SFT 6 Precourse 1

Rough lecture notes

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1 Contact structures on 3-manifolds

A plane field ξ on a 3–manifold V is a (smooth) map associating to each point p of V a 2-dimensional subspace $\xi(p)$ of T_pV. All plane fields considered here will be coorientable, it means one can continuously choose one of the half spaces cut out by $\xi(p)$ in T_pV. In this situation, ξ can be defined as the kernel of some nowhere vanishing 1–form α: $\xi(p) = \ker \alpha(p)$. The coorientation is given by the sign of α. We will always assume that V is oriented. In this situation a coorientation of ξ combines with the ambient orientation to give an orientation on ξ.

1.1 The canonical contact structure on the space of contact elements

Let S be a surface and $\pi : ST^*S \to S$ the bundle of cooriented lines tangent to S. It can be seen as the bundle of rays in T^*S, hence the notation. The canonical contact structure on ST^*S at a point d is defined as the inverse image under π_* of $d \subset T_{\pi(d)}S$, see Figure 1.

Suppose first that S is the torus $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. Let x and y be the canonical $\mathbb{S}^1$-valued coordinates on T^2. A cooriented line tangent to T^2 at some point (x, y) can be seen as the kernel of some 1–form λ which has unit norm with

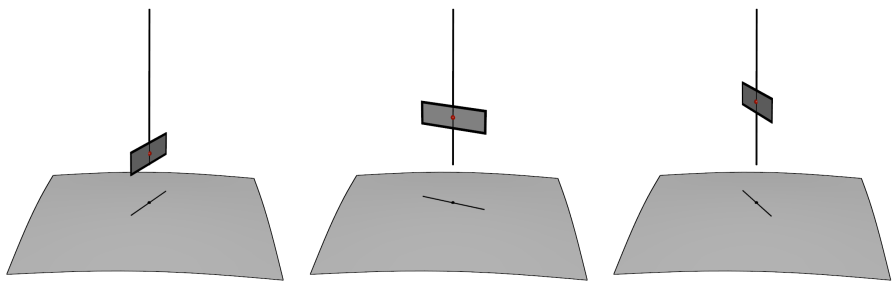

Figure 1: Canonical contact structure on the bundle of cooriented lines. At bottom is a portion of S with a tangent line at some point. Above that point one gets the fiber by gluing top and bottom of the interval. The contact structure is shown at the point of the fiber corresponding to the line drawn below.

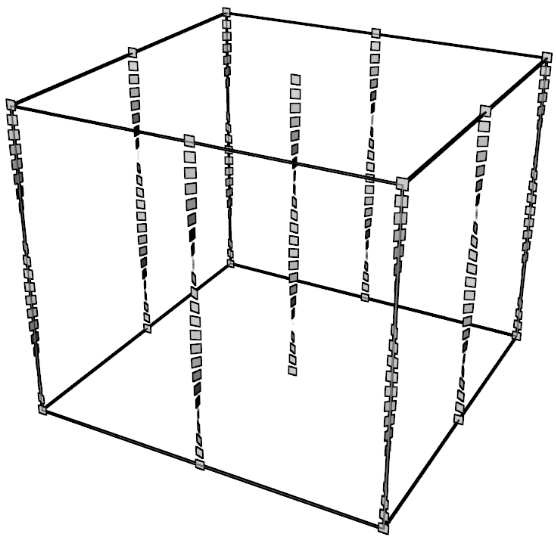

Figure 2: Canonical contact structure on T^3. Opposite faces of the cube are glued to get T^3

respect to the canonical flat metric. So there is some angle z such that $\lambda = \cos(z)dx - \sin(z)dy$. So we have a natural identification of ST^*T^2 with T^3. In addition the canonical contact structure can be defined by $\cos(z)dx - \sin(z)dy$ now seen as a 1–form on T^3 called the canonical contact form on T^3, see Figure 2.

When S is the sphere $\mathbb{S}^2$, ST^*S can be identified with $\mathbb{R}P^3$ so there is a two-fold covering map from S^3 to ST^*S. The lifted contact structure is called the canonical contact structure on $\mathbb{S}^3$. Later we will explain a way of thinking of this contact structure as the boundary of the standard symplectic 4-ball.

1.2 Contact structures and contact forms

Definition 1. *A* contact structure *on a 3–manifold is a plane field which is locally diffeomorphic to the canonical contact structure on $ST^*\mathbb{R}^2$. A* contact form *is a 1–form whose kernel is a contact structure. A curve or a vector field is* Legendrian *if it is tangent to a given contact structure.*

As noted above all our manifolds will be oriented and diffeomorphisms in the above definition shall preserve orientations and coorientations of plane fields.

