cup \{\infty\}$. The line bundle given by $g_{01}(z) = z^n$ is denoted by $\mathcal{O}(n)$. Its holomorphic sections corresponds to polynomials of degree at most n. The zero set of a transverse section consists of n points.

2 Connections on hermitian line bundles

A connection on L is a map $\nabla \colon \Gamma(L) \to \Gamma(T^*V \otimes L)$ such that, for any section s of L and any complex-valued function f, $\nabla(fs) = f\nabla s + df \otimes s$. It is a hermitian connection if $d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$.

In each trivializing open set U_i, ∇s becomes $ds_i + A_i s_i$ for some (complex-valued) 1-form A_i. The connection is hermitian if and only if all A_i are purely imaginary.

A short computation shows, in U_{ij}, $A_j = A_i + g_{ij}^{-1}dg_{ij}$. From now on, we will assume that $\mathfrak{U}$ is a good cover: all finite intersections of open sets in this cover are contractible. In particular g_{ij} has a logarithm in U_{ij} so $g_{ij}^{-1}dg_{ij}$ is exact hence closed and the local 2-forms dA_i glue consistently to give a closed 2-form F on V. This form is called the curvature of (L, ∇). It does not depend on the choice of $\mathfrak{U}$, as can be seen by using cover refinements.

The curvature is the obstruction to constructing local non-vanishing sections s which are "horizontal": $\nabla s = 0$. Indeed, if s_i is a non-vanishing function on U_i representing some local section s then

$$\nabla s = 0 \iff A_i = -s_i^{-1}ds_i \implies dA_i = 0.$$

Conversely, if A_i is closed (hence exact) then one can foliate $U_i \times \mathbb{C}$ by graphs of horizontal sections so the plane field defined by ∇ on L is not "curved".

3 From line bundles to cohomology

In the following theorem, we use smooth singular homology $H_*(V)$ whose chains are formal linear combinations of smooth maps from simplices to V. In particular one can integrate differential forms on chains. A de Rham cohomology class is called integral if it evaluates to an integer on any cycle with integer coefficients.

If Z^k is a submanifold of V then any $n-k$-cycle is homologous to a linear combination Σ of simplicies meeting Z transversely in their interior. We then have a well-defined intersection number $Z \cdot \Sigma \in \mathbb{Z}$.

Theorem 1. *Let $F = -\mathrm{i}\omega$ be the curvature of a line bundle L with connection ∇. If s is a section of L transverse to the zero section and Z its vanishing locus then the homology class of Z is Poincaré-dual to the de Rham class $[\omega/2\pi]$:*

$$\text{for any 2-cycle } \Sigma \text{ transverse to } Z, \quad \frac{1}{2\pi}\int_\Sigma \omega = Z \cdot \Sigma.$$

In particular $[\omega/2\pi]$ is integral.

Proof. Given a positive number ε, we set $V_\varepsilon = \{|s| \geq \varepsilon\}$. and $\Sigma_\varepsilon = \Sigma \cap V_\varepsilon$. In V_ε, s does not vanish and $\nabla s/s$ is a well defined complex valued 1-form. In addition $-\mathrm{i}\omega = d(\nabla s/s)$.

$$\begin{aligned}\int_\Sigma -\mathrm{i}\omega &= \lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} -\mathrm{i}\omega \\ &= \lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} d\frac{\nabla s}{s} \\ &= \lim_{\varepsilon\to 0}\int_{\partial\Sigma_\varepsilon} \frac{\nabla s}{s}\end{aligned}$$

When ε is small enough, each connected component of $\partial\Sigma_\varepsilon$ is a circle C_ε^x coming from an intersection point $x \in Z \cap \Sigma$ and contained in some U_i. We have

$$\int_{C_\varepsilon^x} \frac{\nabla s}{s} = \int_{C_\varepsilon^x} (s_i^{-1}ds_i + A_i)$$

and A_i is bounded in U_i so it does not contribute to the limit when ε goes to zero.

