On Liouville pairs in dimension 5

Patrick Massot

January 3, 2013

Abstract

We classify Lie groups in dimension 5 which admit a Liouville pair and a cocompact lattice.

Theorem. The only simply connected Lie groups in dimension 5 which have a cocompact lattice and a left invariant Liouville pair are the one considered in [MNW].

According to [DF, Theorem 3.1] there are exactly seven simply connected 5-dimensional Lie groups having a left invariant contact form and a cocompact lattice. We will use the notations in [DF] to name them. Two of them, D18 and D20, are considered in [MNW]. We want to prove that the other ones have no invariant Liouville pair. We now list those groups with a basis $(\alpha_1, \cdots, \alpha_5)$ of their spaces of invariant 1-forms and non-vanishing exterior derivatives.

- D1 where
$$d\alpha_1 = \alpha_4 \wedge \alpha_2 + \alpha_5 \wedge \alpha_3$$
- D2 where
$$d\alpha_1 = \alpha_4 \wedge \alpha_3 + \alpha_5 \wedge \alpha_2 \quad \text{and} \quad d\alpha_2 = \alpha_5 \wedge \alpha_3$$
- D3 where
$$d\alpha_1 = \alpha_4 \wedge \alpha_3 + \alpha_5 \wedge \alpha_2 \quad \text{and} \quad d\alpha_2 = \alpha_5 \wedge \alpha_3 \quad \text{and} \quad d\alpha_3 = \alpha_5 \wedge \alpha_4$$
- D5 where
$$d\alpha_1 = \alpha_3 \wedge \alpha_2 + \alpha_5 \wedge \alpha_4 \quad \text{and} \quad d\alpha_2 = \alpha_5 \wedge \alpha_2 \quad \text{and} \quad d\alpha_3 = \alpha_3 \wedge \alpha_5$$
- D11 where
$$d\alpha_1 = \alpha_3 \wedge \alpha_2 + \alpha_5 \wedge \alpha_4 \quad \text{and} \quad d\alpha_2 = \alpha_3 \wedge \alpha_5 \quad \text{and} \quad d\alpha_3 = \alpha_5 \wedge \alpha_2$$

In all cases, we notice that $\mathrm{vol} := \alpha_1 \wedge d\alpha_1^2$ is a volume form and:

$$\forall \gamma \in \mathrm{span}(\alpha_2, \cdots, \alpha_5), \qquad d\gamma^2 = 0, \quad \gamma \wedge d\alpha_1^2 = 0 \quad \text{and} \quad d\alpha_1 \wedge d\gamma = 0. \quad (*)$$

Suppose now we have a Liouville pair $\alpha_\pm$. We write $\alpha_\pm = a_\pm \alpha_1 + \gamma_\pm$ where $\gamma_\pm$ is in $\mathrm{span}(\alpha_2, \cdots, \alpha_5)$. Our Liouville form on $\mathbb{R} \times G$ is $\beta = e^s \alpha_+ + e^{-s} \alpha_-$ as usual.

Using Equation $(*)$, one computes first that:

$$\alpha_\pm \wedge d\alpha_\pm^2 = a_\pm^3 \mathrm{vol}$$

hence a_+ is positive and a_- is negative. Then one computes:

$$d\beta^3 = (e^s a_+ - e^{-s} a_-)(e^s a_+ + e^{-s} a_-)^2 \, ds \wedge \mathrm{vol}.$$

So the symplectic condition is violated when $s = \frac{1}{2} \ln(-a_-/a_+)$.

References

[DF] A. Diatta and B. Foreman, *Lattices in contact lie groups and 5-dimensional contact solvmanifolds*, arxiv:0904.3113.

[MNW] P. Massot, Klaus N., and C. Wendl, *Weak and strong fillability of higher dimensional contact manifold*, arxiv:1111.6008, Invent. Math Online first.