Theorem 1 (Darboux–Pfaff theorem)**.** *A 1–form α is a contact form if and only if $\alpha \wedge d\alpha$ is a positive volume form.*

The condition $\alpha \wedge d\alpha > 0$ will henceforth be called the contact condition for α.

Proof. We denote by ξ the kernel of α. If ξ is a contact structure then the image of α in the local model is $f\alpha_0$ where f is some nowhere vanishing function and $\alpha_0 = \cos(z)dx - \sin(z)dy$. So

$$\begin{aligned}\alpha \wedge d\alpha &= f\alpha_0 \wedge (f d\alpha_0 + df \wedge \alpha_0) = f^2 \alpha_0 \wedge d\alpha_0 \\ &= f^2\, dx \wedge dy \wedge dz\end{aligned}$$

which is a positive volume form.

Conversely, suppose $\alpha \wedge d\alpha$ is positive. Let p be a point in M. We want to construct a coordinate chart around p such that $\xi = \ker(\cos(z)dx - \sin(z)dy)$. We first choose a small surface S containing p and transverse to $\xi = \ker \alpha$. Then we pick a non-singular vector field X tangent to S and ξ near p and a small curve c in S containing p and transverse to X, see Figure 3. Let y be

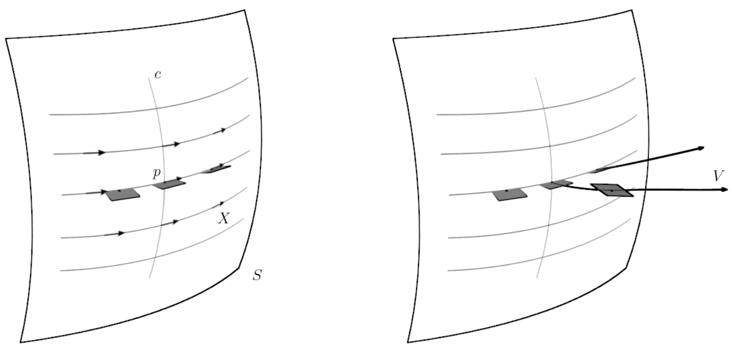

Figure 3: Proof of the Darboux–Pfaff theorem

a coordinate on c. The flow of X at time x starting from c gives coordinates (x, y) on S near p in which $X = \partial_x$.

We now consider a vector field V transverse to S and tangent to ξ. The flow of V at time t starting from S gives coordinates (x, y, t) near p such that $\alpha = f(x, y, t)dx + g(x, y, t)dy$ because $\alpha(\partial_t) = \alpha(V) = 0$. Up to rescaling, one can use instead $\alpha_1 = \cos z(x, y, t)dx - \sin z(x, y, t)dy$ for some function z such that $z(x, y, 0) = 0$. Now is time to use the contact condition. We can compute

$$\alpha_1 \wedge d\alpha_1 = \frac{\partial z}{\partial t} dx \wedge dy \wedge dt$$

so the implicit function theorem guaranties that we can use z as a coordinate instead of t. $\square$

In the above proof, $z(x, y, t)$ was the angle between ξ and the horizontal ∂_x is the plane normal to the Legendrian vector field ∂_t. We saw that the contact condition forces this angle to increase. This means that the contact structure rotates around ∂_t. The above proof essentially says that this rotation along Legendrian vector fields characterizes contact structures.

We now focus on the difference between contact structures and contact forms. The data of a contact form is equivalent to a contact structure and either a choice of a Reeb vector field or a section of its symplectization.

Definition 2. *A Reeb vector field for a contact structure ξ is a vector field which is transverse to ξ and whose flow preserves ξ.*

If one has a Riemannian metric on a surface S then the bundle of contact elements of S can be identified with the unit tangent bundle STS and the geodesic flow is then a Reeb vector field for the canonical contact structure.

One can easily prove that each contact form α comes with a canonical Reeb vector field R_α which is characterized by $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$. All Reeb vector fields arise this way.

Next, recall that for any co-oriented hyperplane field ξ on a manifold V, one can consider the annihilator of ξ in T^*V:

$$S\xi := \left\{ \lambda \in T^*V \;\middle|\; \ker \lambda = \xi \text{ and } \lambda(v) > 0 \text{ if } v \text{ is positively transverse to } \xi \right\}.$$

The field ξ is a contact structure if and only if $S\xi$ is a symplectic submanifold of $(T^*V, \omega_{\text{can}})$, and in this case $S\xi$ is called the *symplectization* of ξ. Any contact form α is a section of this $\mathbb{R}$-bundle, and thus determines a trivialization $\mathbb{R} \times V \to S\xi$ given by $(t, v) \mapsto (v, e^t\alpha(v))$. In this trivialization, the restriction of the canonical symplectic form ω_{can} becomes $d(e^t\alpha)$.