On the other hand, since s vanishes transversely along Z, one has polar coordinates $(r_1, \theta_1, \dots, r_n, \theta_n)$ near x such that $s_i = r_1 e^{\mathrm{i}\theta_1}$. So we get

$$\int_{C_\varepsilon^x} s_i^{-1} ds_i = \int_{C_\varepsilon^x} d \ln r_1 + \mathrm{i} d\theta_1 = -2\mathrm{i}\pi (Z \cdot \Sigma)_x$$

where $(Z \cdot \Sigma)_x = \pm 1$ is the contribution of x to the signed count of intersection points. The minus sign arises because C_ε^x is oriented as the boundary of Σ_ε whereas $(Z \cdot \Sigma)_x$ is the linking number between Z and the boundary of a small disk surrounding x in Σ. □

Remark. A purely cohomological way of saying that $[Z]$ is Poincaré dual to $[\omega/2\pi]$ in the above theorem is to say that, for any closed form β,

$$\frac{1}{2\pi} \int_V \omega \wedge \beta = \int_Z \beta.$$

This follows from general principles and the above theorem but one can also prove it directly using the same kind of ideas as above.

4 From cohomology to line bundles

Theorem 2. *Let ω be a closed 2-form on V. If $[\omega/2\pi]$ is integral then there is a hermitian line bundle with connection $(L, \nabla, |.|)$ such that $-\mathrm{i}\omega$ is the curvature of ∇. One can enforce as connection 1-forms $A_i = -\mathrm{i}\beta_i$ for any a priori given primitives β_i of ω.*

Proof. It suffices to build transition functions $g_{ij} : U_{ij} \to U(1)$ satisfying the cocycle conditions (Equation (1)) and such that $A_j = A_i + g_{ij}^{-1} dg_{ij}$.

On each U_{ij}, $d(\beta_i - \beta_j) = \omega - \omega = 0$ and U_{ij} is contractible so there is some function $f_{ij} \colon U_{ij} \to \mathbb{R}$ such that $df_{ij} = \beta_i - \beta_j$. On U_{ijk},

$$d(f_{jk} - f_{ik} + f_{ij}) = (\beta_j - \beta_k) - (\beta_i - \beta_k) + (\beta_i - \beta_j) = 0$$

and U_{ijk} is connected so $f_{jk} - f_{ik} + f_{ij}$ is a constant that we denote by a_{ijk}.

Claim. *The hypothesis of integrality of $[\omega/2\pi]$ ensures that one can choose functions f_{ij} above such that all a_{ijk} are in $2\pi\mathbb{Z}$.*

Assuming the above claim for a while, we can then define $g_{ij} = \exp(\mathrm{i} f_{ij})$. These functions satisfy the cocycle condition because all a_{ijk} are in $2\pi\mathbb{Z}$.

Regarding connection 1-forms, we have

$$A_j - A_i = \mathrm{i}\beta_i - \mathrm{i}\beta_j = \mathrm{i} df_{ij} = g_{ij}^{-1} dg_{ij}$$

so we proved the theorem modulo the above claim.

We now prove the claim. In order to get rid of irrelevant factors of 2π, we rename $\omega/2\pi$ as ω and $\beta_i/2\pi$ as β_i. The most naive way of defining functions f_{ij} would be to choose a base point x_{ij} in each non-empty U_{ij} and set $f_{ij}(x) = \int_{x_{ij}}^x (\beta_i - \beta_j)$ (which makes sense and gives the right derivative since $\beta_i - \beta_j$ is closed and U_{ij} is contractible). However this choice leads to some a_{ijk} related to integrals of β_i, β_j and β_k along arcs which do not close up so we cannot even use Stokes's formula to relate a_{ijk} to the integral of ω on some 2-chain. We will remedy this by adding to the naive choice of f_{ij} some constant integrals on arcs.