2 Isotopic contact structures and Gray's theorem

Clearly we want to consider two contact structures to be the same if they are conjugated by some diffeomorphism. One can restrict this by considering only diffeomorphism corresponding to deformations of the ambient manifold. An isotopy is family of diffeomorphisms φ_t parametrized by $t \in [0, 1]$ such that $(x, t) \mapsto \varphi_t(x)$ is smooth and $\varphi_0 = Id$. The time-dependant vector field generating φ_t is defined as $X_t = \frac{d}{dt}\varphi_t$. One says that two contact structures ξ_0 and ξ_1 are isotopic if there an isotopy φ_t such that $\xi_1 = (\varphi_1)_*\xi_0$. In particular such contact structures can be connected by the path of contact structures $\xi_t := (\varphi_t)_*\xi_0$. It is then natural to consider the seemingly weaker equivalence relation of homotopy among contact structures. The next theorem says in particular that, on closed manifolds, this equivalence relation is actually the same as the isotopy relation.

Theorem 2 (Gray)**.** *For any path $(\xi_t)_{t\in[0,1]}$ of contact structures on a closed manifold, there is an isotopy φ_t such that $\varphi_t^*\xi_t = \xi_0$.*

The vector field X_t generating φ_t can be chosen in $\lim_{\varepsilon\to 0} \xi_t \cap \xi_{t+\varepsilon}$ at each time t.

Proof. The proof of this theorem can be found in many places but without much geometric explanations so we now explain the semi-heuristic picture behind it. The key is to be able to construct an isotopy pulling back $\xi_{t+\varepsilon}$ to ξ_t for infinitesimally small ε. It means we will construct the generating vector field X_t rather than φ_t directly. The compactness assumption will guaranty that the flow of X_t exists for all time.

At any point p, if the plane $\xi_{t+\varepsilon}$ coincides with ξ_t then we have nothing to do and set $X_t = 0$. Otherwise, these two planes intersect transversely along a line $d_{t,\varepsilon}$. The natural way to bring $\xi_{t+\varepsilon}$ back to ξ_t is to rotate it around $d_{t,\varepsilon}$. Since we know from the proof of Theorem 1 that the flow of Legendrian vector fields

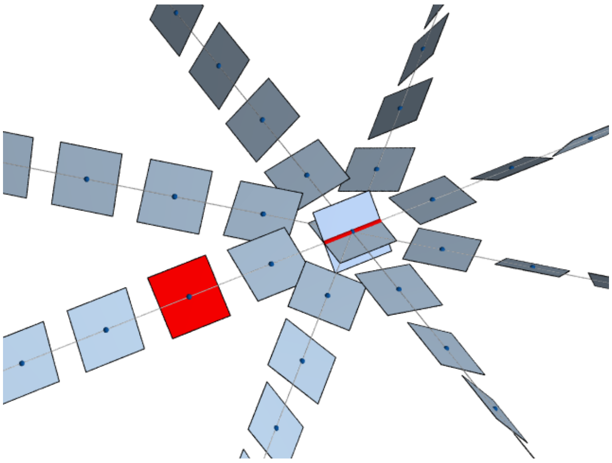

Figure 4: Proof of Gray's theorem

rotate the contact structure, we will choose X_t in the line $d_t := \lim_{\varepsilon \to 0} d_{t,\varepsilon}$, see Figure 4. Let us compute $d_{t,\varepsilon}$:

$$d_{t,\varepsilon} = \{v \mid \alpha_{t+\varepsilon}(v) = \alpha_t(v) = 0\} = \{v \in \xi_t \mid \tfrac{1}{\varepsilon}(\alpha_{t+\varepsilon} - \alpha_t)(v) = 0\}$$

which gives, as ε goes to zero: $d_t = \xi_t \cap \ker(\dot{\alpha}_t)$.

The contact condition for α_t is equivalent to the fact that $(d\alpha_t)_{|\xi_t}$ is non-degenerate. So X_t belongs to $\xi_t \cap \ker(\dot{\alpha}_t)$ if and only if it belongs to ξ_t and $\iota_{X_t} d\alpha_t = f_t \dot{\alpha}_t$ on ξ_t for some function f_t.

Moreover, we want X_t to compensate the rotation expressed by $\dot{\alpha}_t$. A natural guess is then to pick X_t such that $\iota_{X_t} d\alpha_t = -\dot{\alpha}_t$.

We now have a precise candidate for X_t and we can compute to prove that it does the job. □

3 Symplectic fillings and cobordisms

3.1 Definitions

Definition 3. *Let V be a closed oriented 3-manifold with a positive and co-oriented contact structure ξ. Let W be a compact symplectic 4-manifold such that $\partial W = V$ as oriented manifolds. We say that (W, ω) is a*

- weak filling *of (V, ξ) if $\omega|_\xi$ is positive*
- strong filling *of (V, ξ) if and ω admits a primitive λ (a* Liouville form*) near ∂W which restricts to V as a contact form for ξ.*
- exact filling *of (V, ξ), or a* Liouville domain *with boundary (V, ξ), if the Liouville form λ extends globally over W.*

The definitions above are ordered from the weakest to the strongest. We will consider two examples of exact fillings. The first one is the ball $B^4 \subset R^4$

with Liouville form $xdy + ydx$ which is an exact filling of the standard contact structure on the sphere $\mathbb{S}^3$.

The second example is the disk cotangent bundle $DT^*S = \{(q,p)\,; q \in S, p \in T^*_q S, \|p\| \leq 1\}$ where we use an auxiliary metric on a surface S. The Liouville form is the canonical form $\lambda = pdq$. The boundary is the contact manifold ST^*S which, as the notation suggests, can be seen as the space of cooriented contact elements of S. In particular we get an exact filling of the canonical contact structure on T^3. This contact structure is the first one in the family

$$\xi_n := \ker\bigl(\cos nz\; dx - \sin nz\; dy\bigr)\;, n \geq 1.$$

Eliashberg proved in [Eli96] that ξ_1 is the only strongly fillable contact structure in this family, this will be explained in Chris Wendl's lectures. However, Giroux previously observed they are all weakly fillable. The product symplectic form ω_p on $T^2 \times D^2$, which is very different from the canonical symplectic structure on T^*T^2, is obviously positive on the foliation $\ker dz$ of T^3 if z is seen as the angular coordinate on D^2. Each contact structure in the above family is isotopic to a contact structure which is C^∞–close to $\ker dz$. Indeed

$$\xi_n^s := \ker\bigl(sdz + \cos nz\; dx - \sin nz\; dy\bigr)\;, s \in [0, S]$$

defines a homotopy of contact structures that can be converted to an isotopy by Gray's theorem. Since positivity of ω_p on a plane field is an open condition, the contact structure ξ_n^s is weakly filled when s is large enough. Actually, closer inspection reveals that any positive s is enough.

This way of proving weak fillability can be generalized to other torus bundles over the circle. There is an alternative way of seeing it which can be generalized to circle bundle over surfaces (or orbifold surfaces). The crucial observation is that the kernel of the restriction of ω_p to T^3 is spanned by ∂_z. Taking orientations into account, one sees that a contact structure on T^3 is weakly filled by $(T^2 \times D^2, \omega_p)$ if and only if it is positively transverse to ∂_z. Let X be a Legendrian vector field for ξ_n which is tangent to $\ker dz$. The contact condition guaranties that the flow of X rotates ξ around X. Hence it instantaneously isotopes ξ to a contact structure transverse to ∂_z (here one has to choose to flow in forward or backward time to get positive transversality).

The above definitions can be adapted to the setting of cobordisms. A cobordism from V_- to V_+ is a compact symplectic manifold W such that $\partial W = -V_- \sqcup V_+$ where $-V_-$ denotes V_- with reversed orientations and we can ask for the same relation between symplectic and contact as in the absolute case, hence getting the notions of weak, strong and exact cobordisms. One says that V_+ is the positive or top or convex end of the cobordism and V_- is the negative or bottom or concave end. A important example of symplectic cobordism will be provided by Legendrian surgery.

3.2 Legendrian surgery

In differential topology, surgery of index $k-1$ on a m-manifold is an operation which removes a domain $\mathbb{S}^{k-1} \times \mathbb{D}^{m-k+1}$ called the attaching region and replaces it with $\mathbb{D}^k \times \mathbb{S}^{m-k}$ (those domains have the same boundary). This operation comes with a cobordism from the original manifold to the new one obtained from the thickening $M \times [0,1]$ by attaching the so-called handle $\mathbb{D}^k \times \mathbb{D}^{m-k+1}$

along $(\mathbb{S}^{k-1} \times \mathbb{D}^{m-k}) \times \{1\}$. The submanifold $\mathbb{D}^k \times \{0\}$ is called the core of the handle. It is attached to the sphere $\mathbb{S}^{k-1} \times \{0\}$ inside the original manifold.