We denote by N the simplicial complex which as one vertex per open set U_i, an edge for each non-empty U_{ij} and a face for each non-empty U_{ijk}, with the obvious incidence relations. We denote by N' the first barycentric subdivision of N. We now construct a map from N' to V which is smooth on each simplex. We choose base points x_i, x_{ij} and x_{ijk} in each U_i, U_{ij} and U_{ijk} then paths between them and triangles as in Figure 1. We denote by γ_i^{ij} the path chosen

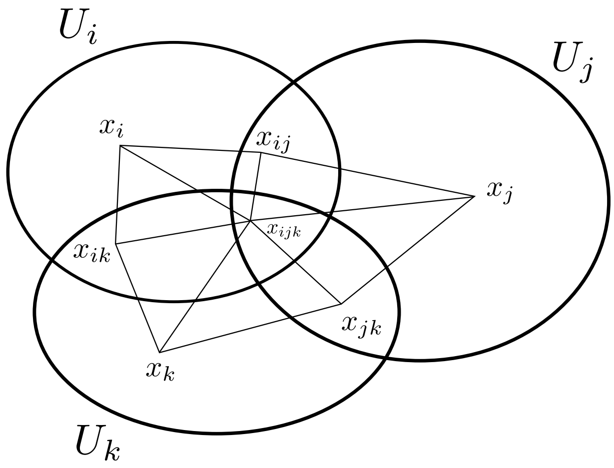

Figure 1: Embedding the complex N'

between x_i and x_{ij}. Stokes's formula guaranties that if we set

$$f_{ij}(x) = \int_{x_{ij}}^{x} (\beta_i - \beta_j) + \int_{\gamma_i^{ij}} \beta_i - \int_{\gamma_j^{ij}} \beta_j$$

then the corresponding number a_{ijk} is the integral of ω on the hexagon of Figure 1. This hexagon is not a cycle so we cannot yet apply the hypothesis on ω. However, integration of ω defines a 2-cocycle ω_N in $C^2(N, \mathbb{R})$, the simplicial cochain complex of N with real coefficients. The hypothesis on $[\omega]$ guaranties that ω_N takes integral values on integral cycles of N. We then need some piece of pure linear algebra.

Sub-claim. *There exists a simplicial 1-cochain b_N such that $\omega_N + db_N$ has integer coefficients.*

Proof of the subclaim. The boundary operator ∂ from 2-chains to 1-chains in N has integer coefficients in the canonical bases so the reduction theory for integer matrices gives an automorphism A of $C_2(N, \mathbb{R})$ with integer coefficients which sends $\{0\} \times \mathbb{R}^d$ to $\ker \partial$. We set $E = A(\mathbb{R}^{n-d} \times \{0\})$ so that $C_2(N, \mathbb{R}) = \ker \partial \oplus E$ and the projection to $\ker \partial$ in this decomposition sends integral chains to integral chains. Let p be any projection from $C_1(N, \mathbb{R})$ onto $\operatorname{im} \partial$. Let $\varphi\colon\ \operatorname{im} \partial \to E$ be the inverse of $\partial\colon\ E \to \operatorname{im} \partial$. We set $b_N = -\omega_N \circ \varphi \circ p$. For any σ in $C_2(N, \mathbb{Z})$,

$\sigma = \sigma_0 + \sigma_E$ where σ_0 is in $\ker \partial$, σ_E in E and,

$$\begin{aligned}
(\omega_N + db_N)(\sigma) &= \omega_N(\sigma) + b_N(\partial\sigma) \\
&= \omega_N(\sigma) - \omega_N \circ \varphi \circ p(\partial\sigma) \\
&= \omega_N(\sigma) - \omega_N(\sigma_E) \\
&= \omega_N(\sigma_0) + \omega_N(\sigma_E) - \omega_N(\sigma_E) \\
&= \omega_N(\sigma_0) \in \mathbb{Z} \qquad \square
\end{aligned}$$

Finally we add to our previous choice of functions f_{ij} the value b_{ij} of b_N on the edge of N corresponding to U_{ij}. $\square$