In contact topology, Legendrian surgery on a contact $(2n-1)$-manifold (M, ξ) is surgery along a Legendrian sphere S^{n-1} where the handle is symplectic and its core is Lagrangian. Weinstein's tubular neighborhood theorem for Lagrangian submanifolds naturally leads to think of the handle $D^n \times D^n$ as the unit disk cotangent bundle DT^*D^n (with respect to the flat metric on D^n). The boundary of this handle is $DT^*D|_S \cup ST^*D$. We want the attaching region $DT^*D|_S$ to be concave and the new region ST^*D to be convex so, instead of the standard Liouville form on T^*D, we use $\lambda_H = 2pdq + qdp$.

Weinstein's theorem for Legendrian submanifolds allows to get the neighborhood[1] $DT^*S \times [-1, 1]$ for the Legendrian sphere S in M. In this model the contact form is $dt + \lambda$ where t is the coordinate in $[-1, 1]$ and λ is the canonical Liouville form on T^*S so we have part of the contactization of (T^*S, λ). We want to glue the attaching region of the symplectic handle onto this region. For this and for considerations of Reeb dynamics later, we use the Euclidean structure on $\mathbb{R}^n$ to identify cotangent bundles of D and S with their tangent bundle. We also use standard embeddings of S and D into $\mathbb{R}^n$ to get the isomorphism $TD|_S = TS \oplus \nu S$ where νS is the normal bundle of S in $\mathbb{R}^n$ with outward orientation. Since this normal bundle is trivial, we get a diffeomorphism from $DTD|_S$ to $N = \{((q, u), t) \in TS \times [-1, 1] \ ; \|u\|^2 + t^2 \leq 1\}$. This identification is the handle attaching map.

When $n = 2$ the inverse of this map is especially easy to write down using the parametrization of TS^1 by $\mathbb{R}/(2\pi)\mathbb{Z} \times \mathbb{R}$ and of $\mathbb{R}^2$ by $\mathbb{C}$:

$$\Phi(\theta, y, t) = \left(e^{i\theta},\, ie^{i\theta}y + e^{i\theta}t\right) \qquad (\theta, y) \in TS, \quad t \in [-1, 1]$$

The above formula allows to check very easily that

$$\Phi^*\lambda_H = dt + yd\theta$$

so our attaching map is indeed a contactomorphism.

We now want to understand the effect of this surgery on the contact manifold (M, ξ). A convenient way of seeing it is through Reeb dynamics, see Figure 5. In $TS \times [-1, 1]$ the Reeb vector field of $dt + \lambda$ is ∂_t so Reeb trajectories are vertical segments. Suppose now we perform surgery and start a Reeb orbit at $(q, u, -1)$. This orbit hits the boundary of the attaching region at $(q, u, -\sqrt{1 - \|u\|^2})$. There it gets identified with a unit vector w tangent to D at q. More precisely, $w = u + v$ where $v \in \nu S$ is inward pointing $\|u + v\| = 1$. The Reeb orbit then goes through the new region STD following the geodesic flow there. It leaves this region at some point $(q', w) \in STD|_S$ where it gets identified back to the projection u' of w onto $T_{q'}S$. So the Reeb orbit exits the attaching region at $(q', u', \sqrt{1 - \|u'\|^2})$. The map $(q, u) \mapsto (q', u')$ is called the symplectic Dehn twist (or Dehn-Arnold-Seidel twist).

In dimension 3 in particular, the attaching region is a solid torus which is replaced by another solid torus and one can see the gluing back map as defined on the double of an annulus as the identity on one annulus and a Dehn twist on the other one. In general, one can describe the effect of Legendrian surgery on (M, ξ) as follows. There is a unique function h on DT^*S which vanishes

[1]Strictly speaking, one get norm ε instead of 1 in general but we will ignore this detail.

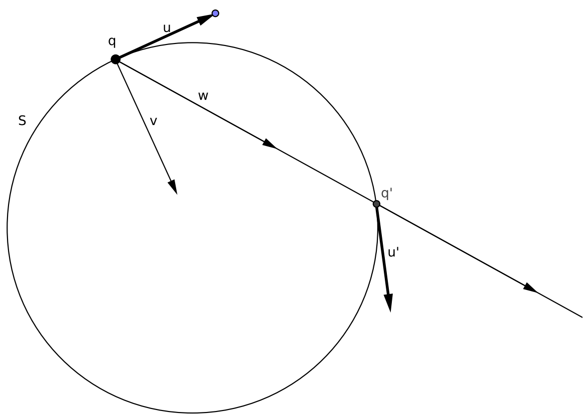

Figure 5: The Dehn twist on DTS^1

along the boundary and satisfies $\tau^*\lambda = \lambda - dh$ where τ is the Dehn twist. In dimension 3, one can check that $\tau(q,p) = (q+(p+1)\pi, p)$ and $h(q,p) = (1-p^2)\frac{\pi}{2}$. In particular $((q,p),t) \mapsto (\tau(q,p), t + h(q,p))$ is a contactomorphism. So one can cut open the region $DT^*S \times [-1,1]$ along $DT^*S \times \{0\}$ and then glue in $\{((q,p),t)\ ; t \in [0, h(q,p)]\}$ by the obvious map on $t = 0$ and the Dehn twist on the other side.

4 The tight vs overtwisted dichotomy

Definition 4 (Eliashberg)**.** *A contact manifold is* overtwisted *if it contains an embedded disk along which the contact structure is as in Figure 6. Otherwise it is called tight.*

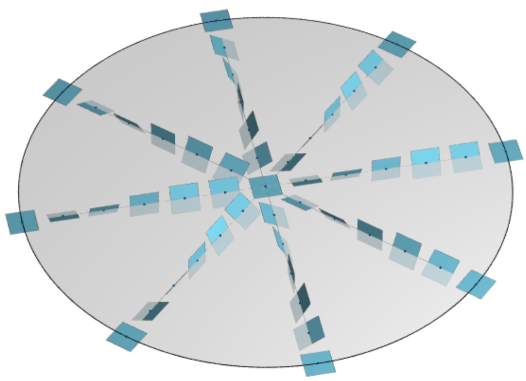

Figure 6: An overtwisted disk

Any contact structure can be modified to become overtwisted using an operation called the (half) Lutz twist. Given a knot K transverse to ξ, one can find a neighborhood $K \times \varepsilon D^2$ with coordinates (z, r, θ) where $\xi = \ker dz + r^2 d\theta$. Any thickened torus $\eta \leq r \leq 2\eta$ is isomorphic to some $T^2 \times [a, b]$ in the standard T^3. One can then replace this piece with a larger $T^2 \times [a', b']$ so that the contact structure rotates half a turn between K and the boundary of a solid torus $K \times \varepsilon' D^2$. The result is called a Lutz tube, it's an S^1-family of Figure 6.

The class of overtwisted contact structures is truly remarkable. They have special properties with respect to fillings, Reeb dynamics and flexibility. Sam Lisi will prove in his lectures the following result.

Theorem 5 (Eliashberg-Gromov [Gro85,Eli90])**.** *Fillable contact structures are tight.*

In particular one recovers Bennequin's theorem [Ben83] saying the standard contact structures on $\mathbb{R}^3$, T^3 or $\mathbb{S}^3$ are tight. This theorem shows in particular that the standard contact structure on $\mathbb{R}^3$ is not isomorphic to the overtwisted structure of Figure 6, although they are homotopic through plane fields. This contrasts sharply with the following result.

Theorem 6 (Eliashberg [Eli89])**.** *If two overtwisted contact structures on a closed manifold are homotopic through plane fields then they are homotopic through contact structures, hence isotopic.*

From this theorem, it follows that any kind of subtle invariant of contact structures like *SFT* or *HF*/*ECH*/*SWF* should completely miss overtwisted contact structures: there is nothing to see there. Indeed, *SFT* vanishes and the contact invariant is trivial in the other theories.

We will focus on one example of contact structures which are homotopic through plane fields. On T^3, one can consider for any positive integer n, the contact structure $\xi_n = \ker(\cos(nz)dx - \sin(nz)dy)$. They are all homotopic to the same (integrable) plane field $\ker dz$, hence homotopic to each other. One possible homotopy consists in rotating ξ_n around $\xi_n \cap \big(T^2 \times \{z\}\big)$ to make it tangent to all tori $T^2 \times \{z\}$. Initially ξ_n is perpendicular to those tori and we can rotate through an angle $s\frac{\pi}{2}$ where $s \in [0, 1]$ is the homotopy parameter. This is the same homotopy as in the weak filling discussion but pushed all the way to the foliation $\ker dz$. Giroux [Gir99] (and then Kanda [Kan97] using different techniques) proved that the structures ξ_n are pairwise non-homotopic among contact structures.

As a variant of the above situation, consider the same plane fields on $T^2 \times [0, 1]$. By reparametrizing slightly the interval, one can arrange that all ξ_n coincide near the boundary and the above homotopy can then be cutoff to prove that all ξ_n are homotopic relative to the boundary. In particular, this prove that full Lutz twists along transverse knots (i.e. applying two half Lutz twists) do not change the homotopy class of a contact structure. So Theorem 6 proves that any overtwisted contact manifold contains a Lutz tube since one can perform a full Lutz twist in the complement of an overtwisted disk without changing the isotopy class of contact structure (there is a more explicit way of exhibiting a Lutz tube in any neighborhood of an overtwisted disk but it requires more technology).

We start our sketch of proof of Theorem 6 with a proposition which holds in any contact manifold. Recall that the characteristic foliation of a surface S

in a contact manifold (V, ξ) is the (singular) foliation ξS tangent to $\xi \cap TS$ and singular where $\xi = \pm TS$.

Proposition 7 ([Eli89])**.** *Let ξ_s, $s \in [0, 1]$ be a homotopy of plane fields on a closed manifold V such that ξ_0 and ξ_1 are contact. Suppose K_s is a family of compact subsets in V such that ξ_s is contact on K_s and $K_0 = K_1 = V$. Then the family ξ_s is homotopic relative to $\cup K_s$ to a homotopy of plane fields such that, for some ball B in V with boundary S and for every s:*

- *the characteristic foliation $\xi_s S$ has exactly two singularities $p_s^\pm$ where $\xi_s = \pm TS$;*
- *along any closed leaf in $\xi_s S$, ξ_s is cooriented towards p_s^+;*
- *ξ_s is contact on $V \setminus B$.*

Very rough sketch of proof of Proposition 7. We will construct B as an embedded boundary connected sum of myriads of small balls. All construction below work continuously with respect to the parameter s so we suppress it from the notations. The key idea is that, in a chart, if S is a strictly convex sphere which is sufficiently small compared to variations of ξ then the first two properties above automatically hold. The second one is intuitively clear. To prove the first one, we consider the Gauss map $G_S : S \to \mathbb{S}^2$ given by the outward normal to S. By convexity of S, this is a diffeomorphism. If S is sufficiently small then its curvature is high so $\|DG_S^{-1}\|$ is small and the map $\pm G_\xi \circ G_S^{-1}$ comparing the normals to S and ξ is a contraction so there is exactly one fixed point $G_S(p_\pm)$. We will call p_+ the North pole and p_- the South pole.

Of course, if the ball B is contained in a region where ξ is integrable like the one appearing in the above homotopy on $T^2 \times [0, 1]$ then the singular foliation ξS will look like the left-hand side of Figure 7. This ξ cannot be perturbed to become contact in a neighborhood of S while keeping the same foliation. However we can first perturb ξ near S to get the right-hand side of Figure 7. The

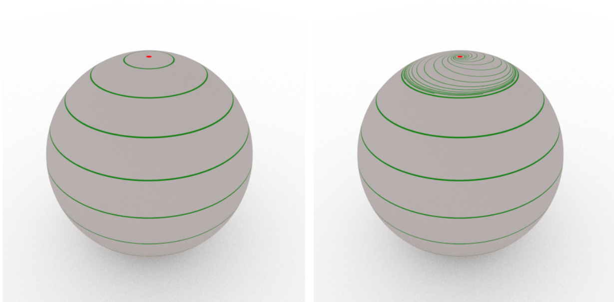

Figure 7: Local perturbation of singularities

plane field ξ can then be perturbed to become contact on some neighborhood $U(S)$ keeping ξS. Suppose now that V is filled by small balls B_i so that the neighborhoods $U(S_i)$ cover $V \setminus \bigcup B_i$. One can then use arcs a_i positively transverse to ξ in $V \setminus \bigcup B_i$ to get a chain of balls, each arc connecting a North pole to a South pole. The ball B is then a small regular neighborhood of the union of all B_i's and a_i's.

Warning: actually proving this proposition is a lot more painful than the previous sketch, see [Eyn09, Chapter 8]. □

The above proposition didn't assume existence of an overtwisted disk. The reason why this proposition alone can't get flexibility is there is no way the characteristic foliation on the right-hand side of Figure 7 can be the foliation printed on S by some contact structure defined on the whole ball B bounded by S.

Suppose now that Δ is an overtwisted disk for ξ_0 and ξ_1. It can always be arranged by homotopy that all ξ_s agree near Δ. Applying Proposition 7 with K_s containing Δ, we can then connect B with a regular neighborhood of Δ to get a new ball B' which have the same properties than B except that two closed leaves, coming from Δ, have the wrong orientation. A typical movie of $\xi_s B'$ is shown in Figure 8. The magic of Eliashberg's argument is this movie is indeed

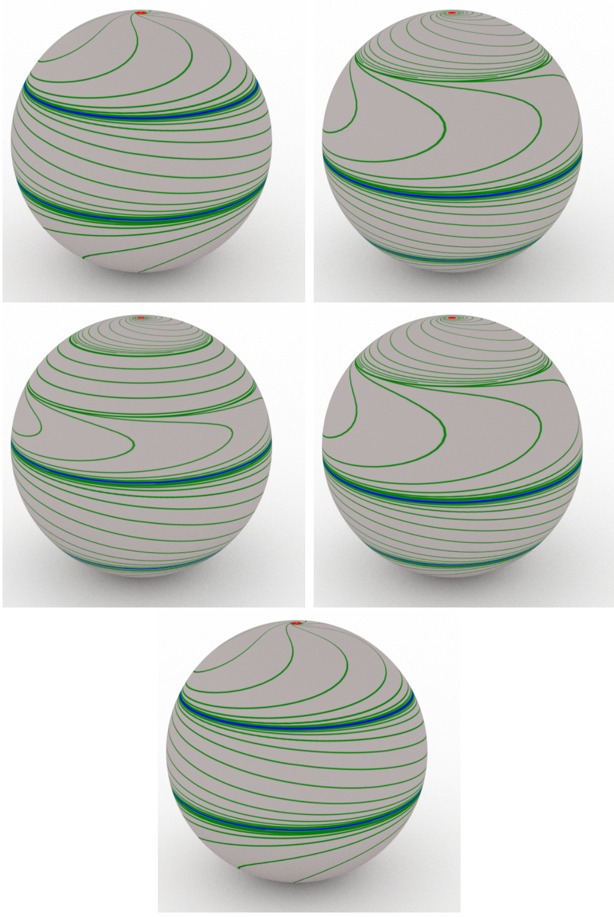

Figure 8: A typical movie of foliations $\xi_s B'$. First B has a North-South dynamics hence contributes nothing. Then a closed leaf appears in $\xi_s B$ then a whole interval of them. Then it disappears the same way it came.

the movie of foliations printed on the boundary of a family of balls B_s^{model} living in any neighborhood of Figure 6. So one can throw away the restriction of ξ_s to B' and replace it with the restriction of ξ_{OT} to B_s^{model} (which is homotopic to the thrown away family). The balls B_s^{model} are bounded by surfaces of revolution obtained from the curves of Figure 9.

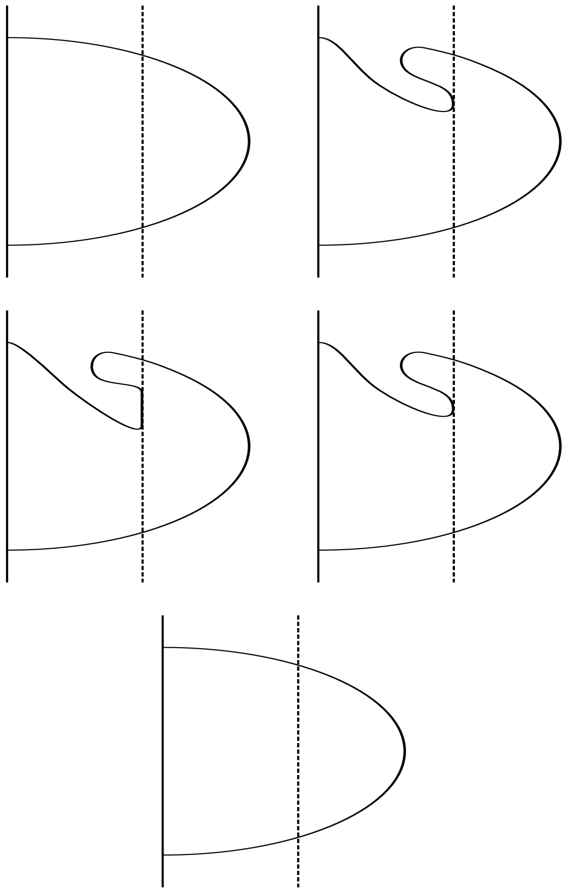

Figure 9: The family of spheres corresponding to Figure 8. The solid vertical line is the rotation axis. The dashed one indicates the cylinder where ξ is horizontal again.

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